# HSC Mathematics Extension 2

Steve Howard

Under construction - currently including:

Chapter 1 Introduction to complex numbers
Chapter 2 The nature of proof (no revision yet)
Chapter 4 Further integration

A textbook for the Mathematics Extension 2 (2024) NSW Syllabus

Last updated: October 2026
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This version first published in 2026-2027 based on versions first published in 2019-2020 by Steve Howard - Howard and Howard Education.

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Published by
Steve Howard
Willowvale Road
COWRA NSW 2794 Australia

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**About the Author**

Steve is a Mathematics Teacher, having taught at Cowra High School for 30 years, and now mainly working from home creating Mathematics resources but also teaching extension courses online with Mount View High School. He has also taught gifted and talented students online through xsel and Aurora College, and is a Senior Marker for the Extension 2 HSC.

He has a particular passion for Mathematics Extension 2. This textbook reflects many years of developing teaching resources and approaches for the course. The textbook is completely free - see howardmathematics.com for the most up-to-date version. The version of the textbook for the 2024 syllabus combines the original textbook with the 1000 Revision Questions, expanded and adjusted as needed.

Steve writes commercial online professional development courses for teachers covering the NSW Senior Mathematics courses through Teacher Training Australia (TTA). The online courses for teachers cover this material in much greater depth and include thousands of recorded examples. For the courses currently offered see stevehoward.tta.edu.au. He also writes the Trial HSCs and Year 11 Final Exam papers for TTA, also available at the same website.

Steve loves finding more efficient techniques for solving mathematical questions, by trawling through other teachers' solutions or making up his own approaches when there must be a better way. Many of the approaches you see here are unique, with emphases on both understanding the work, plus little tricks to help students succeed in exams.

Steve did 4 Unit Mathematics as a student way back in 1988, gaining a mark of 198/200 and training as an actuary. Working in an office in the city wasn’t for him, so he went back to uni to retrain as a teacher, then headed to the country, where he lives in an owner-built mud house, with chickens, goats and parrots (which you will sometimes hear in his recordings)!!

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## Introduction

Welcome! I hope all students and teachers find this book useful and enjoyable in your journey in Extension 2 Maths. This textbook will be continually changed and updated over time. For a free download of the latest version, please visit howardmathematics.com.

This textbook is primarily a passion project. It is supported indirectly through school copyright licensing and by increasing traffic to my commercial online professional development courses for teachers through TTA.

Some of the features of this textbook that students and teachers may find useful:

- The questions are in the style of past HSC questions where possible, so students are learning the right type of questions throughout the course.
- The questions are graded in difficulty so students can work at their own level.
- Each exercise starts with Basic questions largely matching the examples, so students can ease themselves into the new concepts if they need to. More confident students can skip these first questions and start with Medium and then Challenging.
- Each chapter has 100 revision questions, largely based on the matching exercise from the 1000 Revision Questions from the 2017 syllabus.
- All questions have fully worked solutions.
- There are lots of diagrams that help students understand concepts that would normally only be dealt with algebraically.
- There are many hints and tips to help students answer questions more efficiently, while also highlighting common techniques and recurring tricks used in HSC exam questions.

The textbook should be set out over roughly 44 lessons - the final number will be determined by how the material is best bundled into lessons. The aim is for students to be able to finish the course as soon as possible, so that there are months available for revision before the HSC to consolidate and master the content.

If you find any mistakes or have any ideas that would make the textbook better, please contact me via email below.

Cheers
Steve Howard
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Thanks to all the teachers and students who have let me know errors or improvements for the textbook or 1000 Revision Questions for the 2017 syllabus, which have flowed through to this updated textbook.

Thank you to the following students and teachers for letting me know of errors or improvements for the 2024 version of the textbook: Bryce T, Angela R, Wafik Z.

# Chapter 1
## *Introduction to complex numbers*

Complex numbers is a great topic for Extension 2 as:

- it provides us with a completely new type of number system to wrap our heads around
- it closely links geometry and algebra, but in unexpected ways
- it introduces us to a branch of mathematics that seems to be purely theoretical yet has massively important practical uses
- it stretches our imaginations as we try to visualise how numbers that seemingly don't exist behave under different conditions.

**Lessons**

1.1 A background to complex numbers

1.2 Cartesian form of a complex number

1.3 Modulus and square roots in Cartesian form

1.4 Modulus-argument form of a complex number

1.5 Conjugate theorems

1.6 Complex numbers as vectors

1.7 Lines, curves and regions

1.8 de Moivre's theorem

1.9 $n^{\text{th}}$ roots of complex numbers

Chapter 1 - 100 revision questions

# 1.1 A background to complex numbers

In Lesson 1 we cover:

- How complex numbers fit with other number systems
- Imaginary numbers

Syllabus reference: This lesson covers the 1st to 7th dot points from Arithmetic of complex numbers, from the 1st dot point from Geometric representation of complex numbers, from the 1st and 2nd dot points in Solving equations with complex numbers, and from the 5th and 6th dot points in Powers and roots of complex numbers, in *Introduction to complex numbers*.

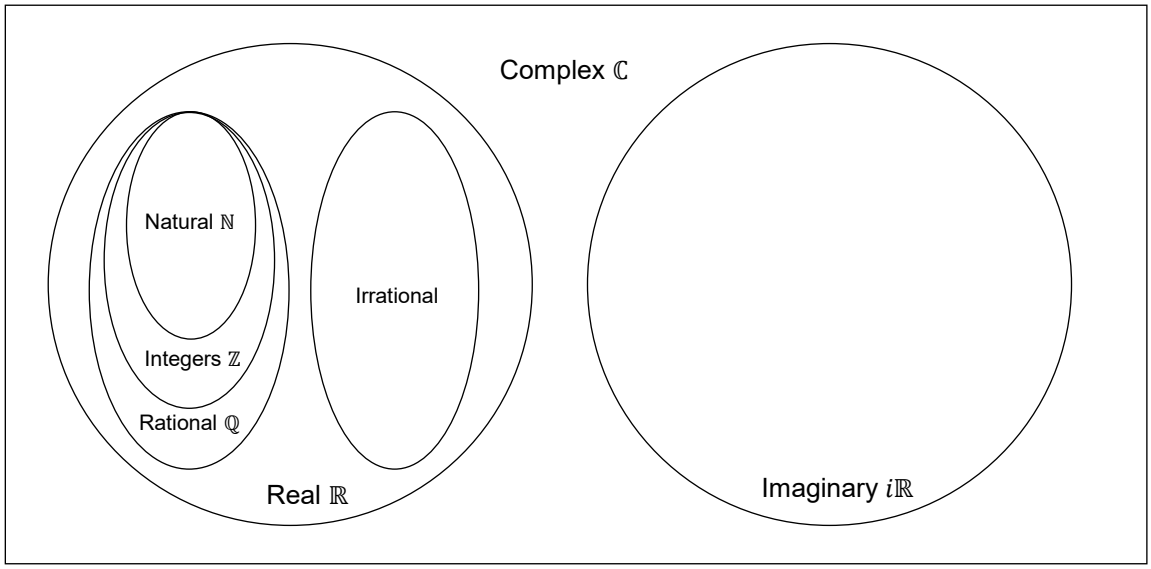
## How complex numbers fit with other number systems

We first learned about natural numbers $\mathbb{N}$ $(1,2,3\ldots)$ when we were growing up. Over the years we then added numbers like zero and the negative numbers to create the integers, $\mathbb{Z}$. The natural numbers are the positive integers which are often written as $\mathbb{Z}^+$ rather than $\mathbb{N}$. We then added fractions (including terminating or recurring decimals) to create the rational numbers, $\mathbb{Q}$. All of the rational numbers can be written as fractions.

The next step was to learn that there are numbers that cannot be written as fractions, such as $\pi$ or $\sqrt{2}$. These are the irrational numbers (non-recurring decimals), which together with rational numbers, form the larger group real numbers, $\mathbb{R}$. A number cannot be both rational and irrational, but it can have a rational and an irrational part, for example, the real number $1+\sqrt{3}$.

You are about to learn that there is another set of numbers, imaginary numbers, $i\mathbb{R}$, that together with real numbers form the larger group complex numbers, $\mathbb{C}$. The set notation $i\mathbb{R}$ is not used in the syllabus, but all the others listed above are. Imaginary numbers are most often expressed as the square root of a negative number, such as $\sqrt{-4}$, or written as multiples of $i$, such as $2i$. We will look at what $i$ represents shortly.

A number cannot be real and imaginary, but it can have a real and an imaginary part, for example, $1+\sqrt{3}i$.




So, the number $1$, the first number we learned about as children, is natural, an integer, rational, real and complex, but it is not irrational or imaginary.

All our work in mathematics up to now has solely involved the real number system. We most commonly use the pronumeral $x$ to represent an unknown real number. When we work with complex numbers, we most commonly use the pronumeral $z$ to represent a complex number.

Complex numbers, $z$, can be written in the form $\mathrm{z} = a + ib$, where $a$ and $b$ are real numbers.

- We refer to $a$ as the real part of the complex number $z = a + ib$ and use the notation $\mathrm{Re}(z)$.
- We refer to $b$ as the imaginary part of the complex number $z = a + ib$ and use the notation $\mathrm{Im}(z)$. Note that the imaginary part of $z$ is $b$, NOT $ib$.

Complex numbers can also be written in modulus-argument form which we look at starting in Lesson 3, and exponential form which is no longer in the syllabus.

When we are placing a complex number on a complex plane, which we will look at shortly, we will often use the form $z = x + iy$.

Complex numbers are:

- purely real if $b = 0$
- purely imaginary if $a = 0$.

A special case is zero, which is written as $0 + 0i$. This is the only complex number that is purely real and purely imaginary.

## Example 1

Consider the following sets: natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), real numbers ($\mathbb{R}$) and complex numbers ($\mathbb{C}$). Classify each of the following numbers using the smallest possible set.

**a** $7.2$

This number is a terminating decimal and can be written as a fraction.
It is rational, real and complex. The smallest set is $\mathbb{Q}$.

**b** $-4$

This number is a negative integer. It is an integer, rational, real and complex.
The smallest set is $\mathbb{Z}$.

**c** $\sqrt{100}$

$\sqrt{100} = 10$.
This number is a positive whole number. It is natural, an integer, rational, real and complex. The smallest set is $\mathbb{N}$.

**d** $6i$

This is a purely imaginary number. It is imaginary and complex.
From the options listed, the smallest set is $\mathbb{C}$.

**e** $\sqrt{3}$

This is a non−recurring decimal. This number is irrational, real and complex. The smallest set is $\mathbb{R}$.

**f** $i \sin \frac{\pi}{6}$

$i \sin \frac{\pi}{6} = \frac{i}{2}$.
This number is imaginary and complex. From the options listed, the smallest set is $\mathbb{C}$.

**g** $2 - \sqrt{3}$

This number includes a rational and irrational part.
It is real and complex. The smallest set is $\mathbb{R}$.

**h** $5 + \sqrt{-3}$

This number includes a real and imaginary part. It is complex. The smallest set is $\mathbb{C}$.

## Imaginary numbers

Although what we now know as imaginary numbers have been commonly accepted for a few hundred years, the associated difficulties were encountered by the Ancient Greeks.

French mathematician and philosopher Rene Descartes ('I think, therefore I am') helped develop analytical geometry including the Cartesian plane, allowing us to link geometry and algebra, and also derisively named numbers involving the square root of negatives as 'imaginary'. It is a poor choice of name that misleads us to this day.

He didn't think negative numbers existed either and called solutions which were negative 'false'!

We should also note that the 'complex' in complex numbers means 'made of parts' rather than 'complicated' - another misleading name. A complex number is made of a real part and an imaginary part.

So, keep in mind that complex numbers are not hard and they do exist.

Imaginary numbers are a bit like irrational numbers, negative numbers and even zero - they are non-tangible numbers that have useful properties. You cannot hold $\sqrt{2}$ marbles, $-2$ marbles, $0$ marbles or $2i$ marbles in your hand, but irrational, negative, zero and imaginary numbers allow us to solve more difficult problems.

At various points in history in different cultures all of these non-tangible numbers have been seen as fictitious or useless!

**Definition of $i$**

The definition of $i$ from the syllabus is $i^2 = -1$, so $i$ is one of the square roots of $-1$.

The definition is not the more commonly recognised $i = \sqrt{-1}$, which is not mentioned in the syllabus at all. It is no big deal if you use both definitions for $i$, as we will certainly gain a better understanding from $i = \sqrt{-1}$.

Note that using the square root sign with negative numbers is problematic. The square root sign means the principal root, for example, the principal square root of positive $1$ is $\sqrt{1} = 1$, and the other square root is $-\sqrt{1} = -1$.

When we take the square roots of $-1$, which we may write as $\pm\sqrt{-1}$, it is slightly problematic as there is no clear agreement on which root is the principal root. Obviously if we are looking at both square roots it makes no difference which is the principal root.

It is important to note that $i$ is similar to a surd, so there are some similarities between the properties of complex numbers and surds, but not all the rules for complex numbers follow the surd rules.

So, let's look at a few examples of using complex numbers algebraically only, before we look at a geometric explanation to improve our understanding.

## Example 2

**a** Factorise $z^2 + 9$.

$$\begin{aligned} & z^2 + 9 \\ &= z^2 - 9i^2 \\ &= z^2 - (3i)^2 \\ &= (z + 3i)(z - 3i) \end{aligned}$$

**b** Find the square roots of $-9$.

Let $z$ be a square root of $-9$

$$\therefore z^2 = -9$$

$$z^2 + 9 = 0$$

$$\therefore (z + 3i)(z - 3i) = 0$$

$$z = \pm 3i$$

Check: $(3i)^2 = 9i^2 = 9\times(-1) = -9$

$(-3i)^2 = 9i^2 = 9\times(-1) = -9$

**Some things to note for later**

The square roots of a complex number (including the square roots of a purely real or purely imaginary number) are the first of the $n^{\text{th}}$ roots that we will see in the chapter. The last lesson looks at cube roots, fourth roots etc. in great detail. When we take the $n^{\text{th}}$ root of any number except zero, there are $n$ distinct roots - in the case of square roots where $n = 2$, there are two square roots for any number except zero.

As we will see later in the course, when the coefficients of a polynomial equation are all real ($1$ and $9$ in the quadratic equation $z^2 + 9 = 0$ above), then any non-real solutions occur in conjugate pairs. Two complex numbers are conjugates if their real parts are equal and their imaginary parts have the same magnitude but are of opposite sign - they can be written in the form $a + ib$ and $a - ib$. The two solutions, $-3i$ and $3i$ are complex conjugates, as they can be written as $0 - 3i$ and $0 + 3i$. We start looking at conjugates in more detail in Lesson 2, and they are an incredibly powerful tool throughout the chapter.

## Example 3

**a** Factorise $z^2 + 4z + 10$.

$$\begin{aligned} & z^2 + 4z + 10 \\ &= z^2 + 4z + 4 + 6 \\ &= (z+2)^2 + 6 \\ &= (z+2)^2 - 6i^2 \\ &= (z + 2 + \sqrt{6}i)(z + 2 - \sqrt{6}i) \end{aligned}$$

**b** Solve $z^2 + 4z + 10 = 0$.

$$z^2 + 4z + 10 = 0$$

$$\left(z + 2 + \sqrt{6}i\right)\left(z + 2 - \sqrt{6}i\right) = 0$$

$$z = -2 \pm \sqrt{6}i$$

*Alternatively:*

$$z = \frac{-4 \pm \sqrt{4^2 - 4\times1\times10}}{2\times1}$$

$$= \frac{-4 \pm \sqrt{-24}}{2}$$

$$= \frac{-4 \pm 2\sqrt{-6}}{2}$$

$$= \frac{-4 \pm 2\sqrt{6}i}{2}$$

$$= -2 \pm \sqrt{6}i$$

Again we can see that the coefficients of the polynomial equation $z^2 + 4z + 10 = 0$ are all real ($1, 4$ and $10$), and the non-real solutions have occurred as a conjugate pair, since $-2 - \sqrt{6}i$ is the conjugate of $-2 + \sqrt{6}i$.

## Example 4

Solve $z^2 + 4iz + 5 = 0$**:**

**a** by completing the square

$$\begin{aligned} & z^2 + 4iz + 5 = 0 \\ & z^2 + 4iz + (2i)^2 - (2i)^2 + 5 = 0 \\ & (z + 2i)^2 + 4 + 5 = 0 \\ & (z + 2i)^2 + 9 = 0 \\ & (z + 2i)^2 - 9i^2 = 0 \\ & (z + 2i + 3i)(z + 2i - 3i) = 0 \\ & (z + 5i)(z - i) = 0 \\ & z = -5i, i \end{aligned}$$

**b** by factorising

$$\begin{aligned} & z^2 + 4iz + 5 = 0 \\ & z^2 + 4iz - 5i^2 = 0 \\ & (z + 5i)(z - i) = 0 \\ & z = -5i, i \end{aligned}$$

A couple of things to note from this example.

- Here the coefficients are not all real ($1$, $4i$ and $5$), so any non-real solutions do not necessarily occur in conjugate pairs. The two solutions, $-5i$ and $i$ are not conjugates.
- One method was much more efficient than the other. As you go through the Extension 2 course you will see this frequently. One of the biggest differences between students who gain an E4 and those who are limited to an E3 is how efficiently they can handle the question up to say Q14, which determines how much time they have left for the harder questions in the paper.

**Geometric explanation of $i$**

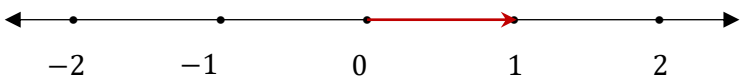
We have just used $i$ in calculations, without really understanding what it is. Let's have a look at a geometric explanation of $i$ to improve our understanding of what we are doing.

Up until now the one-dimensional number line has been adequate for all our work with real numbers, but it has also restricted our thinking unnecessarily.

Let's have a look at the equation $x^2 = -1$, and make sure we understand what squaring, the minus sign and $1$ really mean.

**The number $1$**

Well, the $1$ is easy - it just means one unit more than zero, which we can represent on the number line as one unit to the right, or by an arrow (vector) stretching from $0$ to $1$.




**The minus sign**

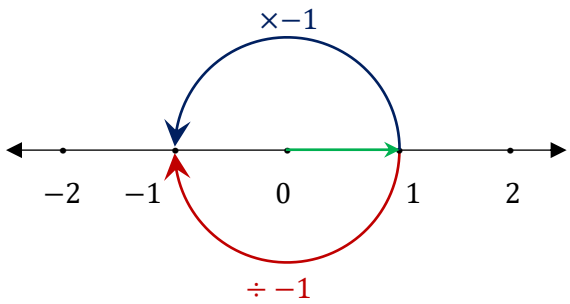
Now with the minus sign we see a restriction to our thinking caused by the number line.

When we place a minus in front of any real number then the only one-dimensional operation we can perform is a reflection about $0$, so $-1$ is the reflection (or opposite) of $1$.



But what if we weren't restricted to movement in one dimension? Is there another operation that could take us to the same place?

The answer is yes, because a rotation either anticlockwise or clockwise through $180°$ would get us to the same point.




Now to be consistent with our work from other topics, we will define multiplying by $-1$ as causing an anticlockwise rotation by $180°$. We can also define dividing by $-1$ as causing a clockwise rotation by $180°$.

We can view $-1$ as $1\times(-1)$, where $1$ is our starting point and multiplying by $-1$ is the operation of rotating $180°$ anticlockwise.

**Squaring**

Squaring normally means that a number is multiplied by itself, but if $x$ represents an operation then it means that the same operation will be repeated twice in a row.

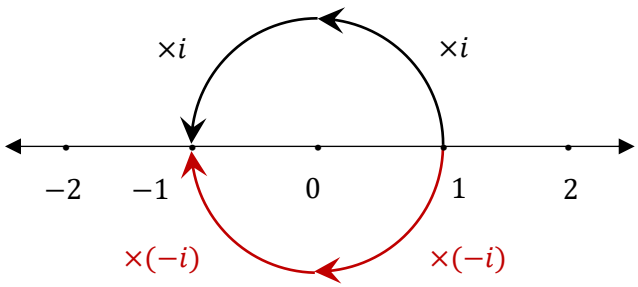
To solve $x^2 = -1$, we first view the equation as $1\times x\times x = -1$. So starting at $1$, what operation can we perform twice to end up at $-1$?

Reflection about $0$ will not work, as reflecting twice brings us back to our starting point of $1$.

There are two possible solutions:

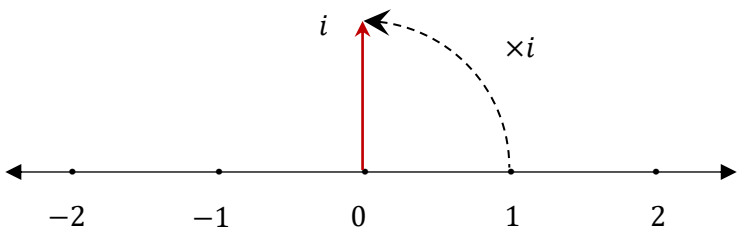
- $x$ could represent a $90°$ anticlockwise rotation
- $x$ could represent a $90°$ clockwise rotation

Now either one is just as valid to be the definition of $i$, but by tradition we let $i$ represent the anticlockwise rotation. There is no reason we couldn't have defined it the other way around, but it is more consistent with our work in other topics.




So if the anticlockwise rotation is $i$, what is the clockwise rotation? It is $-i$, as $(-i)^2 = i^2 = -1$. The two solutions to $x^2 = -1$ are $i$ and $-i$.

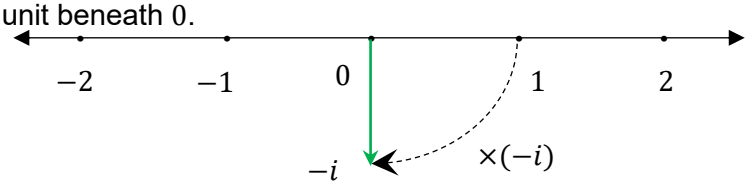
Now $i = 1 \times i$, so starting with the vector to $1$ and rotating it $90°$ anticlockwise, we see that $i$ must be a number that is off the number line, $1$ unit above $0$.




So multiplication by $i$ rotates a complex number anticlockwise by $\frac{\pi}{2}$, while division by $i$ rotates a complex number clockwise by $\frac{\pi}{2}$. Adding or subtraction by $i$ moves a complex number one unit perpendicular to the number line, up or down respectively.

It is the ease with which complex numbers can either move objects at right angles or rotate them that makes them so important in science.

Similarly $-i$ is one unit beneath $0$.




Later in the lesson we will extend the number line to form the complex plane, but I always like showing $i$ as a point somewhere off the number line so that we can better see the link between the number line and the complex plane, rather than being confused between the Cartesian plane and the complex plane.

**Surd laws**

We mentioned that complex numbers act like surds in some cases.

It is important to note that our surd laws only work with complex numbers when certain restrictions are put in place. Since these restrictions are beyond the course it is safest to simply avoid the surd laws.

As an example of how the surd laws don't always work with complex numbers, consider this false proof:

$$\begin{aligned} -1 &= i^2 \\ &= i \times i \\ &= \sqrt{-1} \times \sqrt{-1} \\ &= \sqrt{(-1) \times (-1)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

**Real world uses of complex numbers**

Like many concepts in mathematics complex numbers have been found to have uses far beyond those they were initially created for.

Complex numbers can be used:

- As a quicker way to solve questions that could still be solved using only real numbers - this is the most common application
- To simplify calculations for determining the position of objects that are rotating or being transformed
- To solve equations where the first and last steps are purely real but intermediary steps can only be solved using complex numbers
- In a small number of cases to model situations that could not be handled by real numbers

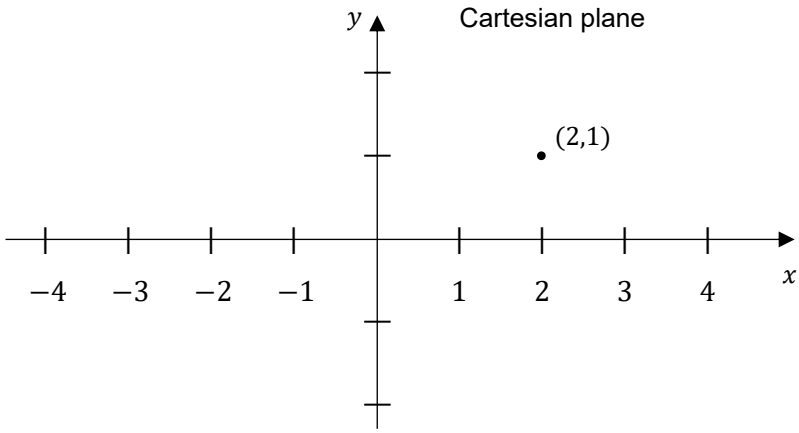
Complex numbers are particularly useful as:

- Circular motion can be modelled quite easily, so the study of any object that involves waves or oscillations, such as electricity, electromagnetism, earthquakes, sound waves and light waves is simpler.
- Anything with two real components can be represented as one complex number, with one real number assigned to the real component and the other to the imaginary, such as the state of an electric circuit which is defined by voltage $V$ and current $I$ and can be represented by $z = V + Ii$ (normally using $j$ instead of $i$ to avoid confusion).
- Anything with two components that are at right angles to each other can be represented as one complex number, since the real and imaginary components are at right angles, such as a force being split into its vertical and horizontal components.
- Objects which repeatedly change direction can be modelled by imaginary numbers, since $i^2 = -1$ represents a change of direction, such as for alternating current in electricity

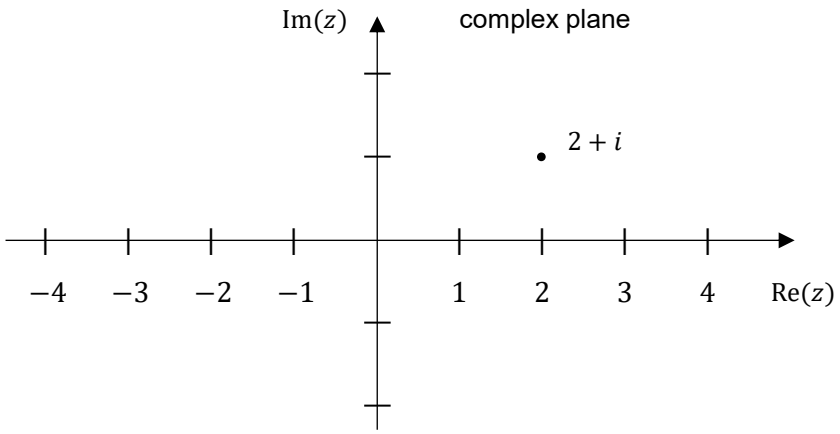
Most technology relies on complex numbers for its invention or to speed up calculations. If you are taking a selfie (CCD) on your phone while texting (signal), then switch to a game stored on your phone (magnetic disk and graphics), under a fluorescent light (particle control), then you are surrounded by advances that would not be possible without complex numbers.

We have barely scratched the surface of how we use complex numbers in the real world.

Although the complex plane looks like a Cartesian plane, it is more closely related to the number line and there are some important differences.




Each point on the Cartesian plane above represents two different real numbers that are somehow related, like the point $(2,1)$ shown above. This could be the input and output of a function, like $y = x^2 - 3$, or the distance travelled and time taken.




Each point on the complex plane above represents one number which has a real and an imaginary part, so $(2,1)$ now represents the single complex number $2 + i$.

Where the two axes cross on the complex plane is the number zero, rather than the pair of zeros $(0,0)$, although we may still refer to it as the origin.

The complex plane has lots of interesting properties, and although it is in many ways more related to a number line, it shares properties with the Cartesian plane and the unit circle.

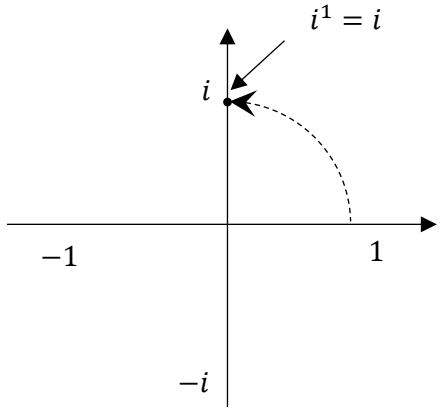
The complex plane is sometimes called the Argand diagram, after one of its early proponents.

**Powers of $i$**

Powers of $i$ can be simplified algebraically or geometrically. Algebraically we can use index laws to break the powers into $i, i^2$ and $i^4$, noting $i^2 = -1$ and $i^4 = \left(i^2\right)^2 = 1$. We see a cyclical pattern repeating every four integers, $i, -1, -i, 1$, representing the intersections of the axes with the unit circle.

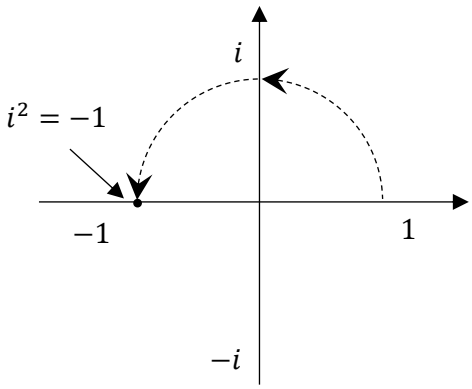
$i^1 = i$

It represents one anticlockwise $90°$ turn




$i^2 = -1$

It represents two anticlockwise $90°$ turns



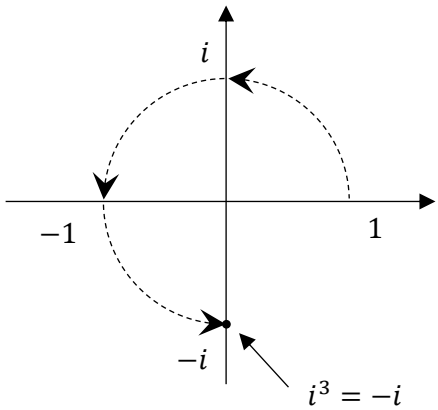

$i^3 = i^2 \times i$

$= -1 \times i$

$= -i$

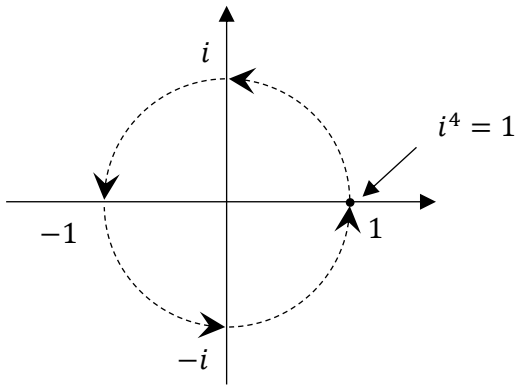
It represents three anticlockwise $90°$ turns

Note that since $i^3 = -i$, we will view multiplication by $-i$ as one clockwise $90°$ turn



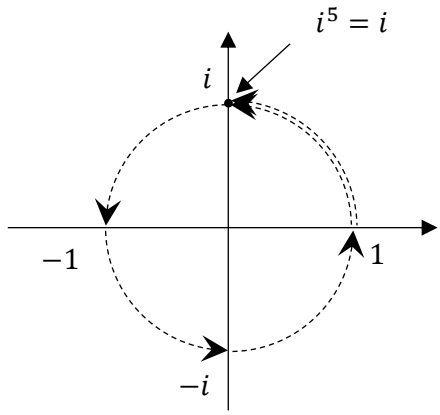
$$\begin{aligned} i^4 &= \left(i^2\right)^2 \\ &= (-1)^2 \\ &= 1 \end{aligned}$$

It represents four anticlockwise $90°$ turns, bringing us back to $1$.




$$\begin{aligned} i^5 &= i^4 \times i \\ &= 1 \times i \\ &= i \end{aligned}$$

It represents five anticlockwise $90°$ turns, ending up in the same place as only one turn.




Higher powers of $i$ continue to cycle around the circle, a property that makes calculations with complex numbers so useful in some fields.

If the power is:

- a multiple of $4$ then it is equal to $1$
- one more than a multiple of $4$ then it is equal to $i$
- two more than a multiple of $4$ and it is equal to $-1$
- three more than a multiple of $4$ and it is equal to $-i$.

We can also use negative powers of $i$ with similar results, as the point cycles clockwise in a similar manner.

We can simplify powers of $i$ by looking at how many revolutions have occurred, and where we end up on the unit circle.

## Example 5

Simplify the following powers of $i$:

**a** $i^{10}$

$= (i^4)^2 \times i^2$
$= 1^2 \times -1$
$= -1$

Ten lots of $90°$ is $2.5$ revolutions, ending at $-1$.

**b** $i^{99}$

$= (i^4)^{24} \times i^3$
$= 1^{24} \times -i$
$= -i$

$99$ lots of $90°$ is $24.75$ revolutions, ending at $-i$.

**c** $i^{-1}$

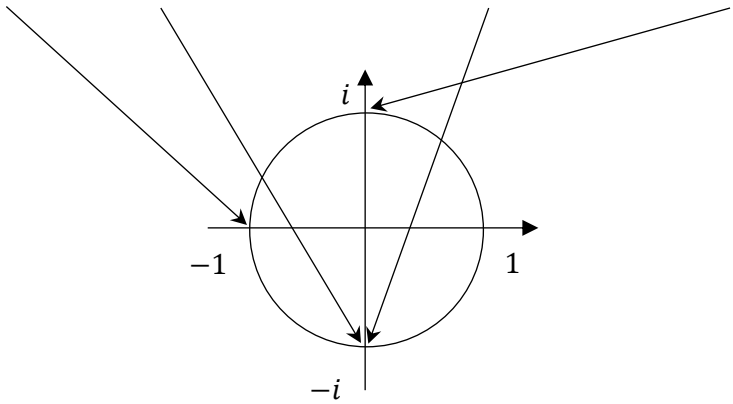
$= (i^2)^{-1} \times i$
$= (-1)^{-1} \times i$
$= \frac{1}{-1} \times i$
$= -i$

$1$ lot of $90°$ backwards ends at $-i$.

**d** $i^{-11}$

$= (i^4)^{-3} \times i$
$= 1^{-3} \times i$
$= i$

$11$ lots of $90°$ backwards is $-2.75$ revolutions ending at $i$.




## Example 6

Simplify:

**a** $i^6 + i^4$

$= i^4(i^2 + 1)$
$= i^4(-1 + 1)$
$= 0$

**b** $i^3(i^5 + i^8)$

$= i^8(1 + i^3)$
$= (i^4)^2(1 - i)$
$= 1 - i$

We have seen that $i$ and $-i$ are the two square roots of $-1$, and that we can find them as $90°$ rotations clockwise or anticlockwise from $1$, since $\frac{1}{2}\times 180° = 90°$. Notice that they are spaced equally around the unit circle.

Using a similar method, where are the three cube roots $-1$? Not surprisingly, since $\frac{1}{3}\times 180° = 60°$, we can find two of the cube roots by rotating $60°$ clockwise or anticlockwise from $1$. But where is the third cube root of $-1$?

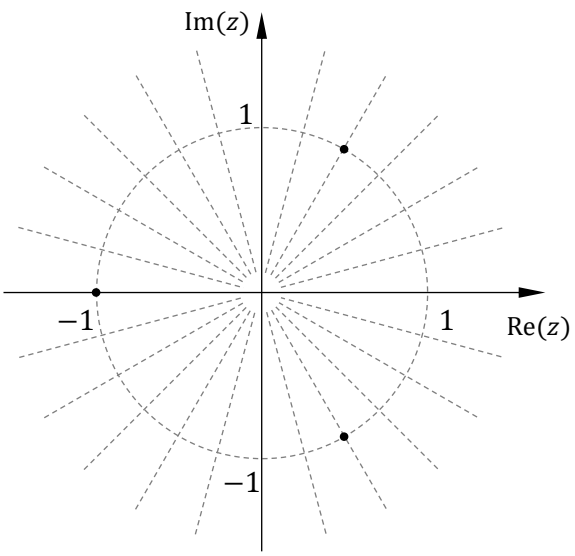
We need to realise that rotating $180°$ is equivalent to rotation $180° + 360° = 540°$, so the third cube root can be found by rotating $\frac{1}{3}\times 540° = 180°$ clockwise or anticlockwise, and so $-1$ is a cube root of $-1$. Notice that this is the single answer we get if we had asked to find the cube root of $-1$ before we started thinking in terms of complex numbers. We will see shortly that the three cube roots are equally spaced around the unit circle as well, as are all $n^{\text{th}}$ roots of any complex number.

Later in the chapter we will look at $n^{\text{th}}$ roots in a more formal manner.

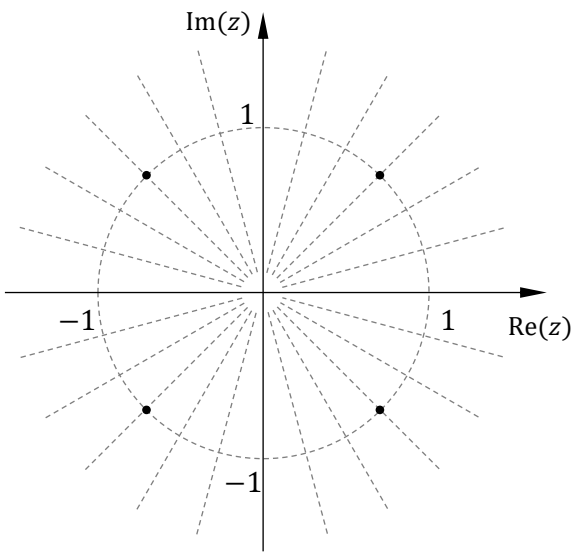
## Example 7

The dashed lines on the complex planes below are spaced every $15°$. On the complex planes below mark the following points:

**a** the cube roots of $-1$




**b** the fourth roots of $-1$



Basic

**1a** Factorise $z^2 + 25$.

**b** Hence find the square roots of $-25$.

**2a** Factorise $z^2 + 2z + 5$.

**b** Hence solve $z^2 + 2z + 5 = 0$.

**c** Solve $z^2 + 2z + 5 = 0$ using the quadratic formula.

**3** Factorise $z^2 + 6iz + 7$:

**a** by completing the square

**b** by factorising

**c** Hence solve $z^2 + 6iz + 7 = 0$.

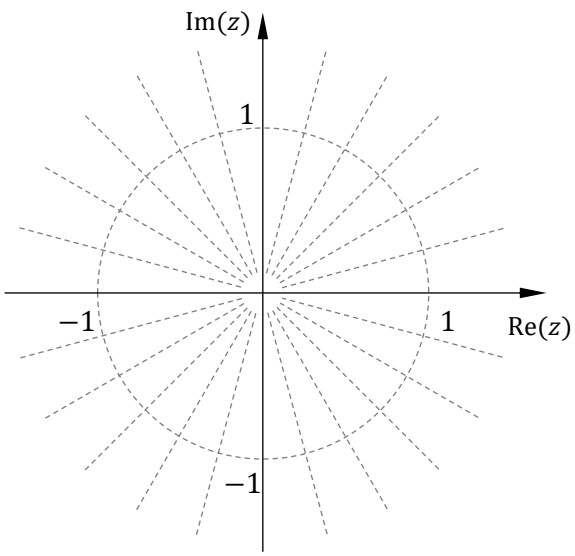
**4** Simplify the following powers of $i$:

**a** $i^6$ **b** $i^{400}$ **c** $i^{63}$ **d** $i^{-3}$ **e** $i^{-10}$

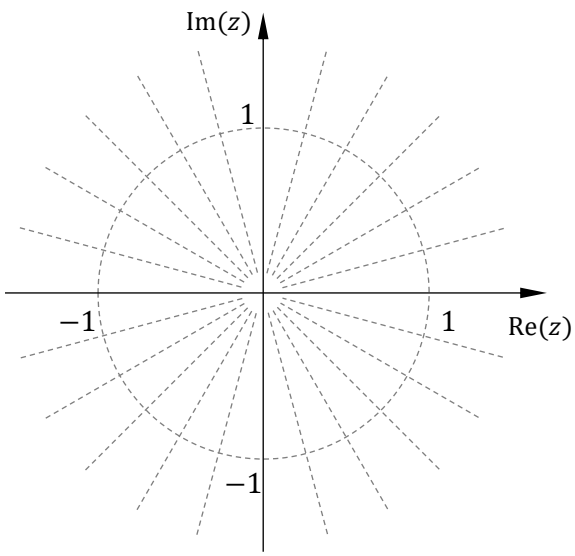
**5** Write $i^9$ in the form $a + ib$ where $a$ and $b$ are real.

**6** The dashed lines on the complex planes below are spaced every $15°$. On the complex planes below mark the following points:

**a** the sixth roots of $-1$




**b** the twelfth roots of $-1$



**7** Simplify:

**a** $i^5 + i^7$

**b** $i^6(i^2 + i^3)$

**8** Find the square roots of $-36$.

## Medium

**9** The three cube roots of $8$ are the solutions to $z^3 - 8 = 0$. The cube roots include $z = 2$ plus two complex roots. Given $z^3 - 8 = (z - 2)(z^2 + 2z + 4)$ find the complex cube roots of $8$.

**10** Factorise $z^4 + z^2 - 12$ over:

**a** the rational field, $\mathbb{Q}$

**b** the real field, $\mathbb{R}$

**c** the complex field, $\mathbb{C}$.

**11** Solve $z^2 + 2z + 5 = 0$:

**a** by completing the square

**b** by replacing $\sqrt{b^2 - 4ac}$ with $\lambda$, where $\lambda^2 = \Delta$

**12** Solve the quadratic equation $z^2 + (2 + 3i)z + (1 + 3i) = 0$, giving your answers in the form $a + bi$, where $a$ and $b$ are real numbers.

**13** Solve the quadratic equation $4z^2 + 4z + 5 = 0$.

**14** Solve the equation $z^2 + (z + 1)^2 = 0$, where $z$ is a complex number.

**15** Find $\sum_{r=2}^{6} i^{3r}$.

## Challenging

**16** Given $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, factorise $z^3 - i$.

**17** Given $(m + n)^3 = m^3 + 3m^2n + 3mn^2 + n^3$, expand $(a - ib)^3$ and write it in the form $X + iY$, where $X, Y \in \mathbb{R}$.

**18** The roots of a cubic polynomial are $\alpha, \beta$ and $\gamma$, where:

$$\alpha + \beta + \gamma = 1$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = 9$$
$$\alpha\beta\gamma = 9.$$

By first writing the cubic polynomial in the form $z^3 + bz^2 + cz + d = 0$ then factorising over the complex field, find the values of the three roots.

**19** The solutions to $f(x) = 0$, where $f(x)$ is a monic quadratic in the form $x^2 + bx + c$, are $x = 1$ and $x = 3$. The solutions to $f(x + m) + n = 0$ are $x = \pm 2i$. Find the values of $m$ and $n$.

In Lesson 2 we cover:

- Forms of a complex number
  - Cartesian (rectangular) form of a complex number
  - Exponential (Euler's) form of a complex number *(not in the 2024 syllabus)*
  - Modulus-argument (polar) form of a complex number
- Addition, subtraction and multiplication in Cartesian form
- The conjugate of a complex number and division in Cartesian form
- Mixed calculations
- Equal complex numbers

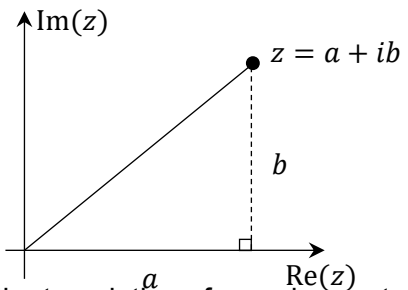
Syllabus reference: This lesson covers the 8th, 9th, 10th, 12th and 13th dot points from Arithmetic of complex numbers, and from the 1st dot point in Solving equations with complex numbers, in *Introduction to complex numbers*.

## Forms of a complex number

Complex numbers can be represented in three forms - Cartesian (or rectangular) form, exponential form (no longer in the syllabus) and modulus-argument (polar) form.

**Cartesian (rectangular) form of a complex number**

As we have seen, a complex number can be written in the form $z = a + ib$ where $a$ and $b$ are real numbers.



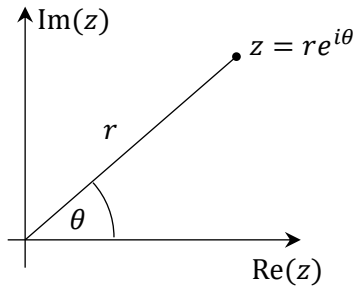

Cartesian form is:

- the easiest form to understand
- the best form to use for addition and subtraction, including the translation of complex vectors
- capable of doing multiplication, powers and division, though it can get complicated for any but the simplest questions

The rest of this lesson deals with calculations in Cartesian form.

**Exponential (Euler's) form of a complex number - *no longer in the Extension 2 syllabus***

We can also write the same number as $z = re^{i\theta}$, where $r$ is the modulus (the distance from zero) and $\theta$ is the argument (the angle measured anti-clockwise from the $x$-axis). This form has been removed from the 2024 syllabus, which is a great shame. Although you won't see this form used in the HSC exam paper, it should still be accepted in the working for many HSC questions.




Make sure you put your final answer in Cartesian form or mod-arg form (over the page), but best to avoid using it at all.

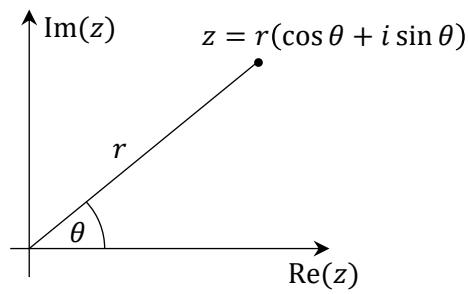
Exponential form is:

- the best form to use for multiplication, powers and division, including the rotation of complex vectors
- capable of doing addition and subtraction, especially in conjunction with the conjugate rules we will learn later in the chapter.

We will not mention exponential form again, except in passing.

**Modulus-argument (polar) form of a complex number**

We can also write the same number as $z = r(\cos\theta + i\sin\theta)$, where $r$ is the modulus and $\theta$ is the argument. This form is more commonly called mod-arg form than polar form.




Mod-arg form is:

- almost as good as exponential form for multiplication, powers and division
- capable of doing addition and subtraction, especially in conjunction with the conjugate rules we learn later in the chapter
- the best way to convert the real and imaginary parts of a complex number into its modulus and argument, or vice versa.

We can abbreviate $r(\cos\theta + i\sin\theta)$ as $r\,\mathrm{cis}\,\theta$ for convenience. We will expand on this next lesson - it is perfectly acceptable to use this form through the body of your solution, though I would always write the final answer in the full mod-arg form, $r(\cos\theta + i\sin\theta)$, just to be on the safe side. The exception is when we are listing multiple complex numbers, such as when we list the complex roots of a number, where leaving answers in cis form is sensible.

We use the terminology $|z|$ for the modulus of the complex number $z$, and $\arg(z)$ for its argument. Next lesson we will look at mod-arg form in more detail, and will also introduce the principal argument, which uses the notation $\text{Arg}(z)$ using a capital A - a common cause of error!

When $z$ is purely real, $|z|$ can mean either the absolute value of $z$ or the modulus of $z$, as they both refer to the distance from zero. The rules for operations with absolute values and moduli are equivalent, and some solutions will have vertical bars change meaning from modulus to absolute value as the expression inside changes, for example, 2025 HSC Q16(a).

## Addition, subtraction and multiplication in Cartesian form

For the remainder of this lesson, we will concentrate on basic calculations in Cartesian form.

Complex numbers follow the usual algebraic laws for real numbers, although some familiar properties of real numbers do not extend to complex numbers - we cannot order compex numbers and so cannot use them in inequalities.

**Addition and subtraction of complex numbers in Cartesian form**

For addition and subtraction of complex numbers in Cartesian form, we simply collect like terms.

$$\begin{aligned} z_1 + z_2 &= (a_1 + ib_1) + (a_2 + ib_2) \\ &= (a_1 + a_2) + (b_1 + b_2)i \end{aligned}$$

$$\begin{aligned} z_1 - z_2 &= (a_1 + ib_1) - (a_2 + ib_2) \\ &= (a_1 - a_2) + (b_1 - b_2)i \end{aligned}$$

Learn these techniques, do NOT remember them as formulas.

## Example 1

Simplify:

**a** $(4 + 2i) + (2 + 3i)$

$= 6 + 5i$

**b** $(4 + 2i) - (2 + 3i)$

$= 2 - i$

**Multiplication of complex numbers in Cartesian form**

For multiplication of complex numbers, expand any brackets, change $i^2$ to $-1$ if needed, then collect the like terms. This works regardless of whether we are multiplying by a real number, a purely imaginary number, or any other complex number. Always write the first step in terms of $i^2$, then simplify on following lines - this avoids a common cause of simple errors in the HSC.

Squaring a complex number is a special case of multiplying two complex numbers and can be done using the expansion of a perfect square formula.

Although we can easily show $(a+ib)^2 = a^2 - b^2 + 2abi$, which we will use to find square roots shortly, it can be a bit safer to use the perfect square formula from scratch each time.

$$\begin{aligned} z_1 z_2 &= (a_1 + ib_1) \times (a_2 + ib_2) \\ &= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 \\ &= a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i \end{aligned}$$

Learn these techniques, do NOT try and remember them as formulas. The techniques work when any of the real or imaginary parts are zero.

## Example 2

Simplify:

**a** $4(2+3i)$

$= 8 + 12i$

**b** $-3(5i-3)$

$= -15i + 9$

$= 9 - 15i$

## Example 3

Simplify:

**a** $-3i(4+5i)$

$= -12i - 15i^2$

$= 15 - 12i$

**b** $(2+i)(1+2i)$

$= 2 + 4i + i + 2i^2$

$= 2 + 4i + i - 2$

$= 5i$

## Example 4

Simplify:

**a** $(1+i)^2$

$= 1 + 2i + i^2$

$= 1 + 2i - 1$

$= 2i$

Alternatively:

$= 1^2 - 1^2 + 2(1)(1)i$

$= 2i$

**b** $(3-2i)^2$

$= 9 - 12i + 4i^2$

$= 9 - 12i - 4$

$= 5 - 12i$

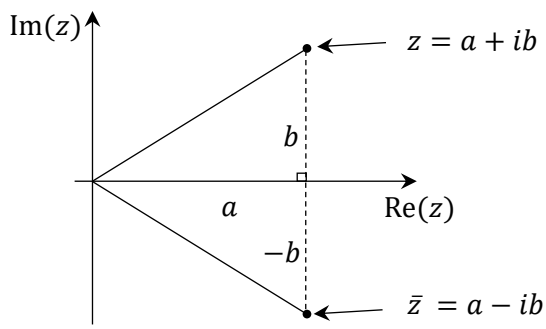
Alternatively:

$= 3^2 - 2^2 + 2(3)(-2)i$

$= 5 - 12i$

## Conjugate of a complex number and division in Cartesian form

The conjugate of a complex number has the same real part while its imaginary part has the same magnitude but is of the opposite sign. Geometrically it is its reflection over the real axis. The conjugate is written as $\bar{z}$ which is pronounced 'z bar'.




In Cartesian form the conjugate has the same real part, but the imaginary part is of the opposite sign. The conjugate of $a + ib$ is $a - ib$.

For example, $\overline{3+2i} = 3 - 2i$. Note the similarity to surds, where the conjugate of $3 + \sqrt{2}$ is $3 - \sqrt{2}$.

In Lesson 5 we will look at many important properties of conjugates.

## Example 5

Find the conjugate of each complex number.

**a** $-3 + 2i$

$\overline{-3+2i}$

$= -3 - 2i$

**b** $4i$

$\overline{4i}$

$= -4i$

We could add the following lines in here:
$= \overline{0+4i}$
$= 0 - 4i$

### Division of complex numbers in Cartesian form

When dividing complex numbers and rearranging them into the form $a + ib$, we must realise the denominator - it must end up with no imaginary component. This is very similar to rationalising the denominator with surds. We multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2}$$

$$= \frac{a_1a_2 - a_1b_2i + a_2b_1i - b_1b_2i^2}{(a_2)^2 - (ib_2)^2}$$

$$= \frac{a_1a_2 + b_1b_2 + (a_2b_1 - a_1b_2)i}{(a_2)^2 + (b_2)^2}$$

Learn these techniques, do NOT remember them as formula. It is worth noting that the denominator is the sum of the squares of $a_2$ and $b_2$, to reduce simple errors.

## Example 6

Simplify:

**a** $\frac{3 + 2i}{2i}$

$$= \frac{3 + 2i}{2i} \times \frac{i}{i}$$

$$= \frac{3i + 2i^2}{2i^2}$$

$$= \frac{-2 + 3i}{-2}$$

$$= 1 - \frac{3}{2}i$$

**b** $\frac{3 + 2i}{1 + i}$

$$= \frac{3 + 2i}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{3 - 3i + 2i - 2i^2}{1^2 + 1^2}$$

$$= \frac{5 - i}{2}$$

The denominator will always be the sum of the squares of the real and imaginary parts of the divisor.

**Relationships between $z, \bar{z}, \mathrm{Re}(z)$ and $\mathrm{Im}(z)$**

Conjugates are an underutilised tool in complex numbers - we will use the rules in more detail later in the chapter. For the moment we look at what happens when we find the sum or difference of a complex number and its conjugate.

$$\begin{aligned} z + \bar{z} &= (a + ib) + (a - ib) \\ &= 2a \end{aligned}$$

$$\therefore z + \bar{z} = 2\mathrm{Re}(z) \quad \rightarrow \quad \mathrm{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\begin{aligned} z - \bar{z} &= (a + ib) - (a - ib) \\ &= 2bi \end{aligned}$$

$$\therefore z - \bar{z} = 2i\mathrm{Im}(z) \quad \rightarrow \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$$

We will also explore these relationships later in the chapter, where we link them to the parallelogram rule.

## Example 7

**a** The complex number $z$ and its conjugate $\bar{z}$ both lie on the line $\mathrm{Re}(z) = 3$ on the complex plane. Evaluate $z + \bar{z}$.

$$\begin{aligned} z + \bar{z} &= 2\mathrm{Re}(z) \\ &= 2(3) \\ &= 6 \end{aligned}$$

**b** What is the average of $z = 5 - i$ and its conjugate?

$$\begin{aligned} \frac{z + \bar{z}}{2} &= \mathrm{Re}(z) \\ &= 5 \end{aligned}$$

## Example 8

**a** Evaluate the difference between $z = -2 + 3i$ and its conjugate.

$$\begin{aligned} z - \bar{z} &= 2i\text{Im}(z) \\ &= 2i(3) \\ &= 6i \end{aligned}$$

**b** For $z = 3 + bi$, $z - \bar{z} = -4i$. What is the value of $b$?

$$\begin{aligned} b &= \text{Im}(z) \\ &= \frac{z - \bar{z}}{2i} \\ &= -\frac{4i}{2i} \\ &= -2 \end{aligned}$$

## Mixed calculations

Let's try some examples involving a mix of calculations, which are very common early questions in Section II of the HSC. We are often given the value of two complex numbers, then given calculations in terms of the two numbers.

## Example 9

Let $z = 3 - 2i$ and $w = 1 + \sqrt{2}i$.

**a** Find $z^2 + w^2$

$$\begin{aligned} &= 9 - 12i + 4i^2 + 1 + 2\sqrt{2}i + 2i^2 \\ &= 9 - 12i - 4 + 1 + 2\sqrt{2i} - 2 \\ &= 4 - \left(12 - 2\sqrt{2}\right)i \end{aligned}$$

**b** Find $\overline{w} + z$

$$\begin{aligned} &= 1 - \sqrt{2}i + 3 - 2i \\ &= 4 - \left(2 + \sqrt{2}\right)i \end{aligned}$$

**c** Find $\frac{w}{z}$ in the form $a + ib$.

$$\begin{aligned} &= \frac{1 + \sqrt{2}i}{3 - 2i} \times \frac{3 + 2i}{3 + 2i} \\ &= \frac{3 + 2i + 3\sqrt{2}i + 2\sqrt{2}i^2}{3^2 + 2^2} \\ &= \frac{3 + 2i + 3\sqrt{2}i - 2\sqrt{2}}{13} \\ &= \frac{3 - 2\sqrt{2} + (2 + 3\sqrt{2})i}{13} \end{aligned}$$

**d** Find $w^2 - \bar{z}$

$$\begin{aligned} &= \left(1 + \sqrt{2}i\right)^2 - (3 - 2i) \\ &= 1 + 2\sqrt{2}i + 2i^2 - 3 + 2i \\ &= 1 + 2\sqrt{2}i - 2 - 3 + 2i \\ &= -4 + \left(2\sqrt{2} + 2\right)i \end{aligned}$$

## Example 10

Let $z = -5 - 12i$ and $\omega = 2 - i$.

**a** Find $(1 + i)\bar{\omega}$.

$$\begin{aligned} &= (1 + i)(2 + i) \\ &= 2 + i + 2i + i^2 \\ &= 2 + i + 2i - 1 \\ &= 1 + 3i \end{aligned}$$

**b** Find $\dfrac{z}{2 - 3i}$ in the form $a + ib$.

$$\begin{aligned} &= -\frac{5 + 12i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i} \\ &= -\frac{10 + 15i + 24i + 36i^2}{2^2 + 3^2} \\ &= -\frac{10 + 15i + 24i - 36}{13} \\ &= -\frac{-26 + 39i}{13} \\ &= 2 - 3i \end{aligned}$$

## Equal complex numbers

An important technique in solving questions is that for two complex numbers in Cartesian form to be equal, their real parts are equal, and their imaginary parts are equal.

In the equation below, the left-hand side is a complex number, and the right-hand side is a complex number. The only way the two sides of the equation can be equal is if the real parts and the imaginary parts of the complex numbers are equal.

## Example 11

Find the value of $z$ in the form $a + ib$ where $a$ and $b$ are real, that solves $z + 3i\bar{z} = -1 + 13i$.

$$\begin{aligned} a + ib + 3i(a - ib) &= -1 + 13i \\ a + ib + 3ai + 3b &= -1 + 13i \\ a + 3b + (3a + b)i &= -1 + 13i \\ \therefore a + 3b &= -1 \qquad (1) \\ 3a + b &= 13 \qquad (2) \end{aligned}$$

$(2) - 3\times(1)$: $-8b = 16$

$$b = -2$$

sub in $(2)$: $3a - 2 = 13$

$$a = 5$$

$$\therefore z = 5 - 2i$$

We can check this answer by substituting it back into the equation, just like we can check the solution to any other equation.

$$\begin{aligned} z + 3i\bar{z} &= 5 - 2i + 3i(5 + 2i) \\ &= 5 - 2i + 15i + 6i^2 \\ &= 5 - 2i + 15i - 6 \\ &= -1 + 13i \ \checkmark \end{aligned}$$

## Example 12

Find the value of $z$ in the form $a+ib$ where $a$ and $b$ are real, that solves

$$\frac{2\bar{z}(1-2i)}{5z}+\frac{i}{1+2i}=\frac{2-3i}{z}.$$

$$\frac{2\bar{z}(1-2i)}{5}+\frac{iz}{1+2i}\times\frac{1-2i}{1-2i}=2-3i$$

$$\frac{2\bar{z}(1-2i)}{5}+\frac{iz+2z}{1^2+2^2}=2-3i$$

$$2\bar{z}-4i\bar{z}+iz+2z=10-15i$$

$$2(a-ib)-4i(a-ib)+i(a+ib)+2(a+ib)=10-15i$$

$$2a-2bi-4ai-4b+ai-b+2a+2bi=10-15i$$

$$4a-5b-3ai=10-15i$$

$$4a-5b=10 \quad (1) \qquad -3a=-15 \quad (2)$$

$$\therefore a=5$$

sub in $(1)$: $\quad 4(5)-5b=10$

$$-5b=-10$$

$$b=2$$

$$\therefore z=5+2i$$

For equations, leave the equation in terms of $z$ for as long as possible, before substituting $z$ for $a+ib$ or one of the other forms. This saves lots of time and reduces algebraic errors.

Basic

**1** Simplify

**a** $(3+2i)+(4+i)$

**b** $(3+2i)-(4+i)$

**2** Simplify

**a** $3(2+4i)$

**b** $-2(4-3i)$

**c** $i(3+i)$

**d** $-2i(3-4i)$

**3** Simplify

**a** $(3+i)(4-2i)$

**b** $(2-i)^2$

**c** $(2+i)^3$

**4** Find the conjugate of each complex number:

**a** $2-2i$

**b** $-6i$

**5** Rewrite the following in the form $a+ib$, where $a$ and $b$ are real.

**a** $\dfrac{2-3i}{2i}$

**b** $\dfrac{3-2i}{1-i}$

**6** Given $z=3-i$ and $w=2i-1$, find $\bar{z}-w$.

**7** Given $z_1=5-2i$ and $z_2=3+4i$, find $z_1\bar{z}_2$.

**8** Given $z=3+2i$ find $\mathrm{Re}(2z+iz)$.

**9** Find the values of $x$ and $y$ given $3x+2iy-ix+5y=7+5i$, where $x,y\in\mathbb{R}$.

**10** Find the real numbers $a$ and $b$ such that $(1+2i)(1-3i)=a+ib$.

**11** Given $z = a + ib$, find the solution to $z - 2i\bar{z} = 3 + 4i$.

**12** Let $z = 1 + i$ and $w = 1 - i$. Express the following in the form $a + ib$.

**a** $z + iw$

**b** $z^2\overline{w}$

**c** $\dfrac{z}{w}$

**13** Let $z = 3 + i$ and $w = -2 + 2i$. Express the following in the form $a + ib$.

**a** $z + w$

**b** $z - w$

**c** $2z$

**d** $-3w$

**e** $iz$

**f** $zw$

**g** $\dfrac{w}{3i}$

**h** $\dfrac{z}{w}$

**14a** The complex number $z$ and its conjugate $\bar{z}$ both lie on the line $\mathrm{Re}(z) = -5$ on the complex plane. Evaluate $z + \bar{z}$.

**b** What is the average of $z = 6 + 2i$ and its conjugate?

**15a** Evaluate the difference between $z = 5 - 2i$ and its conjugate.

**b** For $z = -2 + bi$, $z - \bar{z} = 6i$. What is the value of $b$?

## Medium

**16** Given that $a$ and $b$ are real numbers and $\dfrac{a}{1+i} - \dfrac{b}{2i} = 2$ find the values of $a$ and $b$.

**17a** Show that $(1 - 2i)^2 = -3 - 4i$.

**b** Hence solve the equation $x^2 - 5x + (7 + i) = 0$.

**18** Find the real values of $x$ and $y$ for which $\dfrac{x}{i} - \dfrac{y}{1+i} = -1 - 3i$.

**19** Given $z_1 = 1 + 2i$ and $z_2 = 3 - i$, find the value of ${z_1}^2 \div \overline{z_2}$ in the form $a + ib$.

## Challenging

**20** The complex number $z = a + ib$ for $a, b \in \mathbb{R}$ solves $4z - 3\bar{z} = \dfrac{1 - 18i}{2 - i}$.
Find the values of $a$ and $b$.

**21a** If $z = \dfrac{\lambda + 4i}{1 + \lambda i}$ for the real constant $\lambda$, find the possible values of $\lambda$ if $z$ is purely real.

**b** Verify that $z$ is purely real for these values of $\lambda$.

# 1.3 Modulus and square roots in Cartesian form

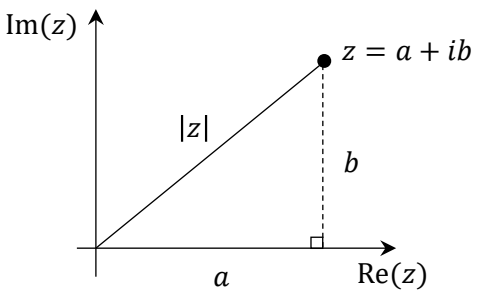
In Lesson 3 we cover:

- Modulus of a complex number in Cartesian form
- Square roots of a complex number in Cartesian form

Syllabus reference: This lesson covers the 11th, 12th and 14th dot points from Arithmetic of complex numbers, from the 7th dot point from Geometric representation of complex numbers, and from the 1st dot point in Solving equations with complex numbers, in *Introduction to complex numbers*.

## Modulus of a complex number in Cartesian form

The modulus of a complex number is its distance from zero. We write the modulus of $z$ as $|z|$, as shown. The modulus is always non-negative, with only the complex number zero having a modulus of $0$. The modulus must be real - a complex number can have an imaginary part but a modulus cannot.



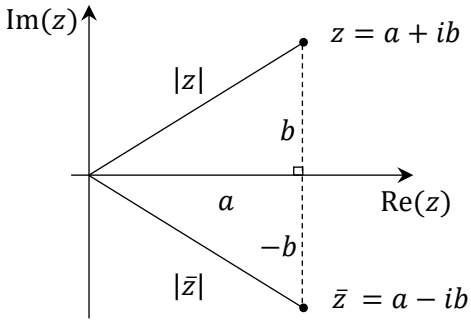

We can find a formula for $|z|$ using a simple application of Pythagoras' theorem, as $|z|$ is the length of the hypotenuse in a right-angled triangle where the short sides have length $|a|$ and $|b|$. Since $|a|^2 = a^2$ and $|b|^2 = b^2$, we can show that when $z = a + ib$, then:

$$\boxed{|z| = \sqrt{a^2 + b^2}}$$

When we use mod-arg form, we often represent the modulus using the pronumeral $r$, which represents the radius, or radial distance, from the origin. Later in the chapter, we see that every point on the complex plane with the same modulus lies on a circle centred at the origin, where the radius of the circle is $r = |z|$.

### Conjugates and moduli

We can easily show that the modulus of a complex number and the modulus of its conjugate are equal, so $|z| = |\bar{z}|$, as shown in the diagram.




Algebraically: $|\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$.

Last lesson we noted that $z\bar{z} = a^2 + b^2$, so an extended formula for the modulus is:

$$\boxed{|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}}.$$

Squaring part of the formula, we see: $\boxed{z\bar{z} = |z|^2 \quad \rightarrow \quad \frac{1}{z} = \frac{\bar{z}}{|z|^2} \text{ for } z \neq 0}$.

We will use this rule in more detail in Lesson 5.

## Example 1

Find the modulus of these complex numbers.

**a** $z = 5 + 2i$

$$|z| = \sqrt{5^2 + 2^2}$$
$$= \sqrt{29}$$

**b** $\omega = -3 - 4i$

$$|\omega| = \sqrt{(-3)^2 + (-4)^2}$$
$$= 5$$

## Example 2

Use the formula $|z| = \sqrt{z\bar{z}}$ to prove that if z lies on the unit circle, then $\bar{z} = \frac{1}{z}$.

$|z| = 1$ since $z$ lies on the unit circle

$\therefore \sqrt{z\bar{z}} = 1$

$\therefore z\bar{z} = 1$

$\therefore \bar{z} = \frac{1}{z}$ since $z \neq 0$ if it lies on the unit circle

If z lies on the unit circle, then $\bar{z} = \frac{1}{z}$.

**Operations with moduli** (the plural of modulus)

The absolute value or modulus of a product of two or more terms, or a term to a power, or the quotient of two terms, can be broken down easily using the fundamental property

$$\boxed{|ab| = |a| \times |b|} \qquad (1)$$

We can say: 'The modulus of a product is the product of the moduli'.

There are two special applications of this formula that we are also interested in:

$$\boxed{\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \text{ where } |b| \neq 0} \qquad (2)$$

We can say: 'The modulus of a quotient is the quotient of the moduli'.

$$\boxed{|a^n| = |a|^n} \qquad (3)$$

We can say: 'The modulus of a term raised to a power is the modulus of the term raised to that power'.

The absolute value of the sum or difference of terms does NOT break down as easily. We will cover this when we look at using the triangle inequality with complex numbers. Just so that we can see how the triangle inequality compares with the rules above, here is the triangle inequality:

$$\boxed{|a + b| \leq |a| + |b|}$$

We can say: 'The modulus of a sum is less than or equal to the sum of the moduli'.

Let's use rules $(1)$, $(2)$ and $(3)$ to simplify expressions.

## Example 3

Which of the following expressions is equivalent to $\left|\frac{\bar{z}}{z^2}\right|$, when $z$ is a non−zero complex number?

(A) $|z|$ (B) $|z|^2$ (C) $\frac{1}{|z|}$ (D) $-|z|$

$$\left|\frac{\bar{z}}{z^2}\right|$$

$$= \frac{|\bar{z}|}{|z^2|}$$

$$= \frac{|z|}{|z|^2}$$

$$= \frac{1}{|z|}$$

**Answer C**

Alternatively:

$$\left|\frac{\bar{z}}{z^2}\right| = \left|\frac{1}{z}\right|$$

$$= \frac{1}{|z|}$$

## Example 4

Simplify $|z^2| + |2z| + 1$.

$$= |z|^2 + 2|z| + 1$$

$$= (|z| + 1)^2$$

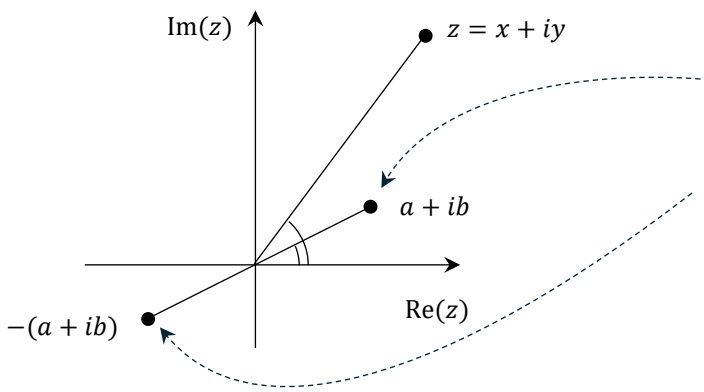
## Square roots of a complex number in Cartesian form

To find the square roots of a complex number in Cartesian form, the easiest method is to use the information from squaring a complex number in Lesson 2 to create simultaneous equations, which we solve by inspection.

We will also look at a formula that works but is not in the syllabus. The formula is good for more complicated examples, but creating simultaneous equations and solving them by inspection is quicker for most of the questions you will encounter in exams. The formula is also often misremembered, costing marks.

As we saw in Lesson 1, there are always two square roots of every non-zero complex number, and one is always the negative of the other. They are in the form $\pm(a+ib)$, where $a, b \in \mathbb{R}$.

If we let $a+ib$ be one of the square roots of $z = x+iy$, then $x+iy = (a+ib)^2$.




As we will see when we look at square roots in mod-arg form:

- the principal argument of one root is half the principal argument of the original number
- the principal argument of the other square root is $\pi$ radians more or less than the first square root
- the modulus of both square roots is the square root of the modulus of the original number

From equating real and imaginary parts when squaring a complex number we found:

$$a^2 - b^2 = x \quad (1)$$

$$2ab = y \quad (2)$$

**Finding square roots using simultaneous equations**

We start with a formal solution of the simultaneous equations, then we will look at some shortcuts to solve the equations. Since each of these pairs of simultaneous equations always have the same form, the shortcuts are the preferred method.

## Example 5

Find the square roots of $5 + 12i$.

$$a^2 - b^2 = 5 \qquad (1)$$

$$2ab = 12 \qquad (2)$$

rearranging $(2)$: $$b = \frac{6}{a} \qquad (3)$$

sub $(3)$ into $(1)$: $$a^2 - \frac{36}{a^2} = 5$$

$$a^4 - 5a^2 - 36 = 0$$

$$(a^2 - 9)(a^2 + 4) = 0$$

$a^2 - 9 = 0$ or $a^2 + 4 = 0$

$a = \pm 3$ no real solution

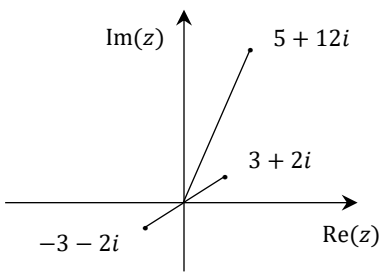
$$b = \frac{6}{\pm 3}$$

$$= \pm 2$$

The square roots of $5 + 12i$ are $\pm(3 + 2i)$.

Note that $a$ and $b$ both have to be real numbers, which is why the equation $a^2 + 4 = 0$ did not give any solutions. Looking at $5 + 12i$ and its two square roots on a complex plane helps us see how they are related.

$3 + 2i$ has half the argument of $5 + 12i$, and its modulus is the square root of the modulus of $5 + 12i$. The other root is $-3 - 2i$.



We just solved the simultaneous equations formally, but we can solve them for simpler examples in our heads, with a little bit of writing. As a shortcut:

- halve the imaginary part of the original number
- find pairs of numbers that multiply to give that number
- find which pair has a difference of their squares equal to the real part of the original number.

See the other adjustments that are needed for more difficult examples that follow.

## Example 6

Find the square roots of $5 + 12i$.

$$a^2 - b^2 = 5 \qquad (1)$$
$$2ab = 12$$
$$ab = 6 \qquad (2)$$

By inspection, the square roots of $5 + 12i$ are $\pm(3 + 2i)$.

Explanation:

- Find pairs of numbers whose product is $6$: $\pm 1$ and $\pm 6$, $\pm 2$ and $\pm 3$.
- Since the real part is positive, $a^2 - b^2$ is positive, so we place the number with the largest absolute value first (otherwise the difference would be negative).
- In our heads: $6^2 - 1^2 \neq 5$; $\quad 3^2 - 2^2 = 5$.
- Thus $3$ and $2$ are the values for $a$ and $b$, so $3 + 2i$ is a solution.
- Therefore $-3 - 2i$ is also a solution.

While this can be quite quick for simple examples, it has its disadvantages, as we will see in the exercises.

## Example 7

Solve $z^2 = -8i$, leaving your answer in the form $a + ib$ for $a, b$ real.

$$a^2 - b^2 = 0 \qquad (1)$$
$$2ab = -8$$
$$ab = -4 \qquad (2)$$

By inspection, the square roots of $-8i$ are $\pm(2 - 2i)$.

*We could write these square roots as $\pm 2 \mp 2i$, using the $\mp$ sign to show that the real and imaginary parts have opposite signs.*

Explanation:

- Find pairs of numbers whose product is $4$: $\pm 1$ and $\mp 4$, $\pm 2$ and $\mp 2$.
- Since the real part is zero then $a^2 - b^2$ is zero, so we need the pair with same magnitude, $2$ and $-2$.
- Thus $2$ and $-2$ are the values for $a$ and $b$, so $2 - 2i$ is a solution.
- Therefore $-2 + 2i$ is also a solution.

If we are trying to find the square roots of a number with an even imaginary part, the shortcut will usually help find the answer easily. In the exercises we will find the square roots of numbers where:

- the imaginary part is a surd in the form $c\sqrt{d}$, where we let either $a$ or $b$ be equal to $\sqrt{d}$
- the imaginary part is odd, where we start by doubling the number, finding the square roots of this number, and dividing each root by $\sqrt{2}$.

**Finding square roots using formulas**

Since the simultaneous equations are always in the same form, we can find formulas which avoid having to find the values of $a$ and $b$ by inspection. Since these formulas are not in the syllabus, we need to use them with care. If you use them in the exam, you should first write down the formula, then substitute in carefully and simplify correctly to find the square roots. If you make any mistake, you would be at risk of earning no marks, so I would not use them.

Equating real parts of $z = x + iy$ and $(a + ib)^2$ we have

$$\text{Re}(z) = a^2 - b^2 \qquad (1)$$

In a later lesson we learn that the modulus of a square is the square of the modulus of its root, so

$$|x + iy| = |a + ib|^2$$
$$\therefore |z| = a^2 + b^2 \qquad (2)$$

We can solve these two equations simultaneously by adding them, and rearranging we get:

$$a = \pm\sqrt{\frac{|z| + \text{Re}(z)}{2}}, \qquad b = \pm\text{sgn}(y)\sqrt{\frac{|z| - \text{Re}(z)}{2}}$$

where $\text{sgn}(y)$ is the signum function, returning the sign of $y$.

## Example 8

Find the square root of $5 + 12i$, using the formulas.

$$a = \pm\sqrt{\frac{\sqrt{5^2 + 12^2} + 5}{2}} = \pm 3$$

$$b = \pm(+)\sqrt{\frac{\sqrt{5^2 + 12^2} - 5}{2}} = \pm 2$$

$\therefore \pm(3 + 2i)$ are the solutions.

Basic

**1** Find the modulus of these complex numbers.

**a** $z = 12 + 5i$

**b** $\omega = 6 - 3i$

**2** Which of the following expressions is equivalent to $\left|\frac{z^3}{\bar{z}}\right|$, when $z$ is a non−zero complex number?

(A) $|z|$ (B) $|z|^2$ (C) $\frac{1}{|z|}$ (D) $-|z|$

**3** Simplify $|z^2| + |w^2| - |2wz|$.

**4** Find the square roots of $3 + 4i$:

**a** By creating simultaneous equations and solving them, showing full working.

**b** Using the formulas.

*The solutions for any other questions finding square roots will solve simultaneous equations by inspection*

**5** Find the square roots of $5 - 12i$.

**6** Find the square roots of $18i$.

Medium

**7** Find the square roots of $2 - 2\sqrt{3}i$.

**8 a** Find the square roots of $-6 + 8i$. Hint: $a$ and $b$ are surds.

**b** Hence, or otherwise, solve the equation $z^2 - \sqrt{2}iz + 1 - 2i = 0$.

**9** Use the formula $|z| = \sqrt{z\bar{z}}$ to prove that $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$, when $z$ is a non−zero complex number.

**10** Solve $z^2 = 21 - 20i$, leaving your answer in the form $a + ib$ for $a, b$ real.

**11a** Find the two square roots of $2i$.

**b** Hence solve $x^2 + 2x + \left(1 - \frac{i}{2}\right) = 0$.

**12** Consider the equation $|z| - z - 3 - 4i = 0$, where $z = a + ib$ where $a, b \in \mathbb{R}$.

**7a** Explain why $b = -4$.

**b** Show that $a = \frac{7}{6}$.

Challenging

**13** Find the square roots of $4 - 3i$.

**14** Solve $z^2 + 4z + 4 - i = 0$, giving your answer(s) in the form $a + bi$, where $a$ and $b$ are real.

**15** Solve $z^2 + (1 - 4i)z - 5 + i = 0$, giving your answer(s) in the form $a + bi$, where $a$ and $b$ are real.

**16** Determine the possible values of the real constant $\lambda$ if $\left|\frac{3+i}{1+2i} + \lambda\right| = \sqrt{\lambda + 2}$.

**17** The complex roots of $iz^2 + \sqrt{3}z - 1 = 0$ are $\alpha$ and $\beta$.

**a** Find $\alpha$ and $\beta$ in Cartesian form.

**b** Show that $\alpha^2\beta^2 + 1 = 0$.

**18** Given that $\mathrm{Re}(z) = a^2 - b^2$ and $|z| = a^2 + b^2$, where $a + ib$ is one of the square roots of $z = x + iy$, prove that $a = \pm\sqrt{\frac{|z| + \mathrm{Re}(z)}{2}}$ and $b = \pm\mathrm{sgn}(y)\sqrt{\frac{|z| - \mathrm{Re}(z)}{2}}$, where $\mathrm{sgn}(y)$ is the signum function, returning the sign of $y$.

**19** The complex number $z = a + ib$ for $a, b$ real solves $z + 2\bar{z} = |z + 2|$. Find the values of $a, b$.

**20** The complex number $z = x + iy$ satisfies $|z| \leq y + 1$ and $|z| \leq x + 1$. What is the maximum possible value of $|z|$?

# 1.4 Modulus-argument form of a complex number

In Lesson 4 we cover:

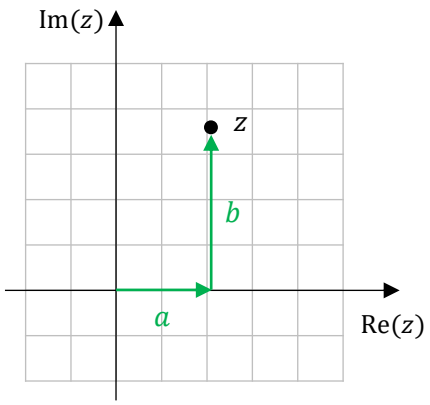
- Modulus, argument and modulus-argument form of a complex number

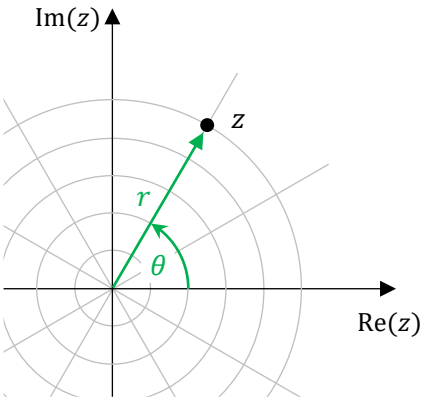
Syllabus reference: This lesson covers the 2nd to 8th dot points from Geometric representation of complex numbers in *Introduction to complex numbers*.

## Modulus, argument and modulus-argument form of a complex number

When we plot points on a Cartesian plane, we use the $x$- and $y$-coordinates of the point to determine its position on a rectangular grid. The $x$-value tells us how far to move left or right from the origin (or zero), while the $y$-value tells us how far to move up or down.

When we use the Cartesian form of a complex number it is very similar. The real part of the complex number, $a$ or $\text{Re}(z)$, tells us how far to move left or right from zero, while the imaginary part of the complex number, $b$ or $\text{Im}(z)$, tells us how far to move up or down. Looking at the diagram below left, we can determine the position of $z$ on the complex plane by starting at zero, moving right $a$ units then up $b$ units. This is a very efficient method for adding or subtracting complex numbers.




There is another method for plotting points on planes called polar form, which relies on the angle and distance from the origin (or zero) to the point. The name ‘polar’ means ‘pivot point’ in this case, as we pivot from the origin (or zero) to measure the angle and distance. It is similar to our work on bearings in trigonometry.

The modulus-argument (mod-arg) form of a complex number, and the exponential form of a complex number (no longer in the syllabus), both use polar coordinates.

For non-zero complex numbers in both forms, the argument, $\theta$, tells us the angle from zero measured anticlockwise from the positive direction of the real axis, while the modulus, $r$ or $|z|$, tells us the distance from zero. Looking at the diagram above right, we can determine the position of $z$ on the complex plane by starting at zero and rotating anticlockwise from the positive direction of the real axis by an angle of $\theta$, then moving $r$ units in that direction. This is a very efficient method for multiplying and dividing complex numbers, including powers and roots.

The number zero has a modulus of zero, but its argument is undefined. Any angle must be formed by two lines - in the case of arguments we use the positive direction of the real axis as one line, and the line from $0$ to $z$ as the second line. If $z = 0$ then the second line does not exist, so no angle is formed - this is why the argument of zero is undefined.

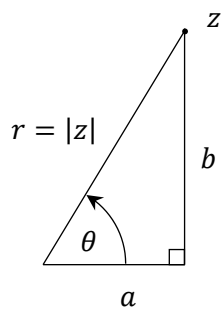
In mod-arg form, a complex number is written in the form $r(\cos\theta + i\sin\theta)$, which is sometimes abbreviated to $r\,\mathrm{cis}\,\theta$. More about this in a moment.

The exponential form is written in the form $z = re^{i\theta}$. The exponential form is a more compact method for writing complex numbers compared to the full mod-arg form, though with the use of $\mathrm{cis}\,\theta$ there is less advantage. Looking closely at the two forms, we see Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$. Letting $\theta = \pi$ and rearranging we find Euler's identity $e^{i\pi} + 1 = 0$. A reminder that exponential form is no longer in the syllabus.

Each complex number has only one modulus, since the distance from zero is fixed. Since we can keep rotating around a circle indefinitely, either anticlockwise or clockwise, each complex number has an infinite number of arguments. Each of these arguments is separated by $2\pi$ radians or $360°$. We will discuss the principal argument over the page.

Combining our diagrams from the previous page, we can see that $a, b$ and $r$ (or $|z|$) form a right-angled triangle. Using simple trigonometry we can link Cartesian and mod-arg form:

$$\begin{aligned} z &= a + ib \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$



For the complex number $z = a + ib$, the argument $\theta$ is ANY angle that satisfies both of these conditions:

$$\boxed{\sin\theta = \frac{b}{|z|} \quad \text{and} \quad \cos\theta = \frac{a}{|z|}}.$$

We need to satisfy both conditions so that we get the angle in the correct quadrant.

For example, in the right-hand diagram at the start of this lesson, we can reach $z$ by rotating anticlockwise from the positive direction of the $x$-axis $\frac{\pi}{3}$ radians, or $\frac{7\pi}{3}$ radians, or $\frac{13\pi}{3}$ radians, and so on. We can also reach $z$ by rotating clockwise from the positive direction of the $x$-axis $\frac{5\pi}{3}$ radians, or $\frac{11\pi}{3}$ radians, and so on. This means that the argument of $z$ can be $-\frac{11\pi}{3}, -\frac{5\pi}{3}, \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}, \ldots$ The argument of a complex number has an infinite number of values.

We are often interested in the unique argument that lies in the interval $(-\pi, \pi]$, which we call the **principal argument**. The notation for the argument and principal argument of a complex number are very similar, and easy to use interchangeably by mistake - we all do this at times!

The notation $\arg(z)$ refers to the argument, while $\text{Arg}(z)$ refers to the principal argument.

Notice that the only difference is whether the 'a' is lowercase or uppercase.

If $\text{Arg}(z) = 0$ or $\pi$ then $z$ is purely real, while if $\text{Arg}(z) = \pm\frac{\pi}{2}$ then $z$ is purely imaginary.

### Using $\text{cis}\,\theta$ notation

We have mentioned that $\cos\theta + i\sin\theta$ can be abbreviated as $\text{cis}\,\theta$, which stands for cos + i sin. For example, $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ can be written more compactly as $2\,\text{cis}\frac{\pi}{3}$. Especially in longer proofs, this is a very sensible abbreviation and to be encouraged - using the full version in every line of every applicable question would waste valuable exam time.

Cis notation rarely, if ever, appears beyond secondary mathematics, for the simple reason that exponential form takes the place of mod-arg form, so $\text{cis}\,\theta$ notation becomes superfluous. If it does come up, your lecturers may discourage it, but it is fine for the HSC, especially since exponential form has been removed from the syllabus.

A common question that arises is whether this notation can be used in the HSC. The short answer is yes, though it is best practice to only use $\text{cis}\,\theta$ notation through the body of your solution, writing out the final answer as $r(\cos\theta + i\sin\theta)$. That is the approach I will use in the solutions of examples and exercises in this textbook, with the exception of listing multiple complex roots in the last lesson, where answers will generally be left in cis form.

In the second last lesson of this chapter, we will use $c$ and $is$ as abbreviations for $\cos$ and $i\sin$ in our working.

### Converting between Cartesian and mod-arg form

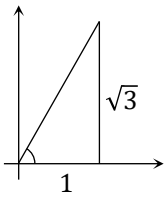
Always draw a neat sketch when you convert from Cartesian form to mod-arg form, as it is the best way to avoid getting an answer in the incorrect quadrant. Look for the $\frac{\pi}{3} - \frac{\pi}{6}$ and $\frac{\pi}{4}$ exact triangles and draw them into your sketch, as this avoids having to make calculations in many cases. You MUST show working when converting between forms - if you write a bald answer expect to get at most $1$ mark in a $2$-mark question.

When we convert from mod-arg form to Cartesian form there is no need for a diagram - just carefully evaluate the real and imaginary parts, often using ratios from exact triangles.

## Example 1

Convert the following complex numbers from Cartesian form into mod-arg form.

**a** $1+\sqrt{3}i$




$r=2, \text{Arg}(z)=\dfrac{\pi}{3}$ (exact Δ)

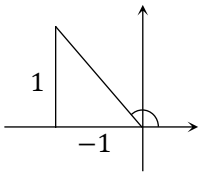
$\therefore 1+\sqrt{3}i=2\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$

Alt: $r=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$

$\text{Arg}(z)=\tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right)=\dfrac{\pi}{3}$

$\therefore 1+\sqrt{3}i=2\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$

**b** $-1+i$




$r=\sqrt{2}$

$\text{Arg}(z)=\pi-\dfrac{\pi}{4}=\dfrac{3\pi}{4}$ (exact Δ)

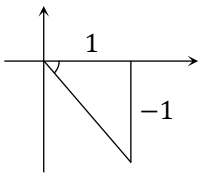
$\therefore -1+i=\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

Alt: $r=\sqrt{1^2+(-1)^2}=\sqrt{2}$

$\text{Arg}(z)=\pi-\tan^{-1}\left(\dfrac{1}{1}\right)=\dfrac{3\pi}{4}$

$\therefore -1+i=\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

**c** $1-i$




$r=\sqrt{2}$

$\text{Arg}(z)=-\dfrac{\pi}{4}$ (exact Δ)

$\therefore 1-i=\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right)+i\sin\left(-\dfrac{\pi}{4}\right)\right)$

Alt: $r=\sqrt{1^2+(-1)^2}=\sqrt{2}$

$\text{Arg}(z)=-\tan^{-1}\left(\dfrac{1}{1}\right)=-\dfrac{\pi}{4}$

$\therefore 1-i=\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right)+i\sin\left(-\dfrac{\pi}{4}\right)\right)$

## Example 2

Convert the following complex numbers from polar into Cartesian form.

**a** $4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$=4\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$

$=2\sqrt{3}+2i$

**b** $2\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

$=2\sqrt{2}\left(-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$

$=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$

$=-2-2i$

**Multiplying complex numbers in mod-arg form**

We can find the product of two complex numbers in mod-arg form by multiplying their moduli and adding their arguments. These rules work for arguments, but not always for principal arguments!

Proof:

$$\begin{aligned} z_1\times z_2 &= r_1(\cos\theta_1+i\sin\theta_1)\times r_2(\cos\theta_2+i\sin\theta_2) \\ &= r_1r_2(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2) \quad \text{commutative law} \\ &= r_1r_2\left(\cos\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2\,i+\sin\theta_1\cos\theta_2\,i+\sin\theta_1\sin\theta_2\,i^2\right) \\ &= r_1r_2\left(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)i\right) \end{aligned}$$

$$z_1\times z_2 = r_1r_2\left(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right) \qquad **$$

$$\boxed{|z_1z_2| = |z_1|\times|z_2| \text{ and } \arg(z_1z_2) = \arg(z_1)+\arg(z_2)}$$

We can find other results from this result. If we let $z_1 = z_2 = z$ then we find that when we square a complex number we square the modulus and double the argument:

$$\boxed{|z^2| = |z|^2 \text{ and } \arg(z^2) = 2\arg z}$$

We can extend this rule to any integer power of a complex number. To find $z^n$ we raise the modulus to the power of $n$ and multiply the argument by $n$:

$$\boxed{|z^n| = |z|^n \text{ and } \arg(z^n) = n\arg z}$$

This is the basis of de Moivre's Theorem which we will look at more deeply in Lesson 8.

When $n$ is not an integer, things get more complicated as we get multiple answers. We will investigate this in our lesson on complex roots.

If we let $r_1 = r_2 = 1$ in $**$ we see a result that will be useful when combined with the conjugate rules we learn next lesson:

$$\boxed{\operatorname{cis}\theta_1 \times \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2)}$$

## Example 3

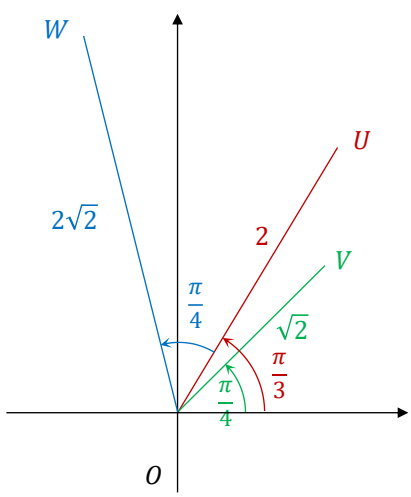
If $u = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $v = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, find $w$ where $w = uv$.

$$\begin{aligned} w &= uv \\ &= 2\operatorname{cis}\frac{\pi}{3} \times \sqrt{2}\operatorname{cis}\frac{\pi}{4} \\ &= \left(2\times\sqrt{2}\right)\operatorname{cis}\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\ &= 2\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) \end{aligned}$$

Note that we used cis notation through the working, but wrote the final answer in full mod-arg form.

It is worthwhile looking at the last example geometrically to increase our understanding of what happens when two complex numbers are multiplied.

Consider the points $U, V$ and $W$ on the complex plane representing $u = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $v = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ and $w = 2\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$ respectively.




The modulus of $W$ is the product of the moduli of $U$ and $V$, $2\sqrt{2} = 2\times\sqrt{2}$.

The argument of $W$ is the sum of the arguments of $U$ and $V$, $\frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$.

**Dividing complex numbers in mod-arg form**

We can find the quotient of two complex numbers in mod-arg form by dividing their moduli and subtracting their arguments.

Proof:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\, i + \sin\theta_1\cos\theta_2\, i - \sin\theta_1\sin\theta_2\, i^2}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + (\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)i}{1}$$

$$= \frac{r_1}{r_2}\Big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\Big)$$

$$\boxed{\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)}$$

## Example 4

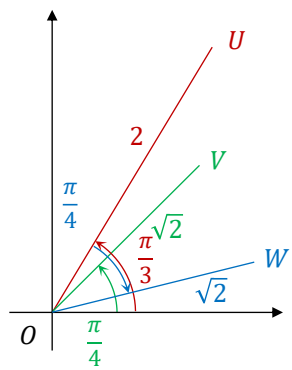
If $u = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $v = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, find $w$ where $w = \frac{u}{v}$.

$$w = \frac{u}{v}$$

$$= \frac{2\,\mathrm{cis}\,\frac{\pi}{3}}{\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}}$$

$$= \frac{2}{\sqrt{2}}\,\mathrm{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$



In the next example we have some expressions which are almost in mod-arg form, but not quite. The modulus must be positive, the angles must be equal and the sign in the middle must be positive.

## Example 5

Convert the following complex numbers into Cartesian form.

**a** $-2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$=-1\times 2\times \operatorname{cis}\frac{\pi}{4}$

$=\operatorname{cis}\pi\times 2\times \operatorname{cis}\frac{\pi}{4}$

$=2\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

**b** $3\left(\cos\frac{5\pi}{3}-i\sin\frac{4\pi}{3}\right)$

$=3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

## Example 6

Simplify the following, leaving the answer in mod-arg form, where $-\pi<\theta\leq\pi$.

**a** $i\times 3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$=\operatorname{cis}\frac{\pi}{2}\times 3\times \operatorname{cis}\frac{\pi}{6}$

$=3\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

**b** $-2i\times\sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)$

$=2\times \operatorname{cis}\left(-\frac{\pi}{2}\right)\times\sqrt{3}\times \operatorname{cis}\left(-\frac{5\pi}{6}\right)$

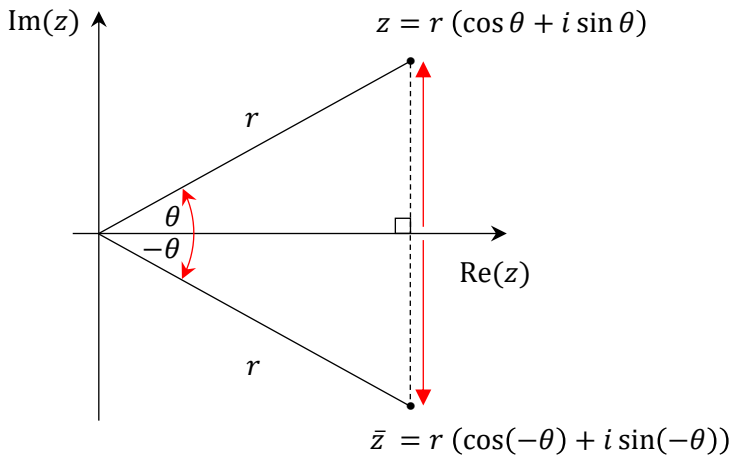
$=2\sqrt{3}\operatorname{cis}\left(-\frac{4\pi}{3}\right)$

$=2\sqrt{3}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

### Conjugates in mod-arg form

In mod-arg form the conjugate of a complex number has the same modulus while the argument is the negative.

The conjugate of $z = r(\cos\theta + i\sin\theta)$ is $\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$.




## Example 7

State the conjugate of each complex number, leaving your answer in mod-arg form.

**a** $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\bar{z} = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

**b** $z = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$

$\bar{z} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

The conjugate rules that we have seen in the last two lessons apply to all complex numbers. We can also prove the results using mod-arg form.

Here we will look at the proof that $z + \bar{z} = 2\text{Re}(z)$ and $z\bar{z} = |z|^2$ using $z = r(\cos\theta + i\sin\theta)$. In the exercises we will prove that $z - \bar{z} = 2i\text{Im}(z)$. As you will see, results involving addition or subtraction are easier to prove using Cartesian form, while results involving multiplication or division are easier to prove using mod-arg form.

$$\begin{aligned} z + \bar{z} &= r(\cos\theta + i\sin\theta) + r(\cos(-\theta) + i\sin(-\theta)) \\ &= r\,(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta) \text{ since cosine is even and sine is odd} \\ &= 2r\cos\theta \\ &= 2\text{Re}(z) \end{aligned}$$

This is a very common exam question, so make sure you memorise the technique, and the similar question in the exercises for $z - \bar{z}$. We will recreate these proofs many times in the revision exercise.

$$\begin{aligned} z\times\bar{z} &= r(\cos\theta + i\sin\theta)\times r(\cos(-\theta) + i\sin(-\theta)) \\ &= r^2(\cos\theta + i\sin\theta)\times r(\cos(-\theta) + i\sin(-\theta)) \\ &= r^2(\cos(\theta-\theta) + i\sin(\theta-\theta)) \\ &= r^2(\cos 0 + i\sin 0) \\ &= r^2(1) \\ &= r^2 \\ &= |z|^2 \end{aligned}$$

**Square roots in mod-arg form**

Finding the square root of a complex number in mod-arg form is much easier than using Cartesian form. We take the square root of the modulus, halve the argument to find the argument of one of the square roots, then add or subtract $\pi$ to find the argument of the other square root. We may need to add or subtract mulitples of $2\pi$ to find the principal arguments.

## Example 8

Find the square roots of the following complex numbers, expressing the roots in the form $r(\cos\theta + i\sin\theta)$, where $\theta$ is the principal argument.

**a** $4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

$r = \sqrt{4} = 2$

$\theta_1 = \frac{\pi}{3} \div 2 = \frac{\pi}{6}$

$\theta_2 = \theta_1 - \pi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$

The square roots are $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$.

**b** $3\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$.

$r = \sqrt{3}$

$\frac{17\pi}{4} \div 2 = \frac{17\pi}{8} \rightarrow \theta_1 = \frac{17\pi}{8} - 2\pi = \frac{\pi}{8}$

$\theta_2 = \theta_1 - \pi = \frac{\pi}{8} - \pi = -\frac{7\pi}{8}$

The square roots are $\sqrt{3}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$ and $\sqrt{3}\left(\cos\left(-\frac{7\pi}{8}\right) + i\sin\left(-\frac{7\pi}{8}\right)\right)$.

Basic

**1** Convert the following complex numbers from Cartesian form into mod-arg form:

**a** $1+i$

**b** $-1+\sqrt{3}i$

**c** $-\sqrt{3}-i$

**2** Convert the following complex numbers from mod-arg into Cartesian form:

**a** $4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

**b** $\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$

**3** If $u=4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $v=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$, find $w$ where $w=uv$.

**4** If $u=4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $v=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$, find $w$ where $w=\frac{u}{v}$.

**5** State the conjugate of each complex number, leaving your answer in mod-arg form:

**a** $z=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

**b** $z=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$

**6** Diagram A shows the complex number $z$ represented in the complex plane.

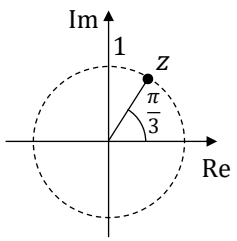



**DIAGRAM A**

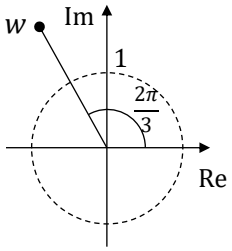
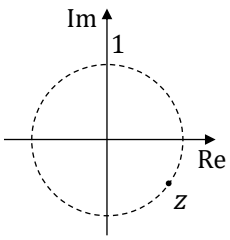


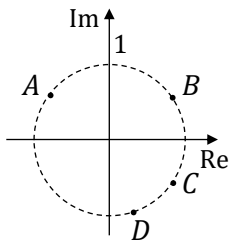

**DIAGRAM B**

Diagram B shows the complex number $w$. Which of the following could be an expression for $w$ in terms of $z$?

**A** $z^2$ **B** $2iz$ **C** $-2z$ **D** $2z^2$

**7** The diagram on the left shows the complex number $z$ on the complex plane.




Which of the points $A, B, C$ and $D$ on the diagram on the right best shows $\frac{z^2}{|z|}$?

**8** Given $z = r(\cos\theta + i\sin\theta)$, prove that $z - \bar{z} = 2i\text{Im}(z)$.

**9** Find the square roots of the following complex numbers, expressing the roots in the form $r(\cos\theta + i\sin\theta)$, where $\theta$ is the principal argument.

**a** $9\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

**b** $8\left(\cos\left(-\frac{11\pi}{4}\right) + i\sin\left(-\frac{11\pi}{4}\right)\right)$

**10** Find $r, \theta$ such that $(\cos\pi + i\sin\pi) - \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = r(\cos\theta + i\sin\theta)$

**11** Convert $\sin\frac{\pi}{6}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\frac{3\pi}{4}\right)$ to mod-arg form.

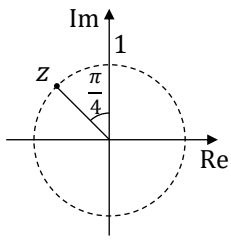
**12** For the complex number $w = 1 - i\sqrt{3}$:

**a** Find $|w|$ and $\arg(w)$

**b** Express $\bar{w}, w^2$ and $\frac{1}{w}$ in the form $a + ib$.

**13** The diagram shows the complex number $z$ represented in the complex plane. Which of the following complex numbers is equal to $i$?

A $\bar{z}$ B $-z$ C $z^2$ D $-z^2$




Medium

**14a** Write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in the form $a + ib$ where $a$ and $b$ are real.

**b** By expressing $\sqrt{3} + i$ and $\sqrt{3} - i$ in mod-arg form, write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in mod-arg form.

**c** Hence find $\sin\frac{\pi}{3}$ in surd form.

**15** Simplify the following, leaving the answers in mod-arg form.

**a** $i\times 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

**b** $-3i\times\frac{1}{\sqrt{3}}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

**16** Simplify the following, leaving the answers in mod-arg form.

**a** $2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \div (2i)$

**b** $i \div \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

**17** Multiplying a non-zero complex number by $\frac{1-i}{-1+i}$ results in a rotation about the origin. What is the angle of rotation, and in what direction?

**18** Given $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ find $(\bar{z})^{-1}$in mod-arg form.

**19** Simplify $-3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, leaving your answer in mod-arg form, where $-\pi < \theta \leq \pi$.

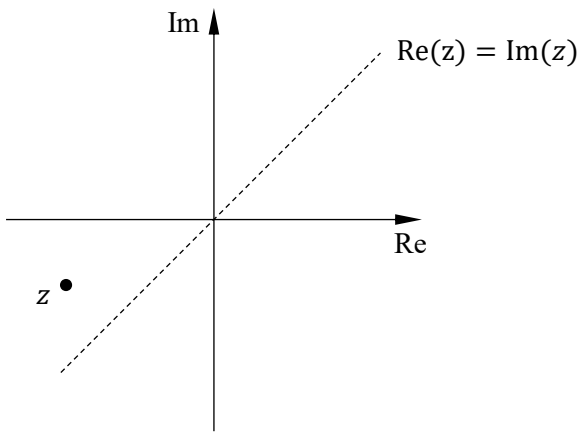
**20** Show that $\sqrt{6} - \sqrt{2}i = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$.

**21** Express $1 + i\sqrt{3}$ and $1 + i$ in modulus-argument form.

**b** Hence find $(1 + i\sqrt{3})(1 + i)$ in modulus-argument form.

**c** Hence find the exact value of $\tan\frac{7\pi}{12}$.

**22** The diagram shows the complex number $z = x + iy$ in the third quadrant, with $x < y$.




Given $z^2 = a + ib$, which of the following statements is true?

A. $a < 0, b < 0$
B. $a < 0, b > 0$
C. $a > 0, b < 0$
D. $a > 0, b > 0$

**23** For the complex numbers $z$ and $w$, it is known that $\frac{z}{w}$ is purely real with a modulus of $2$. If $z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, find $w$ in mod-arg form.

## Challenging

**24** Let $z = 1 - 3i$ and $w = 2 + i$.

**a** Express $zw$ in the form $a + ib$.

**b** Express $zw$ in modulus-argument form.

**c** Hence find $x$ if $\dfrac{\sqrt{2}(\cos x - i\sin x)}{2 + i} = \dfrac{1 - 3i}{5}$.

**25** Given $z = \dfrac{a+3i}{2+ai}$ for real $a$ is a complex number in the first quadrant.

**a** If $a = 4$ find $|z|$.

**b** Prove that if $\arg z = \frac{\pi}{4}$ that $a$ can only equal $1$.

**26** Let $z = 3(\cos\theta + i\sin\theta)$.

**a** Find $\overline{1-z}$.

**b** Express the imaginary part of $\frac{1}{1-z}$ in terms of $\theta$.

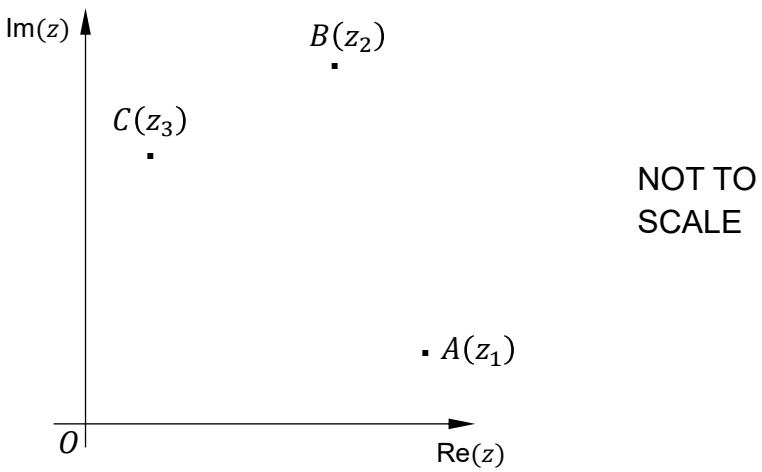
**27** Given $a\,(\cos\alpha + i\sin\alpha) + b(\cos\beta + i\sin\beta) = r(\cos\theta + i\sin\theta)$.

**a** Prove $\cos\theta = \dfrac{a\cos\alpha + b\cos\beta}{r}$ and $\sin\theta = \dfrac{a\sin\alpha + b\sin\beta}{r}$.

**b** Prove $r = \sqrt{a^2 + b^2 + 2ab\cos(\alpha - \beta)}$.

**c** Hence prove $|z_1| + |z_2| \geq |z_1 + z_2|$.

**28** The complex numbers $z_1, z_2$ and $z_3$ are represented by the points $A, B$ and $C$ respectively.




The complex numbers satisfy $z_2 - z_1 = iz_1$ and $z_3 = \left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)z_1$.

By considering the angle $BOC$ prove that $\cos\dfrac{\pi}{12} = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$.

# 1.5 Conjugate theorems

In Lesson 5 we cover:

- Conjugate rules
- Operations with conjugates
- Conjugate root theorem

Syllabus reference: This lesson covers the 10th, 11th and 12th dot points from Arithmetic of complex numbers, from the 9th dot point from Geometric representation of complex numbers, from the 3rd and 4th dot points in Solving equations with complex numbers, in *Introduction to complex numbers*.

## Conjugate rules

The conjugate rules are an incredibly powerful yet underutilised set of tools that we can use in a wide variety of proofs. *Always keep an eye out for an opportunity to use one of these rules and simplify the rest of the proof.*

Where possible use the rules leaving the complex number and its conjugate as $z$ and $\bar{z}$. Only substitute $z = a + ib$ and $\bar{z} = a - ib$, or $z = r\,\text{cis}\,\theta$ and $\bar{z} = r\,\text{cis}(-\theta)$, late in the proof if needed.

We will state the rules involving conjugates here, leaving their proofs for the exercises. We have already seen some of these rules in earlier lessons. When doing the proofs remember that we will generally find Cartesian form easiest if the proof involves addition or subtraction and mod-arg form easiest if the proof involves multiplication or division.

| | |
|---|---|
| Sum of a complex number and its conjugate: | $z + \bar{z} = 2\text{Re}(z) \quad \rightarrow \quad \text{Re}(z) = \dfrac{z + \bar{z}}{2}$ |
| Difference of a complex number and its conjugate: | $z - \bar{z} = 2i\text{Im}(z) \quad \rightarrow \quad \text{Im}(z) = \dfrac{z - \bar{z}}{2i}$ |
| Product of a complex number and its conjugate: | $z\bar{z} = \lvert z\rvert^2 \quad \rightarrow \quad \dfrac{1}{z} = \dfrac{\bar{z}}{\lvert z\rvert^2}, z \neq 0$ |
| Reciprocal of a complex number where $\lvert z\rvert = 1$: | $\dfrac{1}{z} = \bar{z}$ |

In Cartesian form, when $z = a + ib$:

$z + \bar{z} = 2a$

$z - \bar{z} = 2bi$

$z\bar{z} = a^2 + b^2$

$\frac{1}{z} = \frac{a - ib}{a^2 + b^2}$

$\frac{1}{z} = a - ib$ when $|z| = 1$

In mod-arg form, where $z = r(\cos\theta + i\sin\theta)$:

$z + \bar{z} = 2r\cos\theta$

$z - \bar{z} = 2r\sin\theta\, i$

$z\bar{z} = r^2$

$\frac{1}{z} = \frac{1}{r}$

$\frac{1}{z} = \cos(-\theta) + i\sin(-\theta)$ when $|z| = 1$

## Example 1

Find $z + \bar{z}$ for each of the following complex numbers.

**a** $z = 3 - 4i$

$$\begin{aligned} z + \bar{z} &= 2\text{Re}(z) \\ &= 2(3) \\ &= 6 \end{aligned}$$

**b** $z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$\begin{aligned} z + \bar{z} &= 2\text{Re}(z) \\ &= 2\times 4\cos\frac{\pi}{3} \\ &= 8\times\frac{1}{2} \\ &= 4 \end{aligned}$$

## Example 2

Find $z - \bar{z}$ for each of the following complex numbers.

**a** $z = 3 - 4i$

$$\begin{aligned} z - \bar{z} &= 2i\text{Im}(z) \\ &= 2i(-4) \\ &= -8i \end{aligned}$$

**b** $z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$\begin{aligned} z - \bar{z} &= 2i\text{Im}(z) \\ &= 2i\left(4\sin\frac{\pi}{3}\right) \\ &= 8i\times\frac{\sqrt{3}}{2} \\ &= 4\sqrt{3}i \end{aligned}$$

## Example 3

Find $z\bar{z}$ for each of the following complex numbers.

**a** $z = 3 - 4i$

$$\begin{aligned} z\bar{z} &= |z|^2 \\ &= a^2 + b^2 \\ &= 3^2 + (-4)^2 \\ &= 25 \end{aligned}$$

**b** $z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$\begin{aligned} z\bar{z} &= |z|^2 \\ &= 4^2 \\ &= 16 \end{aligned}$$

## Example 4

Find $\frac{1}{z}$ for each of the following complex numbers, leaving the answer in the same form as $z$.

**a** $z = 3 - 4i$

$$\begin{aligned} \frac{1}{z} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{3 + 4i}{3^2 + (-4)^2} \\ &= \frac{3}{25} + \frac{4}{25}i \end{aligned}$$

**b** $z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$\begin{aligned} \frac{1}{z} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)}{4^2} \\ &= \frac{1}{4}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \end{aligned}$$

## Example 5

If $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ find $\frac{1}{z}$ in the form $a + ib$.

$$\begin{aligned} |z| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= 1 \\ \therefore \frac{1}{z} &= \bar{z} \\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{aligned}$$

## Operations with conjugates

We will now look at the conjugate of a sum and the conjugate of a product, but these rules extend to differences and quotients, so we see that:

The conjugate of a sum equals the sum of the conjugates:

The conjugate of a product equals the product of the conjugates:

The conjugate of a difference equals the difference of the conjugates:

The conjugate of a quotient equals the quotient of the conjugates:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1} \times \overline{z_2}$$

$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

$$\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$$

The rules for sums and products extend to multiple complex numbers.

$$\overline{z_1 + z_2 + \ldots + z_n} = \overline{z_1} + \overline{z_2} + \ldots + \overline{z_n}$$

$$\overline{z_1 \times z_2 \times \ldots \times z_n} = \overline{z_1} \times \overline{z_2} \times \ldots \times \overline{z_n}$$

## Example 6

If $\alpha$ and $\beta$ are the roots of $2z^2 + (1+i)z + (10-3i) = 0$, find:

**a** $\overline{\alpha} + \overline{\beta}$

$$\begin{aligned}\overline{\alpha} + \overline{\beta} &= \overline{\alpha + \beta} \\ &= \overline{-\frac{b}{a}} \\ &= \overline{-\left(\frac{1+i}{2}\right)} \\ &= -\frac{1-i}{2}\end{aligned}$$

**b** $\overline{\alpha} \times \overline{\beta}$

$$\begin{aligned}\overline{\alpha} \times \overline{\beta} &= \overline{\alpha \times \beta} \\ &= \overline{\frac{c}{a}} \\ &= \overline{\frac{10-3i}{2}} \\ &= \frac{10+3i}{2}\end{aligned}$$

## Conjugate root theorem

If $z$ is a non-real root of a polynomial with real coefficients, then $\bar{z}$ is also a root.

We will prove this result in the exercise, using the rules for operations with conjugates.

**Important note:** the conjugate root theorem only applies to polynomials where each coefficient is real. This is a common cause of mistakes in exams.

For example, the conjugate root theorem applies to $P(z) = z^2 + 4z + 8$, but NOT to $P(z) = z^2 + 4iz + 2$ or $P(z) = z^2 + (2 + 3i)z - 4i$.

Consider the roots of $P(z) = z^2 + 4z + 8$:

$$z = \frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{-16}}{2}$$

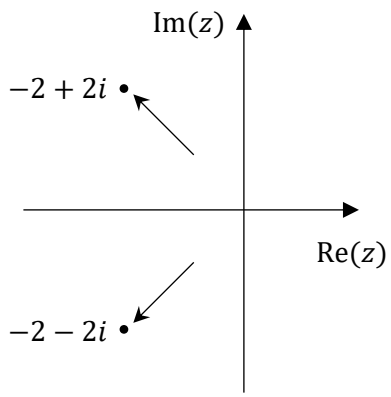
$$= \frac{-4 \pm 4i}{2}$$

$$= -2 - 2, -2 + 2i$$

Looking at this example:

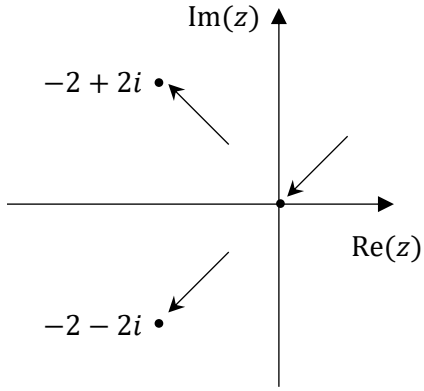
- the coefficients of $P(z) = z^2 + 4z + 8$ are all real: $1, 4$ and $8$
- $-1 - i$ is a non-real root (i.e. it cannot be written in the form $z = k$ where $k \in \mathbb{R}$)
- its conjugate $-1 + i$ is also a root

On the diagram at right we can see that the roots are non-real and occur as a conjugate pair.



Some other important points that often cause confusion:

- a polynomial with real coefficients can also have one or more real roots - it is only if there is a non-real root that its conjugate will also be a root. For example, the diagram on the right shows the roots of $P(z) = z^3 + 4z^2 + 8z$, with a single real root at $z = 0$, as well as roots at $z = -2 \pm 2i$




If one or more of the coefficients are not real, then any non-real roots might still occur as a conjugate pair by coincidence, but there is no guarantee.

A standard HSC-style question is in two parts. In the first part you are given one non-real root and prove that it is a root by substituting into the polynomial - usually a cubic or quartic. In the second part you find any other roots, and may also need to factorise the polynomial.

Once you have the non-real root and its conjugate, there are multiple options to find the other roots:

- the easiest method is to use the sum and product of roots
- alternatively you can find a quadratic factor by expanding $(z - \alpha)(z - \bar{\alpha}) = z^2 - 2\text{Re}(\alpha)z + |\alpha|^2$, then divide the polynomial by the quadratic factor to find the other linear or quadratic factor.

The fundamental theorem of algebra states that a polynomial of degree $n$ has exactly $n$ complex roots, including multiple roots. So a cubic must have three complex roots and a quartic has four roots - use that fact to make sure you have found all roots.

Depending on the question, there may be alternative methods, as we will see in Example 8.

## Example 7

**a** Prove that $2+i$ is a root of $P(z)=z^3-3z^2+z+5$.

$$(2+i)^3-3(2+i)^2+(2+i)+5$$
$$=2^3+3(2)^2i+3(2)(i)^2+i^3-3\left(2^2+4i+i^2\right)+(2+i)+5$$
$$=8+12i-6-i-12-12i+3+2+i+5$$
$$=0$$
$\therefore 2+i$ is a root.

Substitute clearly and show sufficient working - one step for each line is best.

**b** Hence find all roots of $P(z)$.

$\therefore 2-i$ must be a root since all coefficients are real.

$$\therefore \sum \alpha = -\frac{b}{a}$$
$$2+i+2-i+\gamma=-\frac{-3}{1}$$
$$\therefore 4+\gamma=3$$
$$\therefore \gamma=-1$$
$\therefore$ The roots of $z^3-3z^2+z+5=0$ are $-1, 2\pm i$.

Let's have a look at Question 12 (e) from the 2023 HSC. Due to the nature of part (ii), we will see four possible solutions - some much easier than others.

## Example 8

The complex number $2+i$ is a zero of the polynomial

$$P(z) = z^4 - 3z^3 + cz^2 + dz - 30$$

where $c$ and $d$ are real numbers.

**a** Explain why $2-i$ is also a root of the polynomial $P(z)$.

Since all coefficients are real, non-real roots occur in conjugate pairs.

$\therefore \overline{2+i} = 2-i$ is also a root.

**b** Find the remaining zeros of the polynomial $P(z)$ using the sum and product of roots.

$$\alpha+\beta+\gamma+\delta = -\frac{b}{a}$$

$$\therefore \alpha+\beta+(2+i)+(2-i) = 3$$

$$\alpha+\beta = -1 \qquad (1)$$

$$\alpha+\beta+\gamma+\delta = \frac{e}{a}$$

$$\alpha\times\beta\times(2+i)\times(2-i) = -30$$

$$\alpha\beta\times(2^2+1^2) = -30$$

$$\alpha\beta = -6 \qquad (2)$$

By inspection from $(1)$ and $(2)$ $\alpha = -3, \beta = 2$.

The other roots are $-3$ and $2$.

Using the sum and product of roots saves a massive amount of time for this question.

Solve simple simultaneous equations by inspection - do not solve them using full working.

**c** Find the remaining zeros of the polynomial $P(z)$ by first finding a quadratic factor using $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(z) + |\alpha|^2$, then using polynomial division.

$$\big(z-(2+i)\big)\big(z-\overline{2+i}\big) = z^2 - 2\text{Re}(2+i)z + |2+i|^2$$

$$= z^2 - 4z + 5 \qquad \text{is a factor.}$$

$$\frac{z^4 - 3z^3 + cz^2 + dz - 30}{z^2 - 4z + 5} = \frac{z^2\big(z^2-4z+5\big) + z\big(z^2-4z+5\big) - 6\big(z^2-4z+5\big)}{z^2-4z+5}$$

$$= z^2 + z - 6$$

$$= (z+3)(z-2)$$

The other roots are $-3$ and $2$.

We don't need to know the values of $c$ and $d$ - the other three coefficients are sufficient to find the other quadratic factor.

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## Example 8 (continued)

**d** Find the remaining zeros of the polynomial $P(z)$ by first finding a quadratic factor using $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(z) + |\alpha|^2$, then letting the other quadratic factor be $z^2 + az + b$.

$\big(z-(2+i)\big)\big(z-\overline{2+i}\big) = z^2 - 2\text{Re}(2+i)z + |2+i|^2$

$= z^2 - 4z + 5$ is a factor.

$\therefore \big(z^2-4z+5\big)\big(z^2+az+b\big) \equiv z^4 - 3z^3 + cz^2 + dz - 30$

Coefficients of $z^3$: $a - 4 = -3 \quad \rightarrow \quad a = 1$

Constants: $5b = -30 \quad \rightarrow \quad b = -6$

$\therefore z^2 + z - 6 = (z+3)(z-2)$ is also a factor.

The other roots are $-3$ and $2$.

**e** Find the remaining zeros of the polynomial $P(z)$ by finding expressions for $P(2+i)$ in terms of $c$ and $d$.

$P(2+i) = 0$

$\therefore (2+i)^4 - 3(2+i)^3 + c(2+i)^2 + d(2+i) - 30 = 0$

$(2+i)^2 = 4 + 4i - 1 = 3 + 4i$

$(2+i)^3 = (3+4i)(2+i) = 6 + 3i + 8i - 4 = 2 + 11i$

$(2+i)^4 = (3+4i)^2 = 9 + 24i - 16 = -7 + 24i$

$\therefore (-7+24i) - 3(2+11i) + c(3+4i) + d(2+i) - 30 = 0$

$(3c + 2d - 43) + (4c + d - 9)i = 0$

$\therefore 3c + 2d = 43 \quad (1)$

$4c + d = 9 \quad (2)$

$(1) - 2(2)$: $-5c = 25 \quad \rightarrow \quad c = -5$

sub in $(2)$: $4(-5) + d = 9 \quad \rightarrow \quad d = 29$

$\big(z-(2+i)\big)\big(z-\overline{2+i}\big) = z^2 - 2\text{Re}(2+i)z + |2+i|^2 = z^2 - 4z + 5$ is a factor of $z^4 - 3z^3 - 5z^2 + 29z - 30$

$$\frac{z^4 - 3z^3 - 5z^2 + 29z - 30}{z^2 - 4z + 5} = \frac{z^2\big(z^2-4z+5\big) + z\big(z^2-4z+5\big) - 6\big(z^2-4z+5\big)}{z^2 - 4z + 5}$$

$= z^2 + z - 6$

$= (z+3)(z-2)$

The other roots are $-3$ and $2$.

Now we can see why we should always use the sum and product of roots for these questions.

Basic

**1** Consider (i) $z = 2 - 3i$ and (ii) $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. For each complex number, find:

**a** $z + \bar{z}$

**b** $z - \bar{z}$

**c** $z\bar{z}$

**d** $\frac{1}{z}$

**2** If $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ find $\frac{1}{z}$.

**3** If $p$ and $q$ are the roots of $3z^2 + (2 - i)z + 6i = 0$, find:

**a** $\overline{p} + \overline{q}$ **b** $\overline{p} \times \overline{q}$

**4 a** Prove that $1 + i$ is a root of $P(z) = z^3 - 2z + 4$.

**b** Hence find all roots of $P(z)$.

**5** Which polynomial could have $2 + i$ as a zero, given that $k$ is a real number?

A. $z^3 - 4z^2 + kz$

B. $z^3 - 4z^2 + kz + 5$

C. $z^3 - 5z^2 + kz$

D. $z^3 - 5z^2 + kz + 5$

**6** A monic polynomial, $f(z)$, of degree $3$ with real coefficients has $3$ and $2 + i$ as two of its roots.

A. $f(z) = z^3 - 7z^2 - 17z + 15$

B. $f(z) = z^3 - 7z^2 + 17z - 15$

C. $f(z) = z^3 + 7z^2 - 17z + 15$

D. $f(z) = z^3 + 7z^2 + 17z - 15$

**7** The sum of the non-real complex numbers $z$ and $w$ is zero. Which of the following statements is FALSE?

A. $\frac{w}{z} = -1$

B. $z - w = 2z$

C. $w - z = 2w$

D. $z + \overline{w} = 0$

**8** Prove $\overline{z_1 + z_2 + \ldots + z_n} = \overline{z_1} + \overline{z_2} + \ldots + \overline{z_n}$. Hint: let $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$ etc

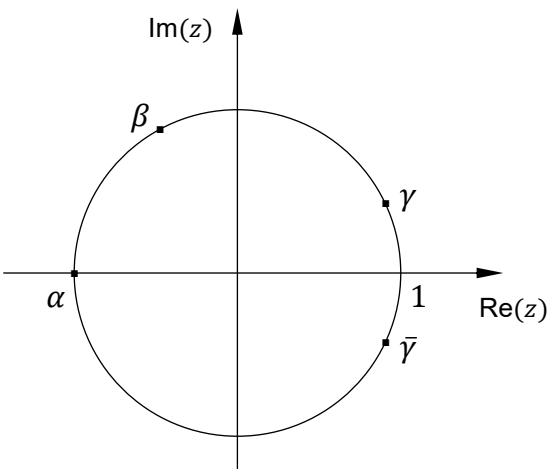
**9** Prove $\overline{z_1 \times z_2 \times \ldots \times z_n} = \overline{z_1} \times \overline{z_2} \times \ldots \times \overline{z_n}$. Hint: let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ etc.

**10** The polynomial $P(z)$ has real coefficients, and $z = 2 + 3i$ is a root of $P(z)$. What quadratic must be a factor of $P(z)$?

**11** The complex number $1 + i$ is a zero of the polynomial $P(z) = z^4 - 2z^3 + z^2 + 2z - 2$. Find the remaining zeros of the polynomial.

**12** Consider the complex number $z = a + ib$. The product of $z$ and $\bar{z}$ is $25$. The sum of $z$ and $\bar{z}$ is $8$. Find $z$.

**13** Four roots of the polynomial $P(x)$ lie on the unit circle on a complex plane, as shown. There may be other roots.




NOT TO SCALE

Which statement about $P(x)$ MUST be TRUE.

**A** If $P(x)$ is a quartic then all its coefficients are real

**B** If $P(x)$ is a quartic then at least one of its coefficients is real

**C** If all of the coefficients of $P(x)$ are real then its degree is at least $5$

**D** $P(x)$ is in the form $z^n + 1 = 0$ where $n$ is odd and at least $5$

**14** The polynomial $P(z)$ has real coefficients, and the non-real numbers $\alpha$ and $-i\alpha$ are zeros of $P(z)$, where $\bar{\alpha} \neq i\alpha$. Explain why $\bar{\alpha}$ and $i\bar{\alpha}$ are also zeros of $P(z)$.

**15a** Given $z$ is a root of $az^3 + bz^2 + cz + d$ where $a, b, c, d$ are real, prove $\bar{z}$ is also a root.

**b** Find all three roots of $z^3 - 6z^2 + 13z - 20$ given $1 + 2i$ is one of the roots.

**16** It is known that a particular complex number $z$ is NOT a real number. Which of the following could be true for this number $z$?

A. $\bar{z} = iz$ B. $\bar{z} = |z|^2$ C. $\text{Re}(iz) = \text{Im}(z)$ D. $\arg(z^3) = \arg(z)$

**17** Let $z = a + ib$ where $a$ and $b$ are real and non-zero. Which if the following is **not** true?

A. $z + \bar{z}$ is real B. $\frac{z}{\bar{z}}$ is non−real C. $z^2 - (\bar{z})^2$ is real D. $z\bar{z}$ is real and positive

**18** Show that the solutions to the equation $2z^3 - z = \bar{z}$ are in the form $z^3 = \text{Re}(z)$, and hence find the three solutions.

**19** Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where $p$ and $q$ are real, what are the values of $p$ and $q$?

Challenging

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**20** If $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then find the value of $z_1^2 + z_2^2 + z_3^2$.

**21** Given $|z|^2 = z\bar{z}$, prove $|z_1 z_2| = |z_1| \times |z_2|$.

**22** Given the complex number $z = \cos\theta + i\sin\theta$, show that $w = \dfrac{z^2 - 1}{z^2 + 1}$ is purely imaginary, where $z^2 + 1 \neq 0$.

**23** Find all the complex numbers $z_1, z_2, z_3$ that satisfy the following three conditions simultaneously.

$$\begin{cases} |z_1| = |z_2| = |z_3| \\ z_1 + z_2 + z_3 = 1 \\ z_1 z_2 z_3 = 1 \end{cases}$$

**24** Let $\alpha, \beta$ and $\gamma$ be the roots of the cubic equation $z^3 + bz^2 + cz + 8 = 0$, where $b, c$ are real. Furthermore $\alpha^2 + \beta^2 = 0$ and $\beta^2 + \gamma^2 = 0$.

**a** Explain why $\beta$ is real and $\alpha$ and $\gamma$ are not real.

**b** Show that $\alpha$ and $\gamma$ are purely imaginary.

**c** Find $b$ and $c$.

**25** It is known that $\sin n\theta = \dfrac{z^n - z^{-n}}{2i}$ and $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$. (Do NOT prove this.) Prove that $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.

# 1.6 Complex numbers as vectors

In Lesson 6 we cover:

- Complex numbers as vectors
- Rotation and dilation
- Midpoint
- Triangle inequality

Syllabus reference: This lesson covers the 10th dot point from Geometric representation of complex numbers, and from the 8th to 11th dot points in Powers and roots of complex numbers, in *Introduction to complex numbers*.

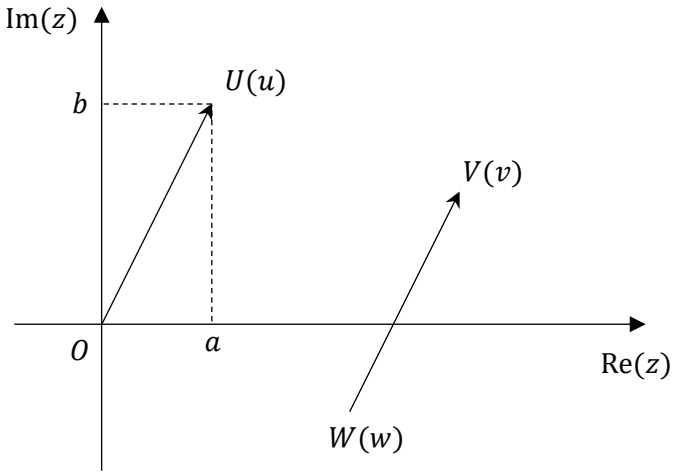
## Complex numbers as vectors

Representing complex numbers as vectors helps us understand relationships between complex numbers more easily.

It is important to note that two-dimensional vectors (which we cover in Extension 1 *Introduction to vectors* and Extension 2 *Further work with vectors*) and complex vectors (which we cover here in Extension 2 *Introduction to complex numbers*) behave the same way for addition and subtraction. We can also multiply and divide complex vectors, as well as raising them to powers or finding their roots, which we cannot do with two-dimensional vectors.

In *Introduction to vectors* and *Further work with vectors* we are only interested in the direction of a two-dimensional vector as a concept, whereas in complex numbers we are interested in the argument of complex vectors, allowing us to solve problems involving rotation.

We can look at a complex number as a vector. For example, $\overrightarrow{OU}$ represents the complex number $u = a + ib$, which can be viewed as a vector starting at zero which travels $a$ units to the right and goes up by $b$ units. Its tail is at zero and its tip is at $U$.




Since $\overrightarrow{OU}$ and $\overrightarrow{WV}$ have the same magnitude and direction they are equal vectors, so $\overrightarrow{WV}$ also equals $a + ib$.

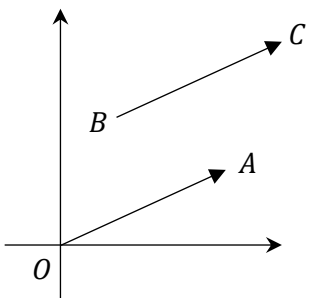
**Tip minus tail**

Any complex vector equals the complex number at its tip minus the complex number at its tail. When we label the vector we put the pronumeral at the tail first followed by the pronumeral at the tip.

So $\overrightarrow{WV} = v - w$, and similarly, $\overrightarrow{OU} = u - 0 = u$.

**Position and displacement vectors**

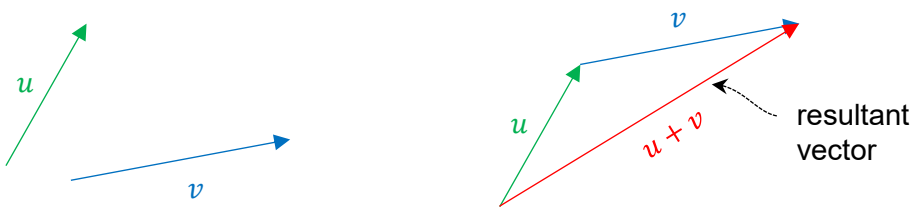
Consider the vectors $\overrightarrow{OA}$ and $\overrightarrow{BC}$ at right.




- $\overrightarrow{OA}$ is a position vector as it has its tail at the origin and its tip at the position of $A$. A position vector tells us where a complex number is compared with zero
- $\overrightarrow{BC}$ is a displacement (or relative) vector as it gives us the displacement of $C$ from $B$. A displacement vector tells us the difference between two complex numbers, equal to the final number minus the initial number.

**Adding vectors**

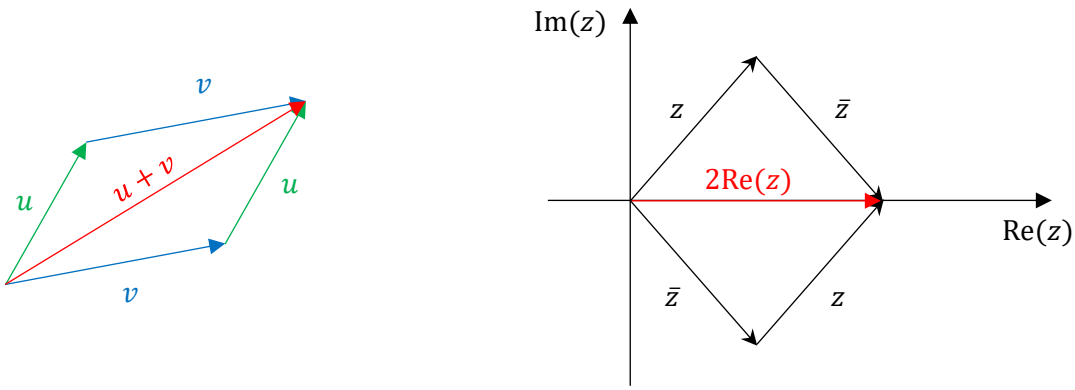
When we add two vectors the resultant vector can be found as the third side of the triangle formed by moving the tail of one vector to the tip of the other. This is known as the triangle law. The resultant vector goes from the tail of the first vector to the tip of the second as shown below, and in this case equals $u + v$.




We can add the vectors in any order and achieve the same result as shown middle and right, as vectors follow the commutative law $u + v = v + u$. Note that the resultant vectors $u + v$ (above) and $v + u$ (below) are the same length and point in the same direction, so represent equal complex numbers.

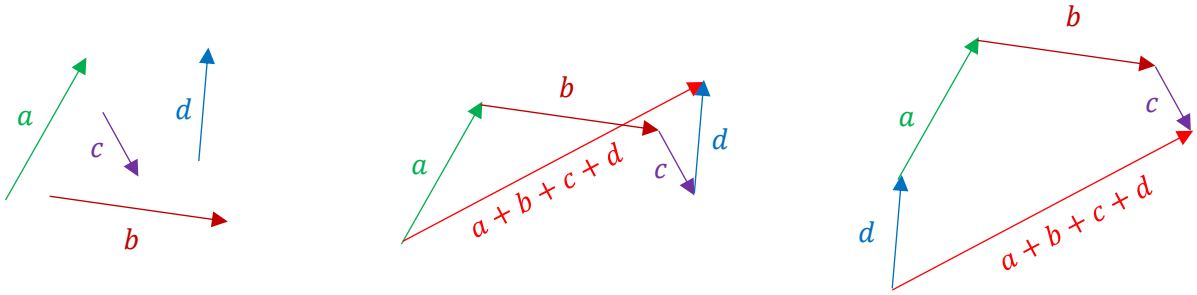



The triangle law for addition leads to a few other important results, the first being the parallelogram law. We start by arranging the two triangles shown above into the parallelogram as shown below left - the diagonal of a parallelogram (from tail to tip) with sides $u$ and $v$ is the sum of the vectors.




In an earlier lesson, we saw that $z + \bar{z} = 2\mathrm{Re}(z)$. We can see that this is an application of the parallelogram law, as shown above right.

We can add any number of vectors, leading to the polygon law. Consider the vectors representing the complex numbers $a, b, c$ and $d$ shown below left. If we arrange them in any order, tail to tip, we get the resultant vector $a + b + c + d$. Two of the $4!$ ways of adding the vector are shown.



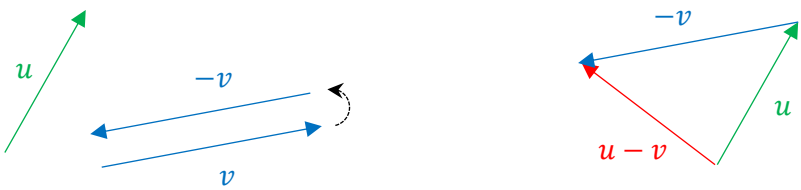

This is an important concept for the $n^{\text{th}}$ roots of a complex number, as the sum of the $n^{\text{th}}$ roots of any complex number ($n \geq 2$) is always equal to zero. We can show this using the polygon law - if we join vectors representing all $n^{\text{th}}$ roots tip to tail they form a loop, ending back where they started.

**Subtracting vectors**

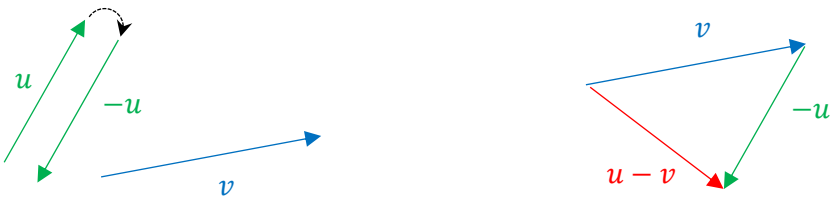
There are two ways to subtract vectors, each of which is useful in different circumstances.

*Method 1 - using the negative of a vector and the triangle law*

We can view $u - v$ as $u + (-v)$. We first find $-v$ by reversing $v$ (below left), then move the tail of this new vector to the tip of $u$ (right). The vector from tail to tip is $u - v$.

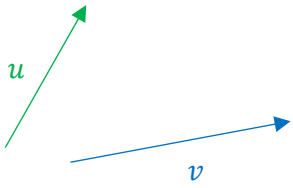


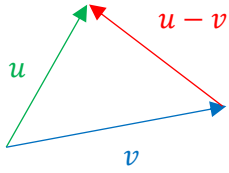

If we wanted $v - u$ then we find $-u$ and move its tail to the end of $v$ as below. Note that $u - v$ (above) and $v - u$ (below) are negative vectors (the same modulus with opposite directions).



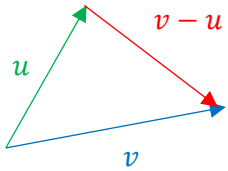
*Method 2 - using the vector between the tips when tails coincide*

If the tails of the two vectors are at the same point (move one or both of them if they are not) then we can find $u - v$ as the vector from the tip of $v$ to the tip of $u$. Notice this is 'tip minus tail'.

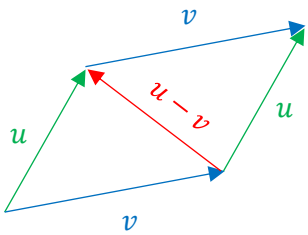


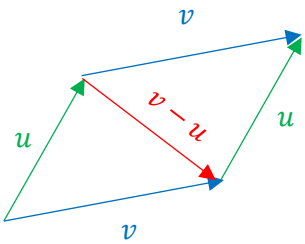




Similarly we can find $v - u$ as the vector from the tip of $u$ to the tip of $v$. Again we see that $u - v$ and $v - u$ are negative vectors.

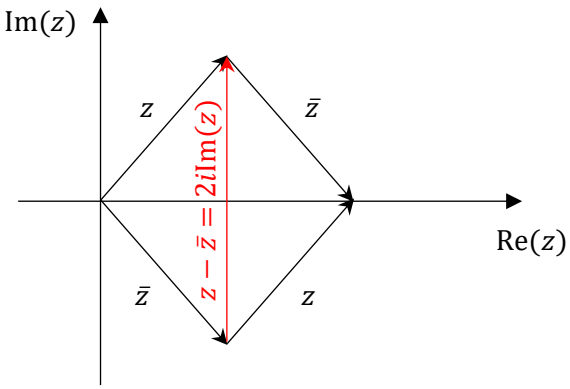



Extending the two triangles above into parallelograms, we can see that the diagonal running across the parallelogram gives the difference of the two vectors. Depending on which way we draw the diagonal we get either $u - v$ or $v - u$.



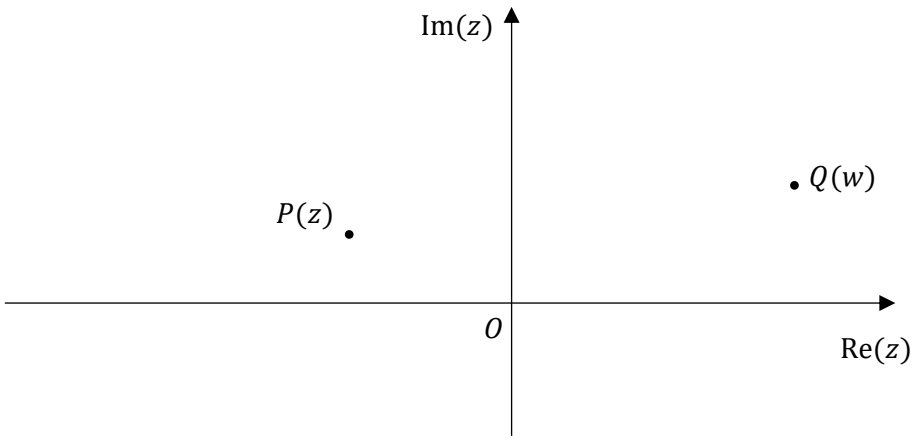




In an earlier lesson, we saw that $z - \bar{z} = 2i\text{Im}(z)$. We can see that this is an application of the parallelogram law, as shown below.



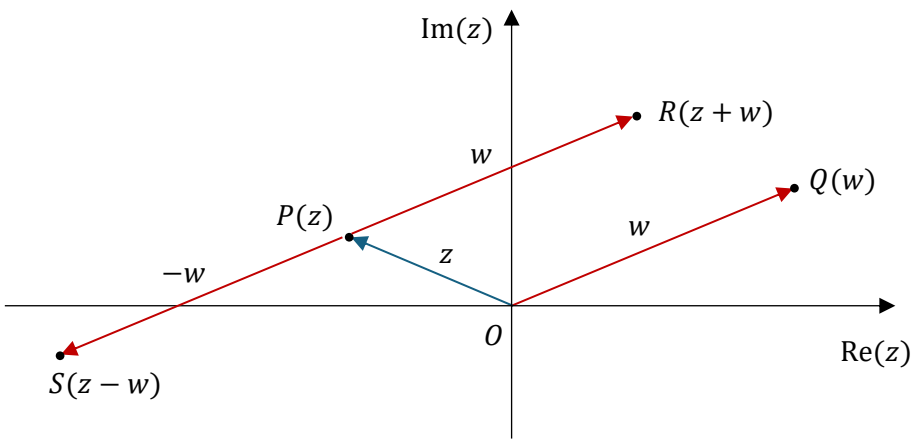
## Example 1

The points $P$ and $Q$ on the complex plane represent the complex numbers $z$ and $w$ respectively.




By first drawing the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, mark the points $R$ representing $z+w$, and $S$ representing $z-w$.

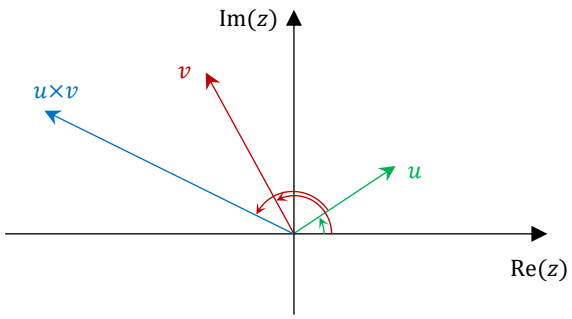
Solution:



## Rotation and dilation

When a complex number is multiplied by another complex number, then starting with the vector representing the original complex number:

- dilate (scale) the modulus by the modulus of the second number
- rotate the vector anticlockwise by the argument of the second number.




The work on complex vectors is far more likely to involve multiplication than division, however division is the inverse operation of multiplication, so the effects occur in the opposite directions:

- dilate (scale) the modulus by the reciprocal of the modulus of the second number
- rotate the vector clockwise by the argument of the second number.

Let's have a look specifically at multiplying by a real number, an imaginary number and a complex number.

**Multiplying complex vectors by a real constant - parallel vectors**

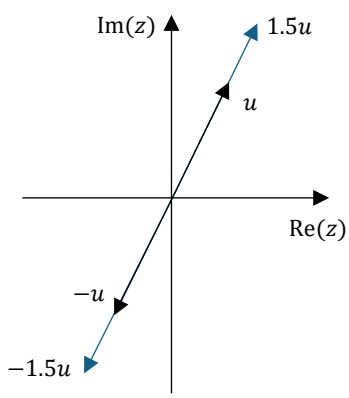
When we multiply a complex vector by a constant real number $k$:

- the modulus is dilated by a factor of $|k|$
- the direction stays the same if $k > 0$, or becomes the opposite direction if $k < 0$, so the new vector is parallel to the original vector.

This matches what we have just said about multiplying by a complex number, as a real number has an argument of $0$ if it is positive (so no rotation) or $\pi$ if it is negative (so reverses direction).

Multiplying a complex number by a positive constant leaves the argument the same (no rotation) and dilates the modulus.

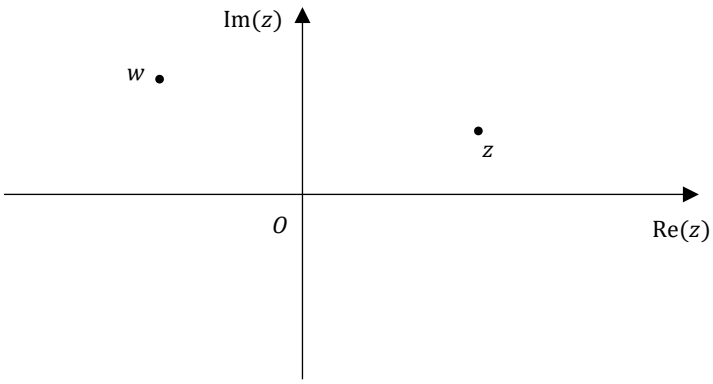
Multiplying a complex number by $-1$ rotates the vector by $\pi$ and leaves the modulus the same (the vector does not change length, it just rotates).



Multiplying a complex number by any other negative constant rotates the vector by $\pi$ and scales the modulus – the point lies on the line through the original point and zero but on the other side of zero.

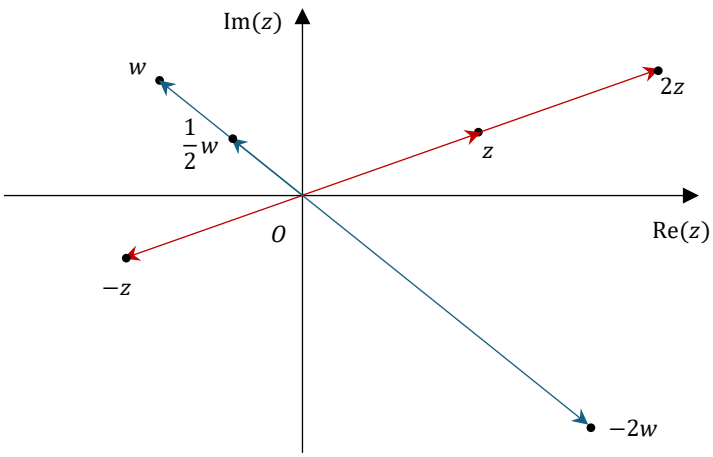
## Example 2

The complex plane shows the complex numbers $w$ and $z$.




Mark possible positions for

**a** $2z$ **b** $\frac{1}{2}w$ **c** $-z$ **d** $-2w$



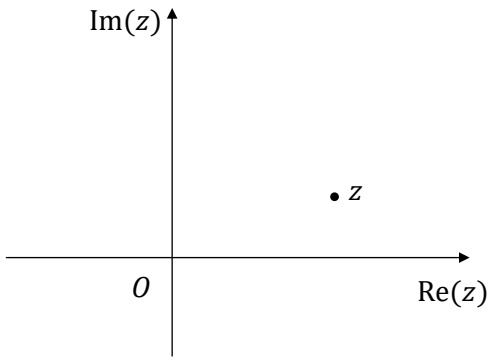
**Rotating anticlockwise by $\frac{\pi}{2}$ - perpendicular vectors**

We have seen that an essential property of $i$ is that when we multiply by $i$ we rotate anticlockwise by $\frac{\pi}{2}$. When we multiply by $-i$ we rotate clockwise by $\frac{\pi}{2}$.

This matches what we have just seen about multiplying by a complex number, as $i$ has an argument of $\frac{\pi}{2}$ and a modulus of 1, so we rotate by $\frac{\pi}{2}$ and scale the modulus by 1 - it is unchanged.

## Example 3

The complex plane shows the complex number $z$.



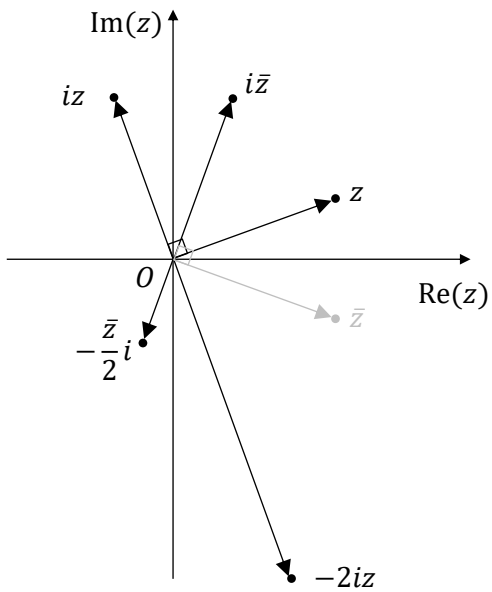

Mark possible positions for:

**a** $iz$

**b** $-2iz$

**c** $i\bar{z}$

**d** $-\frac{\bar{z}}{2}i$

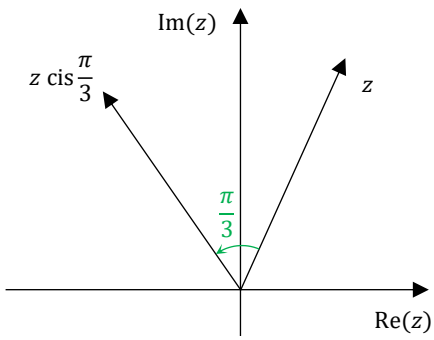


**Rotating anticlockwise by $\alpha$**

We rotate a complex vector anticlockwise by multiplying by a complex number which has an argument equal to the angle we wish to rotate anticlockwise by.

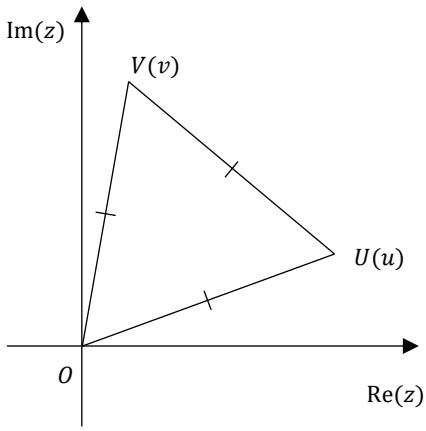
If we want to rotate a vector without changing its length then we want the modulus of the multiplier to be $1$ and the argument to be the angle we want the vector to rotate, so $\cos\theta + i\sin\theta$ or its equivalent in Cartesian form.

So, to rotate anticlockwise by $\frac{\pi}{3}$ radians, we would multiply by $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ or $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.




## Example 4

The complex numbers $0, u$ and $v$ form the vertices of an equilateral triangle in the complex plane, as shown.




Show that $u^2 + v^2 = uv$.

$\angle VOU = \frac{\pi}{3}$ angle in an equilateral triangle

Let $v = u\ \text{cis}\frac{\pi}{3}$

$u^2 + v^2$

$= u^2 + u^2\left(\text{cis}\frac{\pi}{3}\right)^2$

$= u^2\left(\text{cis}\,0 + \text{cis}\frac{2\pi}{3}\right)$

$= u^2\,\text{cis}\frac{\pi}{3}\left(\text{cis}\left(-\frac{\pi}{3}\right) + \text{cis}\frac{\pi}{3}\right)$

$= u^2\,\text{cis}\frac{\pi}{3}\left(2\text{Re}\left(\text{cis}\frac{\pi}{3}\right)\right)$

$= u^2\,\text{cis}\frac{\pi}{3}\left(2\cos\frac{\pi}{3}\right)$

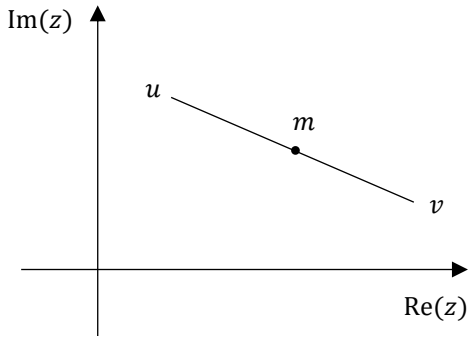
$= u\times u\,\text{cis}\frac{\pi}{3}\left(2\times\frac{1}{2}\right)$

$= u\times v$

From this step we take out a common factor of $\text{cis}\frac{\pi}{3}$, since $\frac{\pi}{3}$ is the average of $0$ and $\frac{2\pi}{3}$. This leaves the sum of a complex number and its conjugate in the brackets, which we can simplify using $z + \bar{z} = 2\text{Re}(z)$ - much simpler than the alternatives. This is a useful technique whenever we are trying to turn a sum into a product.

## Midpoint

The midpoint of two points on the complex plane is often needed in proofs. The midpoint can be found in two different ways depending on the form of the complex numbers. In the diagram below we see that the midpoint of the complex numbers $u$ and $v$ is $m$.




Given that the complex plane is an extension of the number line, the midpoint is the average of the two complex numbers:

$$\boxed{m = \frac{u+v}{2}}$$

If the complex numbers are in Cartesian form, we can use the midpoint formula. If $u = x_1 + iy_1$ and $v = x_2 + iy_2$ then:

$$\boxed{m = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \frac{x_1+x_2}{2} + \frac{y_1+y_2}{2}i}$$

## Example 5

Find the midpoint of $u = 1 + 2i$ and $v = 3 - 4i$.

$$\frac{1+3}{2} + \frac{2-4}{2}i$$
$$= 2 - i$$

## Triangle inequality

In Lesson 3 we looked at rules for working with the modulus of a product, a quotient and a power, and saw that we could easily break them up using the rules $|ab| = |a| \times |b|$, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ where $|b| \neq 0$ and $|a^n| = |a|^n$.

The rule for the modulus of a sum isn't as straightforward - we CANNOT say that $|a + b| = |a| + |b|$. The value $|a| + |b|$ is actually the maximum possible value, and we say that:

$$\boxed{|a + b| \leq |a| + |b|}.$$

This is the triangle inequality - it gets its name as the length of any side in a triangle is less than or equal to the sum of the other two sides.

If the sum of the two sides equals the third side then the three sides form one straight line, which is sometimes called a degenerate triangle, and is not a triangle as we know it. It is important that students use $\leq$, not $<$, when using the triangle inequality.

The rule can be applied to the sum of any number of complex numbers:

$$\boxed{|z_1 + z_2 + z_3 + \ldots + z_n| \leq |z_1| + |z_2| + |z_3| + \ldots + |z_n|}.$$

Memorise this rule as 'the modulus of a sum is less than or equal to the sum of the moduli'.

The triangle inequality only applies to non-negative real numbers, such as lengths. It does NOT apply to any number that could be negative or non-real. So we can apply the triangle inequality to the moduli of complex numbers, but NOT to the complex numbers themselves. We also cannot use the triangle inequality with trigonometric functions, which can be positive or negative, unless we take their absolute values.

## Example 6

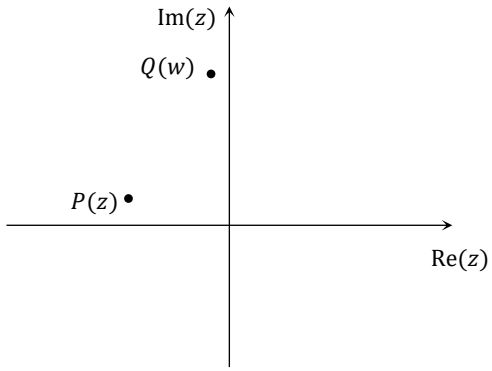
Given $|z| < \frac{1}{2}$, show that $\left|(1+i)z^3 + iz\right| < \frac{3}{4}$.

$$\begin{aligned}
\left|(1+i)z^3 + iz\right| &\leq \left|(1+i)z^3\right| + |iz| && \text{since the modulus of a sum} \leq \text{the sum of the moduli} \\
&= |1+i||z|^3 + |i||z| && \text{modulus rules} \\
&= \sqrt{1^2+1^2}|z|^3 + |z| && \text{since the modulus of } i \text{ is } 1 \\
&< \sqrt{2}\left(\frac{1}{2}\right)^3 + \frac{1}{2} && \text{since } |z| < \frac{1}{2} \\
&= \frac{\sqrt{2}+4}{8} \\
&< \frac{2+4}{8} && \text{since } \sqrt{2} < 2 \\
&= \frac{3}{4}
\end{aligned}$$

$\therefore \left|(1+i)z^3 + iz\right| < \frac{3}{4}$.

Basic

**1** The points P and Q on the complex plane represent the complex numbers $z$ and $w$ respectively. Mark the points $R$ representing $z + w$, and $S$ representing $z - w$.

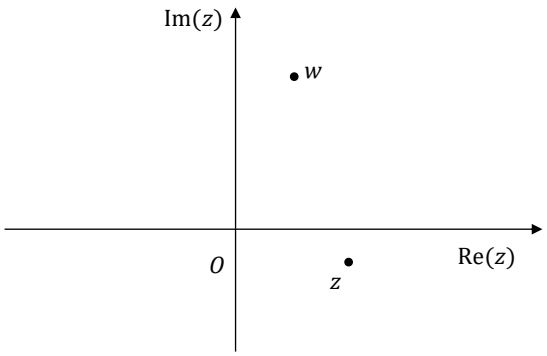



**2** The complex plane shows the complex numbers $w$ and $z$. Mark possible positions for

**a** $2z$

**b** $\frac{1}{2}w$

**c** $-z$



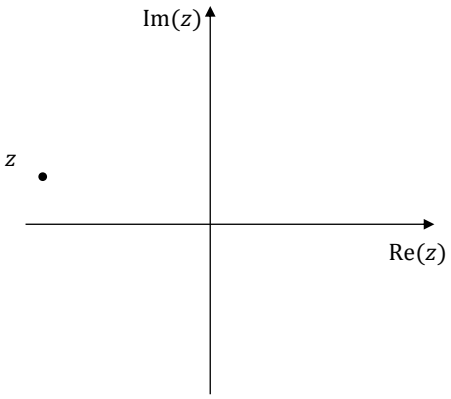

**3** The complex plane shows the complex number $z$. Mark possible positions for

**a** $iz$

**b** $-\frac{i}{2}z$

**c** $i\bar{z}$

**d** $-i\bar{z}$

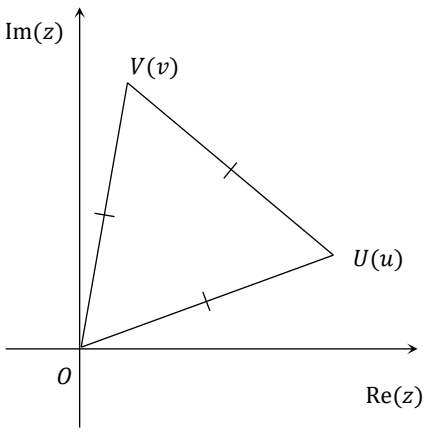



**4** Find the midpoint $M$ of $u = -1 + 3i$ and $v = 5 - i$.

**5** The complex numbers $0, u$ and $v$ form the vertices of an equilateral triangle in the complex plane.

It is given that $u = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$.

Show that $u^2 + v^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

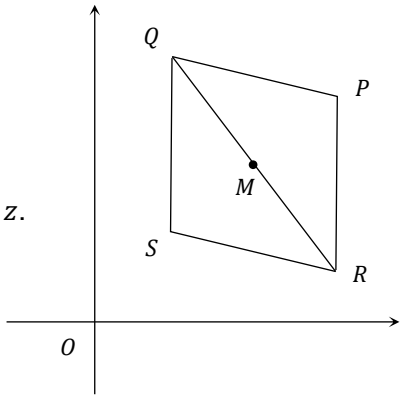


**6** The point $P$ on the Argand diagram represents the complex number $z$. The points $Q$ and $R$ represent the points $\omega z$ and $\bar{\omega} z$ respectively, where $\omega = \text{cis}\frac{\pi}{3}$.

The point $M$ is the midpoint of $QR$.

a Find the complex number representing $M$ in terms of $z$.

b The point $S$ is chosen so that $PSQR$ is a parallelogram. Find the complex number represented by $S$.

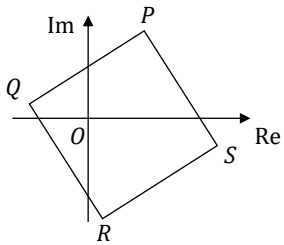



**7** Given $|z| < 1$, show that $\left|(1-i)z^2 + \sqrt{2}iz\right| < 3$.

**8** In the complex plane vectors $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}, \overrightarrow{OS}$ represent the complex numbers $p, q, r, s$ respectively where $PQRS$ is a square.

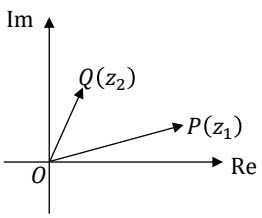
Which of the following statements is correct?

**A** $q - s = i(p - r)$
**B** $q - p = i(s - r)$
**C** $s - q = i(p - r)$
**D** $q - r = i(p - s)$



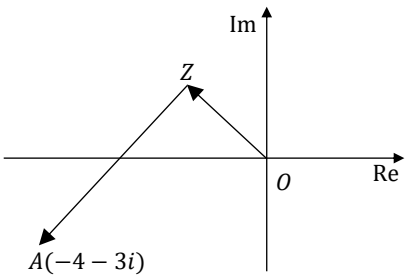

**9** The points $P$ and $Q$ in the first quadrant represent the complex numbers $z_1$ and $z_2$ respectively, as shown in the diagram.
Which statement about the complex number $z_2 - z_1$ is true?

**A** It is represented by the vector $\overrightarrow{QP}$
**B** Its principal argument lies between $\frac{\pi}{2}$ and $\pi$
**C** Its real part is positive
**D** Its modulus is greater than $|z_1 + z_2|$



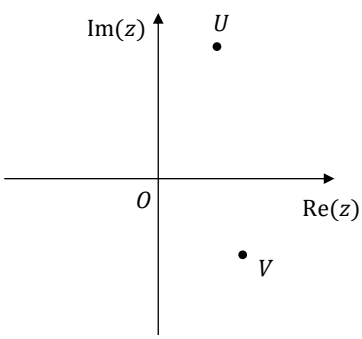

**10** The point $A$ represents the complex number $-4 - 3i$. $\angle OZA = 90°$ and $|ZA| = 2|z|$. Find the complex number represented by the point $Z$.

**A** $-1 + \sqrt{2}i$
**B** $-1 + 2i$
**C** $-2 + i$
**D** $-\sqrt{2} + i$

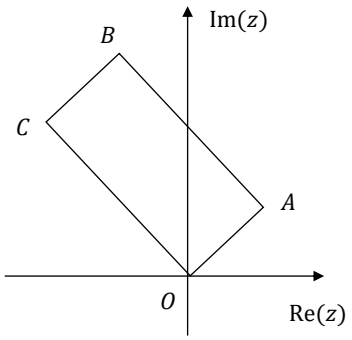



Medium

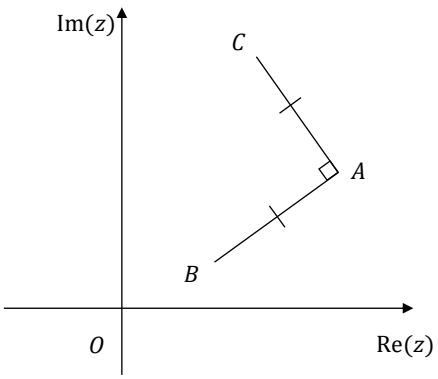
**11** The complex numbers $u$ and $v$ are indicated on the complex plane by $U$ and $V$ respectively. Given $u + v + w = 0$, sketch a possible location for $W$, representing the complex number $w$.



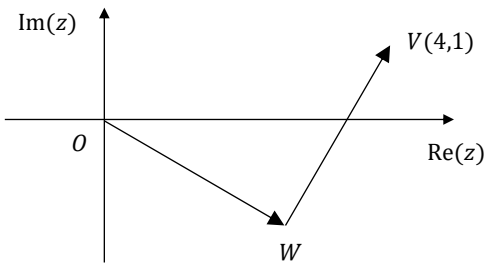
**12** $OABC$ is a rectangle in a complex plane where $O$ is the origin and point $A$ corresponds to the complex number $1+i$. $C$ lies in the second quadrant as shown. Given the length of the rectangle is three times its breadth and $OA$ is one of the shorter sides, find the complex number represented by $C$.



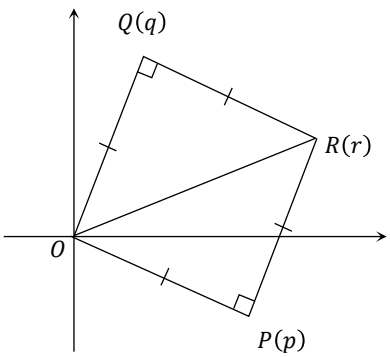

**13** The point $A$ represents the complex number $a$ and the point $B$ represents the complex number $b$. The point $B$ is rotated clockwise about $A$ through a right angle to take up the position $C$, representing the complex number $c$. Show that $c=(1+i)a-ib$.



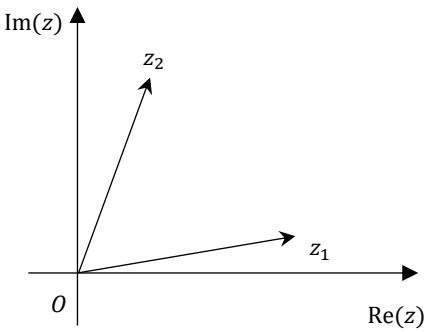

**14** The point $V$ represents the complex number $4+i$. $\angle OWV=\frac{\pi}{2}$ and $|VW|=|OW|$. Find the complex number represented by the point $W$.



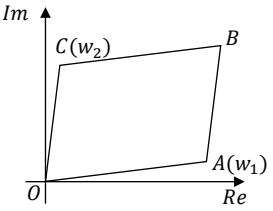

**15** The points $P, Q$ and $R$ on the complex plane represent the complex numbers $p, q$ and $r$ respectively. The triangles $POR$ and $OQR$ are right angled isosceles with right angles at $P$ and $Q$ as shown. Show that $pq=\frac{r^2}{2}$




**16** Given that the sum of any two sides of a triangle must be greater than or equal to the length of the other side, prove for two complex numbers $z_1, z_2$ that $|z_1+z_2|\leq|z_1|+|z_2|$.



**17** In the complex plane, $OABC$ is a rhombus with $\angle COA = \frac{\pi}{3}$. The points $A$ and $C$ represent the complex numbers $\omega_1$ and $\omega_2$ respectively.



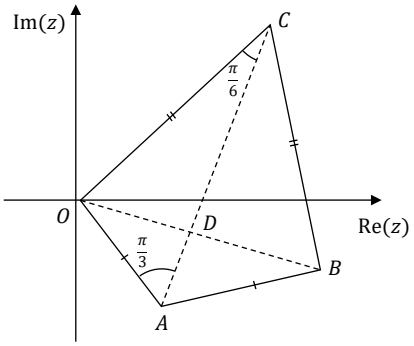

**a** Explain why $\omega_2 = \omega_1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

**b** Write down, in terms of $\omega_1$ only, the complex numbers represented by the vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$.

**c** By considering $i(\omega_1 + \omega_2)$, show that the diagonals $OB$ and $AC$ of the rhombus are perpendicular.

**18** The points $A$ and $B$ represent the complex numbers $z_1 = 2 - i$ and $z_2 = 8 + i$ respectively. Find all possible complex numbers $z_3$, represented by $C$, such that $\Delta ABC$ is isosceles and right angled at $C$.

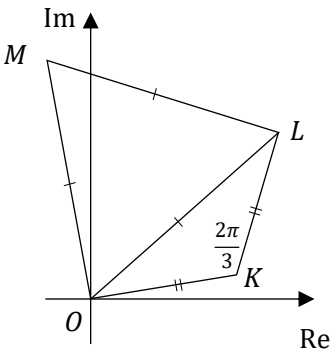
**19** In the complex plane, $OABC$ is a kite. $OA = AB, OC = CB$, $\overrightarrow{OA} = w, \angle OAC = \frac{\pi}{3}$ and $\angle OCA = \frac{\pi}{6}$.
Find the complex numbers $\overrightarrow{OB}$ and $\overrightarrow{OC}$ in terms of $w$.




**20** The points $A(\omega)$ and $B(\phi)$ are on the Argand diagram such that $\angle AOB = \frac{2\pi}{3}$ and $\Delta AOB$ is isosceles with $|OA| = |OB|$. Prove that $(\omega + \phi)^2 = \omega\phi$

**21** The points $K$ and $M$ in a complex plane represent the complex numbers $\alpha$ and $\beta$ respectively. The triangle $OKL$ is isosceles and $\angle OKL = \frac{2\pi}{3}$. The triangle $OLM$ is equilateral.

Show that $3\alpha^2 + \beta^2 = 0$.

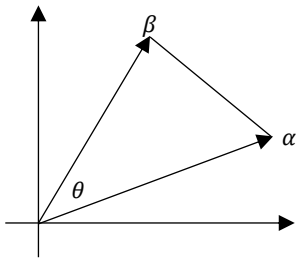



**22** The complex number $z$ satisfies $\left|z - \frac{4}{z}\right| = 2$. Using the triangle inequality show that $|z| \leq \sqrt{5} + 1$.

**23** For the complex numbers $z$ and $w$, it is known that $\arg\left(\frac{z}{w}\right) = -\frac{\pi}{2}$. Find $\left|\frac{(z-w)}{(z+w)}\right|$.

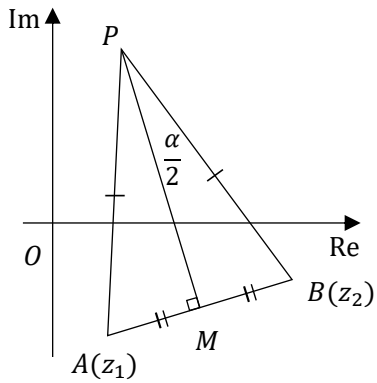
## Challenging

**24a** Use the results $z + \bar{z} = 2\text{Re}(z)$ and $|z|^2 = z\bar{z}$ for the complex number $z$ to show that $|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 = 2\text{Re}(\alpha\bar{\beta})$.




**b** The diagram shows the angle $\theta$ between the complex numbers $\alpha$ and $\beta$. Prove that $|\alpha||\beta|\cos\theta = \text{Re}(\alpha\bar{\beta})$.

**25** The diagram shows an isosceles triangle $ABP$ in the complex plane, with base $AB$ and $\angle APB = \alpha$. $PM$ is the perpendicular bisector of $AB$ and so bisects $\angle APB$. Suppose that $A$ and $B$ represent the complex numbers $z_1$ and $z_2$ respectively.




**a** In terms of $z_1, z_2$ and $\alpha$, find the complex numbers represented by :
(i) the vector $AM$ (ii) the vector $MP$

**b** Hence show that $P$ represents the complex number $\frac{1}{2}\left(1 - i\cot\frac{\alpha}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)z_2$.

**26** Given $z_1 = i\sqrt{2}$ and $z_2 = \dfrac{2}{1-i}$.

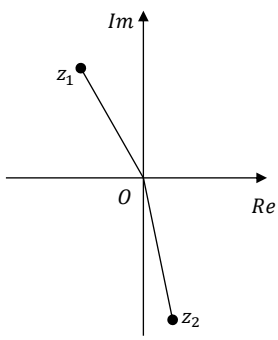
**a** Express $z_1$ and $z_2$ in modulus-argument form.

**b** If $z_1 = wz_2$, find $w$ in modulus-argument form.

**c** On an Argand diagram, plot the points $P, Q$ and $R$, where $P$ represents $z_1$, $Q$ represents $z_2$ and $R$ represents $(z_1 + z_2)$.

**d** Show that $\arg(z_1 + z_2) = \dfrac{3\pi}{8}$ and hence find the exact value of $\tan\dfrac{3\pi}{8}$.

**27** $z_1, z_2$ and $z_3$ are three complex numbers which satisfy $z_1 + z_2 + z_3 = 0$
$z_1$ and $z_2$ are indicated in the Argand diagram as shown in the figure below.




**a** Copy the diagram and sketch on the same diagram a possible location for $z_3$. Explain your decision.

**b** Given that the arguments of $z_1, z_2$ and $z_3$ are $\alpha, \beta$ and $\gamma$ respectively, and their moduli are $1, k$ and $2 - k$ respectively, where $0 < k < 2$. Express $z_1, z_2$ and $z_3$ in modulus-argument form.

**c** Prove that $\begin{cases} \cos\alpha + k\cos\beta + (2-k)\cos\gamma = 0 \\ \sin\alpha + k\sin\beta + (2-k)\sin\gamma = 0 \end{cases}$.

**d** From (iii), by eliminating $\alpha$ or otherwise, prove that
$$k^2 + (2-k)^2 + 2k(2-k)\cos(\beta - \gamma) = 1.$$

**e** By considering $|\cos(\beta - \gamma)| \leq 1$, find the range of values of $k$.

**28** The equation $z^4 - 10z^3 + (36 + i)z^2 - 5(11 + i)\text{z} + 4(7 + i) = 0$ has roots $\alpha, \beta, 4\alpha$ and $\bar{\beta} + 1$, which represent the vertices $A, B, C$ and $D$ respectively of a quadrilateral in the complex plane. The vertices are labelled in a clockwise direction around the quadrilateral. It is known that $4\alpha - \beta = \bar{\beta} + 1 - \alpha$. Using the above information, or otherwise, find the four roots of the equation.

In Lesson 7 we cover:

- Lines on the complex plane
- Circles
- Rays and sectors
- Miscellaneous examples
- Arcs and intervals using arguments

Syllabus reference: This lesson covers the 1st to 5th dot points from Describing lines, curves and regions in *Introduction to complex numbers*..

## Lines on the complex plane

In this lesson we look at lines, curves and regions on the complex plane. Let's start with lines and associated regions, which are quite simple.

**Horizontal and vertical lines**

$\text{Re}(z) = a$ is the vertical line though $a$ on the real axis.

$\text{Re}(z) < a$ is the region to the left of the vertical line through $a$ on the real axis.

$\text{Re}(z) > a$ is the region to the right of the vertical line through $a$ on the real axis.

$\text{Im}(z) = b$ is the horizontal line though $b$ on the imaginary axis.

$\text{Im}(z) < b$ is the region below the horizontal line through $b$ on the imaginary axis.

$\text{Im}(z) > b$ is the region above the horizontal line through $b$ on the imaginary axis.

## Example 1

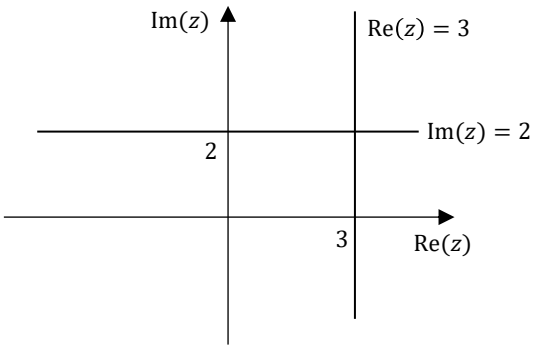
Sketch the following lines on a complex plane.

**a** $\text{Re}(z) = 3$

A vertical line though $3$ on the real axis.

**b** $\text{Im}(z) = 2$

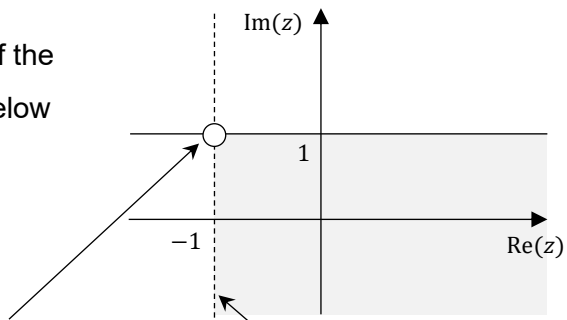
A horizontal line though $2$ on the imaginary axis.




## Example 2

Sketch the region where $\text{Re}(z) > -1$ and $\text{Im}(z) \leq 1$.

This is the region to the right of the line $\text{Re}(z) = -1$ and on and below the line $\text{Im}(z) = 1$.

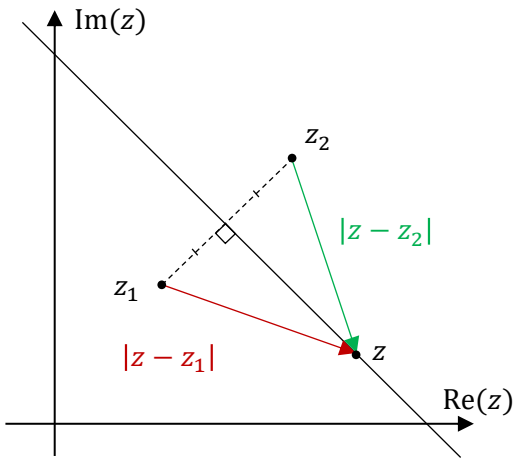



Where the two lines intersect is not part of the region, which should be shown as a point of discontinuity.

This must be a dotted line, as the line itself is not part of the region.

**Perpendicular bisectors**

The length of the vectors from any two points representing the complex numbers $z_1$ and $z_2$, to any point on their perpendicular bisector have equal moduli, so $|z - z_1| = |z - z_2|$.




All complex numbers on the same side as $z_1$ are closer to $z_1$ than $z_2$, so satisfy $|z - z_1| < |z - z_2|$.

Similarly, all complex numbers on the same side as $z_2$ are closer to $z_2$ than $z_1$, so satisfy $|z - z_1| > |z - z_2|$.

Hint: to sketch the perpendicular bisector, mark the midpoint of the two points, and draw a line through it whose gradient is the negative reciprocal of the gradient between the two points.

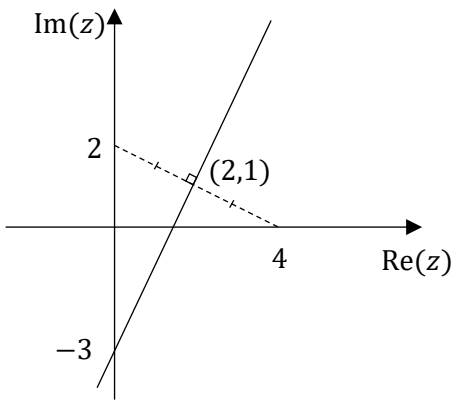
## Example 3

Sketch $|z - 4| = |z - 2i|$.

$|z - (4 + 0i)| = |z - (0 + 2i)|$

This is the perpendicular bisector of the interval joining $(4,0)$ and $(0,2)$.

The midpoint is $(2, 1)$ and the gradient is $-\frac{1}{2}$, so draw a line through $(2, 1)$ with $m = 2$.



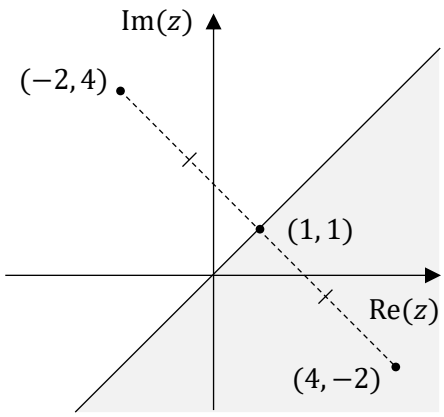

## Example 4

Sketch $|z - 4 + 2i| \leq |z + 2 - 4i|$.

$|z - (4 - 2i)| \leq |z - (-2 + 4i)|$

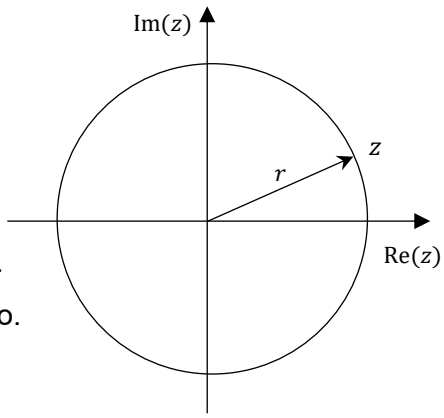
This is the perpendicular bisector of the interval joining $(4, -2)$ and $(-2, 4)$ and the region including $(4, -2)$.

The midpoint is $(1, 1)$ and the gradient is $-1$, so draw a solid line through $(1, 1)$ with $m = 1$, and shade lower right-hand side.



## Circles

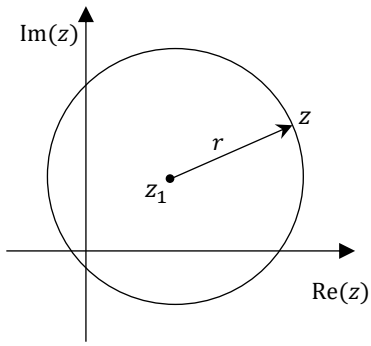
The length of a vector from the centre of a circle to any point on the circumference has a modulus equalling the radius.




$|z| = r$ is a circle of radius $r$ centred at zero.

$|z| < r$ is the region inside the circle of radius $r$ centred at zero.

$|z| > r$ is the region outside the circle of radius $r$ centred at zero.




$|z - z_1| = r$ is a circle of radius $r$ centred at $z_1$.

$|z - z_1| < r$ is the region inside the circle of radius $r$ centred at $z_1$.

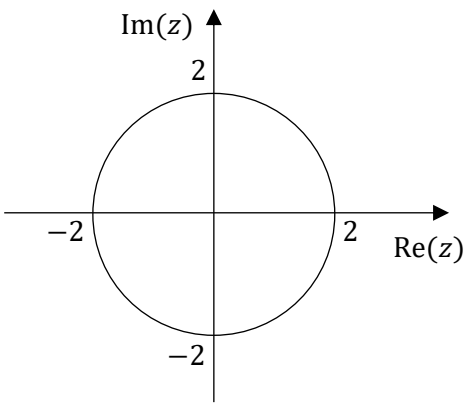
$|z - z_1| > r$ is the region outside the circle of radius $r$ centred at $z_1$.

## Example 5

Illustrate the following on a complex plane:

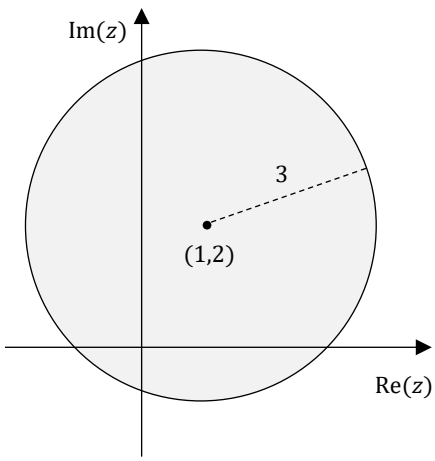
**a** $|z| = 2$

This is a circle centred at zero with radius $2$.




**b** $|z - 1 - 2i| \leq 3$

This is the inside and circumference of a circle centred at $(1,2)$ with radius $3$ units.



## Rays and sectors

A ray is a line that starts from a point and continues in one direction. The argument of any point on the ray equals the angle the ray makes with the real axis. Note that there should be an open circle on the starting point itself, as there can be no angle formed at the point itself, so the argument is undefined.

A ray starting at zero at an angle of $\theta$ is $\arg(z) = \theta$, though for most questions the principal argument is assumed, so $\text{Arg}(z) = \theta$.

A ray starting at $z_1$ at an angle of $\theta$ is $\arg(z - z_1) = \theta$.

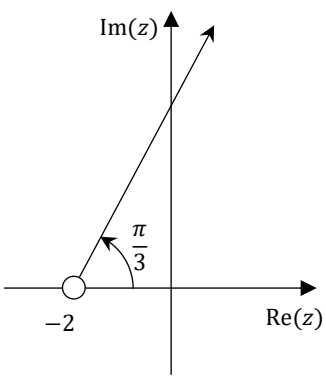
A sector with centre $z_1$ and arms at angles to the $x$-axis of $\alpha$ and $\beta$ is $\alpha \leq \arg(z - z_1) \leq \beta$. Again there must be an open circle at the centre, as the angle is undefined at that point. The arms may need to be solid or dashed, depending on which inequality sign is used.

## Example 6

Sketch the following on a complex plane:

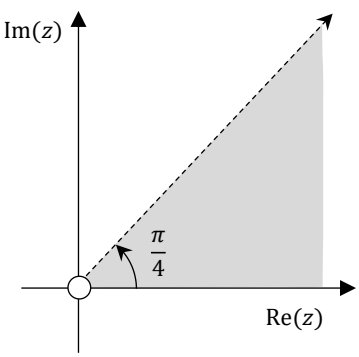
**a** $\arg(z + 2) = \dfrac{\pi}{3}$

A ray from the point $(-2,0)$ at an angle of $\frac{\pi}{3}$.




**b** $0 \leq \arg(z) < \dfrac{\pi}{4}$

A sector centred at zero, with rays at 0 radians (solid) and $\frac{\pi}{4}$ radians (dashed).



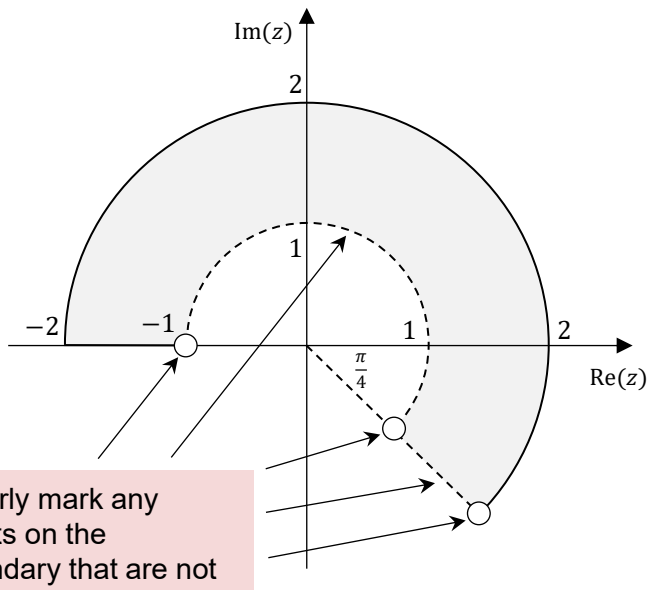
## Example 7

Sketch the region on the complex plane that satisfies $\text{Arg}(z) > -\frac{\pi}{4}$ and $1 < |z| \leq 2$.

The region is the intersection of the sector $-\frac{\pi}{4} < \text{Arg}(z) \leq \pi$ and the annulus formed by the circles centred at zero with radii $1$ unit and $2$ units.

The region includes the circle of radius $2$ units and the area between the circles centred at zero of radius $1$ unit and $2$ units.

The region does NOT include any point on the circle of radius $1$ unit or any point on the ray $\text{Arg}(z) = -\frac{\pi}{4}$.




Clearly mark any points on the boundary that are not included in the region.

This question would not make sense if we used the argument, $\arg(z)$, instead of the principal argument, $\text{Arg}(z)$, as every point on the complex plane except zero satisfies $\arg(z) \geq -\frac{\pi}{4}$, as we could just keep revolving around zero until we find an argument at the point which is greater than $-\frac{\pi}{4}$.

## Miscellaneous examples

Many other curves will need to be simplified algebraically or geometrically until we can recognise their details.

## Example 8

Sketch the curve with equation $|z-1|=2|z+1|$.

Every point is twice as far from 1 as it is from $-1$.

Let $z=x+iy$

$$\therefore |x+iy-1|=2|x+iy+1|$$

$$\sqrt{(x-1)^2+y^2}=2\sqrt{(x+1)^2+y^2}$$

$$x^2-2x+1+y^2=4\left(x^2+2x+1+y^2\right)$$

$$x^2-2x+1+y^2=4x^2+8x+4+4y^2$$

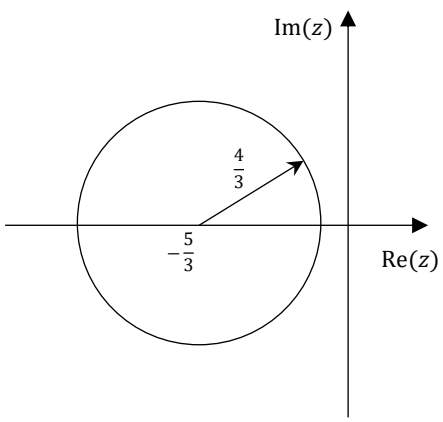
$$3x^2+10x+3y^2+3=0$$

$$x^2+\frac{10}{3}x+y^2=-1$$

$$x^2+\frac{10}{3}x+\frac{25}{9}+y^2=-1+\frac{25}{9}$$

$$\left(x+\frac{5}{3}\right)^2+y^2=\frac{16}{9}$$

Which is a circle with centre $\left(-\frac{5}{3},0\right)$ with radius $\frac{4}{3}$.

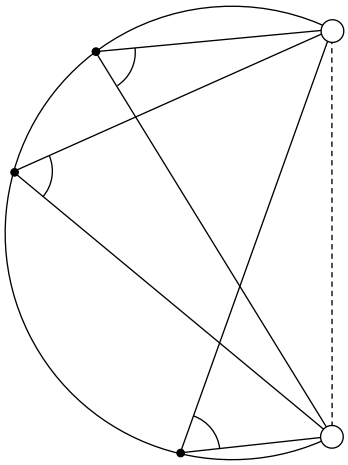


## Arcs and intervals using arguments

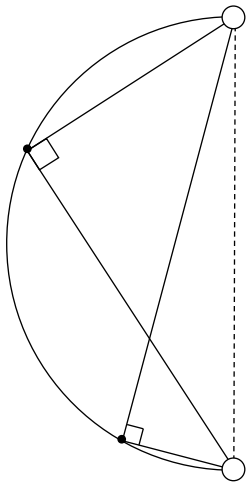
We will now look at lines and curves based on circle geometry theorems.

**Angles in the same segment are equal**

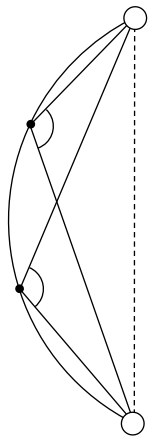
This is the most important rule that we will use. All angles with their vertex on the circumference and arms ending at the ends of a chord are equal. In a major segment the angles will be acute (left), in a semicircle the angle is a right-angle (centre) and in a minor segment the angle is obtuse (right).



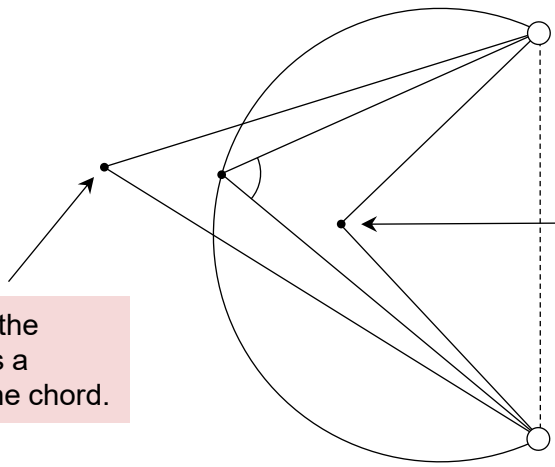
The angle in a major segment is acute.



The angle in a semicircle is a right-angle.



The angle in a minor segment is obtuse.

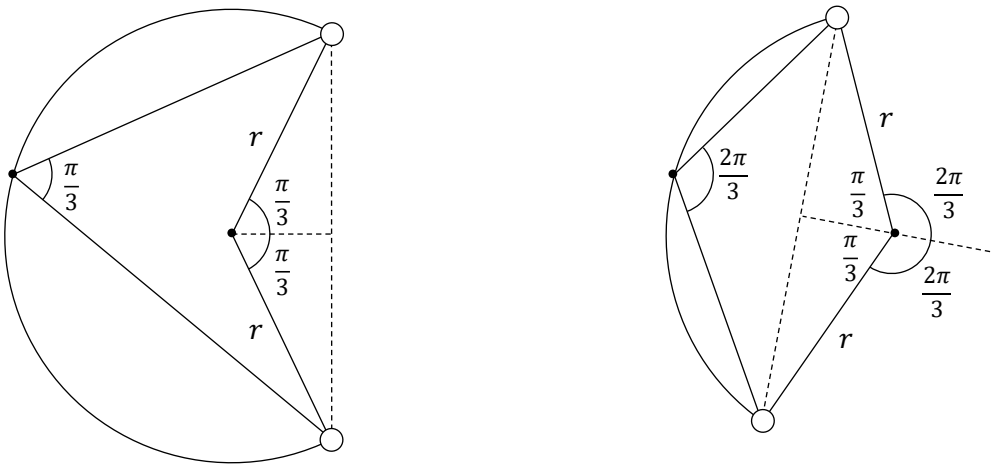


Any point outside the segment subtends a smaller angle to the chord.

Any point inside the segment subtends a larger angle to the chord.

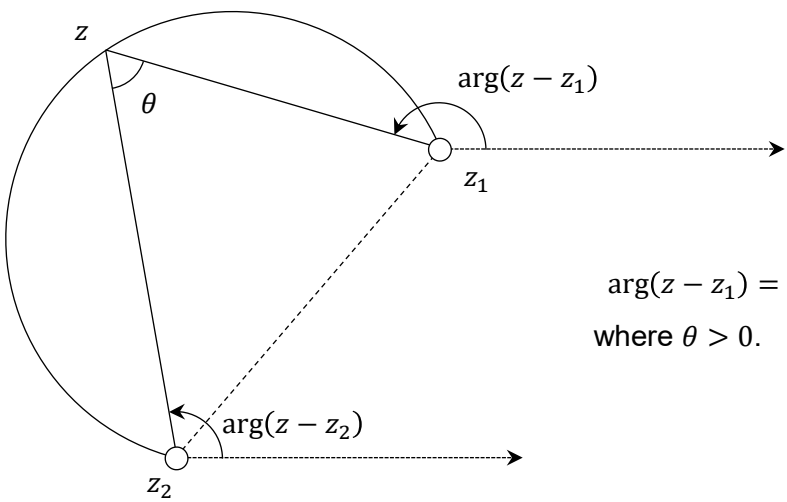
**The angle at the centre is twice the angle at the circumference standing on the same arc**

This rule is mainly used for finding the centre of the circle for drawing arcs. Since the angle at the centre is twice the angle at the circumference, we can bisect the angle at the centre to form two right-angled triangles as shown. The angle in the right-angled triangle at the centre equals the acute angle in a major segment (left) or the supplement of the obtuse angle in a minor segment (right). This helps us find the centre and radius of the arc. Expect that these right-angled triangles will be one of the exact triangles, to reduce excessive calculations, as shown below where the right-angled triangles involve the $\frac{\pi}{3} - \frac{\pi}{6}$ exact triangle.




**Arcs on the complex plane**

Now let's see how we can use these concepts with arcs on the complex plane. Consider the diagram below, where $z$ represents any complex number on the major arc, $z_1$ and $z_2$ represent the complex numbers at each end of the chord (or the ends of the major arc), and the angle at the circumference is $\theta$.




$$\arg(z - z_1) = \arg(z - z_2) + \theta$$

where $\theta > 0$.

We easily show using parallel lines that $\arg(z - z_1)$ is larger than $\arg(z - z_2)$ by $\theta$. This will be true for every point on the major arc.

We use the argument in the interval $[0, 2\pi)$, rather than the principal argument, for most questions involving arcs. This avoids having to add or subtract $2\pi$ for arcs where either of the vectors could have an argument either side of $\pi$ radians.

We can show the relationship from the previous diagram in several ways, each of which can be used to define an arc.

$$\arg(z - z_1) = \arg(z - z_2) + \theta$$
$$\arg(z - z_1) - \arg(z - z_2) = \theta$$
$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \theta$$

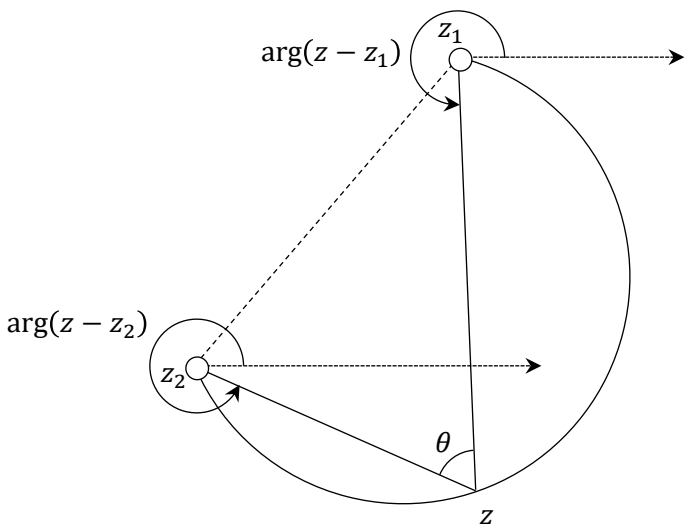
Note that the second and third versions are an application of our rule for dividing complex numbers - the argument of the quotient of two complex numbers is the argument of the numerator minus the argument of the denominator.

**On which side of the chord does the arc lie?**

In the diagram at the bottom of the last page, the argument of $z - z_1$ was greater than the argument of $z - z_2$ by $\theta$, where $\theta$ was positive. If you imagine standing at $z_2$ and looking at $z_1$, the arc lies on the left-hand side of the chord joining $z_1$ and $z_2$. This will always be true if $\theta$ is positive in any of the forms above, regardless of the relative positions of $z_1$ and $z_2$ on the complex plane.

Not surprisingly, if the argument of $z - z_1$ is less than the argument of $z - z_2$, then in the equations above there is a minus in front of $\theta$. In this case, if you imagine standing at $z_2$ and looking at $z_1$, the arc lies on the right-hand side of the chord joining $z_1$ and $z_2$.

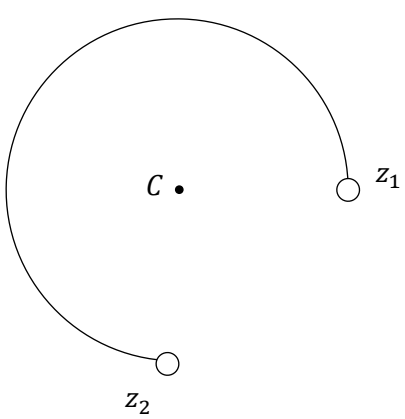
In the diagram below we can see that the argument of $z - z_1$ is less than the argument of $z - z_2$ by $\theta$.

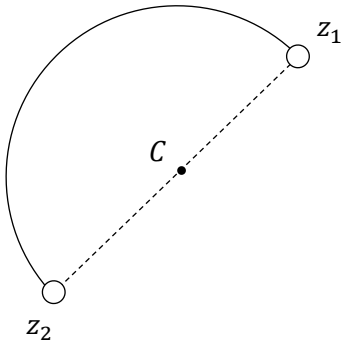


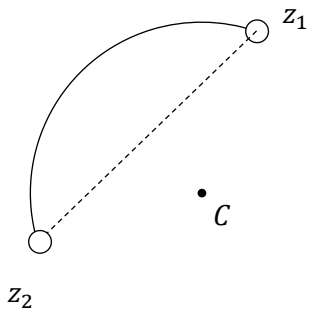

$\arg(z - z_1) = \arg(z - z_2) - \theta$
where $\theta > 0$.

To draw the arc, we first need to find the centre of the circle.

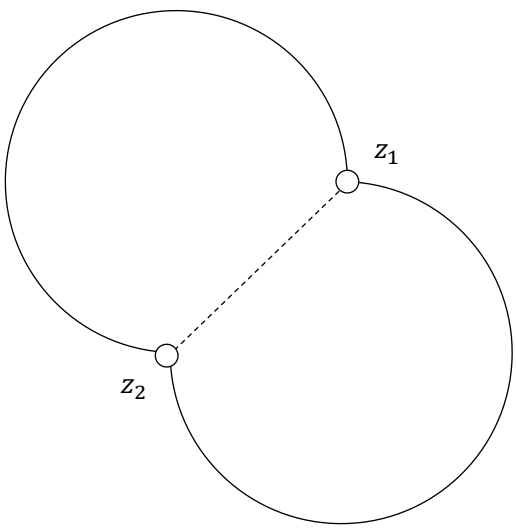
- If $\theta$ is acute, we will draw a major arc, and the centre of the circle will be on the same side of the chord as the arc (below left).
- If $\theta = \frac{\pi}{2}$ then the centre of the circle is the midpoint of the chord (below centre).
- If $\theta$ is obtuse, we will draw a minor arc, and the centre of the circle will be on the opposite side of the chord to the arc (below right).

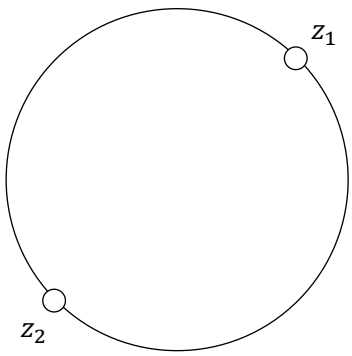


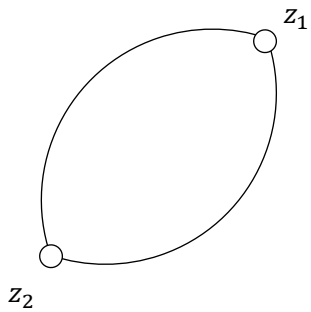




There can be cases where we have $\pm\theta$, in which case we want two equal arcs on each side of the chord. Note that the two arcs will not form a circle unless the angle is $\frac{\pi}{2}$, in which case each arc will be half the circumference, and the two arcs will form a circle except for the points of discontinuity (centre diagram below).



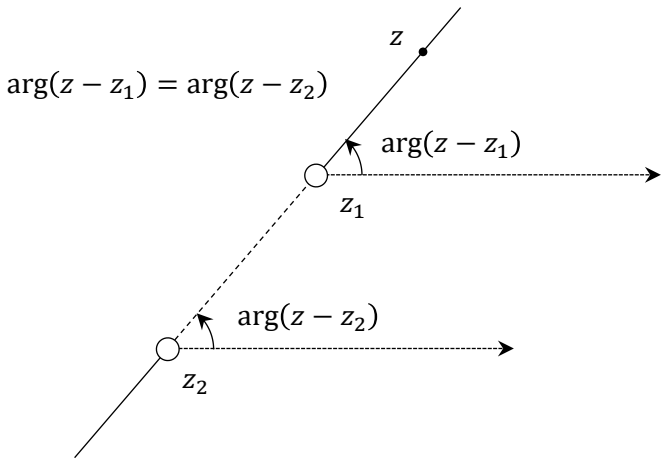



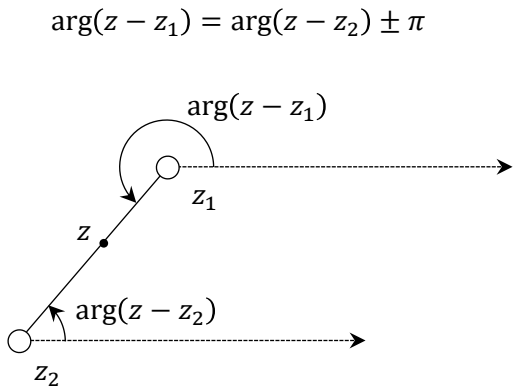




If $\theta < \frac{\pi}{2}$ then we will have two major arcs forming a snowman (above left), or if $\theta > \frac{\pi}{2}$ we will have two minor arcs (above right).

We also have two similar equations that do NOT form arcs.

- If $\arg(z-z_1)=\arg(z-z_2)$ (i.e. $\theta=0$) then the vectors to $z$ from $z_1$ and from $z_2$ have the same argument. This only occurs when $z$ is on the line passing through the two points, but **outside** the two points, as shown below left.
- If $\arg(z-z_1)=\arg(z-z_2)\pm\pi$ (i.e. $\theta=\pm\pi$) then the vectors to $z$ from $z_1$ and $z_2$ point in opposite directions. This only occurs on the line passing through the two points **between** the points, as shown below right.

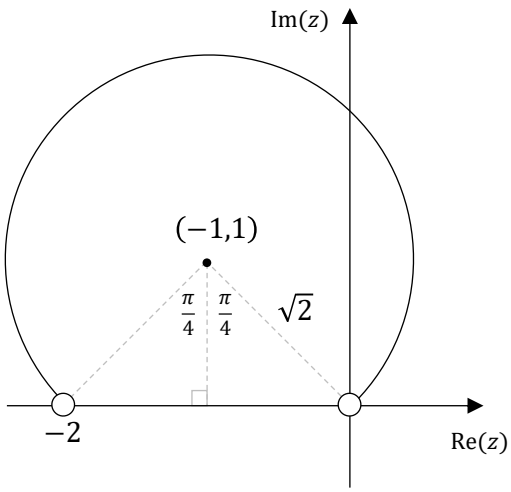



Even simple questions on arcs and intervals using arguments can get complicated, so don't expect these questions to become too involved.

## Example 9

Sketch $\arg\left(\frac{z}{z+2}\right)=\frac{\pi}{4}$, showing the centre and radius of the arc.



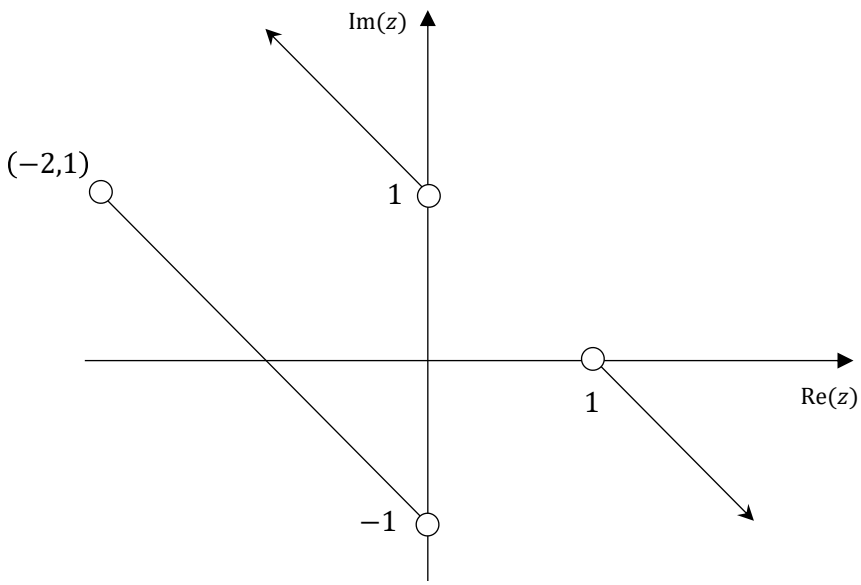

- The ends of the chord are $z=0$ and $z=-2$.
- Standing at the second point, $z=-2$, and facing the first point, $z=0$, we want the arc on the left since $\frac{\pi}{4}$ is positive.
- The centre of the arc is on the same side of the chord, since $\frac{\pi}{4}$ is acute.
- Find the centre using the $\frac{\pi}{4}$ exact triangle.

## Example 10

Sketch the following intervals on the axes below.

**a** $\arg\left(\dfrac{z+i}{z+2-i}\right)=\pi$

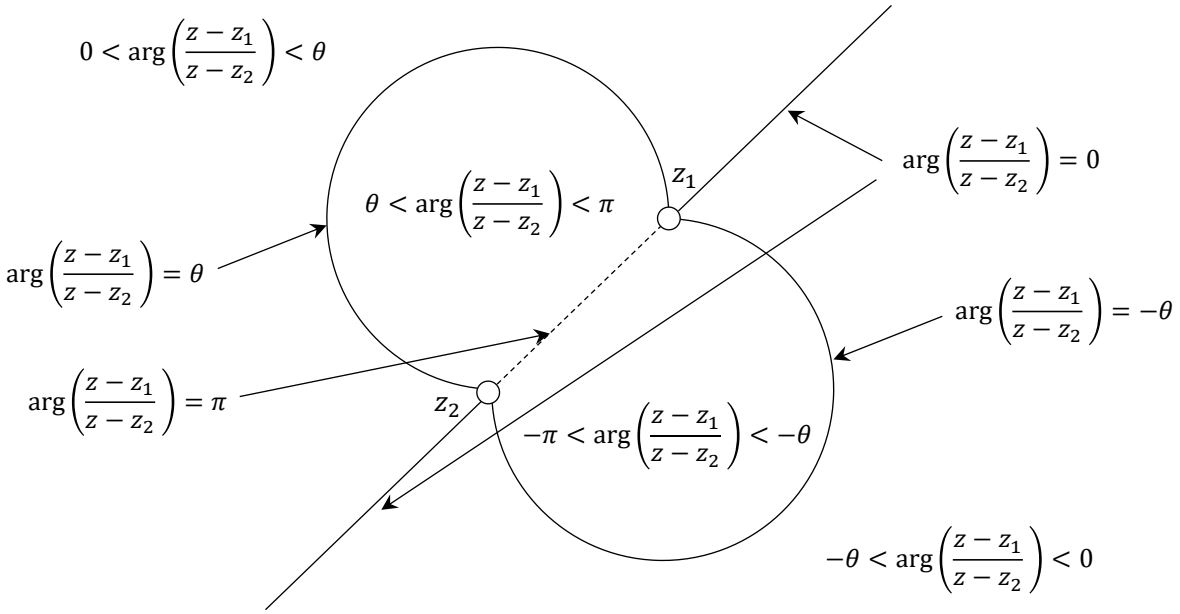
**b** $\arg\left(\dfrac{z-i}{z-1}\right)=0$




**Regions involving arcs**

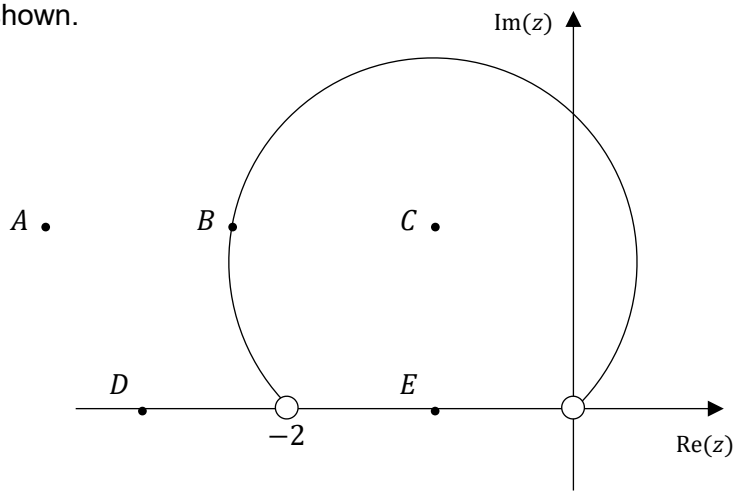
Given the wording of the dot point in the syllabus, we could also be asked to define regions on the complex plane using arcs. Again, expect any questions to be simple.

In the diagram on the next page we see the chord joining $z_1$ and $z_2$, which can be defined as $\arg\left(\frac{z-z_1}{z-z_2}\right)=\pi$. The extension of the chord beyond the points is $\arg\left(\frac{z-z_1}{z-z_2}\right)=0$. The arc top-left is $\arg\left(\frac{z-z_1}{z-z_2}\right)=\theta$, with any point inside this segment defined by $\theta<\arg\left(\frac{z-z_1}{z-z_2}\right)<\pi$, while anything outside the segment on the same side of the chord being defined by $0<\arg\left(\frac{z-z_1}{z-z_2}\right)<\theta$. We see similar definitions for regions involving the other arc.




## Example 11

The arc defined by $\arg\left(\frac{z}{z+2}\right) = \frac{\pi}{4}$ is shown.




Which of the points $A, B, C, D$ and $E$ satisfy the following?

**a** $0 < \arg\left(\frac{z}{z+2}\right) < \frac{\pi}{4}$

A (outside the segment on the same side of the chord)

**b** $\arg\left(\frac{z}{z+2}\right) = 0$

D (on the extension of the chord beyond the points)

**c** $\frac{\pi}{4} < \arg\left(\frac{z}{z+2}\right) < \pi$

C (inside the segment on the same side of the chord)

**d** $\arg\left(\frac{z}{z+2}\right) > 0$

A, B, C and E (for all these points the argument of $z$ is greater than the argument of $z+2$)

Basic

**1** Illustrate the following lines on a complex plane.

**a** $\text{Re}(z) = -2$

**b** $\text{Im}(z) = 1$

**2** Sketch the region where $-1 < \text{Re}(z) \leq 2$ and $\text{Im}(z) > -2$.

**3** Sketch $|z + 1 - i| = |z - 1 + i|$

**4** Sketch $|z + 1 - 2i| \leq |z - 2|$.

**5** Illustrate the following regions on a complex plane.

**a** $|z| = 3$

**b** $|z + 1 - i| \leq 1$

**6** Sketch the following on a complex plane.

**a** $\arg(z) = -\dfrac{\pi}{4}$

**b** $0 \leq \arg(z - 1) \leq \dfrac{\pi}{3}$

**7** Which of the following corresponds to the set of points in the complex plane defined by $|z + 2i| = |z|$?

**A** the point given by $z = -i$

**B** the line $\text{Im}(z) = -1$

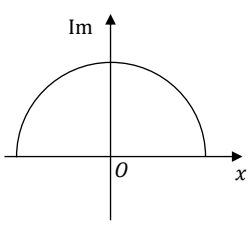
**C** the circle with centre $-2i$ and radius $1$

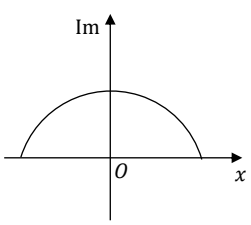
**D** the line $\text{Re}(z) = -1$

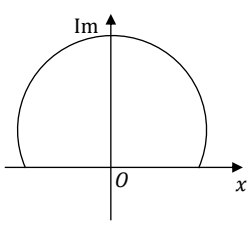
**8** Sketch the region for $z$ defined by $|z - 1 + i| < 2$ and $-\dfrac{\pi}{4} \leq \arg(z - 1 + i) < \pi$.

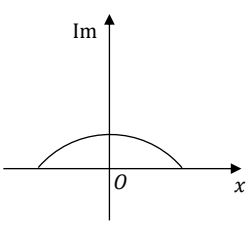
**9** Sketch the region such that $|z + 1| \leq |z - i|$ and $|z| \geq 1$.

**10** Which of the following diagrams could show $\arg(z - 2) - \arg(z + 2) = \dfrac{\pi}{4}$?



## Medium

**11** Give a geometric description of $|z+1|=3|z|$.

**12** Sketch $\arg(z+1)-\arg(z-1)=\dfrac{\pi}{4}$.

**13** Sketch $\mathrm{Im}\left(z^2\right)=1$.

**14** Sketch the region where the inequalities $|z-\bar{z}|<1$ and $|z-i|\leq 2$ hold simultaneously.

**15** The point $P$ represents the complex number $z$, where $z$ satisfies $\dfrac{1}{z+1}+\dfrac{1}{\bar{z}+1}=1$. Give a geometrical description of the position of $P$ as $z$ varies.

**16** Sketch $\arg\left(\dfrac{z+1+i}{z-1}\right)=\pm\dfrac{\pi}{4}$.

**17** Sketch $\arg(z-1-i)=\arg(z+1-i)$.

**18** Sketch $\arg(z-2)-\arg(z+2i)=\pi$.

**19** Show all complex numbers represented by $z=1+2(\cos\theta+i\sin\theta)$ for $-\dfrac{\pi}{6}\leq\theta\leq\dfrac{\pi}{3}$.

**20** It is given that $|z-1-i|=1$. What is the maximum value of $\mathrm{Arg}\,(z)$?

## Challenging

**21** The points $z=x+iy$ on the curve $x^2-2x+y^2-2y-2=0, y\neq 1$ can also be represented by $\arg\left(\dfrac{z-z_1}{z-z_2}\right)=\pm\theta$. Find $z_1$, $z_2$, and $\theta$.

**22** $Z$ and $W$ represent the complex numbers $z$ and $w$ respectively. If $|z|=2$ and $w=\dfrac{z+3}{z}$, describe the possible positions of $W$ geometrically.

**23a** Sketch $z$, where $|z-2|=1$.

**b** Hence find the range of possible values of $|z|$ and $\arg(z)$.

**c** The points $P_1$ and $P_2$ represent the complex numbers $z_1$ and $z_2$ respectively, such that $OP_1$ and $OP_2$ are tangents to $|z-2|=1$. Express $z_1$ and $z_2$ in modulus-argument form.

**d** Evaluate ${z_1}^2+{z_2}^2$, give your answer in its simplest form.

**24** If $z_1 = 2i$ and $z_2 = 1 + 3i$ are two complex numbers, describe $z = z_1 + k(z_2 - z_1)$, when:

**a** $k = 1$

**b** $0 < k < 1$

**c** $k$ is any real number

**d** When $0 < k < 1$, $z$ can also be written in the form $\arg\left(\frac{z-\alpha}{z-\beta}\right) = \theta$. Find $\alpha, \beta$ and $\theta$.

**25** Which of the following equations is equivalent to the intersection of $|z| = |z - 2k - 2ki|$ and $|z - k - ki| < \sqrt{2}k$, where $k$ is a positive constant?

A. $\arg(z-k) = \arg(z-ki)$

B. $\arg(z-k) = \arg(z-ki) + \pi$

C. $\arg(z-2k) = \arg(z-2ki)$

D. $\arg(z-2k) = \arg(z-2ki) + \pi$

**26** Sketch the region of the complex plane defined by $\text{Arg}(z) < \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$.

In Lesson 8 we cover:

- de Moivre's theorem
- Trigonometrical applications of de Moivre's theorem

Syllabus reference: This lesson covers the 1st to 4th dot points in Powers and roots of complex numbers in *Introduction to complex numbers*.

## de Moivre's theorem

de Moivre's theorem was developed by Abraham de Moivre, a French mathematician who spent most of his adult life in England. His surname has a few different pronunciations, with 'duh MWAHV-ruh' being the most common. de Moivre's theorem is sometimes abbreviated as DMT. Since his surname starts with a lowercase 'd', it should be left as lowercase at the start of sentences and headings.

de Moivre's theorem states that if we raise a complex number to an integer power then we raise the modulus to that power and multiply the argument by the power.

$$\boxed{\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)}$$

This is an extension of the work we did earlier in the chapter when we squared or cubed a complex number.

We will prove de Moivre's theorem for non-negative integers using induction, and negative integers using a direct proof, in two questions in the exercise. They are fairly simple proofs.

It doesn't matter whether we use the principal argument of a complex number or any other argument - we get an answer which is equal. Let's have a look at why, which will help us understand why we have to be careful when we use fractional powers next lesson.

Consider the complex number $z = r(\cos\theta + i\sin\theta)$, where $\theta$ is the principal argument. The complex number is also equal to $z = r(\cos(2k\pi + \theta) + i\sin(2k\pi + \theta))$ where $k \in \mathbb{Z}$.

Now $z^n = [r(\cos(2k\pi + \theta) + i\sin(2k\pi + \theta))]^n$

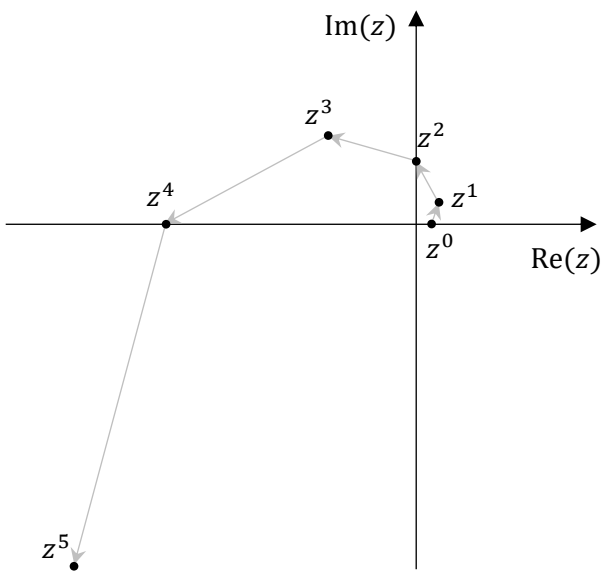
$= r^n(\cos(2kn\pi + n\theta) + i\sin(2kn\pi + n\theta))$ from de Moivre's theorem

$= r^n(\cos n\theta + i\sin n\theta)$ since $kn \in \mathbb{Z}$

Although $z$ can be represented using infinitely many arguments, applying de Moivre's theorem to any of these representations gives the same value of $z^n$. We can therefore use any argument, although we will generally express the final answer using the principal argument.

Next lesson we will see that we can use a variation of de Moivre's theorem for fractional powers. If we use de Moivre's theorem for $n = \frac{p}{q}$ where $p, q \in \mathbb{Z}, \text{HCF}(p, q) = 1$ and $q \neq 0$, we will find one of the $q$ distinct solutions, but miss the other $q - 1$ solutions. Again, although $z$ can be represented using infinitely many arguments, applying de Moivre's theorem to any of these representations gives the same $q$ values of $z^{\frac{p}{q}}$. We can therefore use any argument, although we will generally express the final answers using their principal arguments. We will explore this next lesson.

To help our understanding of what happens with de Moivre's theorem for integer powers, let's have a look at what happens to $2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ as we raise it to successively higher non-negative integer powers, starting at zero. In the diagram we see that the powers form a spiral, starting at $z^0 = 1$.



Since each successive power has its modulus scaled by a factor of $2$ and its argument increased by $\frac{\pi}{4}$, the powers spiral out in an anticlockwise direction. If the modulus of the original number is between $0$ and $1$ the spiral will move in towards zero. If the argument of the original number is negative the spiral will be clockwise.

de Moivre's theorem is often used in calculations as an intermediate step. In the next example we will use de Moivre's theorem to evaluate $(1+i)^5$. Later in the lesson we will use de Moivre's theorem in combination with binomial expansions to prove relationships between trigonometric ratios.

## Example 1

Let $z = 1 + i$.

**a** Express $z$ in mod-arg form.

$$\begin{aligned} |1+i| &= \sqrt{1^2+1^2} \\ &= \sqrt{2} \\ \arg(1+i) &= \tan^{-1}\frac{1}{1} \\ &= \frac{\pi}{4} \\ \therefore\ 1+i &= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \end{aligned}$$

**b** Express $z^5$ in mod-arg form.

$$\begin{aligned} (1+i)^5 &= \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^5 \\ &= \left(\sqrt{2}\right)^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \\ &= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \end{aligned}$$

**c** Hence express $z^5$ in the form $x + iy$.

$$\begin{aligned} \therefore (1+i)^5 &= 4\sqrt{2}\left(-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) \\ &= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \\ &= -4 - 4i \end{aligned}$$

We can also use de Moivre's theorem to simplify more complicated power expressions.

## Example 2

Write $\left(1+\sqrt{3}i\right)^4(1-i)^2$ in the form $a+ib$.

$$1+\sqrt{3}i = 2\,\text{cis}\frac{\pi}{3}$$

$$1-i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

$$\therefore \left(1+\sqrt{3}i\right)^4(1-i)^2 = \left(2\,\text{cis}\frac{\pi}{3}\right)^4\left(\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^2$$

$$= 2^4\times\text{cis}\frac{4\pi}{3}\times\sqrt{2}^2\times\text{cis}\left(-\frac{2\pi}{4}\right)$$

$$= 32\,\text{cis}\frac{5\pi}{6}$$

$$= 32\left(-\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

$$= 32\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$

$$= -16\sqrt{3}+16i$$

## Example 3

**a** Show that for every positive integer $n$, $(1+i)^n + (1-i)^n = 2\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}$.

$1+i = \sqrt{2}\text{cis}\frac{\pi}{4}$ and $1-i = \sqrt{2}\ \text{cis}\ \left(-\frac{\pi}{4}\right)$

$$\therefore (1+i)^n + (1-i)^n = \left(\sqrt{2}\ \text{cis}\frac{\pi}{4}\right)^n + \left(\sqrt{2}\ \text{cis}\ \left(-\frac{\pi}{4}\right)\right)^n$$

$$= 2^{\frac{n}{2}}\ \text{cis}\frac{n\pi}{4} + 2^{\frac{n}{2}}\ \text{cis}\left(-\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}}\left(\text{cis}\frac{n\pi}{4} + \overline{\left(\text{cis}\frac{n\pi}{4}\right)}\right)$$

The conjugate rules can give more efficient solutions than other methods

$$= 2^{\frac{n}{2}}\left(2\text{Re}\left(\text{cis}\frac{n\pi}{4}\right)\right)$$

$$= 2^{\frac{n}{2}}\left(2\cos\frac{n\pi}{4}\right)$$

$$= 2\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4} \text{ as required}$$

**b** Hence, show that for every positive integer $n$ divisible by 4,

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots + \binom{n}{n} = (-1)^{\frac{n}{4}}\left(\sqrt{2}\right)^n.$$

$(1+i)^n + (1-i)^n$

$$= \left[\binom{n}{0} + \binom{n}{1}i + \binom{n}{2}i^2 + \binom{n}{3}i^3 + \ldots + \binom{n}{n}i^n\right] + \left[\binom{n}{0} - \binom{n}{1}i + \binom{n}{2}i^2 - \binom{n}{3}i^3 + \ldots + \binom{n}{n}i^n\right]$$

$$= 2\left(\binom{n}{0} + \binom{n}{2}i^2 + \binom{n}{4}i^4 + \ldots + \binom{n}{n}i^n\right)$$

$$= 2\left(\binom{n}{0} - \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{n}\right)$$

$$\therefore 2\left(\binom{n}{0} - \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{n}\right) = 2\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4} \text{ from (a)}$$

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{n} = \left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}$$

but since $n$ is a multiple of $4$, $\frac{n\pi}{4}$ is equal to $0$ or $\pi$, and thus $\cos\frac{n\pi}{4}$ is equal to $1$ or $-1$,

which we can say mathematically as $(-1)^{\frac{n}{4}}$.

$$\therefore \binom{n}{0} - \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{n} = (-1)^{\frac{n}{4}}\left(\sqrt{2}\right)^n \text{ as required.}$$

## Trigonometrical applications of de Moivre's theorem

Combining de Moivre's theorem and binomial expansion allows us to find links between powers of trig ratios. For brevity in the examples below we will let $c = \cos\theta$ and $is = i\sin\theta$, but leave the final answers in terms of $\cos\theta$ and $\sin\theta$.

## Example 4

**a** Expand $(\cos\theta + i\sin\theta)^5$.

Let $c = \cos\theta$ and $is = i\sin\theta$.

$(c + is)^5$

$= c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$ ←

$= c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$

$= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

The coefficients are $1, 5, 10, 10, 5$ and $1$ from Pascal's triangle.

**b** Hence find an expression for $\cos 5\theta$ in terms of $\cos\theta$.

From de Moivre's Theorem $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$.

Equating real parts with part (a) we get:

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta\left(1 - \cos^2\theta\right)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Basic

**1** Let $z = 1 - i$.

**a** Express $z$ in mod-arg form.

**b** Express $z^5$ in mod-arg form.

**c** Hence express $z^5$ in the form $a + ib$.

**2** Simplify $\left(1 - \sqrt{3}i\right)^4 (1 + i)^2$, leaving the answer in the form $a + ib$.

**3a** Write $\sqrt{3} + i$ in mod-arg form.

**b** Hence evaluate $\left(\sqrt{3} + i\right)^6$.

**4** Evaluate $\left(1 - \sqrt{3}i\right)^{21}$.

**5** Which of the following is equivalent to $\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^3$?

A. $-2 - 2i$ B. $-2 + 2i$ C. $2 - 2i$ D. $2 + 2i$

**6** Show that $(3 + i)^n + (3 - i)^n$ is purely real.

**7** Use de Moivre's theorem to prove that if $|z| = 1$, then $\bar{z} = \frac{1}{z}$.

**8** *You need to have done Series before you can answer this question.*
Express the sum of the geometric series $1 + (1 + i) + (1 + i)^2 + \ldots + (1 + i)^{99}$ in the form $x + iy$, $x, y \in \mathbb{R}$.

Medium

**9a** Expand $(\cos\theta + i\sin\theta)^5$.

**b** Hence show that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

**10** For what values of $n$ is $(1 - i)^n$ purely imaginary?

**11** Prove $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ for non-negative integers by induction.

**12** Use the result from Q11 and conjugate rules to prove that $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ for negative integers.

**13a** Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^4$.

**b** Hence show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

**c** Hence, or otherwise, find the smallest positive solution of $8\cos^4\theta - 8\cos^2\theta = -1$.

**14a** Let $z = \cos\theta + i\sin\theta$. Show that $z^n + z^{-n} = 2\cos n\theta$.

**b** Hence prove that $8\cos^4\theta - 10\cos^2\theta + 2 = \cos 4\theta - \cos 2\theta$ .

**15a** Let $z = \cos\theta + i\sin\theta$. Show that $z^n - z^{-n} = 2i\sin n\theta$.

**b** By expanding $(z - z^{-1})^3$, show that $\sin^3\theta = \dfrac{3\sin\theta - \sin 3\theta}{4}$.

**16** Show that $\left(1 - \sqrt{3}i\right)^6 + \left(1 + \sqrt{3}i\right)^6 = 128$.

**17a** Express $z = \sqrt{3} + i$ in mod-arg form.

**b** Show that $z^7 + 64z = 0$.

**18a** Express $z = -3\sqrt{3} + 3i$ in mod-arg form.

**b** Hence find the smallest possible positive integer $n$ so that $z^n$ is real.

**19a** Express $-2 + 2i$ in mod-arg form.

**b** Simplify $(-2 + 2i)^{8k}$, where $k \in \mathbb{Z}$.

Challenging

**20a** Using de Moivre's theorem, or otherwise, show that for every positive integer $n$,

$$(1+i)^n - (1-i)^n = 2\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}i.$$

**b** Hence, or otherwise, show that for every positive integer $n$ divisible by 4,

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \ldots - \binom{n}{n-1} = 0$$

**21a** Show that $(1 + i\tan\theta)^n - (1 - i\tan\theta)^n = \dfrac{2\sin n\theta}{\cos^n\theta}i$, where $\cos\theta \neq 0$ and $n \in \mathbb{Z}^+$.

**b** Hence show that if $z$ is a purely imaginary number, the roots of $(1+z)^4 - (1-z)^4 = 0$ are $z = 0, \pm i$.

**22** *You need to have done Series before you can answer part (b) of this question.*

Given $z = \cos\alpha + i\sin\alpha$, where $\sin\alpha \neq 0$:

**a** Prove that $\dfrac{1}{1 - z\cos\alpha} = 1 + i\cot\alpha$.

**b** Hence, by considering $\displaystyle\sum_{k=0}^{\infty}(z\cos\alpha)^k$, deduce that the sum of the infinite series

$$\sin\alpha\cos\alpha + \sin 2\alpha\cos^2\alpha + \ldots + \sin k\alpha\cos^k\alpha + \ldots = \cot\alpha.$$

**23** Given $|z| = 1$ and $z^n + z^{-n} = 2\cos n\theta$, solve $5z^4 - z^3 + 6z^2 - z + 5 = 0$, leaving answers in the form $z = a + ib$.

**24a** By considering the expansion of $(\cos\theta + i\sin\theta)^3$ and de Moivre's theorem, show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

**b** Deduce that $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$, where $\cos 3\theta = \dfrac{1}{2}$.

**c** Find the roots of $8x^3 - 6x - 1 = 0$ in the form $x = \cos\alpha$, for $0 \leq \alpha \leq 2\pi$.

**d** Hence show that $\cos\left(\dfrac{\pi}{9}\right)\cos\left(\dfrac{2\pi}{9}\right)\cos\left(\dfrac{4\pi}{9}\right) = \dfrac{1}{8}$.

**25** Let $z = \cos\theta + i\sin\theta$ and suppose that $n$ is a positive integer.

**a** Use the identities $z^n + z^{-n} = 2\cos n\theta$ and $2\cos A\sin B = \sin(A+B) - \sin(A-B)$ to show that $\left(z^{2n} + z^{2n-2} + z^{2n-4} + \ldots + z^{-2n}\right)\sin\theta = \sin(2n+1)\theta$.

**b** Use part (a) and the identity $\cos 3A = 4\cos^3 A - 3\cos A$ to deduce that

$$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \frac{\sin 7\theta}{\sin\theta}.$$

**c** Hence show that $\cos\dfrac{2\pi}{7}$ is a root of the equation $8x^3 + 4x^2 - 4x - 1 = 0$.

**26** It is given $z = \cos\theta + i\sin\theta$ and $z^n + z^{-n} = 2\cos(n\theta)$.

**a** If $z + \dfrac{1}{z} = u$, find an expression for $z^3 + \dfrac{1}{z^3}$ solely in terms of $u$.

**b** It can be shown that $z^5 + \dfrac{1}{z^5} = u^5 - 5u^3 + 5u$. (Do NOT prove this). Show that $1 + \cos 10\theta = 2\left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right)^2$.

**27a** Use De Moivre's Theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

**b** Show that the equation $16x^4 - 16x^2 + 1 = 0$ has roots $x = \cos\frac{\pi}{12}, x = -\cos\frac{\pi}{12}$, $x = \cos\frac{5\pi}{12}$ and $x = -\cos\frac{5\pi}{12}$.

**c** By considering this equation as a quadratic equation in $x^2$, prove $\cos\frac{5\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$.

**28** Solve the equation $8x^3 - 6x - \sqrt{2} = 0$ using de Moivre's theorem, leaving answers to 3 decimal places.

# 1.9 $n^{\text{th}}$ roots of complex numbers

In Lesson 9 we cover:

- Complex numbers to a fractional power
- Complex roots of unity
- Complex roots of $-1$
- $n^{\text{th}}$ roots of other complex numbers

Syllabus reference: This lesson covers the 5th, 6th, 7th and 12th dot points from Powers and roots of complex numbers in *Introduction to complex numbers*.

## Complex numbers to a fractional power

Last lesson we saw that de Moivre's theorem only works for integer powers, also mentioning that we can use de Moivre's theorem with fractional powers to find one of the solutions, but missing all the other solutions. Let's prove this algebraically.

Let $\omega$ be any complex number equal to $z^{\frac{p}{q}}$ where $z \neq 0$, $p, q \in \mathbb{Z}^+$, $\text{HCF}(p, q) = 1$, where $\omega = m(\cos\alpha + i\sin\alpha)$ and $z = r(\cos\theta + i\sin\theta)$.

$$\omega = z^{\frac{p}{q}}$$

$$\omega^q = z^p$$

$$(m(\cos\alpha + i\sin\alpha))^q = \big(r(\cos\theta + i\sin\theta)\big)^p$$

$$m^q(\cos q\alpha + i\sin q\alpha) = r^p(\cos p\theta + i\sin p\theta) \text{ from de Moivre's theorem}$$

$$\therefore m^q = r^p \quad \text{and} \quad q\alpha = 2k\pi + p\theta \text{ for } k = 0, 1, 2, \ldots, q-1$$

$$m = r^{\frac{p}{q}} \qquad \alpha = \frac{2k\pi + p\theta}{q}$$

$$\therefore \omega = r^{\frac{p}{q}}\left(\cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)\right) \text{ for } k = 0, 1, 2, \ldots, q-1.$$

Note that we need to use $q$ different consecutive values for $k$, as once we try the next value it will start repeating values for $\alpha$. For example, $k = 0$ gives the argument $\alpha = \frac{p\theta}{q}$, while $k = q$ gives the argument $\alpha = \frac{2q\pi + p\theta}{q} = 2\pi + \frac{p\theta}{q}$ which is equivalent to $\frac{p\theta}{q}$.

Now if we just use a straightforward application of de Moivre's theorem we see:

$$z^{\frac{p}{q}} = \left(r(\cos\theta + i\sin\theta)\right)^{\frac{p}{q}}$$

$$= r^{\frac{p}{q}}\left(\cos\left(\frac{p\theta}{q}\right) + i\sin\left(\frac{p\theta}{q}\right)\right)$$

Notice that this only gives us the result matching $k = 0$ above, and misses $k = 1, 2, \ldots, q-1$. This is why we say that de Moivre's theorem only works for integer powers, but that with a slight adjustment it works for fractional powers as well.

For most of the rest of the lesson we look at the $n^{\text{th}}$ roots of complex numbers, where we raise a complex number to the power of $\frac{1}{n}$. If we used a simple application of de Moivre's theorem we would find one root and miss the other $n-1$ roots.

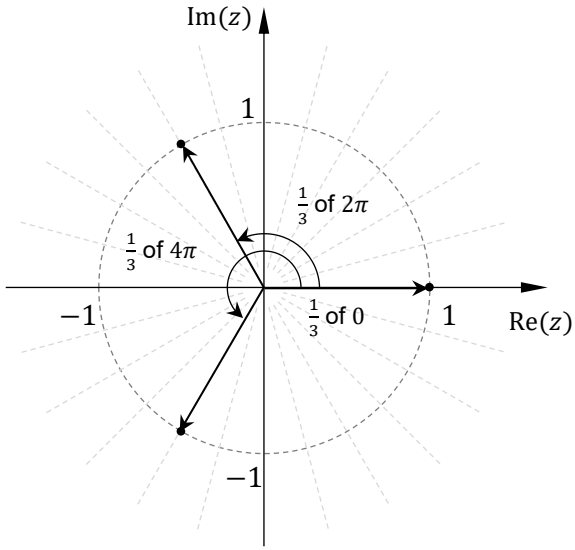
## Complex roots of unity

The complex $n^{\text{th}}$ roots of the number $1$ are called the complex roots of unity. There are $n$ unique $n^{\text{th}}$ roots of unity, each of which solves the equation $z^n = 1$.

If we rearrange the equation we find the polynomial equation $z^n - 1 = 0$. Some important results that flow from this polynomial equation:

- Since the coefficients of $z^{n-1}, z^{n-2}, \ldots, z$ all equal zero, the sum of the $n$ complex roots of unity one one-at-a-time, two-at-a-time all the way to $n-1$-at-a-time all equal zero.
- The product of the $n$ complex roots of unity equals $(-1)^{n-1}$
  - if $n$ is even the product of the roots is $-1$
  - if $n$ is odd the product of the roots is $1$.
- Since the coefficients are real, any non-real complex roots of unity occur in conjugate pairs.

The complex roots of unity are spread equally around the unit circle, starting with the number $1$ itself. For example, the cube roots of unity are shown below.




Note that the argument at $1$ is not just $0$, but also any multiple of $2\pi$. Since we are after the three cube roots we are interested in the first three multiples of $2\pi$, so $0, 2\pi$ and $4\pi$. Any three consecutive multiples of $2\pi$ would also work.

We can think of cube roots of unity as:

- the cube root of $\cos 0 + i \sin 0$, which is $\cos\frac{0}{3} + i \sin\frac{0}{3} = \cos 0 + i \sin 0 = 1$
- the cube root of $\cos 2\pi + i \sin 2\pi$, which is $\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}$
- the cube root of $\cos 4\pi + i \sin 4\pi$, which is $\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}$.

Since the multiples of $2\pi$ are equally spaced, the arguments of the three cube roots are also equally spaced. We can see above that we have the real cube root, $1$, and $\text{cis}\left(\pm\frac{2\pi}{3}\right)$, since $\text{cis}\frac{4\pi}{3} = \text{cis}\left(-\frac{2\pi}{3}\right)$, so the non-real roots do occur in conjugate pairs as expected.

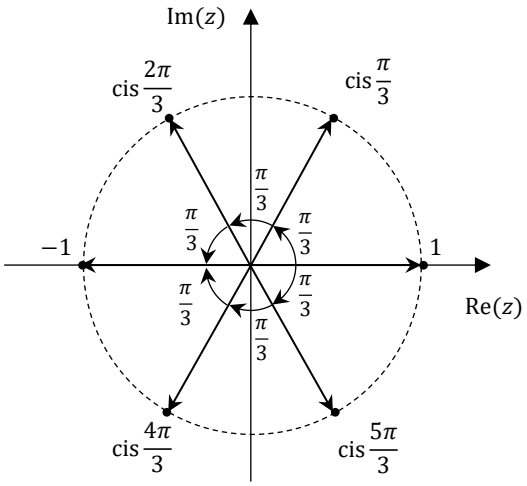
We can check that they are the cube roots of unity by cubing each one, and checking that they equal $1$.

- $1^3 = 1$
- $\left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right)^3 = \cos 2\pi + i \sin 2\pi = \cos 0 + i \sin 0 = 1$
- $\left(\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}\right)^3 = \cos 4\pi + i \sin 4\pi = \cos 0 + i \sin 0 = 1$

When solving questions like this in exams the most successful technique is to start by sketching the roots, then simply write down the roots using the modulus and argument.

## Example 1

Find the sixth roots of 1 by sketching the roots first.




1 is a sixth root of 1, and the other roots must be spaced by $\frac{2\pi}{6} = \frac{\pi}{3}$ as shown. The sixth roots of 1 are $1, \text{cis}\left(\pm\frac{\pi}{3}\right), \text{cis}\left(\pm\frac{2\pi}{3}\right)$ and $-1$.

When there are many roots we do not want to write each of them out in correct mod-arg form - using cis and $\pm$ to save time and effort.

## Example 2

Find the sixth roots of 1 using de Moivre's theorem.

Let one of the sixth roots be $z = r(\cos\theta + i\sin\theta)$

$$\therefore z^6 = 1$$

$$r^6(\cos\theta + i\sin\theta)^6 = \cos 2k\pi + i\sin 2k\pi \quad \text{for } k = -2, -1, 0, 1, 2, 3$$

$$r^6(\cos 6\theta + i\sin 6\theta) = \cos 2k\pi + i\sin 2k\pi$$

$$\therefore r^6 = 1 \quad \text{and } 6\theta = 2k\pi$$

$$\therefore r = 1 \quad \text{and } \theta = \frac{2k\pi}{6}$$

$$\therefore z = \text{cis}\left(-\frac{2\pi}{3}\right), \text{cis}\left(-\frac{\pi}{3}\right), \text{cis}\,0\,, \text{cis}\frac{\pi}{3}, \text{cis}\frac{2\pi}{3}, \text{cis}\,\pi$$

$$= 1, \text{cis}\left(\pm\frac{\pi}{3}\right), \text{cis}\left(\pm\frac{2\pi}{3}\right) \text{ and } -1$$

**Some points to note before we continue**

For $n \geq 3$, all complex roots of unity are powers of the non-real root with the smallest positive argument, $\omega = \text{cis}\left(\frac{2\pi}{n}\right)$, so the other roots are $\omega^2, \omega^3, \omega^4 \ldots \omega^{n-1}$. For the cube roots of unity, if we let $\omega = \text{cis}\left(\frac{2\pi}{3}\right)$ then we see that the other two cube roots of unity are $\omega^2 = \text{cis}\left(\frac{4\pi}{3}\right)$ and $\omega^3 = \text{cis}\left(\frac{6\pi}{3}\right) = \text{cis}\, 2\pi = \text{cis}\, 0 = 1$. This rule only works for complex roots of $1$.

The following formula may be handy to memorise for some proofs - we will prove both of these results using mathematical induction in *The nature of proof*:

$$x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1}\right)$$

$$x^n + y^n = (x+y)\left(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots + y^{n-1}\right) \text{ only if } n \text{ is odd}$$

So if $\omega$ is a complex $n$th root of $1$, i.e. a complex solution of $z^n - 1 = 0$ (which is $z^n - 1^n = 0$) then using the top rule with $x = z$ and $y = 1$ we can say:

$$(\omega - 1)\left(\omega^{n-1} + \omega^{n-2} + \omega^{n-3} + \ldots + \omega^2 + \omega + 1\right) = 0$$

$$\therefore\ 1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^{n-1} = 0 \quad \text{since } \omega \text{ is not real}$$

This is an alternative way of proving that the sum of the $n^{\text{th}}$ roots is equal to zero, though it is simpler to use the sum of the roots one-at-a-time. This also means that when we join the vectors representing the $n^{\text{th}}$ roots using the polygon rule that the tip of the last vector returns to the tail of the first vector.

## Example 3

Prove that if $\omega$ is a non-real cube root of unity, then $\left(1 + \omega - \omega^2\right)^7 = -128\omega^2$.

$$1 + \omega + \omega^2 = 0$$

$$\therefore \left(1 + \omega - w^2\right)^7$$

$$= \left(1 + \omega + \omega^2 - 2\omega^2\right)^7$$

$$= \left(-2\omega^2\right)^7$$

$$= -128\omega^{14}$$

$$= -128\left(\omega^3\right)^4\omega^2$$

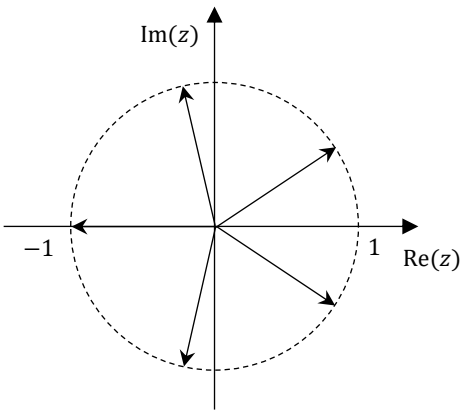
$$= -128(1)^4\omega^2 \quad \text{since } \omega \text{ is a cube root of unity}$$

$$= -128\omega^2$$

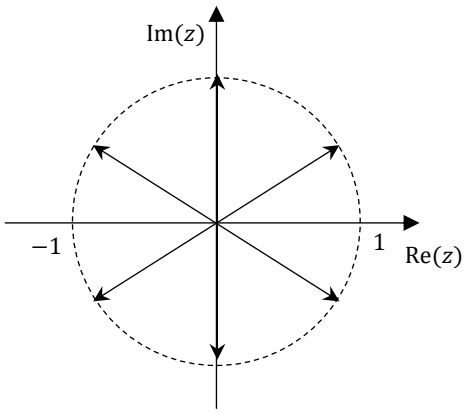
## Complex roots of $-1$

The complex roots of $-1$ are the solutions to the equations $z^n = -1$ or $z^n + 1 = 0$, and are also spread equally around the circle - finding the first root can be tricky if $n$ is even.

- if $n$ is odd then the roots include $-1$
- if $n$ is even then there are no purely real roots, and the two roots either side of $-1$ are spaced equally either side (one is still the conjugate of the other).




Spacing of odd roots

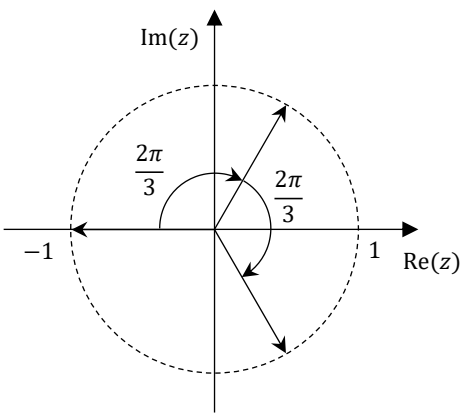



Spacing of even roots

- Since the coefficients of $z^{n-1}, z^{n-2}, \ldots, z$ all equal zero, the sum of the $n$ complex roots of $-1$ one one-at-a-time, two-at-a-time all the way to $n-1$-at-a-time all equal zero.
- The product of the $n$ complex roots of $-1$ equals $(-1)^n$
  - if $n$ is even the product of the roots is $1$
  - if $n$ is odd the product of the roots is $-1$.

## Example 4

Find the cube roots of $-1$ in mod-arg form.




$-1$ is one of the roots, and the others are spread evenly around the unit circle, so there is $\frac{2\pi}{3}$ between each cube root of $-1$.

$\therefore$ the roots are $-1$, $\text{cis}\left(\pm\frac{\pi}{3}\right)$.

## $n^{\text{th}}$ roots of other complex numbers

To find the $n^{\text{th}}$ root of a complex number:

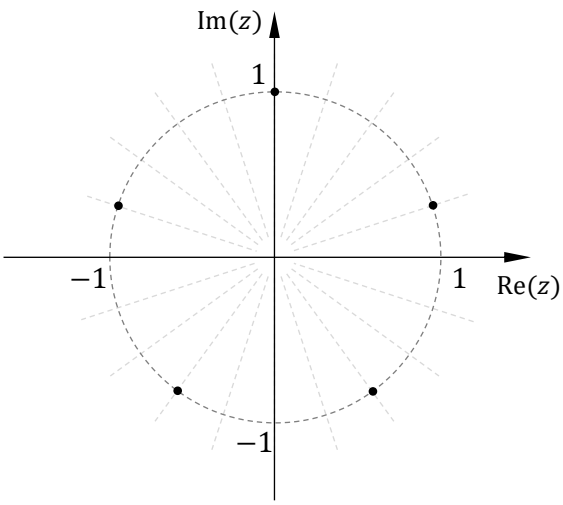
- convert the complex number to mod-arg form if needed
- use de Moivre's theorem with a power of $\frac{1}{n}$ to find one of the $n^{\text{th}}$ roots
- the other $n-1$ roots will have the same modulus, with the arguments of successive roots differing by $\frac{2\pi}{n}$.

The $n^{\text{th}}$ roots of $\omega$ are the solutions to $z^n - \omega = 0$.

- Since the coefficients of $z^{n-1}, z^{n-2}, \ldots, z$ all equal zero, the sum of the $n$ complex roots of $\omega$ one one-at-a-time, two-at-a-time all the way to $n-1$-at-a-time all equal zero.
- The product of the $n$ complex roots of $\omega$ equals $(-1)^{n-1}\omega$
  - if $n$ is even the product of the roots is $-\omega$
  - if $n$ is odd the product of the roots is $\omega$.
- When $\omega$ is a non-real complex number the conjugate root theorem does not apply, as not all coefficients are real, so non-real roots do NOT occur in conjugate pairs.

## Example 5

Find the fifth roots of $i$ in mod-arg form.




$i = 1\,\text{cis}\,\frac{\pi}{2}$

One of the fifth roots of $i$ is $\left(\text{cis}\,\frac{\pi}{2}\right)^{\frac{1}{5}} = \text{cis}\left(\frac{\pi}{10}\right)$

The other roots have arguments which differ by $\frac{2\pi}{5} = \frac{4\pi}{10}$. The fifth roots of $i$ are

$\text{cis}\left(-\frac{7\pi}{10}\right), \text{cis}\left(-\frac{3\pi}{10}\right), \text{cis}\,\frac{\pi}{10}, i$ and $\text{cis}\,\frac{9\pi}{10}$.

If we use the principal argument for the original complex number, then when we use de Moivre's theorem to find one of the roots, it will be a root with its argument closest to zero.

## Example 6

By first finding $-4\sqrt{2}-4\sqrt{2}i$ in mod-arg form and drawing it on the complex plane, find the cube roots of $-4\sqrt{2}-4\sqrt{2}i$, leaving answers in mod-arg form.

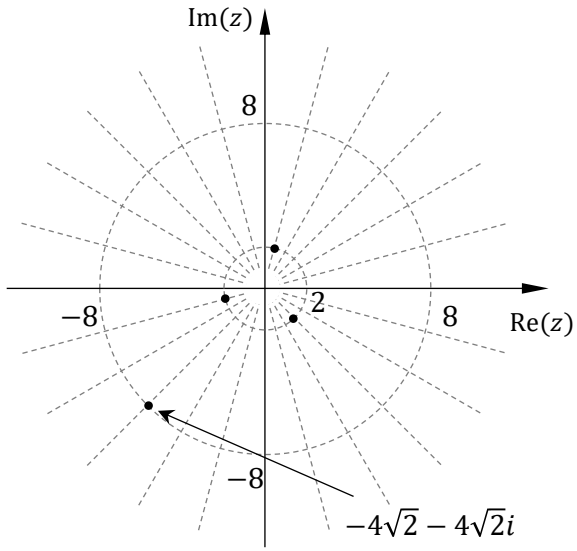
For $-4\sqrt{2}-4\sqrt{2}i$ $r=\sqrt{\left(-4\sqrt{2}\right)^2+\left(-4\sqrt{2}\right)^2}=8$

and $\theta=-\dfrac{3\pi}{4}$.

$-4\sqrt{2}-4\sqrt{2}i=8\,\mathrm{cis}\left(-\dfrac{3\pi}{4}\right)$

One of the cube roots of $-4\sqrt{2}-4\sqrt{2}i$ is

$\left(8\,\mathrm{cis}\left(-\dfrac{3\pi}{4}\right)\right)^{\frac{1}{3}}=2\,\mathrm{cis}\left(-\dfrac{\pi}{4}\right)$




The other roots have arguments which differ by $\dfrac{2\pi}{3}=\dfrac{8\pi}{12}$.

The cube roots of $-4\sqrt{2}-4\sqrt{2}i$ are $2\,\mathrm{cis}\left(-\dfrac{11\pi}{12}\right)$, $2\,\mathrm{cis}\left(-\dfrac{\pi}{4}\right)$ and $2\,\mathrm{cis}\dfrac{5\pi}{12}$.

## Example 7

Find all solutions to $\left(\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\right)^{\frac{4}{3}}$, leaving answers in mod-arg form.

We are looking for the cube roots of $\left(\mathrm{cis}\frac{\pi}{3}\right)^4=\mathrm{cis}\frac{4\pi}{3}$.

One of the cube roots of $\mathrm{cis}\frac{4\pi}{3}$ is $\left(\mathrm{cis}\frac{4\pi}{3}\right)^{\frac{1}{3}}=\mathrm{cis}\frac{4\pi}{9}$.

The other roots have arguments which differ by $\dfrac{2\pi}{3}=\dfrac{6\pi}{9}$.

The solutions to $\left(\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\right)^{\frac{4}{3}}$ are $\mathrm{cis}\left(-\dfrac{8\pi}{9}\right)$, $\mathrm{cis}\left(-\dfrac{2\pi}{9}\right)$ and $\mathrm{cis}\dfrac{4\pi}{9}$.

## Basic

**1** Find the fifth roots of $1$ by sketching the roots first.

**2** Find the fifth roots of $1$ using de Moivre's theorem.

**3** Prove that if $\omega$ is a non-real cube root of unity, then $\left(1-\omega+\omega^2\right)^4 = 16\omega$.

**4** Find the fifth roots of $-1$ in mod-arg form.

**5** Find the cube roots of $i$ in mod-arg form.

**6** Find the eighth roots of $1$ leaving answers in the form $a+ib$ where $a, b \in \mathbb{R}$.

**7** Given the cube roots of unity are $1, \omega$ and $\omega^2$, find the sum of the reciprocals of the roots.

**8** The solutions to the equation $z^3 = 8$ are $z =$

**A** $2, -\frac{1}{2}+\frac{i\sqrt{3}}{2}, -\frac{1}{2}-\frac{i\sqrt{3}}{2}$

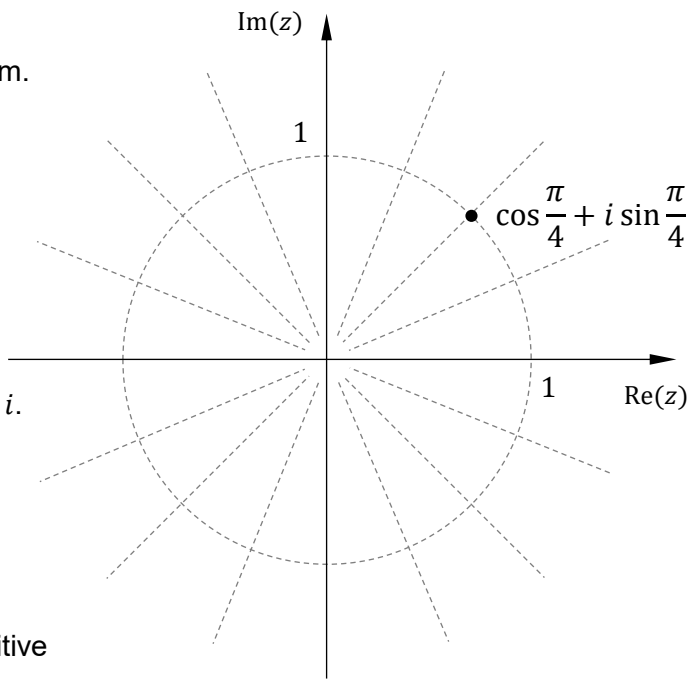
**B** $2, \frac{1}{2}+\frac{i\sqrt{3}}{2}, \frac{1}{2}+\frac{i\sqrt{3}}{2}$

**C** $2, -1+i\sqrt{3}, -1-i\sqrt{3}$

**D** $2, 1+i\sqrt{3}, 1-i\sqrt{3}$

## Medium

**9a** The location of the complex number $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$ is shown on the diagram.




On the diagram, clearly indicate the location of the square roots of $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$.

**b** Verify that the square roots of $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$ are also fourth roots of $i$.

**c** The complex number $z$ lies in the second quadrant and is also one of the fourth roots of $i$. Given that $z^n = \cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$, where $n$ is a positive integer, find the smallest possible value of $n$ that satisfies the equation.

**10** Find the cube roots of $4\sqrt{2} - 4\sqrt{2}i$, leaving answers in mod-arg form.

**11** If $w$ is a non−real cube root of unity the value $\frac{1}{1+w} + \frac{1}{1+w^2}$ is equal to

**A** $-1$ **B** $0$ **C** $1$ **D** none of the above

**12** Consider the complex number $\omega$, which is the non−real solution to $z^5 - 1 = 0$ with the smallest principal argument. Using the fact that $z = \omega^2 + \frac{1}{\omega^2}$ is a root of $z^2 + z - 1 = 0$, show that $\cos\frac{4\pi}{5} = -\frac{\sqrt{5}+1}{4}$.

**13** Find the cube roots of $8$ in Cartesian form.

**14** Find all solutions to $\left((\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^{\frac{3}{4}}$, leaving answers in mod−arg form.

**15** If $w$ is a non−real cube root of unity prove $\frac{1}{1+w} - \frac{1}{1+w^2} = -(1+2w)$.

**16** If $\omega$ is a non−real cube root of unity, prove that $(a-b)(a-\omega b)(a-\omega^2 b) = a^3 - b^3$.

**17** Given $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$ let $w$ be a solution to $z^5 - 1 = 0$ where $w \neq -1$.

**a** Prove that $1 + w^2 + w^4 = -(w + w^3)$.

**b** Hence show that $\cos\frac{2\pi}{5} - \cos\frac{\pi}{5} = -\frac{1}{2}$.

**18** The cube roots of unity are $z_1 = 1, z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

**a** Prove that each non−real cube root of unity is the square root of the other.

**b** Let $\omega$ be the second square root of $z_3$, Find $\omega$, leaving the answer in Cartesian form.

**c** Find the smallest positive value of $n$ that solves $\omega^n = 1$.

**19a** Factorise $z^5 + 1 = 0$ over the real field.

**b** List the roots of $z^5 + 1 = 0$ in $r\text{cis}\,\theta$ form.

**c** Deduce that $2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} - 1 = 0$.

**20** If $\omega$ is one of the complex cube roots of unity prove that if $n$ is a positive integer, then $1 + \omega^n + \omega^{2n} = 3$ or $0$ depending on whether $n$ is or is not a multiple of 3.

**21** $\omega$ is a non-real cube root of unity.

**a** Find the value of $\frac{1}{\omega^2}+\frac{1}{\omega}$.

**b** Show that $\frac{1+2\omega+3\omega^2}{2+3\omega+\omega^2}+\frac{1+2\omega+3\omega^2}{3+\omega+2\omega^2}=-1$.

**22** The equation $z^5=1$ has roots $1,\omega,\omega^2,\omega^3,\omega^4$, where $\omega=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$.

**a** Show that $1+\omega+\omega^2+\omega^3+\omega^4=0$.

**b** Show that $\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega+\frac{1}{\omega}\right)-1=0$.

**c** Hence, show that $\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}$.

Challenging

**23** $1,\omega$ and $\omega^2$ are the cube roots of unity and each are represented by the points $A_1,A_2$ and $A_3$ respectively on an Argand diagram. Find the value of $A_1A_2\times A_1A_3$, where '$A_1A_2$' represents the length $A_1A_2$.

**24** Let $\alpha$ be a non-real root of the polynomial $z^7=1$ with the smallest possible argument. Let $\theta=\alpha+\alpha^2+\alpha^4$ and $\delta=\alpha^3+\alpha^5+\alpha^6$.

**a** Explain why $\alpha^7=1$ and $1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6=0$.

**b** Show that $\theta+\delta=-1$ and $\theta\delta=2$ and hence write a quadratic equation whose roots are $\theta$ and $\delta$.

**c** Show that $\theta=-\frac{1}{2}+\frac{i\sqrt{7}}{2}$ and $\delta=-\frac{1}{2}-\frac{i\sqrt{7}}{2}$.

**d** Write down $\alpha$ in modulus−argument form, and show that

$$\cos\frac{4\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}=-\frac{1}{2} \text{ and } \sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{\pi}{7}=\frac{\sqrt{7}}{2}.$$

**25a** Find the five roots of the equation $z^5=1$ in modulus−argument form.

**b** Show that $z^5-1=(z-1)\left(z^2-2z\cos\frac{2\pi}{5}+1\right)\left(z^2-2z\cos\frac{4\pi}{5}+1\right)$.

**c** Hence show that $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$.

**26** The roots of $z^n = 1, n \in \mathbb{Z}^+$, are $z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, k = 1, 2, \ldots, n$. If $z_k$ is such that $z_k, z_k^2, z_k^3, \ldots, z_k^n$ generates all the roots of $z^n = 1$, then $z_k$ is called a primitive root of $z^n = 1$.

**a** Show that $z_1$ is a primitive root of $z^n = 1$.

**b** Show that $z_5$ is a primitive root of $z^6 = 1$.

**27a** Prove $\dfrac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} = i\tan\dfrac{\theta}{2}$.

**b** Find the five roots of the equation $\omega^5 = 1$ and express your answers in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.

**c** Hence show that the roots of the equation $\left(\dfrac{2+z}{2-z}\right)^5 = 1$ are $2i\tan\left(\dfrac{k\pi}{5}\right)$, where $k = 0, \pm 1, \pm 2$.

**d** By expressing the equation in part (c) in the form $z^5 + mz^3 + nz = 0$, show that $\tan\dfrac{\pi}{5}\tan\dfrac{2\pi}{5} = \sqrt{5}$.

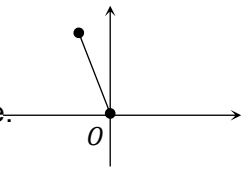
**28** The complex number $w$ lies on the unit circle with $\text{Arg}(w) = \theta$.

**a** Show that $\dfrac{1+w}{1-w} = i\cot\dfrac{\theta}{2}$.

**b** Hence solve $\left(\dfrac{z-1}{z+1}\right)^8 = -1$.

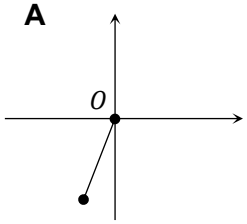
Basic

**1** A particular complex number z is represented by the point on the following complex plane.

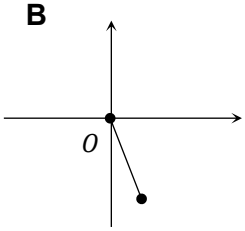



All axes below have the same scale as those in the diagram above.
The complex number $i\bar{z}$ is best represented by

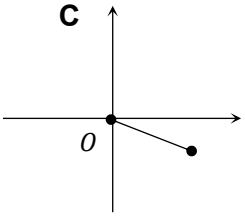
**A**



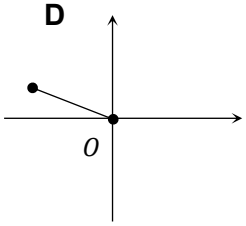

**B**




**C**




**D**




**2** On an complex plane, sketch the locus of the points $z$ such that $|z-1| = |z+i|$

**3** What is $-\sqrt{3}+i$ expressed in modulus-argument form?

**A** $\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ **B** $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

**C** $\sqrt{2}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ **D** $2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

**4** Let $z = 3-i$. What is the value of $i\bar{z}$?

**A** $-1-3i$ **B** $-1+3i$ **C** $1-3i$ **D** $1+3i$

**5** Given $z = 1+2i$ and $w = -2+i$, find:

**a** $|z|$

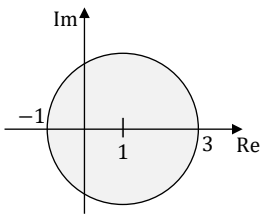
**b** $zw$

**c** $\dfrac{5}{iw}$

**6** Which of the following inequalities is represented by the complex plane?

**A** $|z-1| \le 2$ **B** $|z-i| \le 2$

**C** $|z+1| \le 2$ **D** $|z+i| \le 2$



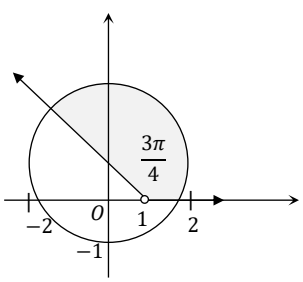

**7** $\dfrac{2-i}{-2-i} = ?$

**A** $-\frac{3}{5}+\frac{4}{5}i$ **B** $-1$ **C** $-1+\frac{4}{3}\mathbf{i}$ **D** $-\frac{5}{3}$

**8** Given that $z = 6i-8$, find the square roots of $z$ in the form $a+ib$.

**9** Prove $\cos\pi + i\sin\pi + 1 = 0$.
Note that this is the equivalent of Euler's identity $e^{i\pi}+1=0$.

**10** Consider the complex plane at right. Which inequality could define the shaded area?




**A** $|z-i| \le 2$ and $0 \le \arg(z-1) \le \dfrac{3\pi}{4}$

**B** $|z+i| \le 2$ and $0 \le \arg(z-1) \le \dfrac{3\pi}{4}$

**C** $|z-i| \le 2$ and $0 \le \arg(z-1) \le \dfrac{\pi}{4}$

**D** $|z+i| \le 2$ and $0 \le \arg(z-1) \le \dfrac{\pi}{4}$

**11** Given that $z = \sqrt{2} - \sqrt{2}i$ and $w = -\sqrt{2}$, find, in the form $x + iy$:

**a** $wz^2$

**b** $\text{Arg}\, z$

**c** $\dfrac{z}{z+w}$

**d** $|z|$

**e** $z^{10}$

**12** Find the values of real numbers $a$ and $b$ such that $(a + ib)^2 = 5 - 12i$.

**13** Draw complex planes to represent the following regions:

**a** $1 \leq |z + 4 - 3i| \leq 3$ .

**b** $\dfrac{\pi}{6} \leq \arg z < \dfrac{\pi}{3}$.

**14** Realising the denominator of $\dfrac{12 - 6i}{4 + 3i}$ gives:

**A** $1.2 + 2.4i$ **B** $\dfrac{30}{7} + \dfrac{60}{7}i$ **C** $\dfrac{30}{7} - \dfrac{60}{7}i$ **D** $1.2 - 2.4i$

**15** Factorise $x^4 - 3x^2 - 10$ over:

**a** the rational field

**b** the real field

**c** the complex field

**16** Find the square roots of $1 + \text{i}\sqrt{3}$.

**17a** Express $z = \dfrac{\sqrt{3}}{2} + \dfrac{i}{2}$ in the form $r(cos\,\theta + i\,sin\,\theta)$.

**b** Hence or otherwise find $z^{15}$ in the form $x + iy$.

**18** Let $z = 3 - i$ and $w = 2 + i$. Find the following in the form $x + iy$.

**a** $\overline{zw}$

**b** $\left|\dfrac{z}{w}\right|$

**19** Simplify $-2i\left(\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right)$.

**20** The complex number $z$ is defined by $|z-(1+2i)|=3$.

**a** Show all points on the complex plane that satisfy the equation.

**b** Find the maximum value of $|z|$.

**21** If $z=1+\sqrt{3}i$, then $z^4=$
**A** $8+8\sqrt{3}i$ **B** $8-8\sqrt{3}i$ **C** $-8+8\sqrt{3}i$ **D** $-8-8\sqrt{3}i$

**22** Convert $1+\sqrt{3}i$ into mod-arg form.

**23** Let $z=3+4i$ and $w=1-2i$. Find in the form $x+iy$:

**a** $\mathrm{Re}(z)-I(w)$

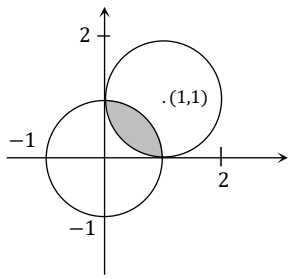
**b** $\dfrac{z}{iw}$

**c** the square roots of $z$

**24** By first sketching the equations on a complex plane, find all values of $z$ which simultaneously satisfy $|z-9|=|z+1|$ and $\arg(z-2i)=\frac{\pi}{4}$.

**25** What is the value of $\mathrm{Arg}\,\bar{z}$ given the complex number $z=1-i\sqrt{3}$?
**A** $-\dfrac{\pi}{3}$ **B** $\dfrac{\pi}{3}$ **C** $\dfrac{-2\pi}{3}$ **D** $\dfrac{2\pi}{3}$

**26** Consider the complex plane at right.




Which inequality could define the shaded area?
**A** $|z|\leq 1$ and $|z-(1-i)|\geq 1$
**B** $|z|\leq 1$ and $|z-(1+i)|\geq 1$
**C** $|z|\leq 1$ and $|z-(1-i)|\leq 1$
**D** $|z|\leq 1$ and $|z-(1+i)|\leq 1$

**27** If $A=3+4i$ and $B=5-13i$ write the following in the form $x+iy$

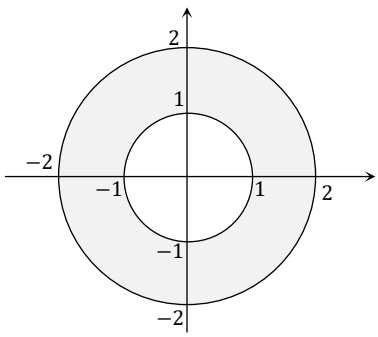
**a** $AB$

**b** $\dfrac{A}{B}$

**c** the square roots of $-A$.

**28** On the complex plane, shade the region where both $|z-1-i|\leq 2$ and $0\leq \arg z<\frac{\pi}{4}$.

**29** Consider the complex plane at right.




Which inequality best describes the shaded area?
**A** $0\leq|z|\leq 2$ **B** $1\leq|z|\leq 2$
**C** $0\leq|z-1|\leq 2$ **D** $1\leq|z-1|\leq 2$

**30** Factorise $x^4 + x^2 - 12$ completely over the field of:

**a** rational numbers

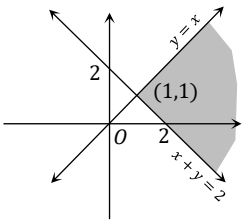
**b** real numbers

**c** complex numbers.

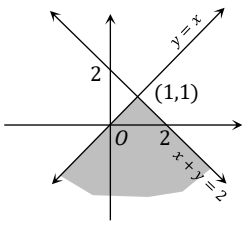
**31** The complex numbers $z_1, z_2, z_3$ and $z_4$ are represented in the complex plane by the points $A, B, C$ and $D$ respectively. If $z_1 + z_3 = z_2 + z_4$ prove $ABCD$ is a parallelogram.

**32** The complex number $z$ satisfies the inequations $|z + 2i| \geq |z + 2|$ and $\text{Im}(z) + \text{Re}(z) \geq 2$ Which of these shows the shaded region in the complex plane that satisfies these inequations?

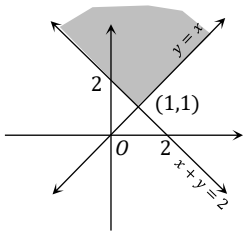
**A**



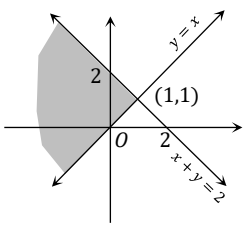

**B**



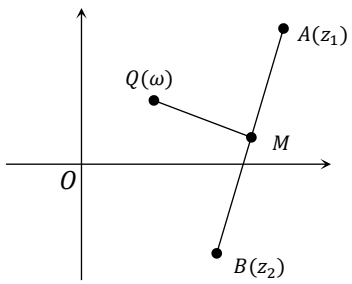

**C**




**D**




**33** The complex numbers $z_1$ and $z_2$ respectively. $M$ is the midpoint of the interval $AB$ and $QM$ is drawn perpendicular to $AB$. $QM = AM = BM$. If $Q$ corresponds to the complex number $\omega$, find $\omega$.




Medium

**34** If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ is equal to
**A** $128\,\omega$ **B** $-128\omega$ **C** $128\omega^2$ **D** $-128\omega^2$

**35** If $z = x + iy$, the points that lie on a circle of radius 2 centred at the origin on the complex plane can be represented by the equation
**A** $z\bar{z} = 2$
**B** $(z + \bar{z})^2 - (z - \bar{z})^2 = 16$
**C** $\text{Re}(z^2) + \text{Im}(z^2) = 4$
**D** $[\text{Re}(z)]^2 + [\text{Im}(z)]^2 = 16$

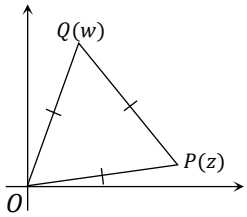
**36** Let $z = 1 + i\sqrt{3}$

**a** Write $z$ in modulus-argument form.

**b** Hence evaluate $z^5 + 16z$.

**37** Given that $z = x + iy$, find the value of $x$ and the value of $y$ such that $z + 3i\bar{z} = -1 + 13i$.

**38** In the complex plane, $OPQ$ is an equilateral triangle. $P$ represents the complex number $z$ and $Q$ represents the complex number $w$. Show that $w^3 + z^3 = 0$



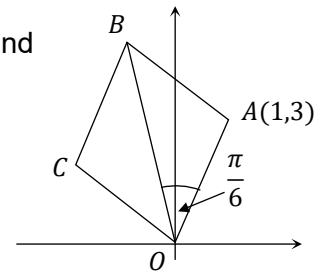

**39** Evaluate $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^9$.

**40a** Show that $(1-3i)^2 = -8-6i$.

**b** Hence solve the equation $2z^2 - 8z + (12+3i) = 0$.

**41** $OABC$ is a rhombus in the complex plane, where $O$ is the origin and point $B$ is in the second quadrant. The point $A$ is $(1,3)$ and the angle $AOB$ is $\frac{\pi}{6}$.

Find the complex numbers represented by the points $B$ and $C$ in the form $a + ib$.




**42** The roots of $z^3 - 1 = 0$ are $1, \Omega$ and $\overline{\Omega}$, where $\Omega$ is one of the non-real roots.

**a** Explain why $1 + \Omega + \Omega^2 = 0$.

**b** Show that $\overline{\Omega} = \Omega^2$.

**43** Solve $1 + i = a\,(\cos(-b) - i\sin b)$ for $b \in (-\pi, \pi]$.

**44** Given $z = \dfrac{\sqrt{3}+i}{1+i}$.

**a** Find the modulus and argument of $z$.

**b** Find the smallest positive integer such that $z^n$ is real.

**45** The complex number $z$ moves such that $\operatorname{Im}\left[\dfrac{1}{\bar{z}-i}\right] = 2$. Show that $z$ lies on a circle.

**46** Sketch the region in the complex number plane where the inequalities $|z+1-i| < 2$ and $-\pi \le \arg(z+1-i) \le \frac{3\pi}{4}$ hold simultaneously.

**47** Find the three different values of $z$ for which $z^3 = \dfrac{1+i}{\sqrt{2}}$.

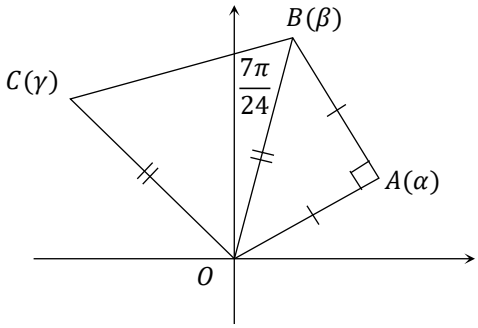
**48** Find the two complex numbers that satisfy $z\bar{z} = 37$ and $\dfrac{z}{\bar{z}} = \dfrac{35}{37} + \dfrac{12i}{37}$.

**49** If $w = \left(-1 + i\sqrt{3}\right)^{2012}$, find $\arg w$.

**50** Write $-3\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right)$ in modulus-argument form.

**51** Points $A$, $B$ and $C$ represent the complex numbers $\alpha, \beta$ and $\gamma$ in the complex plane respectively.

$\Delta OAB$ is right isosceles at $A$, $\Delta COB$ is isosceles with $OB = OC$ and $\angle OBC = \dfrac{7\pi}{24}$.




**a** Find $\angle AOC$.

**b** Explain why $\gamma = \sqrt{2}\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)\alpha$.

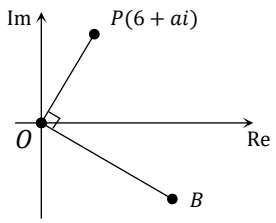
**c** Hence find the value of $2\alpha^2 + \gamma^2 + \alpha\gamma\sqrt{2}$.

**52** Describe the curve on the complex plane where the complex numbers $z$ are defined by $z\bar{z} - 2(z + \bar{z}) = 5$.

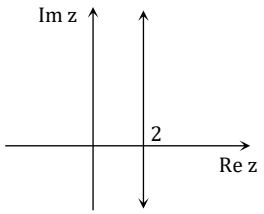
**53** Find the value of $\mathrm{z}^{10}$ in Cartesian form, given that $\mathrm{z} = \sqrt{2} - \sqrt{2}\mathrm{i}$.

**54** In the complex plane, $P$ represents the point $6 + ai$, and $O$ is the origin.

Find the complex number represented by the point $B$, given $\angle POB = 90°$ and $2|OB| = 3|OP|$.




**55** Which if the following is NOT a valid algebraic description of the line shown at right?




**A** $\mathrm{Re}(z) = 2$ **B** $|z| = |z - 4|$
**C** $\mathrm{Arg}(z - 4) + \mathrm{Arg}\, z = \pi$ **D** $z + \bar{z} = 4$

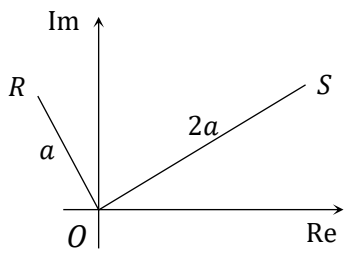
**56a** Write $2 + 2\sqrt{3}i$ in modulus-argument form.

**b** Hence express $\left(2 + 2\sqrt{3}i\right)^3$ in the form $x + iy$.

**c** Find all unique solutions to the equation $z^4 = 2 + 2\sqrt{3}i$, giving answers in modulus-argument form.

**57** Given $z$ is a complex number, sketch on a number plane the line defined by $\mathrm{Arg}\, z = \mathrm{Arg}[z - (1 + i)]$.

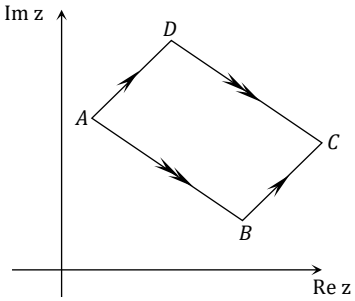
**58** In the complex plane the points $R$ and $S$ represent the complex numbers $w$ and $z$, respectively where $\angle SOR = 90°$. The distance $OS$ is $2a$ units, and distance $OR$ is $a$ units. Which of the following is correct?




**A** $w = 2iz$ **B** $w = i\bar{w}$
**C** $w = -\dfrac{iz}{2}$ **D** $w = -\dfrac{z}{2i}$

**59** Use a binomial expansion and de Moivre's theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

**60** The diagram shows parallelogram $ABCD$ drawn in the first quadrant of the complex plane. The points $A$, $B$ and $C$ represent the complex numbers $z_1, z_2$ and $z_3$ respectively. The vector $DB$ represents the complex number:




**A** $z_1 + z_3 - 2z_2$ **B** $z_2 - z_1 - z_3$
**C** $2z_2 - z_1 - z_3$ **D** $2z_2 - z_1 + z_3$

**61** The non-real complex number $\omega$ is a root of the equation $z^3 + 1 = 0$.

Which of the following is FALSE?

**A** $\bar{\omega}$ is also a root **B** $\omega^2 + 1 - \omega \neq 0$ **C** $\frac{1}{\omega}$ is also a root **D** $(\omega - 1)^3 = -1$.

**62a** Find the two square roots of $16 - 30i$.

**b** Hence solve $z^2 - 2z - (15 - 30i) = 0$.

**63** Use conjugate rules to show that if $z = r(\cos\theta + i\sin\theta)$ that $\cos\theta = \frac{z + \bar{z}}{2}$ and $\sin\theta = \frac{z - \bar{z}}{2i}$.

**64a** On an complex plane, draw and shade the region $R$ given by $|z - 2 - 2i| \leq 2$.

**b** $P$ is the point in $R$ where $|z|$ is at its maximum value. What is the modulus of $OP$?

**c** The tangent to the curve at $P$ cuts the $x$-axis at the point $T$. By using the nature of $\Delta OPT$, or otherwise, find the exact area of $\Delta OPT$

**65** On an complex plane, the points $A$ and $B$ represent the complex numbers $z_1 = -2i$ and $z_2 = 1 - \sqrt{3}i$. Which of the following statements is true?

**A** $\arg(z_1 + z_2) = -\frac{5\pi}{12}$ **B** $|z_1 - z_2| = 2 + \sqrt{3}$
**C** $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{6}$ **D** $\arg(z_1 z_2) = -\frac{\pi}{3}$

**66** It is given that $z = \cos\theta + i\sin\theta$, where $0 < \arg z < \frac{\pi}{2}$.

**a** Show that $z + 1 = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ and $z - 1 = -2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)$.

**b** Hence show that $\text{Re}\left(\frac{z-1}{z+1}\right) = 0$.

**67** Which of the following complex numbers equals $\left(\sqrt{3} + i\right)^4$?

**A** $-2 + \frac{2}{\sqrt{3}}i$ **B** $-8 + \frac{8}{\sqrt{3}}i$ **C** $-2 + 2\sqrt{3}i$ **D** $-8 + 8\sqrt{3}i$

**68** On the complex plane, let $O$ be the origin and $A$ be the point representing the complex number $\alpha = \frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)$. Point $B$ represents the complex number $\beta$, where $\beta = \alpha\text{cis}\left(\frac{\pi}{3}\right)$.

**a** Express $\beta$ in mod-arg form.

**b** Hence find the area of $\Delta OAB$.

**69** The complex number $Z$ moves such that $\text{Im}\left(\frac{1}{\bar{Z}-i}\right) = 1$. Show that $Z$ lies on a circle, and find its centre and radius.

**70** Show that the complex number $z = \frac{1-t^2+2it}{1+t^2}$ lies on a unit circle centre the origin for all values of $t$.

**71** Find the complex number $z = a + bi$, where $a$ and $b$ are real, such that $2\bar{z} - iz = 1 + 4i$.

**72** Verify that $z = \cos\frac{\pi}{9} + i\sin\frac{\pi}{9}$ is a root of the equation $z^6 - z^3 + 1 = 0$.

**73** In an complex plane, $ABCD$ is a quadrilateral such that the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ represent the complex numbers $a, b, c, d$ respectively. $P, Q, R$ and $S$ are the midpoints of $AB, BC, CD$ and $DA$ respectively. $M$ and $N$ are the midpoints of $PR$ and $QS$ respectively.

**a** Show that the vectors $\overrightarrow{OM}$ and $\overrightarrow{ON}$ both represent the complex number $\frac{1}{4}(a + b + c + d)$

**b** Hence explain what type of quadrilateral $PQRS$ is.

**74** Find the fourth roots of $2 + 2\sqrt{3}i$.

**75** The complex number $z = x + iy$ satisfies the relation $(z - \bar{z})^2 + 18(z + \bar{z}) = 36$. Show that $z$ lies on a parabola on the complex plane.

**76** If $z = a(\cos\theta + i\sin\theta)$ when $a$ and $\theta$ are real, show that $\frac{z}{z^2 + a^2}$ is equivalent to $\frac{1}{2a\cos\theta}$.

**77** Suppose $\omega^3 = 1, \omega \neq 1$. What are the two values of $1 + \omega^k + \omega^{2k}$ where $k \in \mathbb{Z}^+$?

**78** Given that $a$ and $b$ are real numbers and $\frac{a}{1+i} + \frac{b}{1+2i} = 1$ find the values of $a$ and $b$.

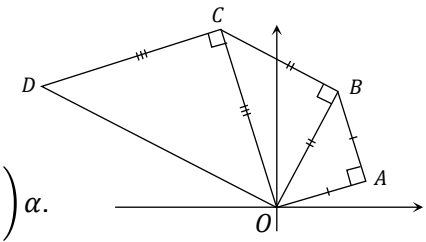
**79** The complex numbers $z_1, z_2, z_3$ and $z_4$ are represented in the complex plane by the points $A, B, C$ and $D$ respectively. $z$ represents a complex number such that $z^5 = 1$, where $z \neq 1$.

**a** Deduce that $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$.

**b** By substituting $x = z + \frac{1}{z}$ reduce the equation in (a) to a quadratic in $x$.

**c** Hence deduce that $\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -\frac{1}{4}$.

**80** The points $A$ and $D$ in a complex plane represent the complex numbers $\alpha$ and $\beta$ respectively. The triangles $AOB, ABC$ and $OCD$ are right angled isosceles triangles as shown.




**a** Show that $B$ represents the complex number $\left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)\alpha$.

**b** Hence show that $\beta = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\times i\alpha$.

**c** Show that $64\alpha^4 + \beta^4 = 0$.

**81** Sketch the following on separate complex planes.

**a** $z^2 - (\bar{z})^2 = 16i$.

**b** $\arg\left(\frac{z-i}{z-2}\right) = \frac{\pi}{2}$

**82** Suppose $z$ is any non-zero complex number.

**a** Explain why $\frac{z}{\bar{z}}$ has modulus 1 and argument twice the argument of $z$.

**b** Find all complex numbers $z$ so that $\frac{z}{\bar{z}} = i$. Give your answers in the form $a + ib$, where $a$ and $b$ are real.

Challenging

**83a** Show that $\cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) = 2\cos n\theta$.

**b** If $z = \cos\theta + i\sin\theta$, show that $(z + z^{-1})^3 = \left(z^3 + z^{-3}\right) + 3(z + z^{-1})$.

**c** Hence prove that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.

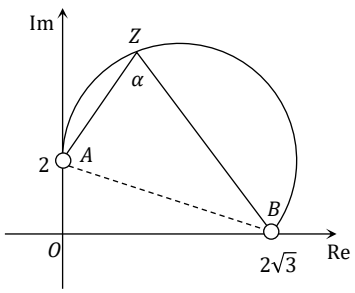
**84** Let $w = \frac{3+4i}{5}$ and $z = \frac{5+12i}{13}$, so that $|w| = |z| = 1$.

**a** Find $wz$ in the form $x + iy$.

**b** Hence, or otherwise, find two distinct ways of writing $65^2$ as the sum of $a^2 + b^2$, where $a$ and $b$ are integers and $0 < a < b$.

**85** Given $|z| < \sqrt{3}$, show that $\left|\left(1 + \sqrt{3}i\right)z^2 + 2iz\right| < 10$.

**86** The complex number $Z$, lies on a circle defined by $\text{Arg}\left(Z - 2\sqrt{3}\right) - \text{Arg}(Z - 2i) = \frac{\pi}{3}$. The angle between the lines from $2i$ to $Z$ and then from $2\sqrt{3}$ to $Z$ is $\alpha$, as shown.




**a** Show that $\alpha = \frac{\pi}{3}$.

**b** Find the centre and radius of the circle.

**87a** Solve $z^5 + 1 = 0$ by de Moivre's theorem, leaving your solutions in modulus-argument form.

**b** Prove that the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^5 + 1 = 0$.

**c** Show that if $z^4 - z^3 + z^2 - z + 1 = 0$ where $z = \text{cis}\,\theta$ then $4\cos^2\theta - 2\cos\theta - 1 = 0$.
Hint: $z^4 - z^3 + z^2 - z + 1 = 0 \Rightarrow z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0$.

**d** Hence find the exact value of $\sec\frac{3\pi}{5}$ in the form $a + b\sqrt{5}$.

**88a** Use de Moivre's theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$.

**b** Hence show that $x^4 - 10x^2 + 5 = 0$ has roots $\pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$.

**c** Deduce that $\tan\frac{\pi}{5}.\tan\frac{2\pi}{5}.\tan\frac{3\pi}{5}.\tan\frac{4\pi}{5} = 5$.

**89a** Find the roots of the equation $z^5 - 1 = 0$, leaving answers in mod-arg form.

**b** Hence find the exact value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$.

**90** When $z = \cos\theta + i\sin\theta$, $\sin n\theta = \frac{z^n - z^{-n}}{2i}$ and $\cos n\theta = \frac{z^n + z^{-n}}{2}$.
(Do NOT prove these.) Prove the identity $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta + 2$.

**91** The roots of the equation $z^5 + 1 = 0$ are $-1, \omega_1, \omega_2, \omega_3, \omega_4$ in cyclic order, anti-clockwise around the complex plane.

**a** Show that $\omega_1 = \overline{\omega_4}$.

**b** Find the values of $a, b$ and $c$ so that $(z+1)(z^4 + az^3 + bz^2 + cz + 1) = z^5 + 1$ and hence show that if $\omega$ is a root of $z^5 + 1 = 0$, not equal to $-1$, then $\omega^4 + \omega^2 + 1 = \omega^3 + \omega$.

**c** Show that ${\omega_1}^3 = \omega_3$.

*(For the rest of this question you may assume :* ${\omega_2}^3 = \omega_1$, ${\omega_4}^3 = \omega_2$ and ${\omega_3}^3 = \omega_4$*)*

**d** Deduce that ${\omega_1}^3 + {\omega_2}^3 + {\omega_3}^3 + {\omega_4}^3 = 1$.

**e** By using the sum of the roots of $z^5 + 1 = 0$ in pairs, or otherwise, prove that
$$\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = -\frac{1}{2}.$$

**92** The fixed complex number $\alpha$ is such that $0 < \arg\alpha < \frac{\pi}{2}$. In an complex plane $\alpha$ is represented by the point $A$ while $i\alpha$ is represented by the point $B$. $z$ is a variable complex number which is represented by the point $P$.

**a** Draw a diagram showing $A$, $B$ and the line defined by $|z - \alpha| = |z - i\alpha|$.

**b** Draw a diagram showing $A$, $B$ and the line defined by $\text{Arg}(z - \alpha) = \text{Arg}(i\alpha)$.

**c** Find, in terms of $\alpha$ the complex number represented by the point of intersection of the line and curve in part (a) and part (b).

**93** The number $c$ is real and non-zero. It is also known that $(1+ic)^5$ is real.

**a** Use binomial theorem to expand $(1+ic)^5$.

**b** Show that $c^4 - 10c^2 + 5 = 0$.

**c** Hence show that $c = \sqrt{5-2\sqrt{5}}, -\sqrt{5-2\sqrt{5}}, \sqrt{5+2\sqrt{5}}, -\sqrt{5+2\sqrt{5}}$.

**d** Let $1+ic = r\text{cis}\,\theta$. Use de Moivre's theorem to show that the smallest positive value of $\theta$ is $\frac{\pi}{5}$.

**e** Hence evaluate $\tan\left(\frac{\pi}{5}\right)$.

**94a** Given that $z = \cos\theta + i\sin\theta$ prove that $z^n + \frac{1}{z^n} = 2\cos n\theta$.

**b** Express $x^5 - 1$ as the product of three factors each containing real coefficients.

**c** Prove that $\left(1-\cos\frac{2\pi}{5}\right)\left(1-\cos\frac{4\pi}{5}\right) = \frac{5}{4}$.

**95** The complex number $v$ has modulus 1 and argument $\frac{\pi}{6}$, and the complex number $w$ has modulus 2 and argument $-\frac{2\pi}{3}$.

**a** Express $wv$ and $iv$ in the modulus−argument form where each argument is between $-\pi$ and $\pi$.

**b** Show that $v$ is a solution of the equation $Z^4 = iZ$. Hence, or otherwise, state the other two non−zero roots of this equation. You may leave your answers in modulus−argument form.

**c** Mark on an complex plane, the points $P, Q, R$ and $S$ representing $v, w, wv$ and $iv$ respectively.

**d** Hence, or otherwise, show that $PS$ is parallel to $RQ$.

**e** Hence, or otherwise, find a real number $u$ such that $iv - v = u(w - wv)$.

**96a** Show that $(1+i\tan\theta)^n + (1-i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ where $\cos\theta \neq 0$ and $n$ is a positive integer.

**b** Hence show that if $z$ is a purely imaginary number, the roots of $(1+z)^4 + (1-z)^4 = 0$ are $z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$.

**97** Let $w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$.

**a** Show that $w^n$ is a root of $z^9 - 1 = 0$, $n$ an integer.

**b** Show that $w + w^8 = 2\cos\frac{2\pi}{9}$.

**c** Show that $(w^3 + w^6)(w^2 + w^7) = w + w^8 + w^4 + w^5$.

**d** Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$. You may assume that $\cos\frac{2\pi}{3} = -\frac{1}{2}$.

**98** Given that $w$ is a root of $z^3 + iz^2 + ikz + 2i = 0$, where $k$ is real, and $(1 - i)w$ is real, find the possible value of $k$.

**99** Let $z = \cos\theta + i\sin\theta$.

**a** Show that $z^n + z^{-n} = 2\cos n\theta$ and find a similar expression for $z^n - z^{-n}$.

**b** Hence prove that $2^5\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.

**100a** Show that $\frac{1 + \cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta} = i\cot\frac{\theta}{2}$.

**b** Hence solve $\left(\frac{z-1}{z+1}\right)^8 = -1$.

# Chapter 2
# *The nature of proof*

*The nature of proof* is a great topic for Extension 2 students as:

- it tests your ability to use mathematical language and plain language to reason and communicate
- it promotes clear, simple and logical thought processes
- it forces you to master your algebra skills - you need to be quick and accurate, and have a deep understanding of how the rules work and why
- the topic has a wide variety of applications so questions do not need to be repetitive.

All topics in Extension 2 rely on your ability to construct logically sound and convincing proofs, so *The nature of proof* is a fitting topic to do as early as possible in the course. We will focus on the logic of proofs, and use alternative ways to prove arguments.

**Lessons**

The nature of proof is covered in 9 lessons.

2.1 The language and notation of proof
2.2 Direct proof, proof by contrapositive and equivalence
2.3 Proof by contradiction
2.4 Disproofs
2.5 Inequalities
2.6 Arithmetic mean - geometric mean inequality
2.7 Other inequalities
2.8 Mathematical induction - inequalities and recursive formulas
2.9 Mathematical induction - proving other results
Chapter 2 - 100 revision questions

In Lesson 1 we cover:

- Statements and their negations
- Implications
- Converse and equivalence
- Quantifiers
- Contrapositive and inverse

Syllabus reference: This lesson covers the 1st to 12th dot points from The language and notation of proof in *The nature of proof*.

## Statements and their negations

A statement, or proposition, is a sentence that can be true or false, but not both.

Some examples of statements are:

- $P$: '$6$ is a composite number'
- $Q$: '$6$ is greater than $10$'
- $R$: 'The square of $6$ is even'
- $P$ and $R$ are true statements, while $Q$ is a false statement.

The negation of a statement covers all possibilities not included in the statement - we can think of the statement and its negation as being similar to complementary events in probability.

- When a statement is true its negation is false, and vice versa.
- The negation of the negation gives the original statement.

For simple statements like the ones above the negation is the exact opposite of the original statement, but things get more subtle as the statements get more involved.

Let's think about some simple events in probability:

- The complement of 'not getting any heads' when you toss one coin is 'getting one head'. In this case the complement is the exact opposite event, as they are the only two possibilities.
- The complement of getting no heads when tossing two coins is not necessarily getting two heads; it is getting at least one head. In this case the complement is NOT the exact opposite event, as there are more than just two possibilities. The complement has to cover everything except the events included in the original, so that all possibilities are covered by the statement, or by its negation, exactly once.

The negation of $P$ is 'not $P$', which we can write as $\neg P$ or $\sim P$. The negations for the three statements above are:

- $\neg P$: '$6$ is not a composite number'
- $\neg Q$: '$6$ is less than or equal to $10$'
- $\neg R$: 'The square of $6$ is not even' [or 'the square of $6$ is odd'].
- $\neg P$ and $\neg R$ are false statements, while $\neg Q$ is a true statement.

We mentioned that the negation of a negation gives the original statement. In symbols we have the first of the five negation rules we will introduce this lesson:

$$\boxed{\neg(\neg P) \equiv P} \qquad (1)$$

This is the basis of proof by contradiction - if we can prove that the negation is false, then the original statement must be true.

If there is a variable or number used in the statement, then we often add it in brackets or as a subscript, such as $P(n)$: The sum of the first $n$ odd integers is $n^2$. You may have used this notation in Extension 1 mathematical induction to represent the propositions in the question.

**Compound statements**

We can combine two or more statements into a compound statement.

1. $P$ and $Q$

- If both parts of a statement have to be correct, we write the statement in the form '$P$ and $Q$', or using logic symbols '$P \wedge Q$'. Some examples are:
  - '$6$ is an even number and $3$ is an odd number' (true since both parts are true).
  - '$6$ is an even number and $4$ is an odd number' (false since one or more parts are false).

2. $P$ or $Q$

- If at least one part of a statement has to be correct, we write the statement in the form '$P$ or $Q$', or using logic symbols '$P \vee Q$'. Note that either one or both parts can be true, not just one. Some examples are:
  - '$6$ is an even number or $4$ is an odd number' (true since at least one part is true).
  - '$6$ is an odd number or $4$ is an odd number' (false since neither part is true).

**Negating compound statements**

When we negate compound statements we need to find the complement carefully. We are no longer finding the opposite of the original statement. It can help to think about the negation of a statement as being the smallest change that could make the original statement break - this gives us the complement. We now look at the next two negation rules, given using words on the left and logic symbols on the right - use either form.

$$\neg(P \text{ and } Q) \equiv \neg P \text{ or } \neg Q \qquad \neg(P \wedge Q) \equiv \neg P \vee \neg Q \qquad (2)$$

Consider a compound statement where **both** parts, $P$ and $Q$, need to be true. The negation is the complement of both parts being true, which is at least one of the parts being false - in other words, at least one of the negations, $\neg P$ or $\neg Q$, being true. The smallest change that would stop both parts being true is one part being false.

$$\neg(P \text{ or } Q) \equiv \neg P \text{ and } \neg Q \qquad \neg(P \vee Q) \equiv \neg P \wedge \neg Q \qquad (3)$$

Consider a compound statement where **at least one** part, $P$ or $Q$, needs to be true. The negation is the complement of at least one part being true, which is both parts being false - in other words, both of the negations, $\neg P$ and $\neg Q$, being true. The smallest change that would stop at least one part being true would be for both parts to be false.

In summary: **to negate a compound statement, negate each of the original statements and swap 'and' for 'or', and vice versa.**

You might come across these rules as De Morgan's Laws, but students do not need to know this.

## Example 1

Consider the statements $P$: '$n$ is a multiple of $5$' $Q$: '$n$ is less than $100$'
Find the following negations.

**a** $\neg P$

'$n$ is not a multiple of $5$'

**b** $\neg Q$

'$n$ is greater than or equal to $100$'

**c** $\neg(P \wedge Q)$

'$n$ is not a multiple of $5$ or is greater than or equal to 100'

**d** $\neg(P \vee Q)$

'$n$ is not a multiple of $5$ and is greater than or equal to 100'

## Example 2

Consider the statement $P$: 'it rained in Cowra on at most two days in January this year'. Which of the following statements is the negation of $P$?

$Q$: 'it didn't rain in Cowra in January this year'
$R$: 'it rained in Cowra on two days in January this year'
$S$: 'it rained in Cowra on at least two days in January this year'
$T$: 'it rained in Cowra on at least three days in January this year'
$U$: 'it rained in Cowra every day in January this year'

$P$ means that there were $0, 1$ or $2$ days on which it rained in Cowra in January. The complement is $3$ to $31$ days of rain in Cowra in January, which is $T$.

## Implications

An implication is an if-then statement, which can be true or false. The implication 'if $P$ then $Q$' or '$P$ implies $Q$', is written as $P \Rightarrow Q$ or $P \rightarrow Q$.

An example of an implication which is true is 'if $x$ is a multiple of $4$, then $x$ is even'. An example of an implication which is false is 'if $x$ is even, then $x$ is a multiple of $4$'.

We use implications in mathematical induction - in the inductive step we prove that if $P(k)$ is true then $P(k+1)$ must be true, and may have finished the inductive step with $P(k) \Rightarrow P(k+1)$.

## Example 3

Consider the statements:

$P$: $n$ is a positive integer. $Q$: $n$ is an even number greater than 0

State whether each implication is true or false.

**a** $P \Rightarrow Q$ **b** $Q \Rightarrow P$ **c** $P$ implies $Q$ **d** $P$ is implied by $Q$

We can list the values of $n$ that satisfy $P$ and $Q$ to help us understand the implication.

$$P: \{1, 2, 3, 4, 5, 6, \ldots\} \quad Q: \{2, 4, 6, 8, 10, 12, \ldots\}$$

Since all values of $n$ satisfying $Q$ also satisfy $P$, we can say that $Q$ implies $P$, $Q \Rightarrow P$, $P$ is implied by $Q$ or $P \Leftarrow Q$, so, **b** $Q \Rightarrow P$ and **d** $P$ is implied by $Q$ are true, and **a** $P \Rightarrow Q$ and **c** $P$ implies $Q$ are false. Note that $P \Rightarrow Q$ and $P$ implies $Q$ are the converse of a true implication, and in this case are false.

## Example 4

Given $P$ and $Q$ below, can we say $P \Rightarrow Q$ or $Q \Rightarrow P$?

$P$: Quadrilateral $ABCD$ is a square  $Q$: Quadrilateral $ABCD$ is a rectangle.

A square is a rectangle, but a rectangle is not necessarily a square, so $P \Rightarrow Q$.

Again, we note that the converse of a true implication is not necessarily true.

**Negating an implication**

When we negate an implication, we need to find the smallest change that will make the implication false - we do NOT find the opposite of the statement, as that is only part of the complement.

So, if the implication $P \Rightarrow Q$ means that when $P$ is true then $Q$ is always true, what is the negation? It is simply $P$ and $\neg Q$ both being true, so $P$ is true and $Q$ is false. This is our fourth negation rule:

$$\boxed{\neg(P \Rightarrow Q) \equiv P \text{ and } \neg Q \qquad \neg(P \Rightarrow Q) \equiv P \wedge \neg Q} \qquad (4)$$

Note that the negation of the implication of $P \Rightarrow Q$ is NOT $P \Rightarrow \neg Q$.

We use (4) to derive the fifth negation rule, which is an important result that we will use next lesson. Negating both sides of (4), we see:

$$\boxed{P \Rightarrow Q \equiv \neg P \text{ or } Q \qquad P \Rightarrow Q \equiv \neg P \vee Q} \qquad (5)$$

So the implication $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$. We will use this result next lesson to prove that an implication is logically equivalent to its contrapositive.

## Example 5

Find the negations of these implications:

**a** $P$: 'if $x$ is an even integer then $x^2$ is an even integer'.

$\neg P$: '$x$ is an even integer and $x^2$ is not an even integer'.

**b** $Q$: 'if $x$ is less than $2$ then $x^2$ is less than $4$'.

$\neg Q$: '$x$ is less than $2$ and $x^2$ is greater than or equal to $4$'.

We can extend this for compound implications, in the form $P \Rightarrow (Q \Rightarrow R)$, where the negation is $P$ and $Q$ and $\neg R$, or $P \wedge Q \wedge \neg R$.

## Example 6

Find the negations of these implications:

**a** $P$: 'if $x$ is odd, then if $x$ is a multiple of $3$, then $\frac{x}{3}$ is odd'.

$\neg P$: '$x$ is odd and $x$ is a multiple of $3$ and $\frac{x}{3}$ is even'.

**b** $Q$: 'if $ABCD$ is a rectangle, then if $ABCD$ is a rhombus, then $ABCD$ is a square'.

$\neg Q$: '$ABCD$ is a rectangle and $ABCD$ is a rhombus and $ABCD$ is not a square'.

## Converse and equivalence

**Converse**

The converse of an implication occurs when we reverse the relationship between the two parts. Consider the implication $P \Rightarrow Q$. Its converse is: $P \Leftarrow Q$ or $Q \Rightarrow P$.

It is important to note that the converse of a true implication is not necessarily true. This is a source of many false direct proofs and lost marks in exams.

## Example 7

Consider the implication 'if the last digit of a positive integer is zero then it is a multiple of ten'. What is the converse of the implication?

'if a positive integer is a multiple of ten then its last digit is zero'.

**Equivalence**

If both the implication $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true, then we say that the two statements are equivalent.

We write this as $P \Leftrightarrow Q$, or in words as '$Q$ if and only if $P$' which we write as '$Q$ iff $P$'.

It is interesting to note that quite often when we write $P$ if $Q$ we could write $P$ iff $Q$, as a lot of our relationships are equivalent.

In equivalent statements we can swap the two statements around and still create a true statement, as the original and its converse are both true. For example, $P$: '$\forall x \in \mathbb{Z}^+, x > 0$' and $Q$: '$\forall x \in \mathbb{Z}^+, 2x > 0$' are equivalent statements, as when $P$ is true $Q$ must be true, and when $Q$ is true $P$ must be true. We can say that '$\forall x \in \mathbb{Z}^+, x > 0$ if and only if $2x > 0$'.

Let's look at an **incorrect** example: '$x^2 = 4$ iff $x = 2$'. This equivalence statement is false, because $x = -2$ would also cause $x^2 = 4$ to be true. One of the two implications in the equivalence statement is correct: 'if $x = 2$ then $x^2 = 4$'. The other implication in the equivalence statement is incorrect, since 'if $x^2 = 4$ then $x = \pm 2$'.

Proving equivalence involves two proofs. We first prove that $P \Rightarrow Q$, and then prove $Q \Rightarrow P$. Quite often one of the two proofs involves an indirect proof - either proof by contrapositive or proof by contradiction - so we will cover equivalence proofs once we have learned these methods of indirect proof.

It is important to note that when we solve an equation we form a string of implications, which are only guaranteed to work in one direction - some algebraic operations preserve equivalence while others do not. We are really saying that if the first line is true, then the second line is true, and if the second line is true, the third line is true, etc. For example, if $x = 4$ then $x^2 + 4 = 20$, as shown below left:

$$\begin{aligned} & x = 4 && \text{Line 1} \\ \Rightarrow \quad & x^2 = 16 && \text{Line 2} \\ \Rightarrow \quad & x^2 + 4 = 20 && \text{Line 3} \end{aligned}$$

$$\begin{aligned} & x^2 + 4 = 20 && \text{Line 4} \\ \Rightarrow \quad & x^2 = 16 && \text{Line 5} \\ \Rightarrow \quad & x = \pm 4 && \text{Line 6} \end{aligned}$$

Does this mean that if $x^2 + 4 = 20$ that $x = 4$? Not quite. Solving $x^2 + 4 = 20$, above right, we see that $x = \pm 4$.

The difference between the two results comes about because when we square a number (line 1 to line 2) there is only one result, but when we remove a square (line 5 to line 6) there are two possible results as we need to consider the positive and negative square roots.

This is another example showing that the converse of a true result is not necessarily true.

**This is why all direct proofs must start with a result known to be true and move towards the result to be proved.**

This avoids potential logical flaws caused by proving the converse of the result to be proved.

## Example 8

The following proof aims to establish that $-3 = 3$.

Let $x = -3$ Line 1

$\Rightarrow \quad x = \sqrt{x^2}$ Line 2

$\Rightarrow \quad = \sqrt{(-3)^2}$ Line 3

$\Rightarrow \quad = \sqrt{9}$ Line 4

$\Rightarrow \quad = 3$ Line 5

At which line is the implication incorrect?

The implication on line 2 is incorrect, as $x = -\sqrt{x^2}$ for $x < 0$.

This is a misapplication of the rule $\sqrt{x^2} = |x|$.

## Quantifiers

**For all** $\forall$

Some statements are true or false for all values of a variable within a domain or set, and we can abbreviate this with the symbol $\forall$, an upside down A. We can call these statements universal statements. For example, '$\forall x \in \mathbb{R}, x^2 \geq 0$' means that the square of every real number is non-negative.

**There exists** $\exists$

Some statements are true for at least one value of a variable within a domain or set, and we can abbreviate this with the symbol $\exists$, a back to front E. We can call these statements existential statements. For example '$\exists x \in \mathbb{Z}, x^2$ is odd' is saying there is at least one integer value of $x$ for which $x^2$ is odd.

**Multiple quantifiers**

When two quantifiers are used for two variables, they are read left to right, and the order matters. Consider $P$: '$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x - y = 6$' and $Q$: '$\exists y \in \mathbb{R}\ \forall x \in \mathbb{R}\ x - y = 6$'.

- Statement $P$ is true, since every value of $x$ has one value of $y$ such that $x - y = 6$.
- Statement $Q$ is false, since there is not any value of $y$ for which $x - y = 6$ for every real value of $x$.

## Example 9

Choose ∀ if the statement is true for every value, or ∃ if it is true for at least one value but not every value.

**a** $\square\ m \in \mathbb{Z}, m^2 \geq 0$

∀ since the square of every integer is non-negative

**b** $\square\ m \in \mathbb{Q}, m = 0$

∃ since there is at least one rational number that equals zero

**c** $\square\ m \in \mathbb{N}, \frac{m}{2} \in \mathbb{Z}$

∃ since there is at least one natural number that is an integer when halved

**d** $\square\ m \in \mathbb{R}, -1 \leq \sin m \leq 1$

∀ since the sine of all real numbers lies in the interval $[-1, 1]$

## Example 10

Consider the following statement.

'It is possible to pick a positive number $p$, for any real value of $d$, such that $\frac{p}{d} = 1$'.

When this statement is written in the formal language of proof, which of the following is obtained?

**A** $\forall d \in \mathbb{R}\ \exists p > 0\ \frac{p}{d} = 1$

**B** $\forall d > 0\ \exists p \in \mathbb{R}\ \frac{p}{d} = 1$

**C** $\exists p \in \mathbb{R}\ \forall d > 0\ \frac{p}{d} = 1$

**D** $\exists p > 0\ \forall d \in \mathbb{R}\ \frac{p}{d} = 1$

We start by picking any real value for $d$, $(\forall d \in \mathbb{R})$, then find at least one positive value of $p$ $(\exists p > 0)$ such that $\frac{p}{d} = 1$. Answer A.

The statement is false because no positive value of $p$ satisfies the equation when $d$ is negative, and the expression is undefined when $d = 0$.

**Negating quantified statements**

When we negate a universal statement, the smallest change is finding at least one value for which the negation is true, within the same domain or set. So, $\neg(\forall x \in \mathbb{R}, P) = \exists x \in \mathbb{R}(\neg P)$. We swap $\forall$ for $\exists$, then take the negation of the remainder of the statement as normal.

When we negate an existential, the smallest change is having the negation true for all values - in other words the statement cannot be true for any value. So, $\neg(\exists x \in \mathbb{R}, P) = \forall x \in \mathbb{R}(\neg P)$. We swap $\exists$ for $\forall$, then take the negation of the remainder of the statement as normal.

In summary, when we negate statements involving quantifiers, then the original statement and its negation swap quantifiers, the domain or set is unchanged, and we negate any other part of the statement.

For example:

- The negation of '$\forall x \in \mathbb{R}, x^2 \geq 0$' is '$\exists x \in \mathbb{R}, x^2 < 0$'.
  - The original statement is saying that $x^2 \geq 0$ for every real value of $x$, while the negation is just saying, hang on, there is at least one value where that isn't true.
  - Notice that the negation is not saying that the original statement is false for every value of $x$, just for one or more values of $x$. In this case the original statement is true and its negation is false.
- The negation of '$\exists x \in \mathbb{R}, x^2 - 4 = 0$' is '$\forall x \in \mathbb{R}, x^2 - 4 \neq 0$'.
  - The original statement is saying that $x^2 - 4 = 0$ for at least one value of $x$, while the negation is just saying, hang on, there are no values of $x$ for which it is true, in other words $x^2 - 4 \neq 0$ for every value of $x$.
  - Notice that the negation is not saying that there is one or more values of $x$ for which $x^2 - 4 \neq 0$. Again the original statement is true and its negation is false.

## Example 11

Find the negation of the following statements:

**a** $\forall x \in \mathbb{Z}^+, 2x$ is even.

$\exists x \in \mathbb{Z}^+, 2x$ is odd

**b** $\exists x \in \mathbb{R}, \exists m \in \mathbb{Z}, \quad x = 4m$

$\forall x \in \mathbb{R}, \forall m \in \mathbb{Z}, \quad x \neq 4m$

## Example 12

Find the negation of the following statements:

**a** $\forall x \in \mathbb{Z}^+, 2x$ is even and $2x + 1$ is odd.

$\exists x \in \mathbb{Z}^+, 2x$ is odd or $2x + 1$ is even

**b** $\exists x \in \mathbb{R}, x^2 = 9$ or $x > 2$

$\forall x \in \mathbb{R}, x^2 \neq 9$ and $x \leq 2$

## Contrapositive and inverse

**Contrapositive**

To find the contrapositive of an implication, swap the order of the two parts and negate each part. The contrapositive of the implication 'if $P$ then $Q$' is 'if not $Q$ then not $P$', so the negation of $Q$ implies the negation of $P$. Using symbols, the contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$.

As we will see next lesson, the contrapositive is true if and only if the statement itself is also true. The implication and its contrapositive are logically equivalent, so to prove an implication we can prove its contrapositive instead.

**Inverse**

To find the inverse of an implication, negate each part, leaving them in the same order. The inverse of the implication 'if $P$ then $Q$' is 'if not $P$ then not $Q$', so the negation of $P$ implies the negation of $Q$. Using symbols the inverse of $P \Rightarrow Q$ is $\neg P \Rightarrow \neg Q$.

The inverse is the contrapositive of the converse, so the converse and inverse are logically equivalent.

The term 'inverse' is not mentioned in the syllabus, so you will not see it again after this lesson. We will sometimes prove an inverse when we prove equivalence, without using that term.

## Example 13

Find the contrapositive of the following implications.

**a** If $x$ is even then $x+1$ is odd.

If $x+1$ is even then $x$ is odd.

**b** If $x^2 = 25$ then $x = \pm 5$

If $x \neq \pm 5$ then $x^2 \neq 25$

## Example 14

Find the inverse of the following implications.

**a** If $x$ is even then $x+1$ is odd.

If $x$ is odd then $x+1$ is even.

**b** If $x^2 = 25$ then $x = \pm 5$

If $x^2 \neq 25$ then $x \neq \pm 5$

Basic

**1** Consider the statements $P$: '$n$ is a multiple of $3$' $Q$: '$n$ is at most $20$'
Find the following negations.

**a** $\neg P$

**b** $\neg Q$

**c** $\neg(P \wedge Q)$

**d** $\neg(P \vee Q)$

**2** Consider the statement $P$: 'a triangle has two sides the same length and one side a different length'. Which of the following statements is the negation of $P$?

$Q$: 'a triangle is scalene'

$R$: 'a triangle is isosceles'

$S$: 'a triangle is equilateral'

$T$: 'a triangle is scalene or isosceles'

$U$: 'a triangle is scalene or equilateral'

**3** What is the converse of the statement 'if $x$ is less than $2$ then $x^2$ is less than $4$'?

**4** Consider the statements: $P$:$n$ is a multiple of $3$. $Q$:$n$ is a multiple of $9$.
Which of the following is true?

**a** $P \Rightarrow Q$

**b** $P \Leftarrow Q$

**c** $P$ implies $Q$

**d** $P$ is implied by $Q$

**5** Given $P$ and $Q$ below, can we say $P \Rightarrow Q$ or $Q \Rightarrow P$?

$P$: Quadrilateral $ABCD$ is a parallelogram $Q$: Quadrilateral $ABCD$ is a rectangle.

**6** The following proof aims to establish that $-1 = 1$.

| | | |
|---|---|---|
| | Let $x = -1$ | Line $1$ |
| $\Rightarrow$ | $x^2 = 1$ | Line $2$ |
| $\Rightarrow$ | $x = \sqrt{1}$ | Line $3$ |
| $\Rightarrow$ | $= 1$ | Line $4$ |

At which line is the implication incorrect?

**7** Find the contrapositive of the following:

**a** If $x$ is prime, then $2x + 1$ is composite.

**b** If $x^2 = 1$ then $\frac{1}{x} = 1$.

**8** Find the inverse of the following:

**a** If $x$ is rational then $x^3$ is rational.

**b** If $x^2 < 1$ then $-1 < x < 1$.

**9** Find the contrapositive of $P$: 'All birds are ducks.', and decide whether the contrapositive is true or false.

**10** What is the contrapositive of the statement: '$\forall n \in Z$, if $n^2$ is odd then $n$ is odd'?

## Medium

**11** Find the negations of these implications:

**a** $P$: 'if $x$ is a positive integer then $-3x$ is a negative integer'.

**b** $Q$: 'if $x$ is an odd integer then $4x$ is an even integer'.

**c** $R$: 'if an animal lives in the ocean then it is a fish'.

**12** Find the negations of these implications:

**a** $P$: 'if $x$ is a multiple of $6$, then if $x$ is a multiple of $8$, then $x$ is a multiple of $24$'.

**b** $Q$: 'if $ABCD$ is a parallelogram, then if $ABCD$ is a kite, then $ABCD$ is a rhombus'.

**13** Choose $\forall$ if the statement is true for every value, or $\exists$ if it is true for at least one value but not every value.

**a** $\square$ $m \in \mathbb{Z}, 2m$ is even

**b** $\square$ $m \in \mathbb{R}, e^m = 4$

**c** $\square$ $m \in \mathbb{N}, \left(\frac{m}{3}\right)^2 \leq 2$

**d** $\square$ $m \in \mathbb{R}, 1 + \cos m \geq 0$

**14** Consider the following statement.

'For any integer, $d$, from 0 to 10, a positive number $p$ can be chosen, such that $p > d$'.

When this statement is written in the formal language of proof, which of the following is obtained?

**a** $\exists p > 0 \ \forall \{d \in \mathbb{Z} | 0 \leq d \leq 10\} \ p - d > 0$

**b** $\forall p > 0 \ \exists \{d \in \mathbb{Z} | 0 \leq d \leq 10\} \ p - d > 0$

**c** $\forall \{d \in \mathbb{Z} | 0 \leq d \leq 10\} \ \exists p > 0 \ p - d > 0$

**d** $\exists \{d \in \mathbb{Z} | 0 \leq d \leq 10\} \ \forall p > 0 \ p - d > 0$

**15** Find the negation of the following:

**a** $\exists x \in \mathbb{Z}^+, 2x = x^2$

**b** $\forall x \in \mathbb{R}, \exists m \in \mathbb{Z}^+, x < 2m + 1$

**16** Consider the statements:

$P$: 'A quadrilateral has four equal sides.'

$Q$: 'A quadrilateral has four right angles.'

Find the following negations, expressing each answer in words.

**a** $\neg P$

**b** $\neg Q$

**c** $\neg (P \wedge Q)$

**d** $\neg (P \vee Q)$

**17** Consider the following implications. For each implication, state its converse and determine whether the original implication and its converse are true or false.

**a** If an animal is a mammal, then it is warm−blooded.

**b** If a quadrilateral is a rhombus, then its diagonals are perpendicular.

**c** If $n$ is a multiple of 12, then $n$ is a multiple of 4.

**18** Find the negations of the following implications.

**a** If a triangle is right−angled, then if it is isosceles, two of its angles are $45°$.

**b** If $n$ is a positive integer, then if $n$ is divisible by $6$, $n^2$ is divisible by $12$.

**c** If an animal is a bird, then if it cannot fly, it is a penguin.

**19** Add the most relevant quantifier, $\forall$ or $\exists$, to the start of each statement to make it true as often as possible, or state that neither quantifier makes the statement true.

**a** $\square\ n \in \mathbb{Z}, n^2 + n$ is even

**b** $\square\ x \in \mathbb{R}, x^2 + 4x + 5 = 0$

**c** $\square\ x \in \mathbb{R}, |x - 3| = x - 3$

**d** $\square\ n \in \mathbb{Z}^+, n^2 < 2n + 3$

**20** Find the contrapositive of each of the following implications. State whether the original implication is true or false.

**a** If $n^2$ is odd, then $n$ is odd, where $n \in \mathbb{Z}$.

**b** If a quadrilateral is a square, then its diagonals are equal in length.

**c** If $x^2 > 9$, then $x > 3$, where $x \in \mathbb{R}$.

Challenging

**21** Find the negation of the following:

**a** $\forall x \in \mathbb{Z}^+, 2x$ is prime or $2x + 1$ is odd

**b** $\exists x \in \mathbb{R}, x^2 - 7 = 9$ and $x > 2$

**22** Consider the proposition. 'If each term in a geometric sequence is smaller than the previous term, then $0 < r < 1$'. Given the first term and common ratio in each of the following geometric sequences, which one disproves the proposition?

A. $T_1 = 2, \quad r = 2$

B. $T_1 = 8, \quad r = 0.5$

C. $T_1 = -2, \quad r = 2$

D. $T_1 = 8, \quad r = -0.5$

**23** Find the negation of each of the following statements, expressing your answers using mathematical symbols.

**a** $\forall x \in \mathbb{R}, x^2 \geq 4 \vee x < 1$

**b** $\exists n \in \mathbb{Z}^+, n^2$ is even $\wedge$ $n^3$ is odd

**c** $\forall n \in \mathbb{Z}^+, (n$ is prime$) \Rightarrow (n$ is odd$)$

**d** $\exists x \in \mathbb{R}, \left(x^2 = 16\right) \wedge (-2 < x < 5)$

**24** Consider the following two statements.

$P$: 'For every positive integer $n$, there exists a positive integer $m$ such that $m > n$.'

$Q$: 'There exists a positive integer $m$ such that $m > n$ for every positive integer $n$.'

**a** Express $P$ and $Q$ using the quantifiers $\forall$ and $\exists$.

**b** Determine whether each statement is true or false, giving reasons.

**c** Find the negation of each statement using mathematical symbols.

**25** Consider the following statement.

'Every animal in a wildlife sanctuary is either a mammal or a bird, but cannot be both, and every bird in the sanctuary can fly.'

Let $A$ be the set of animals in the sanctuary, $M(x)$ mean that $x$ is a mammal, $B(x)$ mean that $x$ is a bird, and $F(x)$ mean that $x$ can fly.

**a** Express the statement using quantifiers and logical symbols.

**b** Find its negation, expressing your answer in words.

**c** Explain why the existence of a flightless mammal would not necessarily disprove the original statement.

**26** The following argument aims to establish that every real number is non−negative.

$$\begin{aligned} &\text{Let } x \in \mathbb{R}. && \text{Line } 1 \\ \Rightarrow\quad & x^2 \geq 0 && \text{Line } 2 \\ \Rightarrow\quad & \sqrt{x^2} \geq 0 && \text{Line } 3 \\ \Rightarrow\quad & x \geq 0 && \text{Line } 4 \end{aligned}$$

**a** Identify the line at which the implication is incorrect.

**b** Explain the error in the reasoning.

**c** Give a counterexample to the conclusion.

**d** State the correct conclusion that follows from line 3.

**27** Consider the proposition:

′If the product of two non−zero real numbers is positive, then both numbers are positive.′

**a** Express the proposition using mathematical symbols, with $x$ and $y$ representing the two non−zero real numbers.

**b** Write the converse of the proposition.

**c** Write the contrapositive of the proposition.

**d** Determine whether the original proposition, its converse and its contrapositive are true or false. Justify your answers.

# 2.2 Direct proof, proof by contrapositive and equivalence

In Lesson 2 we cover:

- Direct proofs and direct reasoning
- Simple proofs involving integers
- Proof by contrapositive
- Proving equivalence

Syllabus reference: This lesson covers the 10th to 12th dot points from The language and notation of proof, and the 3rd dot point from Illustrations of proofs in *The nature of proof*.

## Direct proofs and direct reasoning

In this lesson we start by looking at simple proofs involving integers, which generally involve direct proofs. We then look at proofs by contrapositive, which are an indirect proof involving a direct proof of the contrapositive, which proves the implication. We finish by proving equiavalence, which involves two direct proofs, or one direct proof and one proof by contrapositive.

## Simple proofs involving integers

We can prove some simple results using direct proofs. Some of these results can also be proved using indirect proofs, as we will see later in the chapter, and if they are only true for positive integers they can also be proved by mathematical induction. **If a question does not mention using induction always try a direct proof first to save time.**

### Example 1

Prove that the product of any odd number $m$ and any even number $n$ is even.

Let $m = 2k + 1$ and $n = 2j$ for $j, k \in \mathbb{Z}$

$mn = (2k + 1)(2j)$

$= 2(2jk + j)$

$= 2p$ for $p \in \mathbb{Z}$ since $j, k \in \mathbb{Z}$

$\therefore$ the product of any odd number $m$ and any even number $n$ is even.

## Example 2

Prove that for every integer that is a multiple of 3, $m^2 = 9p$ for $p \in \mathbb{Z}$.

$m^2 = (3k)^2$

$= 9k^2$

$= 9p$ for $p \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore \forall m = 3k$ for $k \in \mathbb{Z}, m^2 = 9p$ for $p \in \mathbb{Z}$.

## Example 3

Prove that the sum of any two consecutive integers is odd.

Let the consecutive integers be $k$ and $k + 1$ for $k \in \mathbb{Z}$.

$k + (k + 1) = 2k + 1$ which is odd since $k \in \mathbb{Z}$.

$\therefore$ the sum of any two consecutive integers is odd.

## Example 4

Prove that $\forall a, b \in \mathbb{Z}$, $(a + b)^2 - (a - b)^2$ is divisible by 4.

$(a + b)^2 - (a - b)^2$

$= \big((a + b) + (a - b)\big)\big((a + b) - (a - b)\big)$

$= (2a)(2b)$

$= 4ab$

$= 4p$ for $p \in \mathbb{Z}$ since $a, b \in \mathbb{Z}$

$\therefore \forall a, b \in \mathbb{Z}$, $(a + b)^2 - (a - b)^2$ is divisible by 4.

## Example 5

Prove that if 1 is added to the product of any 4 consecutive positive integers, the resulting number will always be a square number.

Let the consecutive positive integers be $k, k + 1, k + 2, k + 3$.

$k(k + 1)(k + 2)(k + 3) + 1$

$= \big(k^2 + k\big)\big(k^2 + 5k + 6\big) + 1$

$= k^4 + 5k^3 + 6k^2 + k^3 + 5k^2 + 6k + 1$

$= k^4 + 6k^3 + 11k^2 + 6k + 1$

$= k^4 + 3k^3 + k^2 + 3k^3 + 9k^2 + 3k + k^2 + 3k + 1$

$= \big(k^2 + 3k + 1\big)^2$

$= p^2$ for $p \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore$ if 1 is added to the product of any 4 consecutive positive integers, the resulting number will always be a square number.

## Proof by contrapositive

Last lesson, we briefly mentioned that an implication and its contrapositive are logically equivalent - when one is true the other is true, and when one is false the other is false. Let's prove this using the first and fifth rules for negation from last lesson:

$$\neg(\neg P) \equiv P \quad (1) \qquad P \Rightarrow Q \equiv \neg P \vee Q \quad (5)$$

.

Consider

$$\begin{aligned} \neg Q \Rightarrow \neg P &\equiv \neg(\neg Q) \vee \neg P && \text{from } (5) \\ &\equiv Q \vee \neg P && \text{from } (1) \\ &\equiv \neg P \vee Q && \text{commutative law} \\ &\equiv P \Rightarrow Q && \text{from } (5) \end{aligned}$$

$$\boxed{(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)}$$

So proving that a contrapositive is true is logically equivalent to proving that the original implication is true. For some implications, it is much easier to prove the contrapositive than the implication.

Proof by contrapositive is a type of indirect proof that generally involves a direct proof. We do not actually prove the implication we are asked to prove - we prove that the contrapositive is true and then correctly interpret that as proving the implication. It is only the start and end of a proof by contrapositive that separates them from the simple proofs involving integers we have looked at so far this lesson.

With all indirect proofs, it is very important to finish the proof correctly, and show how it proves the result to be proved. For example, if a solution to a proof finishes by proving that the contrapositive is true, without showing that this proves the original implication, then full marks might or might not be awarded. As a general rule, finish every proof with a sentence stating the original result to be proved.

Proof by contrapositive is generally considered easier than proof by contradiction, which we cover next lesson. In general we should always try a direct proof first, then proof by contrapositive (if it is a conditional statement), then proof by contradiction.

## Example 6

Prove by contrapositive $\forall n \in \mathbb{Z}$, that if $n^2$ is even then $n$ is even.

Suppose by contrapositive that $n$ is odd.

Let $n = 2k + 1$ for $k \in \mathbb{Z}$.

$$\therefore n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$
$$= 2j + 1 \quad \text{for } j \in \mathbb{Z} \text{ since } k \in \mathbb{Z}$$

$\therefore$ if $n$ is odd then $n^2$ is odd.

$\therefore$ if $n^2$ is even then $n$ is even by contrapositive.

It is best to mention 'contrapositive' on the first line, to help guide the reader.

We show that we have proved the contrapositive, then mention that this proves the implication, again being best to mention 'contrapositive' for a more convincing proof.

## Example 7

Prove by contrapositive that if $mn$ is even for $m, n \in \mathbb{Z}$, then $m$ or $n$ must be even.

Suppose by contrapositive that $m$ is odd and $n$ is odd.

Let $m = 2k + 1, n = 2j + 1$ for $k, j \in \mathbb{Z}$

$$mn = (2k+1)(2j+1)$$
$$= 4kj + 2k + 2j + 1$$
$$= 2(2kj + k + j) + 1$$
$$= 2p + 1 \quad \text{for } p \in \mathbb{Z} \text{ since } j, k \in \mathbb{Z}$$

$\therefore$ if $m$ and $n$ are both odd then $mn$ is odd.

$\therefore$ if $mn$ is even then $m$ or $n$ must be even by contrapositive.

## Proving equivalence

To prove a statement in the form $P \Leftrightarrow Q$, or $P$ if and only if $Q$, we have to prove two separate cases:

- Case 1: $P \Rightarrow Q$
- Case 2: $Q \Rightarrow P$

It is best to write the two cases under separate headings as above, or as two separate paragraphs with 'conversely' before the proof of $Q \Rightarrow P$.

Depending on the result to be proved, proving equivalence usually either involves two direct proofs, or one direct proof and one proof by contrapositive.

## Example 8

Prove that for $x \in \mathbb{Z}$, $x$ is odd if and only if $x^2$ is odd

If $x$ is odd, let $x = 2k + 1$ for $k \in \mathbb{Z}$.

Start by proving $P \Rightarrow Q$

$x^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2j + 1$ for $j \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore$ if $x$ is odd then $x^2$ is odd

Conversely, we will show that if $x^2$ is odd then $x$ is odd using proof by contrapositive.

Suppose by contrapositive that $x$ is even

Let $x = 2k$ for $k \in \mathbb{Z}$

Next, prove $Q \Rightarrow P$. In this case we will use proof by contrapositive, so state that clearly

$x^2 = 4k^2$

$= 2(2k^2)$

$= 2p$ for $p \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore$ if $x$ is even $x^2$ is even, hence if $x^2$ is odd then $x$ is odd by contrapositive.

$\therefore$ $x$ is odd if and only if $x^2$ is odd

After we finish the two halves of the proof, we show that we have proved equivalence.

## Basic

**1** Prove that if $m$ is even, then $m^2$ is even.

**2** Prove that the product of any odd number $m$ and any odd number $n$ is odd.

**3** Prove that $\forall m = 4k$ for $k \in \mathbb{Z}, m^2 = 16p$ for $p \in \mathbb{Z}$.

**4** Prove that $\forall a, b \in \mathbb{Z}$, $(a+b)^2 + (a-b)^2$ is divisible by $2$.

**5** Prove by contrapositive $\forall n \in \mathbb{Z}$, that if $n^2$ is odd then $n$ is odd.

**6** Prove by contrapositive for $m, n$ positive integers that if $mn$ is divisible by $5$ then $m$ or $n$ must be divisible by $5$.

**7** Prove that for $x \in \mathbb{Z}$, $x$ is even if and only if $x^2$ is even.

**8** Prove by contrapositive that if $a^3 + 5$ is odd, then $a$ is even.

## Medium

**9** Prove that the difference between the squares of any two consecutive integers equals their sum.

**10** Prove that the product of any three consecutive integers is even, by considering the cases where the middle integer is even or odd.

**11** Prove that the sum of any four consecutive integers is even.

**12** Prove by contrapositive that $\forall m \in \mathbb{Z}$, if $m^2$ is not divisible by $4$ then $m$ is odd.

**13** Prove that $\forall x \in \mathbb{Z}$, $x^2$ is divisible by $9$ if and only if $x$ is a multiple of $3$.

**14** Prove by contrapositive that for $m, n \in \mathbb{Z}$, if $\dfrac{mn}{2}$ is an integer then $m$ or $n$ must be even.

**15** Prove that if $m, n$ are integers that $m^2 - n^2$ is even iff at least one of the sum and difference of $m$ and $n$ are even.

**16** Prove that the product of two integers is even if and only if one or both are even.

**17** Prove that if a rational number is divided bya non−zero rational number the result is rational.

**18** Consider the following sequence, with terms in the form $(k+1)^2 - 1$ where $k$ is a positive integer: $3, 8, 15, 24, 35, 48, \ldots$. Prove that the product of any two consecutive terms of the sequence can be written as the product of 4 consecutive numbers.

**19** Given $a^k - b^k = (a-b)\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 \ldots + b^{k-1}\right)$ for $k \in \mathbb{Z}^+, a, b \in \mathbb{R}$, prove that when $3^k$ is divided by $2$, the remainder is $1$.

**20** If $a + b = 2$ prove $a^2 + 2b = b^2 + 2a$.

**21** Prove the expression $a^3 - a + 1$ is odd $\forall a \in \mathbb{Z}^+$.
You may assume that the product of three consecutive integers is even.

**22** Prove $n^2 - 1$ is divisible by $3$ if $n$ is not a multiple of $3$.

Challenging

**23** Prove $\forall m \in \mathbb{Z}^+$, that $(2^m - 1)^2 - 1$ is divisible by $2^{m+1}$.

**24** Prove by contrapositive that if $2^n - 1$ is prime then $n$ is prime, for $n > 1$.
You are given $a^m - b^m = (a-b)\left(a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \ldots + b^{m-1}\right)$.

**25** Prove that a positive integer greater than or equal to $100$ is divisible by $4$ if and only if the last two digits are a multiple of $4$.

**26a** Given $\forall a \in \mathbb{Z}$ where $a$ is not divisible by $5$, prove the remainder when $a^2$ is divided by $5$ is either $1$ or $4$.

**b** Hence $\forall a, b \in \mathbb{Z}$ where $a, b$ are not divisible by $5$ prove that $a^4 - b^4$ is divisible by $5$.

**27** A triangular number is in the form $T = \dfrac{k(k+1)}{2}$. Prove the square of any odd positive integer greater than $1$ is of the form $8T + 1$ where $T$ is a triangular number.

**28** Prove that $(n+1)^n - n^{n+1}$ is odd for all positive integers $n$.

**29** Prove that an irrational number raised to the power of an irrational number can be rational, by considering whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. You may assume that $\sqrt{2}$ is irrational.

**30** Prove that a positive integer $n$ is divisible by $3$ if and only if the sum of its digits is divisible by $3$. You may assume $10^p = 3q + 1$ for $p, q \in \mathbb{Z}$.

In Lesson 3 we cover:

- Proof by contradiction
- Proving irrationality

Syllabus reference: This lesson covers the 1st and 3rd dot points from Illustrations of proofs in *The nature of proof*.

## Proof by contradiction

We have seen that a statement, $P$, must be either true or false, and so its negation, $\neg P$, must be false or true, respectively.

In a proof by contradiction we simply prove that the negation is false, and interpret this as meaning that the statement is true. It gets its name since the negation normally leads to a fact that contradicts some other fact - either a fact that we assumed earlier, or another fact established during the proof. For example a number cannot be both odd and even, or both rational and irrational.

The logic of a proof by contradiction is as follows:

- we start by assuming that the negation is correct, which we aren't sure about
- we then follow a series of steps that we know are correct, and reach a contradiction
- the only step that can have caused the contradiction is the negation, which must be wrong.

Proof by contradiction is a type of indirect proof that generally involves an attempt at a direct proof that leads to a contradiction. We do not actually prove the statement or implication we are asked to prove - we prove that its negation is false and then correctly interpret that as proving the statement or implication is true.

As with all indirect proofs, it is very important to finish the proof correctly, and show how it proves the result to be proved. For example, if a solution to a proof finishes by proving that the negation is false, without showing that this proves the original result, then full marks might or might not be awarded. As a general rule, finish every proof with a sentence stating the original result to be proved.

Remember that direct proofs and proofs by contrapositive are generally easier, so use them when you can.

To prove that a simple statement $P$ is true using a proof by contradiction, we assume that $\neg P$ is true, show that this leads to a contradiction (a result that cannot be true), then interpret this to mean that the negation is false and so the original statement must be true. In symbols, we show that $\neg P$ is false and so $\neg(\neg P) \equiv P$ must be true.

To prove that a conditional statement $P \Rightarrow Q$ is true using a proof by contradiction, we assume that $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$ is true, show that this leads to a contradiction, then interpret this to mean that the negation is false and so the original statement must be true. In symbols, we show that $P \wedge \neg Q$ is false and so $\neg\big(\neg(P \Rightarrow Q)\big) \equiv P \Rightarrow Q$ must be true.

The start and end of a proof by contradiction are common causes of lost marks. Look at the examples that follow.

## Example 1

Prove $\sqrt{2} + \sqrt{3} < \sqrt{10}$ by contradiction.

Suppose by contradiction that $\sqrt{2} + \sqrt{3} \geq \sqrt{10}$

It is best to mention 'contradiction' on the first line, to help guide the reader.

$\left(\sqrt{2} + \sqrt{3}\right)^2 \geq 10$ since $\sqrt{2}, \sqrt{3}, \sqrt{10} > 0$

$2 + 2\sqrt{6} + 3 \geq 10$

$2\sqrt{6} \geq 5$

$\sqrt{24} \geq \sqrt{25}$

$24 \geq 25$

We clearly show that we have reached a contradiction.

Which is a contradiction, so $\sqrt{2} + \sqrt{3} < \sqrt{10}$.

Interpret this as proving the statement, again mentioning 'contradiction' for a more convincing proof.

## Example 2

Prove by contradiction that if $n$ is an odd integer then $n^2$ is odd.

Suppose by contradiction that $n$ is an odd integer and $n^2$ is even. *

Let $n = 2k + 1$ for $k \in \mathbb{Z}$

$\therefore n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2p + 1$ for $p \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore n^2$ is odd #

If we reach a fact that contradicts the original assumption, then we often mark the assumption with an asterisk and the contradictory fact with #.

Which contradicts (*) since $n^2$ cannot be both odd and even, hence if $n$ is an odd integer then $n^2$ is odd.

## Example 3

Prove by contradiction that for $a, b \in \mathbb{Z}$, if $a + b \geq 9$ then $a \geq 5$ or $b \geq 5$.

Suppose by contradiction that $a + b \geq 9$ and $a < 5$ and $b < 5$ *

$\therefore a + b \leq 4 + 4$ since $a, b \in \mathbb{Z}$

$\leq 8$ #

Which contradicts $(*)$ since $a + b$ cannot be $\geq 9$ and $\leq 8$.

$\therefore$ if $a + b \geq 9$ then $a \geq 5$ or $b \geq 5$.

## Example 4

Prove $\dfrac{1}{x+y} \neq \dfrac{1}{x} + \dfrac{1}{y}$ where $x, y, x + y \neq 0$

Suppose by contradiction that $\dfrac{1}{x+y} = \dfrac{1}{x} + \dfrac{1}{y}$

$$\therefore \frac{1}{x+y} = \frac{y+x}{xy}$$

$$\therefore xy = (x+y)^2$$

$$xy = x^2 + 2xy + y^2$$

$$x^2 + xy + y^2 = 0$$

taking $y$ as a constant $\Delta = y^2 - 4(1)(y^2)$

$= -3y^2$

$\leq 0$

$\therefore$ no solutions

$\therefore \dfrac{1}{x+y} \neq \dfrac{1}{x} + \dfrac{1}{y}$

## Proving irrationality

A common question for proofs by contradiction involves proving that a number is irrational - make sure that you can quickly reproduce these proofs and cover all the pressure points, to save time for other questions in the exam.

We start by supposing that the number is rational by contradiction, then define the number as rational by letting it equal a fraction where:

- the numerator and denominator are integers
- the denominator is not zero
- the highest common factor of the numerator and denominator is one.

When we are proving irrationality using contradiction, the contradiction is usually finding a common factor greater than $1$, which contradicts the highest common factor being $1$.

Make sure that the two pronumerals used for the numerator and denominator do not appear elsewhere in the question. I generally use $p$ and $q$.

If we want to define $x$ as rational, and we know $x$ is positive, the most efficient way is:

$$\text{Let } x = \frac{p}{q} \text{ where } p, q \in \mathbb{Z}^+, \text{HCF}(p, q) = 1.$$

If we do not know if $x$ is positive, then we need something like:

$$\text{Let } x = \frac{p}{q} \text{ where } p, q \in \mathbb{Z}, q \neq 0, \text{HCF}(p, q) = 1.$$

The meaning of '$\text{HCF}(p, q) = 1$' is that the highest common factor of $p$ and $q$ is $1$. For example, they cannot both be even, as they would have a HCF that is at least $2$, which would be the contradiction we are after.

Alternatives to '$\text{HCF}(p, q) = 1$' are:

- '$\text{gcd}(p, q) = 1$' (the greatest common divisor is $1$)
- '$p$ and $q$ are coprime' (which means the same thing).
  - Just be careful using 'coprime', as it does NOT mean that the numbers are prime. For example, $8$ and $9$ are coprime, as their highest common factor is $1$, yet they are both composite numbers.

The most common numbers that are used in irrationality proofs are square roots. Most commonly we will look at the square root of a prime number, such as $\sqrt{2}$ or $\sqrt{23}$, or the square root of a composite number with single prime factors. These numbers can be abbreviated as square-free, since they are not the multiple of a perfect square other than $1$. We need a slightly different technique for composite numbers which are the multiple of a square number other than $1$.

An important step when we prove $\sqrt{k}$ is irrational occurs when we reach an equation of the form $kq^2 = p^2$. We need to be able to interpret it quickly and correctly to gain full marks.

Let's have a quick look at Euclid's lemma - it is not part of the syllabus, but helps us cover the harder examples proving the irrationality of surds. Here is a summary where we are proving $\sqrt{k}$ is irrational:

- When $k$ is square-free (prime OR a composite that has NO repeated prime factors):
  - If $k$ divides $p^2$ then $k$ divides $p$.
    - For example, $7$ is prime, so if $7$ divides $p^2$ then $7$ divides $p$.
    - For example, $30 = 2\times3\times5$ ($2, 3$ and $5$ are prime factors that are not repeated), so if $30$ divides $p^2$ then $30$ divides $p$. Technically, Euclid's lemma applies to each prime factor separately, but we can combine them for our work here.
- When $k$ is not square-free (composite WITH at least one repeated prime factor):
  - If $k$ divides $p^2$ then $k$ DOES NOT NECESARILY divide $p$.
    - Consider $k = 12$. Now $12 = 2^2\times3$, so $2$ is a repeated prime factor.
    - Now consider $p = 6$ and $p^2 = 36$. $12$ divides $36$, but obviously $12$ does not divide $6$.
    - So, in this case, we see that if $k$ has a repeated prime factor, then if $k$ divides $p^2$ then $k$ does not necessarily divide $p$.
    - Now, the individual prime factors of $12$ are $2$ and $3$, so we can say that $2$ and $3$ both divide $36$, so $2$ and $3$ divide $6$. More easily, we can say that $6$ divides $36$ so $6$ divides $6$.

When we reach this step in the proof, there are three essential steps in interpreting this equation, plus one optional step, which can be covered in one line. Assume that $k$ is prime or composite with no repeated prime factors ($k$ is square-free):

1. '$k$ divides the LHS' [or 'the LHS is even' if $k = 2$)]
2. 'so $k$ divides $p^2$' [or '$p^2$ is even' if $k = 2$)]
3. 'so $k$ divides $p$.' [or '$p$ is even' if $k = 2$)]
4. (optional) 'since $k$ is square-free'. [we can skip this step for even numbers]

If $k$ is composite with at least one repeated prime factor, i.e. it is a multiple of a perfect square, then we would find the product of the single prime factors and use that number in place of $k$ in the four steps above.

It is important that we cover all three steps, as otherwise marks will normally be lost in an exam.

Let's look at some examples.

## Example 5

Prove that $\sqrt{2}$ is irrational.

Suppose by contradiction that $\sqrt{2}$ is rational.

Let $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$, HCF$(p, q) = 1$ $\quad *$

$$\therefore 2 = \frac{p^2}{q^2}$$

$$2q^2 = p^2$$

Alternatively: '2 divides the LHS so 2 divides $p^2$ so 2 divides $p$, since 2 is prime'.

The LHS is even so the RHS must be even, so $p^2$ is even and so $p$ is even.

Let $p = 2k$ for $k \in \mathbb{Z}$

$$\therefore 2q^2 = (2k^2)$$

$$q^2 = 2k^2$$

So similarly $q$ is even.

$\therefore p$ and $q$ have a common factor of 2 which contradicts HCF$(p, q) = 1$ $\quad (*)$.

$\therefore \sqrt{2}$ must be irrational.

We can use 'similarly' here, since $2q^2 = p^2$ and $q^2 = 2k^2$ are in the same form. This will not be the case if we are not dividing by the integer in the first equation.

## Example 6

Prove that $\sqrt{10}$ is irrational.

Suppose by contradiction that $\sqrt{10}$ is rational.

Let $\sqrt{10} = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$, HCF$(p, q) = 1$ $\quad *$

$$\therefore 10 = \frac{p^2}{q^2}$$

$$10q^2 = p^2$$

We could use divisibility by $2$, or divisibility by $5$, but that involves slightly more work later in the proof, as we cannot use 'similarly'. See the next example.

$10$ divides the LHS, so $10$ divides $p^2$, so $10$ divides $p$, since $10$ is square−free.

Let $p = 10k$ for some $k \in \mathbb{Z}$

$$\therefore 10q^2 = (10k^2)$$

$$q^2 = 10k^2$$

So similarly $10$ also divides $q$.

$\therefore p$ and $q$ have a common factor of $10$ which contradicts HCF$(p, q) = 1$ $\quad (*)$.

$\therefore \sqrt{10}$ must be irrational

## Example 7

Prove that $\sqrt{18}$ is irrational.

Suppose by contradiction that $\sqrt{18}$ is rational.

Let $\sqrt{18} = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$, HCF$(p, q) = 1$ $\quad *$

$$\therefore 18 = \frac{p^2}{q^2}$$

$$18q^2 = p^2$$

$6$ divides the LHS, so $6$ divides $p^2$, so $6$ divides $p$ since $6$ is square−free.

Let $p = 6k$ for some $k \in \mathbb{Z}$

$$\therefore 18q^2 = (6k^2)$$

$$q^2 = 2k^2$$

We cannot use 'similarly' as the equations are not in the same form.

The RHS is even, so $q^2$ is even, so $q$ is even.

$\therefore p$ and $q$ have a common factor of $2$ which contradicts HCF$(p, q) = 1$ $\quad (*)$

$\therefore \sqrt{18}$ must be irrational

## Example 8

Prove that $\log_2 n$ is irrational when $n$ is an odd integer greater than or equal to $3$.

Suppose by contradiction $\log_2 n$ is rational when $n$ is an odd integer greater than or equal to $3$.

Let $n = 2k + 1$ for $k \in \mathbb{Z}^+$

$$\therefore \log_2(2k+1) = \frac{p}{q} \text{ for } p, q \in \mathbb{Z}^+, \text{HCF}(p,q) = 1$$

$$\therefore q\log_2(2k+1) = p$$

$$\log_2(2k+1)^q = p$$

$$(2k+1)^q = 2^p$$

Make sure that you explain why each side of the equation is odd - just saying they are odd is generally not sufficient with reason.

LHS is odd, since an odd number to any power is odd.

RHS is even, since an even number to any power is even.

This is a contradiction, so $\log_2 n$ must be irrational for all odd $n \geq 3$.

## Basic

**1** Prove by contradiction that $2\sqrt{3} - \sqrt{5} < \sqrt{2}$.

**2** Prove by contradiction that there are no integers $a$ and $b$ such that $2m + 4n = 51$.

**3** Prove by contradiction that if $n$ is an even integer then $n^2$ is even.

**4** Prove by contradiction $\forall n \in \mathbb{Z}$, if $n^2 - 1$ is even then $n$ is odd.

**5** Prove by contradiction that if the product of any two integers is odd, then both of them must be odd.

## Medium

**6** Prove $\sqrt{5} + \sqrt{7} < 5$ by contradiction.

**7** Prove that $\sqrt{3}$ is irrational.

**8** Prove that $\log_2 5$ is irrational.

**9** Prove by contradiction that if $a, b$ are integers and $a + b \leq 5$ then $a \leq 2$ or $b \leq 2$.

**10** Prove by contradiction that there are no integers $m, n$ which satisfy $4n + 8m = 102$.

**11** For the positive real numbers $a$ and $b$, with $a \neq 2b$, prove that $\dfrac{1}{a - 2b} \neq \dfrac{1}{a} - \dfrac{1}{2b}$.

**12** Prove by contradiction that $\forall n \in \mathbb{Z}$, if $n^3 - 1$ is odd then $n$ is even.

**13** Prove by contradiction that the product of a non−zero rational and irrational number is irrational.

**14** Prove by contradiction that if $m^2 - 4n = 3$ then $m$ and $n$ cannot both be integers.

**15** Prove by contradiction that there are infinitely many odd integers.

**16** Prove that $\forall n \in \mathbb{Z}^+$, $\log_2(2n + 1)$ is irrational.

## Challenging

**17** Prove by contradiction that the square root of $\pi$ is also irrational.

**18** Prove $\sin x + \cos x \geq 1$ for all $0 < x < \dfrac{\pi}{2}$ by contradiction.

**19** Prove that $\sqrt{60}$ is irrational.

**20** If $a$ is rational and $b$ is irrational, prove $a + b$ is irrational.

**21** Prove that for positive integers $a, b$ and $a > 1$ that either $b$ is not divisible by $a$ or $b + 1$ is not divisible by $a$.

**22** Prove that there are no positive integers $x, y$ such that $x^2 - y^2 = 1$.

**23** Prove that the sum of two integers is even if and only if they have the same parity (both odd or both even).

**24** Prove that there are no rational solutions to $x^3 + 3x + 5 = 0$ using contradiction.

**25** Prove that all composite integers $n$ have a prime divisor $k$ where $k \leq \sqrt{n}$.

In Lesson 4 we cover:

- Disproof, examples and counterexamples

Syllabus reference: This lesson covers the 2nd and 3rd dot points from Illustrations of proofs in *The nature of proof*.

## Disproof, examples and counterexamples

We will commonly be given a statement and told to prove it is true, but there are times when we are given a statement that is false and have to disprove it, or are given a statement that could be true or false and have to prove or disprove it.

Not being able to prove a statement is true is not enough to prove it is false.

There are different techniques that we can use to disprove a statement, depending on whether it applies to all numbers (∀) or to at least one number (∃).

If we are given a statement that might be true or false, we will have to read the question carefully and use our judgement as to how we should approach it.

**Disproving universal statements**

To disprove a statement that applies to all numbers we can:

- Use a counterexample - a counterexample is an example where the statement is false. This is the simplest method.
- Use a direct proof to show that the negation of the statement is true for at least one value, so the statement is false.
- Attempt a direct proof of the statement that leads to a contradiction, which proves that it is false. This is a type of proof by contradiction, but we do not need to negate the statement.
- Prove by cases to show that it is false for at least one case.
- Disprove an implication $P \Rightarrow Q$, by finding a case in which $P$ is true and $Q$ is false.

For example, to disprove '∀ odd positive integers $x,\ x+1$ is odd' we could:

- Use a counterexample by letting $x=1$. Since $x+1=2$ is even, the statement is false.
- Use a direct proof by cases to show that if $x$ is odd that $x+1$ must be even, which is a contradiction and the statement is false.

Also remember that even though one counterexample is sufficient to prove a universal statement is false, no number of examples are sufficient to prove a universal statement is true, as we have seen in false proofs by mathematical induction in Extension 1.

## Example 1

Prove the following statement is false:

If $a^2-b^2>0$, where $a$ and $b$ are real, then $a-b>0$

Let $a=-2, b=0$

$(-2)^2-0^2=4>0$ yet $-2-0<0$, so

the statement is false.

This is a universal statement, that is easiest to disprove using a counterexample.

Alternatively:

$$a^2-b^2>0$$

$$(a+b)(a-b)>0$$

$\therefore a+b>0$ and $a-b>0$ or $a+b<0$ and $a-b<0$

$\therefore$ the statement is false, since it is false if $a+b<0$.

## Example 2

Prove the following statement is false: There are no prime numbers divisible by $5$.

5 is prime and it is divisible by 5, so the statement is false.

**Disproving existence statements**

To disprove a statement that applies to at least one number we cannot use a counterexample, as it is perfectly acceptable that there are many numbers for which the statement is false, and yet the statement itself is true. We can:

- Use a direct proof to show that the negation of the statement is true for all values, so the original statement is false for all values.
- Use a direct proof of the statement for all values that leads to a contradiction, so the statement is false. If using proof by cases all cases must cause a contradiction.
- Prove that the stated condition is false for every value in the specified domain.

## Example 3

Prove the following statement is false: $\exists$ a real number $x$, $x^2 + 2x + 5 < 0$.

Here we will prove that the statement is false for all real numbers.

$a > 0$

$\Delta = 2^2 - 4(1)(5)$

$= -16$

$< 0$

The quadratic is positive definite, since the leading coefficient is positive and the discriminant is negative, so the statement is false.

## Example 4

Prove the following statement is false. There is a Pythagorean triad consisting of positive integers, where all numbers are odd.

Let the triad of odd numbers be $a, b$ and $c$, such that $a = 2i + 1$, $b = 2j + 1$ and $c = 2k + 1$ for $i, j, k \in \mathbb{Z}$.

$\therefore (2i + 1)^2 + (2j + 1)^2 = (2k + 1)^2$

$4i^2 + 4i + 1 + 4j^2 + 4j + 1 = 4k^2 + 4k + 1$

$4\left(i^2 + i + j^2 + j\right) + 1 = 4\left(k^2 + k\right)$

$4m + 1 = 4n$ for $m, n \in \mathbb{Z}$ since $i, j, k \in \mathbb{Z}$

Now the RHS is a multiple of $4$ yet the LHS isn't so we have a contradiction, so there is no Pythagorean triad where all numbers are odd.

## Example 5

Prove or disprove the following statement:

The sum of the squares of two consecutive even numbers is divisible by $8$

Let the two consecutive even numbers be $2k$ and $2k+2$ for $k \in \mathbb{Z}$.

$(2k)^2+(2k+2)^2$

$= 4k^2 + 4k^2 + 8k + 4$

$= 8k^2 + 8k + 4$

$= 4\left(2k^2 + 2k + 1\right)$

$= 4\left(2\left(k^2 + k\right) + 1\right)$

$= 4(2p + 1)$ for $p \in \mathbb{Z}$

The statement is false, as the sum of the squares of two consecutive even numbers is divisible by $4$ but not $8$, since $2p + 1$ is odd rather than even.

## Example 6

Prove or disprove the following statement: $\exists$ a real number $n$ such that $3^n + 4^n = 5^n$

Let $x = 2$

$3^2 + 4^2 = 5^2$

The statement is true.

This is a rare example where an example does prove a statement, as it is not a universal statement.

Basic

**1** Prove the following statement is false: 'There are no prime numbers divisible by 7'.

**2** Prove the following statement is false: '$\exists x \in \mathbb{R},\ -x^2 + 2x - 2 \geq 0$'.

**3** Prove the following statement is false: 'If $a - b > 0$, where $a, b \in \mathbb{R}$, then $a^2 - b^2 > 0$'.

**4** Prove the following statement is false: 'If $a^2 - b^2 > 0$, where $a, b \in \mathbb{R}$, then $a - b > 0$'.

**5** Which of the following statements is true?

**A** $\forall a, b \in \mathbb{R} \qquad \sin a < \sin b \Rightarrow a < b$

**B** $\forall a, b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \sin a < \sin b \Rightarrow a < b$

**C** $\forall a, b \in \mathbb{R} \qquad \cos a < \cos b \Rightarrow a < b$

**D** $\forall a, b \in [0, \pi] \qquad \cos a < \cos b \Rightarrow a < b$

**6** Prove or disprove the following statement: '$\exists n \in \mathbb{R}$ such that $3^n + 4^n < 5^n$'.

**7** Prove the following statement is false: '$\forall \in \mathbb{R}, |2x + 1| \leq 5 \Rightarrow |x| \leq 2$'.

Medium

**8** Prove or disprove the following statement: 'The sum of the squares of three consecutive even numbers is divisible by 4'.

**9** Prove the following statement is false: 'There is a Pythagorean triad where the two smallest numbers are even and the largest number is odd.'

**10** Consider the statement: 'There are no prime numbers divisible by 6'. Prove that it is true, or find a counterexample to prove that it is false.

**11** Consider the statement: '$3^n + 2$ is prime for all positive integers $n$'. Prove that it is true, or find a counterexample to prove that it is false.

**12** Find the negation of the following statement, and decide whether the negation is true or not.
$P: \exists m > 0, m^2 - 9 = 0$

**13** Prove or disprove that if $x$ and $y$ are irrational and $x \neq y$, then $xy$ is irrational.

**14** Consider the following proposition.

'If a positive integer is not a perfect square, then it is not prime.'

Which of the following would provide a counterexample to the proposition?

**A** A positive integer that is not a perfect square and is prime.

**B** A positive integer that is not a perfect square and is not prime.

**C** A positive integer that is a perfect square and is prime.

**D** A positive integer that is a perfect square and is not prime.

Challenging

**15** Consider the proposition '$\forall n \in Z, \mathrm{z} \in \mathbb{C}$, if $z^n$ is not purely imaginary then $z$ is not purely imaginary'. Given that each of the following statements is true, which statement disproves the proposition?

**A** $(0+i)^3 = -i$

**B** $(0+i)^4 = 1$

**C** $(1+i)^2 = 2i$

**D** $(1+i)^3 = -2+2i$

**16** Prove or disprove the following statement:

$\forall n \in \mathbb{Z}$, if $n^2 + 3n + 1$ is prime, then $n$ is even.

**17** Prove or disprove the following statement:

There exist three consecutive positive integers such that the sum of their squares is a perfect square.

**18** Prove or disprove the following statement:

$\forall n \in \mathbb{Z}^+$, if $n^2 + 2$ is divisible by $3$, then $n^2 + 5$ is divisible by $9$.

**19** Prove or disprove the following statement:

There exist two irrational numbers $a$ and $b$ such that $a + b$ and $ab$ are both rational.

**20** Prove or disprove the following statement:

$\exists a, b \in \mathbb{Z}^+$ such that $a^2 + b^2 = 4ab + 3$.

In Lesson 5 we cover:

- Inequalities
- The square of a real number is non-negative
- The triangle inequality

Syllabus reference: This lesson covers the 1st to the 4th and 6th and 7th dot points from Proof of inequalities in *The nature of proof*.

## Inequalities

Inequalities are a great source of hard yet simple proofs that stretch the mind and separate out the very top students. In the old syllabus these formed the basis of the hardest marks on average of any topic, and yet the final solution was always short and simple.

Why are they so hard? Firstly, there is generally little, if any, scaffolding to provide hints as to how to solve them. Secondly, students must have superb algebra skills, and a deep understanding of any limitations to methods they might be considering.

Let's have another look at how to set out a proof before we delve into inequalities.

**Setting out the proof**

Before we start looking at inequalities, a quick reminder on how to set out proofs, which is particularly important in inequality proofs.

> A proof is an argument that convinces one of your peers that a statement is always true. A better proof also convinces a suspicious peer that it is true, and helps a less able peer understand why it is true.

In order to make a convincing argument you are best to follow one of the following two methods.

**Type 1 - Inequality**

- Start with an inequality or equation that is known to be true (an axiom* or a result given or already proved)
- Progress in clear and logical steps towards the result to be proved.

* An axiom is a basic assumption we accept to be true. Many of the proofs rely on the identity that the square of a real number must be non-negative - which we can write as $\mathbb{R}^2 \geq 0$.

If you are stuck remember that you can use working out paper to work from the conclusion to a true statement, then write it out in the correct order as your answer, but it wastes exam time.

**Type 2 - Expression**

- Start with the LHS, or the RHS, or an expression which equals LHS − RHS or RHS−LHS
- Progress in clear and logical steps until it simplifies to an expression that is clearly positive or negative.
- Interpret this result - explain why it proves the original result.

Do NOT start with the result to be proved and work towards a true statement, as this can lead to logical fallacies such as 'proving' results that are false. Similarly, do NOT rework the result to be proved into a different inequality and then 'prove' this, as depending on how far it has been reworked this may also involve a logical fallacy. Both situations will normally result in some or all marks being lost in an exam.

**Why do we reverse the inequality sign at times and not at others?**

An inequality is a statement showing the relative size of two (or three) real numbers, terms or expressions. For example:

- $a > b$ means that $a$ is larger than $b$, and $a - b > 0$ - we will use this property in many proofs.
- $a > b > c$ means that $a$ is larger than $b$, $b$ is larger than $c$, and importantly $a$ is larger than $c$ - this is another important property we will use in proofs.

When we perform an operation on each part of the inequality the relative sizes might stay the same or might be reversed, so we might leave the inequality sign the same or need to reverse it.

The inequality sign stays the same when:

- We add or subtract the same number or term from each part of the inequality. For example, if $a > b$ then $a \pm c > b \pm c$.
- We add a larger number or term to the larger part, or subtract a smaller number or term from the larger part. For example, if $a > b$ and $c > d$ then $a + c > b + d$ and $a - d > b - c$.
- We multiply or divide each part of the inequality by a number or term which is positive. For example, if $a > b$ and $c > 0$ then $ac > bc$.
- We perform any function that matches a monotonic increasing function within an appropriate domain. For example, if $a$ and $b$ are in the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $a > b$, then $\sin a > \sin b$, since sine is an increasing function in the domain.

The inequality sign is reversed when:

- We multiply or divide each part of the inequality by a number or term which is negative. For example, if $a > b$ and $c < 0$ then $ac < bc$
- The parts of the inequality from the previous line swap sides (or we reverse all three parts)
- We perform any function that matches a monotonic decreasing function within an appropriate domain. For example, if $a$ and $b$ are in the domain $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and $a > b$, then $\sin a < \sin b$, since sine is a decreasing function in the domain.

When we perform some functions we need more information before we can determine whether the sign stays the same or reverses:

- When we take the reciprocal of the parts of an inequality:
  - the sign reverses if the parts have the same sign (both positive or both negative). For example, if $a > b$ and $a, b > 0$ then $\frac{1}{a} < \frac{1}{b}$.
  - the sign stays the same if the parts have opposite sign (one negative and one positive). For example, if $a > b$ and $a > 0$ and $b < 0$ then $\frac{1}{a} > \frac{1}{b}$.
- When we square all parts of an inequality:
  - if $|a| > |b|$ then $a^2 > b^2$ - the sign might stay the same or reverse - we need to check carefully. For example, if $0 < a < 1$ and $-2 < b < -1$ then $a > b$ and $a^2 < b^2$.
  - Similarly, if $|a| < |b|$ then $a^2 < b^2$.

Sometimes we need more information before we can determine whether the inequality sign stays the same or needs reversing, or we need to cover both cases.

- If we multiply by a variable that could be positive or negative, we need to allow for both cases or use a different method. You will have covered this in Extension 1 when you looked at inequalities with an unknown in the denominator.
- If we perform an operation that matches a function that is not monotonic, such as taking a trigonometric ratio of both sides of an inequality with an unrestricted domain, then we need more information.
- If we perform an operation that matches a function that has discontinuities, such as taking the reciprocal of both sides of an inequality above, then we need more information.

## Example 1

When each of the following operations are performed on each side of an inequality, state whether the sign stays the same or is reversed on the next line, or whether we need more information. Justify your answers.

**a** multiplying by 3

stays the same, as multiplying by a positive number is equivalent to an increasing function

**b** dividing by $-2$

reverses, as dividing by a negative number is equivalent to a decreasing function

**c** taking the tangent

we need more information, such as restricting the domain, since tangent is a discontinuous function

**d** taking the negative reciprocal

we need more information, such as whether both expressions have the same sign, since the negative reciprocal function is a discontinuous function

**e** taking the square root

stays the same, assuming both sides are non-negative, as the square root function is an increasing function for non-negative values.

## Example 2

Given $m > n > 0$ and $0 > p > r$, insert the appropriate inequality symbol between each pair of expressions to indicate their relative size, or state why it is not possible to determine their relative size.

**a** $m + p \boxed{>} n + r$

$m > n \quad (1)$
$p > r \quad (2)$
$(1) + (2)$: $m + p > n + r$

**b** $n - r \boxed{?} m - p$

$n < m \quad (1)$
$-r > -p \quad (2)$
There is no way to determine their relative size

**c** $mr \boxed{<} np$

$m > n \quad (1)$
$|r| > |p| \quad (2)$
$(1) \times (2)$: $m|r| > n|p|$
$\therefore mr < np$ since $p, r < 0$

**d** $mp \boxed{?} nr$

$m > n \quad (1)$
$|p| < |r| \quad (2)$
There is no way to determine the relative size of $m|p|$ and $n|r|$, nor $mp$ and $nr$.

**e** $\frac{n}{r} \boxed{>} \frac{m}{p}$

$np > mr$ from (e)
$\frac{np}{r} < m$ since $r < 0$
$\frac{n}{r} > \frac{m}{p}$ since $p < 0$

**f** $m^2 \boxed{>} n^2$

$m > n > 0$
$\therefore m^2 > n^2$

**g** $p^2 \boxed{<} r^2$

$r < p < 0$
$\therefore p^2 < r^2$

**h** $\frac{1}{m} \boxed{<} \frac{1}{n}$

$m > n > 0$
$\therefore \frac{1}{m} < \frac{1}{n}$

**i** $\frac{1}{p} \boxed{<} \frac{1}{n}$

$p < 0 < n$
$\therefore \frac{1}{p} < \frac{1}{n}$

**j** $\tan^{-1}(m - r) \boxed{>} \tan^{-1}(n - p)$

$m + p > n + r$ from $(a)$
$\therefore m - r > n - p$
$\therefore \tan^{-1}(m - r) > \tan^{-1}(n - p)$
since $\tan^{-1}$ is an increasing function

There are many possible methods for proving inequalities, which is part of what makes it such great Extension 2 content. We will solve many of the examples and exercises from this lesson in three different ways, which also apply to the inequalities next lesson.

- LHS−RHS - we start with an expression equal to the LHS − RHS (or vice versa) of the result to be proved, and show that it is positive, etc. This is the most popular method, but can be a little longer than other methods. Make sure that you conclude the proof with the result to be proved, rather than stopping with LHS − RHS $> 0$ or similar.
- Proof by contradiction - this is an underutilised method. Make sure that the correct negation is made, and the proof is concluded correctly and ends with the result to be proved
- We can also start with conditions in the question or an axiom and build towards the result to be proved. This is often the most efficient method, but harder to see in advance.

If you only want to learn one method, then LHS−RHS is probably the best, since so many worked solutions from other sources will use this method. A good alternative is proof by contradiction which is at least as effective, but there are fewer worked solutions done using this method.

## Example 3

Given $a > b > 0$, prove that $(a+1)(a-1) > (b+1)(b-1)$.

LHS − RHS

$= (a+1)(a-1) - (b+1)(b-1)$

$= (a^2 - 1) - (b^2 - 1)$

$= a^2 - b^2$

$> 0$ since $a > b > 0$

$\therefore (a+1)(a-1) > (b+1)(b-1)$

Add short explanatory notes to justify steps where needed.

The proof must end with the result to be proved, which is a common error.

Alternative I:

Suppose by contradiction that

$(a+1)(a-1) \leq (b+1)(b-1)$

$a^2 - 1 \leq b^2 - 1$

$a^2 \leq b^2$

$\therefore a \leq b$ since $a, b > 0$

Which is a contradiction since $a > b$

$\therefore (a+1)(a-1) > (b+1)(b-1)$.

Alternative II:

$a > b$

$a^2 > b^2$ since $a > b > 0$

$a^2 - 1 > b^2 - 1$

$(a+1)(a-1) > (b+1)(b-1)$

## Example 4

Given $a > b > 0$, prove that $2^{-a^2} < 2^{-b^2}$.

LHS − RHS

$= 2^{-a^2} - 2^{-b^2}$

$= \dfrac{1}{2^{a^2}} - \dfrac{1}{2^{b^2}}$

$= \dfrac{2^{b^2} - 2^{a^2}}{2^{a^2+b^2}}$

$< 0$ since $a > b > 0$

$\therefore 2^{-a^2} < 2^{-b^2}$

Alternative I:

Suppose by contradiction that

$2^{-a^2} \geq 2^{-b^2}$

$\dfrac{1}{2^{a^2}} \geq \dfrac{1}{2^{b^2}}$

$2^{b^2} \geq 2^{a^2}$

$b^2 \geq a^2$

$b \geq a$ since $a, b > 0$

Which is a contradiction since $a > b$

$\therefore 2^{-a^2} < 2^{-b^2}$

Alternative II:

$a > b$

$a^2 > b^2$ since $a > b$

$-a^2 < -b^2$

$2^{-a^2} < 2^{-b^2}$

## The square of a real number is non-negative

A fundamental property of real numbers is that their square is non-negative, which we write as $\mathbb{R}^2 \geq 0$.

We will use this property extensively with the arithmetic mean - geometric mean inequality next lesson, but here are a couple of other applications.

## Example 5

Given $a, b, c \in \mathbb{R}$, prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$.

$$\begin{aligned}
\text{LHS} - \text{RHS} &= a^2 + b^2 + c^2 - (ab + bc + ca) \\
&= \frac{1}{2}\left(a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2\right) \\
&= \frac{1}{2}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) \\
&\geq \frac{1}{2}(0 + 0 + 0) \\
&= 0
\end{aligned}$$

$\therefore a^2 + b^2 + c^2 \geq ab + bc + ac$

Alternative I:

Suppose by contradiction that

$$a^2 + b^2 + c^2 < ab + bc + ac$$

$$a^2 + b^2 + c^2 - ab - bc - ac < 0$$

$$\frac{1}{2}\left(a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2\right) < 0$$

$$\frac{1}{2}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) < 0$$

$$(a-b)^2 + (b-c)^2 + (c-a)^2 < 0$$

Which is a contradiction since $\mathbb{R}^2 \geq 0$

$\therefore a^2 + b^2 + c^2 \geq ab + bc + ac$

Alternative II:

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0 \quad \text{since } \mathbb{R}^2 \geq 0$$

$$a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ac + a^2 \geq 0$$

$$2[a^2 + b^2 + c^2 - (ab + bc + ca)] \geq 0$$

$$a^2 + b^2 + c^2 - (ab + bc + ac) \geq 0$$

$$a^2 + b^2 + c^2 \geq ab + bc + ac$$

## Example 6

Given $x, y > 0$, prove that $x^4 + y^4 \geq xy(x^2 + y^2)$.

You may assume that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$$\begin{aligned}
\text{LHS} - \text{RHS} &= x^4 + y^4 - xy(x^2 + y^2) \\
&= x^4 - x^3y + y^4 - xy^3 \\
&= x^3(x - y) + y^3(y - x) \\
&= x^3(x - y) - y^3(x - y) \\
&= (x^3 - y^3)(x - y) \\
&= (x - y)(x^2 + xy + y^2)(x - y) \\
&= (x - y)^2(x^2 + xy + y^2) \\
&\geq 0 \quad \text{since both expressions are non-negative}
\end{aligned}$$

$\therefore x^4 + y^4 \geq x^3y + xy^3$

Alternative I:

Suppose by contradiction that

$$\begin{aligned}
x^4 + y^4 &< xy(x^2 + y^2) \\
x^4 + y^4 &< x^3y + xy^3 \\
x^4 - x^3y + y^4 - xy^3 &< 0 \\
x^3(x - y) + y^3(y - x) &< 0 \\
x^3(x - y) - y^3(x - y) &< 0 \\
(x^3 - y^3)(x - y) &< 0 \\
(x - y)(x^2 + xy + y^2)(x - y) &< 0 \\
(x - y)^2(x^2 + xy + y^2) &< 0
\end{aligned}$$

Which is a contradiction since $\mathbb{R}^2 \geq 0$ and $x, y > 0$

$\therefore x^4 + y^4 \geq x^3y + xy^3$

Alternative II:

$$\begin{aligned}
(x - y)^2(x^2 + xy + y^2) &\geq 0 \text{ since } \mathbb{R}^2 \geq 0 \text{ and } x, y > 0 \\
(x - y)(x - y)(x^2 + xy + y^2) &\geq 0 \\
(x - y)(x^3 - y^3) &\geq 0 \\
x^4 - xy^3 - x^3y + y^4 &\geq 0 \\
x^4 + y^4 &\geq x^3y + xy^3 \\
x^4 + y^4 &\geq xy(x^2 + y^2)
\end{aligned}$$

When variables are defined as positive we have other techniques that cannot be used if the variables are real numbers:

- any variable is positive
- addition, multiplication and division involving one or more variables is positive
- we can take square roots of a variable, as we do not need to worry about unintentionally taking the square root of a negative number
- we can divide by a variable, as we do not need to worry about unintentionally dividing by zero

If the variables are defined as being non-negative, then the first three dot points are still relevant, but we cannot divide by a variable in case we unintentionally divide by zero.

## Example 7

If $a, b, c > 0$ prove that $(a+b+c)^2 > ab+bc+ac$.

$$\begin{aligned}\text{LHS} - \text{RHS} &= (a+b+c)^2 - (ab+bc+ac) \\ &= a^2+b^2+c^2+2(ab+bc+ac) - (ab+bc+ac) \\ &= a^2+b^2+c^2+ab+bc+ac \\ &> 0+0+0+0+0+0 \qquad \text{since } a,b,c>0 \\ &= 0\end{aligned}$$

$\therefore (a+b+c)^2 > ab+bc+ac.$

Remember that
$(\sum\alpha)^2 = \sum\alpha^2 + 2\sum\alpha\beta$

Alternative I:

Suppose by contradiction that

$$(a+b+c)^2 \leq ab+bc+ac$$

$$a^2+b^2+c^2+2ab+2ac+2bc \leq ab+bc+ac$$

$$a^2+b^2+c^2+ab+ac+bc \leq 0$$

Which is a contradiction since $a^2, b^2, c^2, ab, ac, bc > 0$

$\therefore (a+b+c)^2 > ab+bc+ac.$

Alternative II:

$$a^2+b^2+c^2+ab+bc+ac > 0 \text{ since } a,b,c>0$$

$$a^2+b^2+c^2+2(ab+bc+ac)-(ab+bc+ac) > 0$$

$$(a+b+c)^2-(ab+bc+ac) > 0$$

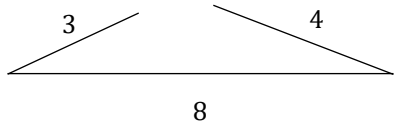
$$\therefore (a+b+c)^2 > (ab+bc+ac)$$

## The triangle inequality

We covered the triangle inequality in complex numbers, and will use it again in vectors. Let's look at how we can use it in proofs involving real numbers.

For three straight lines to form a non-degenerate triangle, each length must be less than the sum of the lengths of the other two sides.

Not convinced? Try making a triangle with sides $3, 4$ and $8$.




We use the triangle inequality in situations other than triangles - if we are using real numbers or vectors the 'sides' of the triangle can lie flat, which is why we need to use $\geq$. This is sometimes referred to as a degenerate triangle.

A reminder that the triangle inequality only works for non-negative real numbers, so we use absolute values when using terms or expressions that could be negative. So we can see that regardless of the values of $x$ and $y$, that

$$\boxed{|x+y| \leq |x| + |y|}$$

We remember this as 'the absolute value of the sum is less than or equal to the sum of the absolute values'. We had a similar saying using modulus in complex numbers.

We proved the triangle inequality in complex numbers, and since all real numbers are also complex then we do not need to prove the result for real numbers, though we will see a proof below. We will also use a much easier proof of this rule in vectors.

**Proof**

$\forall x, y \in \mathbb{R}$

$$-|x| \leq x \leq |x| \quad (1)$$

$$-|y| \leq y \leq |y| \quad (2)$$

$(1) + (2)$: $-(|x| + |y|) \leq x + y \leq |x| + |y|$

$$\therefore x + y \leq |x| + |y|$$

## Example 8

A triangle has sides $x$ cm, $10$ cm and $12$ cm. What are the possible values of $x$?

If $x$ is the smallest side then $x + 10 > 12$, so $x > 2$.

If $x$ is the longest side then $10 + 12 > x$, so $x < 22$

$\therefore 2 < x < 22.$

## Example 9

Prove $\left|a^2 - ab\right| + \left|ab - b^2\right| \geq (a + b)(a - b)$ for $a > b > 0$.

$\left|a^2 - ab\right| + \left|ab - b^2\right|$

$\geq \left|\left(a^2 - ab\right) + \left(ab - b^2\right)\right|$ triangle inequality

$= \left|a^2 - b^2\right|$

$= a^2 - b^2$ since $a > b$

$= (a + b)(a - b)$

$\therefore \left|a^2 - ab\right| + \left|ab - b^2\right| \geq (a + b)(a - b)$

## Example 10

Prove $a - c \leq |a - b| + |c - b|$ for $a > b > c > 0$.

$a - c$

$= |a - c|$ since $a > c$

$= |a - b + b - c|$

$\leq |a - b| + |b - c|$ triangle inequality

$= |a - b| + |c - b|$

Basic

**1** When each of the following operations is performed on each side of an inequality, state whether the sign stays the same, changes, or whether we need more information.

**a** adding $-3$

**b** dividing by $2$

**c** multiplying by $-1$

**d** taking the sine of both sides

**e** taking the natural logarithm of both sides

**2** Given $m > n > 0$ and $0 > p > r$, insert the appropriate inequality symbol between each pair of expressions to indicate their relative size, or state why it is not possible to determine their relative size.

**a** $\sqrt{m} \square \sqrt{n}$

**b** $-p \square -n$

**c** $r-n \square p-m$

**d** $\frac{r}{n} \square \frac{p}{m}$

**e** $-\frac{2}{m} \square -\frac{2}{n}$

**f** $(m-r)^3 \square (n-p)^3$

**g** $mr^2 \square np^2$

**h** $\frac{1}{mp} \square \frac{1}{nr}$

**i** $m^3 \square p^3$

**j** $n^2+p^2 \square m^2+r^2$

**3** Given $a > b > 0$, prove that $-\frac{1}{a} > -\frac{1}{b}$.

**4** Given $a > b > 0$, prove that $(2+a)(2-a) < (2+b)(2-b)$.

**5a** Given $a, b, c > 0$, prove that $(a+b+c)^2 - (ab+bc+ac) > 0$.

**b** Given $a, b, c \in \mathbb{R}$, prove that $a^2 + b^2 + c^2 - (ab+bc+ac) \geq 0$.

**6** Prove $|a-b| + |b-a| > 0$ for $a > b > 0$.

**7** Prove $(a+2b)(a-2b) \leq |a^2 - 2ab| + |4b^2 - 2ab|$ for $a > 2b > 0$.

**8** A triangle has sides $x, 7, 5$. What are the possible values of $x$?

**9** Given $x, y > 0$ prove that $x^5 + y^5 \geq x^4 y + xy^4$.

**10** Prove $1 - \sin 2x \geq 0$ for $x \in \mathbb{R}$.

**11** Given $a > b$ for $a, b \in \mathbb{R}$, prove that $a^2 + b^2 - 2(a+b) + 2 \geq 0$.

## Medium

**12** Given $a = b + c$ for $b, c > 0$, prove that $a^2 - bc > b^2 + c^2$.

**13** Given $a, b > 0$ prove that $a^3 - ab^2 - a^2 b + b^3 \geq 0$.

**14** Given $x > \sin x$ for $x > 0$, prove $\pi x - x^2 > \sin^2 x$ for $0 < x < \pi$.

**15** Two sides of an isosceles triangle are $4$ cm and $6$ cm. Prove the third side can be $4$ cm or $6$ cm.

**16** Given $a > b$ for real $a$ and $b$, prove that $a^2 + b^2 + 8 \geq 4(a+b)$.

**17** Given that $x$ and $y$ are positive real numbers prove $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{x+y}$.

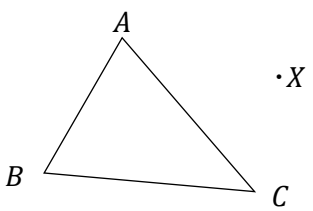
**18** Given $k$ is a positive integer, prove $\dfrac{2k+2}{2k+3} > \dfrac{2k}{2k+1}$.

**19** Prove $\dfrac{x^2+y^2}{2} \geq \left(\dfrac{x+y}{2}\right)^2$ for $x, y > 0$.

**20** Prove for real $x$, that $|x^2 - x| + |x-1| \geq |(x+1)(x-1)|$.

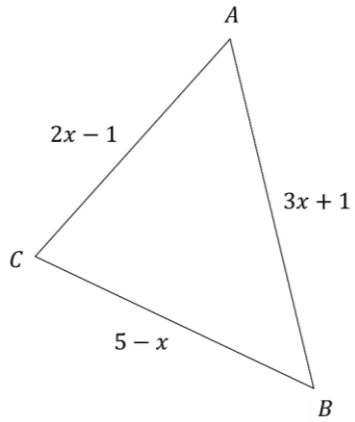
**21** $\Delta ABC$ has sides $a, b, c$. If $a^2 + b^2 + c^2 = ab + bc + ca$, show that $\Delta ABC$ is equilateral.

**22** The diagram shows a point $X$ outside a triangle $ABC$.
Show that $AX + BX + CX > \dfrac{AB + BC + CA}{2}$.




**23** Prove that for $x, y \in \mathbb{R}$, where $x^2 + y^2 \neq 0$, that $\dfrac{(x+y)^2}{x^2+y^2} \leq 2$.

**24** Triangle $ABC$ has sides $a = 5 - x, b = 2x - 1$ and $c = 3x + 1$ as shown, where $x \in \mathbb{R}$.




**a** Prove that $\frac{5}{6} < x < \frac{3}{2}$.

**b** Given $\Delta ABC$ is isosceles, prove that $\cos A = \frac{1}{8}$.

**25** Show that for any real $x$, $(x-3)(x+1)(x-5)(x+3) + 40 \geq 0$.

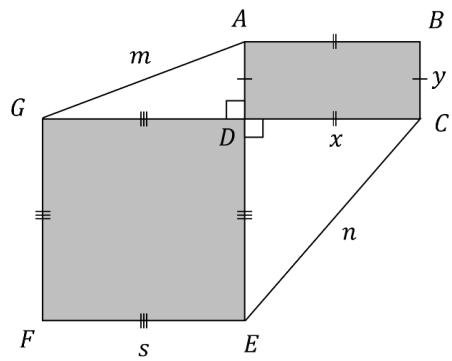
**26** Let $a, b, x, y > 0$. Prove that $(a^2 - b^2)(x^2 - y^2) \leq (ax - by)^2$.

## Challenging

**27a** For the positive real numbers $a$ and $b$, $a \neq b$, prove $a^3 + b^3 > ab(a + b)$.

**b** Hence, or otherwise, show that if $c$ is also a positive real number that is not equal to $a$ or $b$, that $2(a^3 + b^3 + c^3) > a^2(b + c) + b^2(a + c) + c^2(a + b)$.

**28** The diagram below shows the points $A$ to $G$, forming rectangle $ABCD$, square $DEFG$, and right−angled triangles $ADG$ and $CDE$. The area of square $DEFG$ is greater than the area of rectangle $ABCD$.




The length of rectangle $ABCD$ is $x$ units, and its width is $y$ units, where $x > y$, as shown. Square $DEFG$ has side length $s$ units. The hypotenuses of triangles $ADG$ and $CDE$ have lengths $m$ units and $n$ units respectively. Prove by contradiction, or otherwise, that $x + sn > y + sm$.

**29** Prove $a^n - b^n < na^{n-1}(a - b)$ for $0 < b < a$, where $n \in \mathbb{Z}, n \geq 2$.

**30** If $0 < b \leq a$, prove $\frac{(a-b)^2}{8a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{(a-b)^2}{8b}$.

**31** If $a, b > 0$, prove that $\frac{a}{\sqrt{b}} + \frac{b}{\sqrt{a}} \geq \sqrt{a} + \sqrt{b}$.

**32** Prove $a^2 + b^2 + c^2 + d^2 + e^2 \geq a(b + c + d + e)$.

**33** Let $a, b, A$ and $B$ be positive numbers.

**a** Prove that $\dfrac{ab}{AB} \leq \dfrac{1}{2}\left(\dfrac{a^2}{A^2} + \dfrac{b^2}{B^2}\right)$.

**b** Let $A = \sqrt{\sum_{k=1}^{n} a_k^2}$ and $B = \sqrt{\sum_{k=1}^{n} b_k^2}$, where $a_k$ and $b_k$ are positive real numbers.

Use (a) to prove that $\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$.

**c** Let $S = x_1 + x_2 + x_3 + \ldots + x_n$, where $x_k > 0$ for all $1 \leq k \leq n, n \geq 2$. Use (b) to prove that

$$\frac{S}{S - x_1} + \frac{S}{S - x_2} + \frac{S}{S - x_3} + \ldots + \frac{S}{S - x_n} \geq \frac{n^2}{n - 1}.$$

**34a** By substituting $a = 2x, b = y$ in $(a - b)^2 \geq 0$ prove $(x + y)\left(\dfrac{1}{x} + \dfrac{4}{y}\right) \geq 9$ for $x, y > 0$.

**b** By substituting $a = \sqrt{x}, b = \sqrt{y}$ in $(a - b)^2 \geq 0$ then $a = \dfrac{1}{\sqrt{x}}, b = \sqrt{\dfrac{4}{y}}$ prove

$$(x + y)\left(\frac{1}{x} + \frac{4}{y}\right) \geq 8.$$

**c** The results in (a) and (b) are different but both correct. Now $(x + y)\left(\dfrac{1}{x} + \dfrac{4}{y}\right)$ cannot be smaller than $9$. What have we done in (b) that has caused us to obtain a lower bound that is not the greatest possible lower bound?

**35** Given $a, b, c$ are positive real numbers with $a > b$ and $c^2 > ab$, prove by contradiction that

$$\frac{a + c}{\sqrt{a^2 + c^2}} - \frac{b + c}{\sqrt{b^2 + c^2}} > 0.$$

In Lesson 6 we cover:

- The four means
- Arithmetic mean - geometric mean inequality (AM-GM)

Syllabus reference: This lesson covers the 5th, 6th and 7th dot points from Proof of inequalities in *The nature of proof*.

## The four means

When we talk about the mean of a set of data we are normally talking about the arithmetic mean (AM) - the sum of the scores divided by the number of scores. We also call this the average.

$$\text{Arithmetic mean (AM)} = \frac{x_1 + x_2 + x_3 + \ldots x_n}{n}$$

The arithmetic mean is a useful measure of centre when the scores are close together, but is affected by very large values when the numbers are spread.

There are many other types of mean, each with a different purpose. We will use the geometric mean extensively in the rest of this lesson, and harder questions sometimes use the harmonic mean and quadratic mean without naming them.

The geometric mean of a set of non-negative numbers is the $n^{\text{th}}$ root of their product. It is more useful than the arithmetic mean when finding the centre of scores that are widely spread, as large scores don't distort the results as much.

$$\text{Geometric mean (GM)} = \sqrt[n]{x_1 \times x_2 \times x_3 \times \ldots \times x_n}$$

The harmonic mean is the reciprocal of the arithmetic mean of the reciprocals of the scores. It is useful when finding the average of two or more rates. The term harmonic mean is not used in the syllabus.

$$\text{Harmonic mean (HM)} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

For two scores the harmonic mean can be rearranged to give $\frac{2x_1x_2}{x_1+x_2}$, which we will sometimes see in questions.

The quadratic mean (also called the root mean square) of a set of data is the square root of the arithmetic mean of the squares of the scores. It is related to standard deviation. The term quadratic mean is not used in the syllabus, but we will sometimes see questions that involve the expression.

$$\text{Quadratic mean (QM)} = \sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}}$$

There is a relationship between the four means. The inequality of means is not part of the syllabus: QM ≥ AM ≥ GM ≥ HM. It does form the basis of some harder questions.

We will not mention the terms harmonic mean or quadratic mean again.

## Arithmetic mean - geometric mean inequality (AM-GM)

The only part of the inequality of means that is part of the syllabus involves the arithmetic and geometric means.

The arithmetic mean of a set of non-negative numbers is always greater than or equal to their geometric mean, AM ≥ GM:

$$\frac{x_1 + x_2 + x_3 + \ldots x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$$

We refer to this inequality as AM-GM for short. The relationship between the AM and GM holds for all positive values of $n$, not just $n = 2$. There will occasionally be questions where you need to use the AM-GM for $n = 2$ to prove results involving more terms, but unless told otherwise in the question, you can use the AM-GM for any number of terms.

When all numbers are equal then AM=GM, while if one or more is different then AM>GM.

A common exam question involves proving the AM-GM for two terms. We will look at two methods over the page, with the first method being the most efficient.

**Proving the AM-GM for two terms.**

**Method 1 (preferred method)**

$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$ since $\mathbb{R}^2 \geq 0$

$a-2\sqrt{ab}+b \geq 0$

$a+b \geq 2\sqrt{ab}$

$\frac{a+b}{2} \geq \sqrt{ab}$

We can also use this method to prove results in exams. We start by squaring the difference of the square roots of the terms we need in the final result, in this case $a$ and $b$.

**Method 2**

$(a-b)^2 \geq 0$

$a^2-2ab+b^2 \geq 0$

$a^2+2ab+b^2 \geq 4ab$ adding $4ab$ to both sides

$(a+b)^2 \geq 4ab$

$a+b \geq 2\sqrt{ab}$ since $a, b \geq 0$

$\frac{a+b}{2} \geq \sqrt{ab}$

We could make Method 1 even shorter by starting with $\left(\sqrt{\frac{a}{2}}-\sqrt{\frac{b}{2}}\right)^2 \geq 0$, but I find the version used in the proof more useful overall.

When we use the AM-GM in exams, we have a number of approaches we can take.

The first method matches Method 1 above.

- Start with the axiom $\mathbb{R}^2 \geq 0$
- The expression in the brackets should be the difference of the square roots of the two terms in the result to be proved. For example, if we want to prove $\frac{x^2+y^2}{2} \geq xy$ then we start with $(x-y)^2 \geq 0$, as $x$ and $y$ are the square roots of $x^2$ and $y^2$ respectively.
- Solving from the axiom is simpler and more convincing.

The second method involves substituting appropriate terms into the AM-GM.

- Start with $\frac{a+b}{2} \geq \sqrt{ab}$ - it is best to write AM-GM whenever you use the result
- Then make appropriate substitutions for $a$ and $b$.

We can also solve AM-GM questions using the methods from previous lessons.

The third method is to use a proof by contradiction. This is an underutilised method that is often quite quick. Just make sure that you complete the proof by interpreting the contradiction, so that you have proved the original result.

The fourth method is to start with an expression equal to LHS – RHS, or RHS – LHS, and simplify the expression to show that it is positive or negative, then interpret the result as proving the original result. Again, it is important to complete the proof correctly.

Finally, as we have seen with other inequalities, we can use working out paper to work from the conclusion to a true statement, then write the steps out in reverse order as the actual solution. This should only be used as a last resort, with the realisation that unless it is written out in the correct order it is logically invalid and will not earn full marks, and may earn no marks at all.

Let's solve a simple AM-GM result using the first four methods.

## Example 1

For $a > 0$ prove $a + \frac{1}{a} \geq 2$:

**a** Starting with $\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0$.

$$a - 2 + \frac{1}{a} \geq 0$$
$$a + \frac{1}{a} \geq 2$$

**b** By letting $x = a$ and $y = \frac{1}{a}$ in $\frac{x+y}{2} \geq \sqrt{xy}$.

$$\frac{a + \frac{1}{a}}{2} \geq \sqrt{a \times \frac{1}{a}}$$
$$a + \frac{1}{a} \geq 2\sqrt{1}$$
$$= 2$$

**c** Using a proof by contradiction.

Suppose by contradiction that

$$a + \frac{1}{a} < 2$$
$$a^2 + 1 < 2a$$
$$a^2 - 2a + 1 < 0$$
$$(a+1)^2 < 0$$

Which is a contradiction

since $\mathbb{R}^2 \geq 0$, so $a + \frac{1}{a} \geq 2$.

**d** Using LHS – RHS.

$$\text{LHS} - \text{RHS} = a - 2 + \frac{1}{a}$$
$$= \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2$$
$$\geq 0$$
$$\therefore a - 2 + \frac{1}{a} \geq 0$$
$$\therefore a + \frac{1}{a} \geq 2$$

For the examples to follow we will only use Methods 1 and 2, reserving the other methods as back up for questions where we cannot see how to progress with the first two methods.

## Example 2

Prove $x \geq 2\sqrt{x-1}$ for $x \geq 1$.

$$\left(\sqrt{x-1}-1\right)^2 \geq 0 \quad \text{for } x \geq 1$$
$$x-1-2\sqrt{x-1}+1 \geq 0$$
$$x \geq 2\sqrt{x-1}$$

If we look closely we see this can be rewritten as $x-1+1 \geq 2\sqrt{(x-1)(1)}$, which is the AM-GM with $x-1$ and $1$.

Alternatively:

$$\therefore \frac{(x-1)+(1)}{2} \geq \sqrt{(x-1)(1)} \text{ for } x \geq 1$$
$$\frac{x}{2} \geq \sqrt{x-1}$$
$$\therefore x \geq 2\sqrt{x-1}$$

We don't need to mention the AM-GM when we use it, though for more complicated examples it is a good idea.

In harder questions we may be required to use the AM-GM inequality for two terms multiple times, rather than using the AM-GM for three terms, for example.

## Example 3

Prove $\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y} \geq x+y+z$ for $x, y, z > 0$.

Here we have the sum of three terms on both sides of the inequality, so our only option is to use the AM-GM three times. If the right-hand side was a product then the AM-GM for three terms might be suitable.

$$\frac{\frac{xy}{z}+\frac{yz}{x}}{2} \geq \sqrt{\frac{xy^2z}{zx}} \qquad \text{(AM−GM)}$$
$$\frac{1}{2}\left(\frac{xy}{z}+\frac{yz}{x}\right) \geq y \qquad (1)$$

Similarly

$$\frac{1}{2}\left(\frac{yz}{x}+\frac{zx}{y}\right) \geq z \qquad (2)$$
$$\frac{1}{2}\left(\frac{xy}{z}+\frac{zx}{y}\right) \geq x \qquad (3)$$

$(1)+(2)+(3)$:

$$\frac{1}{2}\left(\frac{xy}{z}+\frac{yz}{x}\right)+\frac{1}{2}\left(\frac{yz}{x}+\frac{zx}{y}\right)+\frac{1}{2}\left(\frac{xy}{z}+\frac{zx}{y}\right) \geq y+z+x$$
$$\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y} \geq x+y+z$$

## Example 4

Prove $x^2 + y^2 + z^2 \geq x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2}$ for $x, y, z > 0$.

$$\frac{x^2 + (y^2 + z^2)}{2} \geq \sqrt{x^2(y^2 + z^2)} \qquad \text{(AM} - \text{GM)}$$

$$\frac{x^2 + (y^2 + z^2)}{2} \geq x\sqrt{y^2 + z^2} \qquad (1)$$

Notice that the RHS is asymmetrical as there is no $z\sqrt{x^2 + y^2}$ term, so we only need 2 applications of AM-GM.

Similarly

$$\frac{y^2 + (x^2 + z^2)}{2} \geq y\sqrt{x^2 + z^2} \qquad (2)$$

$(1) + (2)$:

$$x^2 + y^2 + z^2 \geq x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2}$$

## Example 5

If $a, b, c > 0$ then prove $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$

$$\frac{a^3 + a^3 + b^3}{3} \geq \sqrt[3]{a^6b^3}$$

$$\therefore \frac{2a^3 + b^3}{3} \geq a^2b \quad (1)$$

Similarly

$$\frac{2b^3 + c^3}{3} \geq b^2c \qquad (2) \qquad \frac{2c^3 + a^3}{3} \geq c^2a \qquad (3)$$

$(1) + (2) + (3)$: $\qquad a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$

## Example 6

If $a, b, c, d > 0$ then prove $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$.

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}}{4} \geq \sqrt[4]{\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{a}}$$

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}}{4} \geq 1$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$$

Notice that we have the sum of four terms on the LHS and a single term on the RHS, so using AM-GM for 4 terms is the easiest solution.

The great thing about AM-GM questions is that there are so many options to solve many of the questions, so if one method doesn't work you can try something else. In the next example we will solve each part of the question in several ways.

## Example 7

For all non−negative real numbers $x$ and $y$, $\sqrt{xy} \leq \frac{x+y}{2}$. (Do NOT prove this.)

**a** Using this fact, show that for all non−negative real numbers $a, b$ and $c$, $\sqrt{abc} \leq \frac{a^2+b^2+2c}{4}$, in at least three different ways.

**Sub in AM-GM 2 terms**

Let $x = ab, y = c$ in $\sqrt{xy} \leq \frac{x+y}{2}$

$$\begin{aligned}\therefore \sqrt{abc} &\leq \frac{ab+c}{2} \quad * \\ &\leq \frac{\sqrt{a^2b^2}+c}{2} \\ &\leq \frac{\frac{a^2+b^2}{2}+c}{2} \\ &\leq \frac{a^2+b^2+2c}{4}\end{aligned}$$

**Rearranging RHS**

$$\begin{aligned}\frac{a^2+b^2+2c}{4} &= \frac{\frac{a^2+c}{2}+\frac{b^2+c}{2}}{2} \\ &\geq \frac{\sqrt{a^2c}+\sqrt{b^2c}}{2} \quad * \\ &\geq \sqrt{\sqrt{a^2c}\times\sqrt{b^2c}} \\ &\geq \sqrt{abc} \\ \therefore \sqrt{abc} &\leq \frac{a^2+b^2+2c}{4}\end{aligned}$$

**Rearranging LHS**

$$\begin{aligned}\sqrt{abc} &= \sqrt{\sqrt{a^2b^2}c} \\ &\leq \frac{\sqrt{a^2b^2}+c}{2} \quad * \\ &\leq \frac{\frac{a^2+b^2}{2}+c}{2} \\ &\leq \frac{a^2+b^2+2c}{4}\end{aligned}$$

Three more solutions over the page

## Example 7 (continued)

**RHS - LHS**

RHS – LHS

$$= \frac{a^2+b^2+2c}{4} - \sqrt{abc}$$

$$= \frac{\frac{a^2+c}{2}+\frac{b^2+c}{2}}{2} - \sqrt{abc}$$

$$\geq \frac{\sqrt{a^2c}+\sqrt{b^2c}}{2} - \sqrt{abc} \quad *$$

$$\geq \sqrt{\sqrt{a^2c}\sqrt{b^2c}} - \sqrt{abc}$$

$$\geq \sqrt{abc} - \sqrt{abc}$$

$$\geq 0$$

$$\therefore \sqrt{abc} \leq \frac{a^2+b^2+2c}{4}$$

**Contradiction**

Suppose by contradiction that:

$$\sqrt{abc} > \frac{a^2+b^2+2c}{4}$$

rearranging: $\sqrt{\sqrt{a^2c}\sqrt{b^2c}} > \frac{\frac{a^2+c}{2}+\frac{b^2+c}{2}}{2}$ #

Now $\sqrt{\sqrt{a^2c}\times\sqrt{b^2c}} \leq \frac{\sqrt{a^2c}+\sqrt{b^2c}}{2}$ AM−GM

$$\therefore \sqrt{\sqrt{a^2b^2c^2}} \leq \frac{\frac{a^2+c}{2}+\frac{b^2+c}{2}}{2}$$

Which contradicts #

$$\therefore \sqrt{abc} \leq \frac{a^2+b^2+2c}{4}$$

**RHS - LHS**

$$\mathbb{R}^2 \geq 0$$

$$\left(\sqrt{\sqrt{a^2c}} - \sqrt{\sqrt{b^2c}}\right)^2 \geq 0 \quad \text{since } \mathbb{R}^2 \geq 0$$

$$\sqrt{a^2c} - 2\sqrt{\sqrt{a^2c}\sqrt{b^2c}} + \sqrt{b^2c} \geq 0$$

$$\sqrt{a^2c} - 2\sqrt{abc} + \sqrt{b^2c} \geq 0$$

$$2\sqrt{abc} \leq \sqrt{a^2c} + \sqrt{b^2c}$$

$$2\sqrt{abc} \leq \frac{a^2+c}{2} + \frac{b^2+c}{2}$$

$$\sqrt{abc} \leq \frac{a^2+b^2+2c}{4}$$

**b** Using part (a), or otherwise, show that for all non−negative real numbers $a, b$ and $c$,

$\sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}$, in at least three different ways.

**Sub in AM-GM 6 terms**

$$\sqrt[6]{uvwxyz} \leq \frac{u+v+w+x+y+z}{6}$$

Let $u = a^2, v = b^2, w = c^2, x = a, y = b, z = c$

$$\therefore \sqrt[6]{a^2b^2c^2abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6} \quad *$$

$$\therefore \sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}$$

**Use result from (a) 3 times**

Similarly

$$\sqrt{abc} \leq \frac{a^2+c^2+2b}{4}$$

$$\sqrt{abc} \leq \frac{b^2+c^2+2a}{4}$$

Adding the inequalities together:

$$3\sqrt{abc} \leq \frac{a^2+b^2+2c}{4} + \frac{a^2+c^2+2b}{4} + \frac{b^2+c^2+2a}{4} \quad *$$

$$\leq \frac{2a^2+2b^2+2c^2+2a+2b+2c}{4}$$

$$\sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}$$

Three more solutions over the page

## Example 7 (continued)

**Contradiction**

Suppose by contradiction that:

$$\begin{aligned}
\sqrt{abc} &> \frac{a^2+b^2+c^2+a+b+c}{6} \\
&> \frac{1}{6}\left(\frac{a^2+c}{2}+\frac{b^2+c}{2}+\frac{a^2+b}{2}+\frac{c^2+b}{2}+\frac{b^2+a}{2}+\frac{c^2+a}{2}\right) \\
&> \frac{1}{6}\left(\sqrt{a^2c}+\sqrt{b^2c}+\sqrt{a^2b}+\sqrt{c^2b}+\sqrt{b^2a}+\sqrt{c^2a}\right) \quad * \\
&> \frac{1}{6}\left(a\sqrt{c}+b\sqrt{c}+a\sqrt{b}+c\sqrt{b}+b\sqrt{a}+c\sqrt{a}\right) \\
&> \frac{1}{3}\left(\frac{\sqrt{c}(a+b)}{2}+\frac{\sqrt{b}(a+c)}{2}+\frac{\sqrt{a}(b+c)}{2}\right) \\
&> \frac{1}{3}\left(\sqrt{c}\sqrt{ab}+\sqrt{b}\sqrt{ac}+\sqrt{a}\sqrt{bc}\right) \\
&> \frac{1}{3}(\sqrt{abc}+\sqrt{abc}+\sqrt{abc}) \\
&> \sqrt{abc}
\end{aligned}$$

Which is a contradiction so $\sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}$.

**RHS - LHS**

RHS – LHS

$$\begin{aligned}
&= \frac{a^2+b^2+c^2+a+b+c}{6} - \sqrt{abc} \\
&= \frac{1}{6}\left(\frac{a^2+c}{2}+\frac{b^2+c}{2}+\frac{a^2+b}{2}+\frac{c^2+b}{2}+\frac{b^2+a}{2}+\frac{c^2+a}{2}\right) - \sqrt{abc} \\
&\geq \frac{1}{6}\left(\sqrt{a^2c}+\sqrt{b^2c}+\sqrt{a^2b}+\sqrt{c^2b}+\sqrt{b^2a}+\sqrt{c^2a}\right) - \sqrt{abc} \quad * \\
&= \frac{1}{3}\left(\frac{\sqrt{a^2c}+\sqrt{b^2c}}{2}+\frac{\sqrt{a^2b}+\sqrt{c^2b}}{2}+\frac{\sqrt{b^2a}+\sqrt{c^2a}}{2}\right) - \sqrt{abc} \\
&\geq \frac{1}{3}\left(\sqrt{\sqrt{a^2c}\times\sqrt{b^2c}}\times\sqrt{\sqrt{a^2b}\times\sqrt{c^2b}}\times\sqrt{\sqrt{b^2a}\times\sqrt{c^2a}}\right) - \sqrt{abc} \\
&\geq \frac{1}{3}\left(\sqrt{abc}+\sqrt{abc}+\sqrt{abc}\right) - \sqrt{abc} \\
&\geq 0 \\
&\therefore \sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}
\end{aligned}$$

**Rearranging RHS**

$$\begin{aligned}
&\frac{a^2+b^2+c^2+a+b+c}{6} \\
&= \frac{1}{6}\left(\frac{a^2+c}{2}+\frac{b^2+c}{2}+\frac{a^2+b}{2}+\frac{c^2+b}{2}+\frac{b^2+a}{2}+\frac{c^2+a}{2}\right) \\
&\geq \frac{1}{6}\left(\sqrt{a^2c}+\sqrt{b^2c}+\sqrt{a^2b}+\sqrt{c^2b}+\sqrt{b^2a}+\sqrt{c^2a}\right) \quad * \\
&= \frac{1}{3}\left(\frac{\sqrt{a^2c}+\sqrt{b^2c}}{2}+\frac{\sqrt{a^2b}+\sqrt{c^2b}}{2}+\frac{\sqrt{b^2a}+\sqrt{c^2a}}{2}\right) \\
&\geq \frac{1}{3}\left(\sqrt{\sqrt{a^2c}\times\sqrt{b^2c}}\times\sqrt{\sqrt{a^2b}\times\sqrt{c^2b}}\times\sqrt{\sqrt{b^2a}\times\sqrt{c^2a}}\right) \\
&= \frac{1}{3}\left(\sqrt{abc}+\sqrt{abc}+\sqrt{abc}\right) \\
&= \sqrt{abc}
\end{aligned}$$

## Basic

**1** For $a, b \geq 0$ prove $\dfrac{a+b}{2} \geq \sqrt{ab}$ by starting with $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$.

**2** For $a, b \geq 0$ prove $\dfrac{a+b}{2} \geq \sqrt{ab}$ by starting with $(a-b)^2 \geq 0$.

**3** For $x \neq 0$ prove $x^2 + \dfrac{1}{x^2} \geq 2$.

**4** Prove $x^2 \geq 2\sqrt{(x-1)(x+1)}$ for $x \geq 1$.

**5** For $a, b > 0$ prove $(a+2b)^2 \geq 8ab$.

**6** Prove $1 + x \geq 2\sqrt{x}$ for $x \geq 0$.

**7** For $a, b, c > 0$ prove $ac + \dfrac{b}{c} \geq 2\sqrt{ab}$.

**8** For all non-negative numbers $x$ and $y$, $\frac{x+y}{2} \geq \sqrt{xy}$. (Do NOT prove this.)

A rectangle has dimensions $a$ and $b$. Given that the rectangle has perimeter $P$, and area $A$, prove that $P^2 \geq 16A$.

**9** It is known that for all positive real numbers, $x, y \quad x + y \geq 2\sqrt{xy}$. (Do NOT prove this.)

Using this result, or otherwise, show that for all real $a$, that $|a| \leq \dfrac{a^2+1}{2}$.

## Medium

**10** Prove $x^4 + x^2y + 4y^2 \geq 5x^2y$ for $x, y \in \mathbb{R}$.

**11** If $a, b, c, d$ are real numbers, prove $\left(a^2+b^2\right)\left(c^2+d^2\right) \geq (ac+bd)^2$.

**12** For $a, b, c > 0$ prove $a^2c^2 + \dfrac{b^2}{c^2} \geq 2ab$.

**13** Given $\dfrac{a+b}{2} \geq \sqrt{ab}$, prove $\dfrac{x}{yz} + \dfrac{y}{xz} + \dfrac{z}{xy} \geq \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$ for $x, y, z > 0$.

**14** Prove $\dfrac{x^2}{y^2} + \dfrac{y^2}{z^2} + \dfrac{z^2}{x^2} \geq \dfrac{1}{yz}\sqrt{x^2y^2+z^4} + \dfrac{1}{xz}\sqrt{y^2z^2+x^4}$ for $x, y, z > 0$.

**15** For $a, b, c, d \geq 0$, use $\dfrac{a+\mathrm{b}}{2}$ and $\dfrac{c+d}{2}$ in the AM−GM for two terms to prove

$\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

**16** If $a, b, c, d > 0$ then prove $\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{d} + \frac{d^3}{a} \geq 4\sqrt{abcd}$.

**17** For $a, b, c \geq 0$, use $\frac{a+\mathrm{b}}{2}$ and $\frac{c + \frac{a+b+c}{3}}{2}$ in the AM−GM for two terms to prove $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

**18** If $a, b, c > 0$ then prove $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$.

**19** Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2a_3 \ldots a_n = 1$.
Prove that $\left(a_1^2 + a_1\right)\left(a_2^2 + a_2\right) \ldots \left(a_n^2 + a_n\right) \geq 2^n$

**20a** For $a, b, c \geq 0$ prove $a^2 + b^2 + c^2 \geq ab + bc + ca$.

**b** Hence prove $(a + b + c)^2 \geq 3(ab + bc + ca)$.

**21** Using the fact that for all non−negative real numbers $x$ and $y$, $\sqrt{xy} \leq \frac{x+y}{2}$, or otherwise, show that for all real numbers $p$ and $q$, $p^2 + 4q^2 \geq 4pq$, in at least three different ways.

**22a** For positive real numbers, $a$ and $b$, prove that $a + b \geq 2\sqrt{ab}$.

**b** It is known that the graphs of $y = \sec^2 x$ and $y = 2\left|\tan x\right|$ lie on or above the $x$−axis, where they are defined. (Do NOT prove this.) Using the result from (a), or otherwise, prove that the graph of $y = \sec^2 x$ lies on or above the graph of $y = 2\left|\tan x\right|$.

**23** Prove that $a^4 \geq 2a^3 - a^2$ for $a \geq 0$.

Challenging

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**24** Let $a, b, c > 0$. Prove that $a^2b + b^2c + c^2a \geq 2abc$.

**25** Given the arithmetic mean is greater than or equal to the geometric mean, if $a, b, c > 0$ then prove $a^7 + b^7 + c^7 \geq a^4b^3 + b^4c^3 + c^4a^3$.

**26** Let $w, x, y, z > 0$.

**a** Prove that $\frac{x}{y} + \frac{y}{x} \geq 2$.

**b** Deduce that $\frac{x+y+z}{w} + \frac{w+y+z}{x} + \frac{w+x+z}{y} + \frac{w+x+y}{z} \geq 12$.

**c** Hence prove that if $w + x + y + z = 1$, then $\frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 16$.

**27** Let $a+b=1$ prove that $a^4+b^4 \geq \frac{1}{8}$.

**28** Let $a,b,c>0$ such that $abc=1$, prove that $a^2+b^2+c^2 \geq a+b+c$.

**29** If $a,b,c>0$ then prove $a^4+b^4+c^4 \geq a^2bc+b^2ca+c^2ab$.

**30** If $a,b,c>0$ satisfy $abc=1$, prove $\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \geq 3$.

**31** Prove the Harmonic Mean $\leq$ the Geometric Mean $\leq$ the Arithmetic Mean $\leq$ the Quadratic Mean for two positive numbers, i.e.:

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \text{ for } a,b,c>0.$$

**32a** For $a,b,c>0$ prove $3\left(a^2+b^2+c^2\right) \geq (a+b+c)^2 \geq 3(ab+bc+ac)$.

**b** If $a+b+c=3$, hence prove that $a^2+b^2+c^2+ab+bc+ca \geq 6$.

**33a** For $a,b>0$ prove that $\frac{a+b}{2} \geq \sqrt{ab}$.

**b** The generalisation of the result in (a) is $\frac{a_1+a_2+a_3+\ldots+a_n}{n} \geq \sqrt[n]{a_1a_2a_3\ldots a_n}$.

Use this result to prove that $n! \leq \left(\frac{n+1}{2}\right)^n$ for $n \geq 1$. Do not use mathematical induction.

**34a** If $x>0, y>0$ show that $x+y \geq 2\sqrt{xy}$.

**b** Hence show that if $x>0, y>0, z>0$ then $(x+y)(y+z)(z+x) \geq 8xyz$.

**c** If $a,b,c$ are the sides of a triangle with semi−perimeter $S=\frac{1}{2}(a+b+c)$ then Heron's formula states that the area of the triangle is given by $A=\sqrt{S(S-a)(S-b)(S-c)}$.

By choosing suitable values for $x,y,z$ show that $A^2 \leq \frac{(a+b+c)abc}{16}$.

**35** It is given that for positive numbers $x_1,x_2,x_3,\ldots,x_n$ with arithmetic mean $A$, $\frac{x_1 \times x_2 \times x_3 \times \ldots \times x_n}{A^n} \leq 1$. (Do NOT prove this.) Suppose a rectangular prism has dimensions $a,b,c$ and surface area $S$. Show that $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$ in at least three ways.

In Lesson 7 we cover:

- Inequalities involving geometry
- Squeeze theorem
- Inequalities using graphical or calculus techniques

Syllabus reference: This lesson covers the 6th to 9th dot points from Proof of inequalities in *The nature of proof*.

## Inequalities involving geometry

We have already used the triangle inequality and the AM-GM to create inequalities involving geometry. We can also use relationships between the side lengths, perimeter and area of common shapes, or the surface area and volume of a 3D solid, to create or prove other inequalities.

We are often applying skills from Stage 5 and Advanced, so there is no new theory to learn - just lots of question types to search out and practise.

Let's have a look at a couple of examples.

## Example 1

A triangle has side lengths $a$, $b$ and $c$, with $c$ being the longest side.

Prove that the perimeter $P$ satisfies $2c < P \leq 3c$.

$$\begin{aligned} \therefore P &= a + b + c \\ &\leq c + c + c \text{ since } a \leq c \text{ and } b \leq c \\ &= 3c \\ \therefore P &\leq 3c \qquad (1) \end{aligned}$$

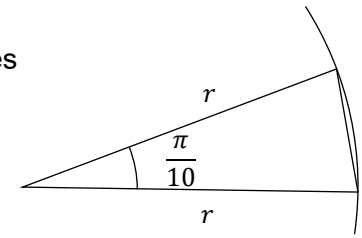
Also,

$$\begin{aligned} \therefore P &= a + b + c \\ &> c + c \quad \text{since } a + b > c \text{ from the triangle inequality} \\ &= 2c \\ \therefore 2c &< P \qquad (2) \end{aligned}$$

$\therefore 2c < P \leq 3c$ from $(1)$ and $(2)$.

## Example 2

By considering the area of a $20$-sided regular polygon inscribed in a circle, show that $\sqrt{5} < \dfrac{2\pi + 5}{5}$. It is given that $\sin\dfrac{\pi}{10} = \dfrac{\sqrt{5}-1}{4}$.




The $20-$sided polygon can be divided into $20$ isosceles triangles with equal sides of $r$ where the included angle is $\dfrac{2\pi}{20} = \dfrac{\pi}{10}$.

$$\text{Area}_{\text{polygon}} = 20 \times \frac{1}{2} \times r \times r \times \sin\frac{\pi}{10}$$

$$= 10r^2 \times \frac{\sqrt{5}-1}{4}$$

$$= \frac{5(\sqrt{5}-1)}{2} r^2 \text{ units}^2$$

$$\text{Area}_{\text{circle}} = \pi r^2$$

$\text{Area}_{\text{polygon}} < \text{Area}_{\text{circle}}$ since the polygon lies on or inside the circle

$$\frac{5(\sqrt{5}-1)}{2} r^2 < \pi r^2$$

$$\frac{5(\sqrt{5}-1)}{2} < \pi$$

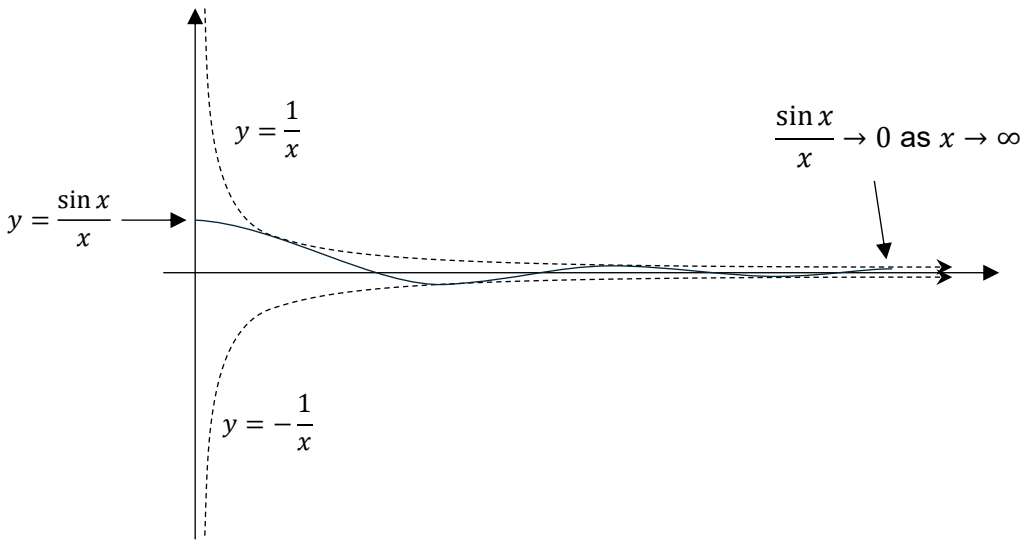
$$\sqrt{5} - 1 < \frac{2\pi}{5}$$

$$\sqrt{5} < \frac{2\pi + 5}{5}$$

## Squeeze theorem

If the value of a function must lie between the values of two other functions, and at some point the two outer functions approach each other, then the function in the middle must also approach this value. This is known as the squeeze theorem, or sandwich theorem.

In the next example we will show that $\frac{\sin x}{x} \to 0$ as $x \to \infty$. Looking at the graph of $y = \frac{\sin x}{x}$, we can see that it lies between the graphs of $y = \frac{1}{x}$ and $y = -\frac{1}{x}$, and as $x \to \infty$ these graphs approach the $x$-axis. This is the basis of the solution.




In general, we can say that if $f(x) \leq g(x) \leq h(x)$ for all $x$ that are near $k$, but not necessarily at $k$, and $\lim_{x \to k} f(x) = \lim_{x \to k} h(x) = L$, then $\lim_{x \to k} g(x) = L$. The theorem also applies to limits at infinity.

Let's look at a couple of examples.

## Example 3

Prove that $\lim_{x \to \infty} \left[\frac{\sin x}{x}\right] = 0$.

$$-1 \leq \sin x \leq 1$$

$$\therefore -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \text{ for } x \neq 0$$

as $x \to \infty, -\frac{1}{x} \to 0, \frac{1}{x} \to 0$

$$\therefore \lim_{x \to \infty} \left[\frac{\sin x}{x}\right] = 0 \text{ by the squeeze theorem.}$$

## Example 4

Evaluate $\lim_{x\to 4}\left[3+(x-4)^2\cos\left(\frac{1}{x-4}\right)\right]$.

$$-1 \le \cos\left(\frac{1}{x-4}\right) \le 1 \text{ for } x \ne 4$$

$$-(x-4)^2 \le (x-4)^2\cos\left(\frac{1}{x-4}\right) \le (x-4)^2$$

$$3-(x-4)^2 \le 3+(x-4)^2\cos\left(\frac{1}{x-4}\right) \le 3+(x-4)^2$$

as $x \to 4,\ 3-(x-4)^2 \to 3, 3+(x-4)^2 \to 3$

$\therefore \lim_{x\to 4}\left[3+(x-4)^2\cos\left(\frac{1}{x-4}\right)\right] = 3$ by the squeeze theorem.

## Inequalities using graphical or calculus techniques

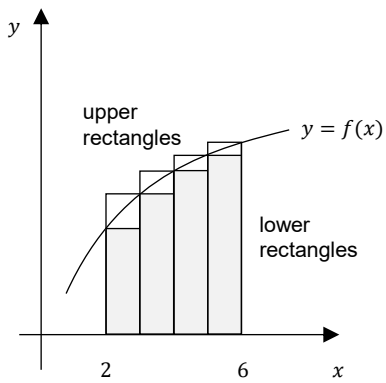
There is a wide variety of inequalities that can be proved using graphical or calculus techniques, or a combination of both. There is no new theory you need to be able to do these questions, just some common sense.

In the next example, we will use the fact that all values of a function lie above its global minimum value.

**Areas and Integrals**

Questions usually involve areas above the $x$-axis, so we can use areas and integrals interchangeably. Questions involving areas usually involve comparing methods of approximating areas with the true area, and ordering their relative size. Students should understand the terms upper rectangles and lower rectangles, and how the concavity of a curve changes the order of the true area with that calculated using the trapezoidal rule.

In the diagram we are interested in the areas formed by the curve $y = f(x)$ between $x = 2$ and $x = 6$, and approximations made with rectangles - the trapezoidal rule is not shown.




The exact area would be given by $\int_2^6 f(x)\,dx$.

Upper rectangles (where the height of each rectangle is determined by either the top-left or top-right corner of the rectangle lying on the function with the other corner above the function) overestimate the area.

Lower rectangles (where the height of each rectangle is determined by either the top-left or top-right corner of the rectangle lying on the function with the other corner below the function) underestimate the area.

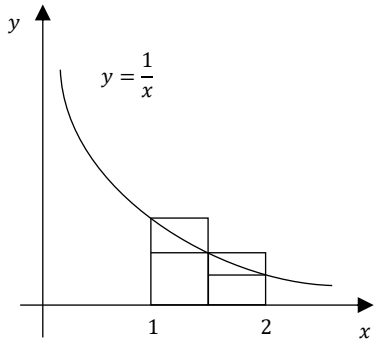
Given that the curve is concave down, an approximation using the trapezoidal rule would also underestimate the area, so would be smaller than the exact area. In this case we could say:

$$\text{lower rectangles} < \text{trapezoidal rule} < \int_2^6 f(x)\,dx < \text{upper rectangles}$$

Let's have a look at an example using a decreasing concave up function.

## Example 5

The hyperbola $y = \frac{1}{x}$ is shown in the diagram, broken into two subintervals.




**a** Using upper and lower rectangles, prove $\frac{7}{12} < \ln 2 < \frac{5}{6}$.

$$\text{Exact Area} = \int_1^2 \frac{dx}{x} = \left[\ln x\right]_1^2 = \ln 2$$

$$\text{Area of Upper Rectangles} = \frac{1}{2} \times \left(\frac{1}{1}\right) + \frac{1}{2} \times \left(\frac{1}{1.5}\right) = \frac{5}{6}$$

$$\text{Area of Lower Rectangles} = \frac{1}{2} \times \left(\frac{1}{1.5}\right) + \frac{1}{2} \times \left(\frac{1}{2}\right) = \frac{7}{12}$$

Lower Rectangles $<$ Exact Area $<$ Upper Rectangles

$$\therefore \frac{7}{12} < \ln 2 < \frac{5}{6}$$

**b** Use the trapezoidal rule to prove $\ln 2 < \frac{17}{24}$.

$$\text{Area by trapezoidal rule} = \frac{2-1}{2 \times 2}\left(\frac{1}{1} + \frac{1}{2} + 2\left(\frac{1}{1.5}\right)\right) = \frac{17}{24}$$

Exact Area $<$ Area by Trapezoidal Rule (since the curve is concave up)

$$\therefore \ln 2 < \frac{17}{24}$$

## Example 6

In $f(x) = ax^2 + bx + c$, we are given $a > 0$. Prove that $\forall x \in \mathbb{R}, f(x) - c + \frac{b^2}{4a} \geq 0$.

The function is a concave up parabola, so all points lie on or above the minimum value, which occurs where $x = -\frac{b}{2a}$.

$$f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$$

$$= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c$$

$$= \frac{ab^2}{4a^2} - \frac{2ab^2}{4a^2} + \frac{4a^2c}{4a^2}$$

$$= \frac{4a^2c - ab^2}{4a^2} \quad f(x) \geq \frac{4ac - b^2}{4a}$$

$$= c - \frac{b^2}{4a}$$

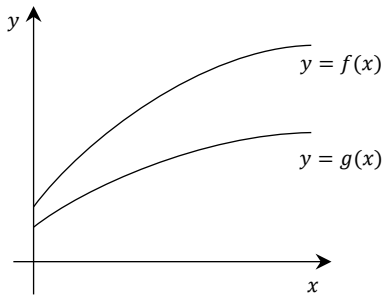
The minimum value is $c - \frac{b^2}{4a}$.

$\therefore \forall x \in \mathbb{R}, f(x) \geq c - \frac{b^2}{4a}$

$\therefore f(x) - c + \frac{b^2}{4a} \geq 0.$

The minimum/maximum value of a quadratic is given by $\frac{4ac - b^2}{4a}$, or $-\frac{\Delta}{4a}$, a handy trick to keep up your sleeve when checking results. It is not in the syllabus so do not use it as your primary method in a question worth more than one mark without proving it.

Another important result leading to inequalities, is that if one function is at the same point or above a second function at the left of the domain, and the gradient of the first function is always greater than or equal to that of the second function, then the first function must always be at least as high as the second function.




In the diagram, $f(0) > g(0)$ and $f'(x) > g'(x)$ for $x \geq 0$, so we can conclude that

$$f(x) > g(x) \text{ for } x \geq 0.$$

A better variation for these types of questions is to create a function which equals the LHS−RHS. If you can prove that the new function is zero or positive at the beginning of the domain, and is an increasing function, this also proves the first function is always greater than the second. We will use this method in the examples.

## Example 7

Prove that $x \geq \sin x$ for $x \geq 0$.

Let $f(x) = x - \sin x$

$\therefore f(0) = 0 - \sin 0$

$= 0$

$f'(x) = 1 - \cos x$

$\geq 0$ since $-1 \leq \cos x \leq 1$

$\therefore f(x)$ is an increasing function

$\therefore f(x) \geq 0$ for $x \geq 0$ since $f(0) = 0$ and $f(x)$ is an increasing function.

$\therefore x - \sin x \geq 0$ for $x \geq 0$

$\therefore x \geq \sin x$ for $x \geq 0$

The gradient or concavity of the whole function, or of a restricted domain of the function, can also give inequalities and methods to solve them.

## Example 8

The cubic $y = ax^3 + bx^2 + cx + d$ has no stationary point. Prove that $b^2 < 3ac$.

$y' = 3ax^2 + 2bx + c$

If there is no stationary point then $y' \neq 0$, so $\Delta < 0$

$$\Delta = (2b)^2 - 4(3a)(c)$$
$$= 4b^2 - 12ac$$

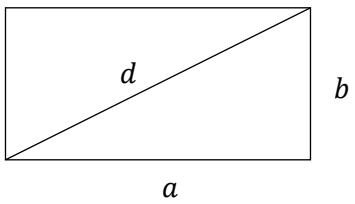
$\therefore 4b^2 - 12ac < 0$

$b^2 - 3ac < 0$

$\therefore b^2 < 3ac$

Basic

**1** A rectangle has length $a$, width $b$, diagonal $d$ and perimeter $P$.

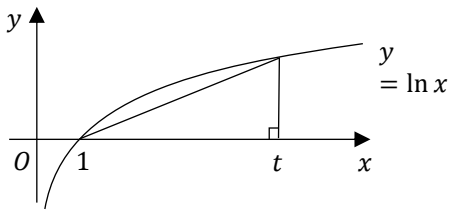



**a** Prove that $2(a^2 + b^2) \geq (a + b)^2$.

**b** Hence, prove that $P \leq 2\sqrt{2}\, d$.

**2** Prove that $\lim\limits_{x \to \infty} \left[ e^{-x} \cos^2 x \right] = 0$.

**3** The diagram shows the graph $y = \ln x$.




By comparing relevant areas in the diagram, or otherwise, show that

$$\ln t > 2\left(\frac{t-1}{t+1}\right), \text{for } t > 1.$$

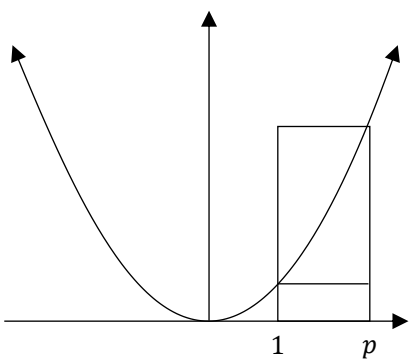
**4** Given that $3 - x^2 \leq f(x) \leq 3 + 2x^2$ for all $x \neq 0$, use the squeeze theorem to find $\lim\limits_{x \to 0} f(x)$.

**5** Prove $e^{\frac{x}{2}} \geq \frac{x}{2} + 1$ for $x \geq 0$.

**6** Evaluate $\lim\limits_{x \to 0} \left[ x \cos\left(\frac{2}{x^2}\right) \right]$.

**7** By considering the graph, show that

$$p - 1 < \frac{p^3}{3} - \frac{1}{3} < p^3 - p^2 \text{ for } p > 1.$$




**8** Using calculus, show that $x \geq \ln(1 + x)$ for $x > -1$.

**9** A triangle has two sides of lengths $a$ and $b$, with included angle $\theta$, and area $A$.

Prove that $A \leq \frac{ab}{2}$.

**10** Evaluate $\lim\limits_{x\to 2}\left[(x-2)^2\sin\left(\dfrac{1}{x-2}\right)\right]$.

Medium

**11** Use the squeeze theorem to prove that $\lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\right]=1$.

You may use the geometrical inequality $\sin x < x < \tan x$ for $0 < x < \dfrac{\pi}{2}$.

**12** A rectangle has perimeter $P$ and area $A$. Prove that $A \le \dfrac{P^2}{16}$.

**13** It is given that $f(x)=x-\ln\left(1+x+\dfrac{x^2}{2}\right)$.

**a** Determine the value of $f(0)$.

**b** Show that $f(x)$ is an increasing function of $x$ for $x<0$.

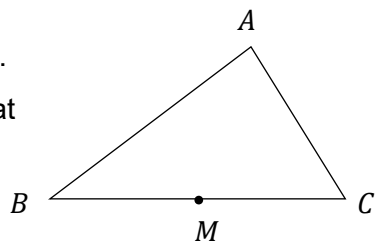
**c** Hence show that $e^x < 1+x+\dfrac{x^2}{2}$ for $x<0$.

**14** Evaluate $\lim\limits_{x\to 0}\left[x^2\left|\sin\left(\dfrac{1}{x}\right)+\cos\left(\dfrac{1}{x}\right)\right|\right]$.

**15** Given that $\cos\left(\dfrac{1}{x^2}\right)\le f(x)\le 1-\sin\left(\dfrac{1}{x^2}\right)$ for $x>0.5$, use the squeeze theorem to find $\lim\limits_{x\to\infty} f(x)$.

**16** In triangle $ABC$, $M$ is the midpoint of $BC$, as shown.

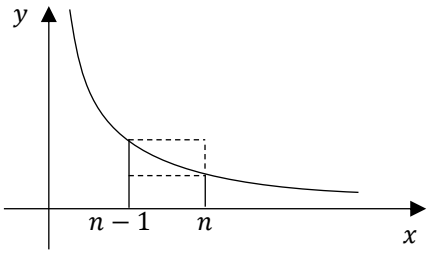
By considering the parallelogram ABDC, prove that $AM < \dfrac{AB+AC}{2}$.




**17** Evaluate $\lim\limits_{x\to 0}\left[\dfrac{x^2\sin\left(\frac{1}{x}\right)}{1+x^2}\right]$.

**18** By considering the area of a $16$-sided regular polygon inscribed in a circle, show that $\dfrac{\pi^2}{16}+\sqrt{2}>2$. It is given that $\sin\dfrac{\pi}{8}=\dfrac{\sqrt{2-\sqrt{2}}}{2}$.

**19** Let $n$ be a positive integer greater than 1. The area under the curve $y = \dfrac{1}{x}$ from $x = n - 1$ to $x = n$ is between the two rectangles, as shown in the diagram.




Show that $e^{-\frac{n}{n-1}} < \left(1 - \dfrac{1}{n}\right)^n < e^{-1}$.

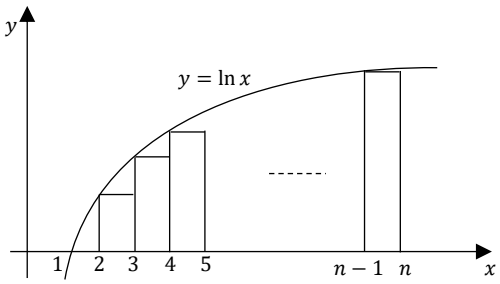
**20** Given that $2(e^{-x} + 2) \leq f(x) \leq \dfrac{3x + 4x^2}{x^2}$ for $x \geq 0$, find $\lim\limits_{x \to \infty} f(x)$.

Challenging

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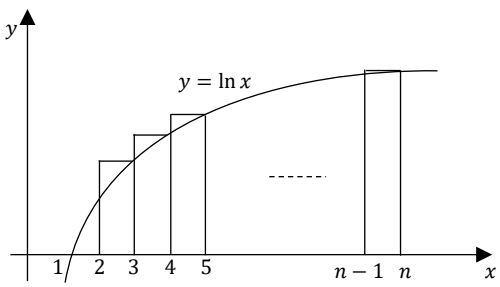
**21a** Evaluate exactly the integral $\displaystyle\int_1^n \ln x \, dx$.

**b** Consider the curve $y = \ln x$. The area under the curve for $1 \leq x \leq n$ is approximated by dividing it into rectangles under the curve each of width 1 unit. See diagram below (not to scale).




Show that the sum of the rectangles, $S_{LR}$, is given by $S_{LR} = \ln\left((n-1)!\right)$

**c** Another approximation, $S_{UR}$, is made by dividing the area into rectangles that lie above the curve. See diagram below (not to scale).




Find a similar expression for $S_{UR}$.

**d** Hence or otherwise show that $(n-1)! < n^n e^{1-n} < n!$.

**22** Consider the function $f(x) = \dfrac{(n+1+x)^{n+1}}{(n+x)^n}, x \geq 0$, where $n \geq 1$ is a fixed positive integer.

**a** Show that for $x > 0$, $f(x)$ is an increasing function of $x$.

**b** It is given that $\left(1+\dfrac{x}{n+1}\right)^{n+1} > \left(1+\dfrac{x}{n}\right)^n$. Deduce that $(n+2)^{n+1}n^n > (n+1)^{2n+1}$.

**23** Use the squeeze theorem to prove that $\lim\limits_{x\to 0}\left[x^2 \sin\left(\dfrac{1}{x^2}\right)\cos\left(\dfrac{1}{x}\right)\right] = 0$.

**24** Suppose that $x \geq 0$ and $n$ is a positive integer.

**a** Show that $1 - x \leq \dfrac{1}{1+x} \leq 1$.

**b** Hence, or otherwise, show that $1 - \dfrac{1}{2n} \leq n \ln\left(1+\dfrac{1}{n}\right) \leq 1$.

**c** Hence, explain why $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$.

**25** Consider $f(x) = \ln x - x + 1$.

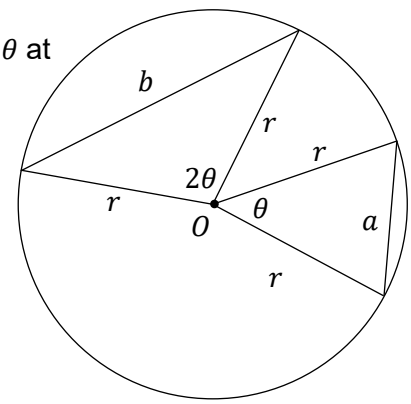
**a** Show that $f(x) \leq 0$ for all $x > 0$.

**b** Consider the set of $n$ positive numbers $p_1, p_2, \ldots p_n$ such that $p_1 + p_2 + \ldots + p_n = 1$.
By using the result in part (a), deduce that $\sum\limits_{r=1}^{n} \log(np_r) \leq np_1 + np_2 + np_3 + \ldots np_n - n$

**c** Show that $\sum\limits_{r=1}^{n} \log np_r \leq 0$

**d** Hence deduce that $0 < n^n p_1 p_2 p_3 \ldots p_n \leq 1$

**26** Two chords of a circle of radius $r$ subtend angles $\theta$ and $2\theta$ at the centre, where $0 < \theta < \dfrac{\pi}{2}$. Their lengths are $a$ and $b$, respectively. Prove that $a < b < 2a$.



**27a** Find the coordinates of the turning points of the curve $y = 27x^3 - 27x^2 + 4$.

**b** By sketching the curve, deduce that $x^2(1-x) \leq \frac{4}{27}$ for all $x \geq 0$

**c** Three real numbers $a, b, c$ lie between 0 and 1. Prove that at least one of the numbers $bc(1-a), ca(1-b)$ and $ab(1-c)$ is less than or equal to $\frac{4}{27}$.

**28** Given that $f(x)$ is a function satisfying $\left|f(x)-2\right| \leq (x-1)^2 \left|\sin\left(\frac{1}{x-1}\right)\right|$ for all $x \neq 1$, prove that $\lim\limits_{x\to 1} f(x) = 2$.

**29** Suppose that $x$ is a positive real number.

**a** Show that $\frac{1}{1+t^2} < 1 - t^2 + t^4 - t^6 + \ldots + t^{4n} < \frac{1}{1+t^2} + t^{4n+2}$ for $t > 0$.

**b** Hence show that

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{x^{4n+1}}{4n+1} < \tan^{-1} x + \frac{x^{4n+3}}{4n+3} \text{ for } 0 < x \leq 1.$$

**c** Explain why $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ for $0 < x \leq 1$.

**d** Deduce that $\pi = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$.

**30a** Given $e^u \geq 1 + u$ prove that $\log_e x \leq x - 1$ for $x > 0$ by making a suitable substitution.

**b** Show that $\ln\left(\frac{c_1 c_2 c_3 \ldots c_n}{\mu^n}\right) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\mu} - n$, where $c_1, c_2, .., c_n > 0$ and $\mu > 0$.

**c** Hence if $\mu = \frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}$, show that $\sqrt[n]{c_1 c_2 c_3 \ldots c_n} \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}$.

**d** Hence use part (c) to find a lower bound for $\frac{101}{103} + \frac{103}{105} + \frac{105}{107} + \ldots + \frac{197}{199} + \frac{199}{101}$.

# 2.8 Mathematical induction - inequalities and recursive formulas

In Lesson 8 we cover:

- Mathematical induction in Extension 2
- Proving results involving inequalities using induction
- Proving results involving first-order recursive formulas using induction

Syllabus reference: This lesson covers the 2nd, 4th and 5th dot points from Further proof by mathematical induction in *The nature of proof*.

## Mathematical induction in Extension 2

Before we delve into mathematical induction in detail, it is worth taking a moment to look at the different ways that mathematical induction is used in the Extension 1 and 2 courses for the 2024 syllabuses.

A lot of the details that used to be included in Extension 2 have moved to Extension 1. All divisibility proofs are now solely in Extension 1. We also see summation notation (sigma notation), base cases other than $1$ and step values other than $1$ included in Extension 1.

There are no other changes to Extension 2, so expect that there will be much greater emphasis on proofs involving inequalities and first-order recursive formulas which we look at this lesson, and unusual proofs that we cover next lesson.

Before you look at induction in Extension 2, make sure that you understand the importance of proving both the base case and the inductive step clearly and logically, showing sufficient detail, with clear setting out.

## Proving results involving inequalities using induction

We now look at induction proofs where we bring our skills from direct inequality proofs into the inductive step. As with any inequality proof, obtaining a deep understanding of inequalities is the key.

The body of the proof for results involving inequalities can be quite difficult, with the pressure points being where we replace an expression with an expression that is clearly larger (or smaller) than the expression on the line before.

The most successful solutions at HSC level involve manipulating the left-hand side of $P(k+1)$ through the body of the proof. At each step the sign should refer to how the line compares with the line above rather than the original LHS. This makes the proof more convincing, and can save adding explanations at the end of each line.

Some other techniques that were appropriate for direct proofs involving inequalities, such as using LHS – RHS, involve too much algebra which wastes exam time and increases the chance of algebraic errors, so are best avoided.

While it is not possible to start with both sides of $P(k+1)$ to prove $P(k+1)$, as it creates logical flaws in the proof, you can start with both sides of $P(k)$, as we assume that it is true. We will use this technique in Example 4.

## Example 1

Prove by induction that $2^n > n^2$ for positive integers $n > 4$.

Let $P(n)$ represent the proposition.

$P(5)$ is true since LHS $= 2^5 = 32$; RHS $= 5^2 = 25$.

If $P(k)$ is true for some arbitrary $k > 4$ then $2^k > k^2$

RTP $P(k+1)$: $2^{k+1} > (k+1)^2$

LHS $= 2(2^k)$

$> 2k^2$ from $P(k)$

$= k^2 + k^2$

$> k^2 + 2k + 1$ for $k > 4$ since $k^2 > 2k + 1$ for $k \geq 3$

$= (k+1)^2$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n > 4$ by induction

## Example 2

Prove by induction that $10^n > 6^n + 8^n$ for $n > 2$.

Let $P(n)$ represent the proposition.

$P(3)$ is true since $10^3 = 1000$ and $6^3 + 8^3 = 728$, so $10^3 > 6^3 + 8^3$.

If $P(k)$ is true for some arbitrary $k > 2$ then $10^k > 6^k + 8^k$

RTP $P(k+1)$: $10^{k+1} > 6^{k+1} + 8^{k+1}$

LHS $= 10(10^k)$

$> 10\left(6^k + 8^k\right)$ from $P(k)$

$= 10(6^k) + 10(8^k)$

$> 6(6^k) + 8(8^k)$

$= 6^{k+1} + 8^{k+1}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n > 2$ by induction.

## Example 3

Prove by induction for positive integers $n$ that $2!\times4!\times6!\times\ldots\times(2n)! \geq \left((n+1)!\right)^n$.

Let $P(n)$ represent the proposition.

$P(1)$ is true since $2! \geq \left((1+1)!\right)^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $2!\times4!\times6!\times\ldots\times(2k)! \geq \left((k+1)!\right)^k$

RTP $P(k+1)$: $2!\times4!\times6!\times\ldots\times(2k)!\times\left(2(k+1)\right)! \geq \left((k+2)!\right)^{k+1}$

LHS $= 2!\times4!\times6!\times\ldots\times(2k)!\times\left(2(k+1)\right)!$

$\geq \left((k+1)!\right)^k \times \left(2(k+1)\right)!$ from $P(k)$

$= \left((k+1)!\right)^k \times \left( \underbrace{(2(k+1)\cdot(2(k+1)-1)\ldots(k+3)}_{k \text{ terms}}\,(k+2)(k+1)! \right)$

$\geq \left((k+1)!\right)^k \times \left((k+2)^k(k+2)!\right)$

$= \left((k+2)(k+1)!\right)^k \times (k+2)!$

$= \left((k+2)!\right)^k \times (k+2)!$

$= \left((k+2)!\right)^{k+1}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

## Proving results involving first-order recursive formulas using induction

A recursive formula, or recurrence relation, is a formula that defines each term of a sequence using a preceding term. You may have already used them in Advanced *Sequences and series*, where they are the basis of arithmetic and geometric series. They are also used in problems in *Financial mathematics*, and a different type of recurrence relation is encountered in Extension 2 *Further integration*, where we relate integrals involving different powers.

A first-order recursive formula creates a sequence where each term is determined by the previous term. The syllabus limits us to first-order recursive formula, so the Fibonacci Sequence cannot be tested as it relies on the two previous terms so is a second order recursive formula. It is still useful and interesting.

There always seem to be two formulas in induction questions involving recursive formulas, but you are given the recursive formula (the one linking $T_n$ and $T_{n-1}$) and must prove the formula linking $T_n$ and $n$. By reading the questions carefully, you will see that you need to prove the result formula linking $T_n$ and $n$.

### Example 4

A sequence $b_n$ is defined by $b_0 = 5$ and $b_n = 4 + b_{n-1}$ for $n \geq 1$.
Use mathematical induction to prove that $b_n = 5 + 4n$ for $n \geq 1$.

Let $P(n)$ represent the proposition $b_n = 5 + 4n$.

$P(1)$ is true since LHS $= b_1 = 4 + b_0 = 4 + 5 = 9$ and RHS $= 5 + 4(1) = 9$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then $b_k = 5 + 4k$

RTP $P(k+1)$: $\quad b_{k+1} = 5 + 4(k+1)$

LHS $= b_{k+1}$

$= 4 + b_k$ from the recursive formula

$= 4 + 5 + 4k$ from $P(k)$

$= 5 + 4(k+1)$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

If we look at this question carefully, we can see that we have an arithmetic sequence where the first term is $5$ and the common difference is $4$. Although we could prove the result using the formula for the $n^{\text{th}}$ term of an arithmetic sequence, this would not earn full marks due to the wording of the question.

## Example 5

A sequence $u_n$ is defined by $u_1 = 2$ and $u_{n+1} = 2u_n + 1$ for $n \geq 1$.

Prove that $u_n = 3(2^{n-1}) - 1$ for $n \geq 1$.

Let $P(n)$ represent the proposition $u_n = 3(2^{n-1}) - 1$.

$P(1)$ is true since LHS $= u_1 = 3(2^{1-1}) - 1 = 2$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then $u_k = 3\left(2^{k-1}\right) - 1$

RTP $P(k+1)$: $u_{k+1} = 3\left(2^{k}\right) - 1$

LHS $= u_{k+1}$

$= 2u_k + 1$ from the recursive formula

$= 2\left(3\left(2^{k-1}\right) - 1\right) + 1$ from $P(k)$

$= 3 \times 2 \times 2^{k-1} - 2 + 1$

$= 3\left(2^{k}\right) - 1$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

## Basic

**1** Prove by induction that $n < 2^n$ for integers $n \geq 1$.

**2** A sequence $a_n$ is defined by $a_1 = 5$ and $a_{n+1} = 3a_n$ for $n \geq 1$.
Prove that $a_n = 5(3^{n-1})$ for $n \geq 1$.

**3** Prove $2^{n+4} > 2n + 9$ for $n \geq 1$ by induction.

**4** A sequence $u_n$ is defined by $u_1 = 4$ and $u_{n+1} = 2u_n - 3$ for $n \geq 1$.
Prove that $u_n = 3 + 2^{n-1}$ for $n \geq 1$.

**5** Prove $2n < (n + 2)!$ for $n \geq 1$ by induction.

**6** Rewrite the following logic statement as an if−then statement.

$$\big[P(1) \wedge (\forall k \geq 1)[P(k) \Rightarrow P(k+1)]\big] \Rightarrow (\forall n \geq 1)\, P(n)$$

What do we know this statement as?

**7** Given $a_1 = 2$ and $a_n = 5a_{n-1}$ for $n \geq 2$, prove that $a_n = 2{\times}5^{n-1}$ for $n \geq 1$.

**8** Prove $2^n + 1 \leq 3^n$ for $n \geq 1$ by induction.

**9** Given $a_0 = A$ and $a_n = (1 + r)a_{n-1}$ for $n \geq 1$, show that $a_n = A(1 + r)^n$ for $n \geq 0$.

**10** Prove $2^{n+4} > (n + 4)^2$ for $n \geq 1$ by induction.

**11** Prove $n^2 > n + 1$ for $n \geq 2$ by induction.

## Medium

**12** Prove $n! > 2n\,, n \geq 4$ by induction.

**13** Given $a_1 = 2$ and $a_n = \dfrac{a_{n-1}}{n}$ for $n \geq 2$, prove that $a_n = \dfrac{2}{n!}$ for $n \geq 1$.

**14** Prove by induction that $6n + 6 < 2^n$ for $n \geq 6$.

**15** Prove $3n^2 + 3n + 1 < 2^n$ for $n \geq 8$ by induction.

**16** Given $u_{n+1} = \sqrt{3u_n}$ and $u_1 = 1$:

**a** Show that $u_2 = 3^{\frac{1}{2}}, u_3 = 3^{\frac{1}{2}+\frac{1}{4}}$ and $u_4 = 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$.

**b** Hence prove by mathematical induction that $u_n = 3^{1-2^{1-n}}$ for $n \geq 1$.

**c** Find the limiting value of $u_n$as $n \to \infty$.

**17** Prove by induction that $n^2 < 4^n$ for $n$ a positive integer.

**18** Given $x > \ln(1+x)$ for all $x > 0$, prove that $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} > \ln(1+n)$ for $n \geq 1$ by induction.

**19** The nth Fermat number, $F_n$, is defined by $F_n = 2^{2^n}+1$ for $n = 0,1,2,3$. Prove by mathematical induction that for all non−negative integers:
$F_0 \times F_1 \times F_2 \times \ldots \times F_{n-1} = F_n - 2$.

**20** Prove by induction that $12^n > 7^n + 5^n$ *for* $n \geq 2$.

Challenging

**21** Given the sequence $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ for $n \geq 1$ and $a_1 = 1$:

**a** Prove $\frac{a_n - \sqrt{2}}{a_n + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}}$ by induction.

**b** Hence find the limiting value of $a_n$ as $n \to \infty$.

**22** Prove by induction for positive integers $n$ that $1! \times 3! \times 5! \times \ldots \times (2n-1)! \geq (n!)^n$.

**23** Prove by induction for $n \geq 2$ that $1^3 + 2^3 + \ldots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \ldots + n^3$.

**24** Prove by induction for positive integers $n$ that $n > \frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n} \geq \frac{n}{2}$.

**25** Prove $x^n + x^{n-2} + x^{n-4} + \ldots + \frac{1}{x^{n-4}} + \frac{1}{x^{n-2}} + \frac{1}{x^n} \geq n+1$, for $x > 0, n > 0$ by induction.
[Hint: separate base cases are required to prove the result for $n$ even or odd].

**26** Prove $\sqrt{n} \leq \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$, $n \geq 1$ by induction.

**27** Prove $1 + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$ for $n \geq 1$ by induction.

**28** Given the sequence $a_{n+1} = a_n(a_n + 1)$ for $n \geq 1$ and $a_1 = 1$:

**a** Prove by mathematical induction that $a_{n+1} = 1 + \sum_{r=1}^{n} a_r^2$.

**b** Show that $(2a_{n+1}+1)^2 = (2a_n+1)^2 + (2a_{n+1})^2$.

**c** Hence deduce that $(2a_{n+1}+1)^2 = (2a_1+1)^2 + \sum_{r=2}^{n+1}(2a_r)^2$.

**d** Find $a_2, a_3$ and $a_4$ and use them to evaluate $a_5$.

**e** Express $a_5$ as the sum of 5 positive integers.

**f** Hence prove that $3^2 + 4^2 + 12^2 + 84^2 + 3612^2 = 3613^2$.

**29** Given $a_{n+1} = \sqrt{2a_n}$ and $a_1 = 1$, prove by mathematical induction that for all integers $n \geq 1$ $a_n = 2^{(1-2^{(1-n)})}$.

In Lesson 9 we cover:

- Choosing a method in unusual questions
- Proving results involving calculus using induction
- Proving results involving combinatorics using induction
- Proving results involving geometry using induction

Syllabus reference: This lesson covers the 1st, 3rd and 6th dot points from Further proof by mathematical induction in *The nature of proof*.

## Choosing a method in unusual questions

In all the following questions, the trick in solving them involves determining how the truth of the statement for $n = k$ leads to it being true for $n = k + 1$. This means we need to find an instance of the smaller problem in the larger problem. In the first five examples we will use algebraic and calculus skills to prove $P(k) \Rightarrow P(k + 1)$, while in the geometry questions we will grow or change the previous diagram.

## Proving results involving calculus using induction

We will start by looking at using induction to prove calculus results. We need to keep in mind that although these results are actually true for all real $n$, induction only proves that they are true for the positive integers.

## Example 1

Prove that for any positive integer $n$, $\frac{d}{dx}(x^n) = nx^{n-1}$.

You may assume that $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$.

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{d}{dx}(x^1) = 1$ and $1x^{1-1} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{d}{dx}(x^k) = kx^{k-1}$

RTP $P(k+1)$: $\frac{d}{dx}(x^{k+1}) = (k+1)x^k$

LHS $= \frac{d}{dx}(x \times x^k)$

$= x \times \frac{d}{dx}(x^k) + x^k \times \frac{d}{dx}(x)$ by the product rule

$= x \times kx^{k-1} + x^k \times 1$ from $P(k)$

$= kx^k + x^k$

$= (k+1)x^k$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

## Proving results involving combinatorics using induction

We start with a result mixing combinations and inequalities from the 2024 HSC. There are many technical algebraic steps in this solution, any of which can cause progress to grind to a halt.

## Example 2

Use mathematical induction to prove that ${}^{2n}C_n < 2^{2n-2}$, for all integers $n \geq 5$.

Let $P(n)$ represent the proposition

$P(5)$ is true since LHS $= {}^{2(5)}C_5 = {}^{10}C_5 = 252$; RHS $= 2^{2(5)-2} = 256$

If $P(k)$ is true for some arbitrary $k \geq 5$ then ${}^{2k}C_k < 2^{2k-2}$

RTP $P(k+1)$: ${}^{2k+2}C_{k+1} < 2^{2k}$

$$\text{LHS} = \frac{(2k+2)!}{(k+1)!\,(k+1)!}$$

$$= \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \times \frac{(2k)!}{k!\,k!}$$

$$= \frac{2(k+1)(2k+1)}{(k+1)(k+1)} \times {}^{2k}C_k$$

$$= \frac{2(2k+1)}{(k+1)} \times {}^{2k}C_k$$

$$< \frac{2(2k+1)}{k+1} \times 2^{2k-2} \qquad \text{from } P(k)$$

$$< \frac{2(2k+2)}{k+1} \times 2^{2k-2}$$

$$= \frac{4(k+1)}{k+1} \times 2^{2k-2}$$

$$= 2^{2k}$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for $n \geq 5$ by induction.

Proving the binomial theorem using mathematical induction is specifically mentioned in the syllabus. We will cover it here in an example, as many students find some of the steps involving manipulating summation notation fairly complicated.

## Example 3

Prove that for any positive integer $n$, $(a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r}b^r$.

Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= (a+b)^1 = a+b$ and

RHS $= \sum_{r=0}^{1} {}^1C_r\, a^{1-r}b^r = {}^1C_0 a^1 b^0 + {}^1C_1 a^0 b^1 = a+b.$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $(a+b)^k = \sum_{r=0}^{k} {}^kC_r\, a^{k-r}b^r$.

RTP $P(k+1)$: $\quad (a+b)^{k+1} = \sum_{r=0}^{k+1} {}^{k+1}C_r\, a^{k-r+1}b^r$

We start by multiplying the $a$ and $b$ into separate sums

The powers of $a$ and $b$ do not match in the two sums, so we increase the values of $r$ by 1 on the right-hand sum and replace $r$ with $r-1$ in the summand.

Separate off the first term of the first sum and last term of the second sum so that the sums have the same limits.

LHS $= (a+b)(a+b)^k$

$= (a+b)\sum_{r=0}^{k} {}^kC_r\, a^{k-r}b^r$ $\quad$ from $P(k)$

$= \sum_{r=0}^{k} {}^kC_r\, a^{k-r+1}b^r + \sum_{r=0}^{k} {}^kC_r\, a^{k-r}b^{r+1}$

$= \sum_{r=0}^{k} {}^kC_r a^{k-r+1}b^r + \sum_{r=1}^{k+1} {}^kC_{r-1} a^{k-r+1}b^r$

$= {}^kC_0 a^{k-0+1}b^0 + \sum_{r=1}^{k} {}^kC_r\, a^{k-r+1}b^r + \sum_{r=1}^{k} {}^kC_{r-1}\, a^{k-r+1}b^r + {}^kC_{(k+1)-1} a^{k-(k+1)+1}b^{k+1}$

$= a^{k+1} + \sum_{r=1}^{k} \left({}^kC_r + {}^kC_{r-1}\right) a^{k-r+1}b^r + b^{k+1}$

$= a^{k+1} + \sum_{r=1}^{k} {}^{k+1}C_r a^{k-r+1}b^r + b^{k+1}$ $\quad$ since ${}^{k+1}C_r = {}^kC_r + {}^kC_{r-1}$

$= {}^{k+1}C_0 a^{k-0+1}b^0 + \sum_{r=1}^{k} {}^{k+1}C_r a^{k-r+1}b^r + {}^{k+1}C_{k+1} a^{k-(k+1)+1}b^{k+1}$

$= \sum_{r=0}^{k+1} {}^{k+1}C_r\, a^{k-r+1}b^r$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

## Proving results involving geometry using induction

Let's have a look at a simple result from geometry proved using induction.

## Example 4

Prove that the sum of the interior angles of an $n$−sided plane convex polygon is $(n-2)180°$, for all integers $n \geq 3$. You may assume that the interior angle sum of a triangle is $180°$.

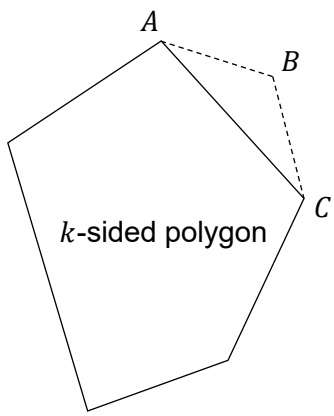
Let $P(n)$ represent the proposition.

$P(3)$ is true since a 3-sided plane convex polygon is a triangle which has an interior angle sum of $180°$, and $(3-2)180° = 180°$.

If $P(k)$ is true for some arbitrary $k \geq 3$, then the interior angle sum of a $k$-sided plane convex polygon is $(k-2)180°$.

RTP $P(k+1)$: The interior angle sum of a $(k+1)$-sided plane convex polygon is $(k-1)180°$.

Consider the $(k+1)$-sided polygon at right, formed by adding $\Delta ABC$ to one side of the $k$-sided polygon.




The interior angle sum of the new polygon will be equal to that of the $k$-sided polygon:

- plus $\angle BAC$ and $\angle BCA$ increase the size of the existing interior angles
- plus $\angle ABC$ is a new interior angle

$\therefore$ the interior angle sum of a $(k+1)$-sided polygon is

$(k-2)180° + \angle BAC + \angle BCA + \angle ABC$ from $P(k)$

$= (k-2)180° + 180°$

$= (k-1)180°$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 3$ by induction.

In this induction proof, the body of the proof starts with $P(k)$ rather than the LHS of $P(k+1)$.

## Basic

**1** Prove that for any positive integer $n$, $\frac{d}{dx}\left(e^{nx}\right) = ne^{nx}$.

You may assume that $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$.

**2** Use mathematical induction to prove that ${}^nC_2 \leq n^2$, for all integers $n \geq 2$.

**3** Prove that the number of diagonals in an $n$−sided plane convex polygon is $\frac{n(n-3)}{2}$, for all integers $n \geq 3$.

**4** Each of $n$ mathematics students (where $n \geq 2$) shakes hands with all the other mathematics students once each.

**a** How many handshakes will there be?

**b** Prove that your answer to (a) is correct using induction.

## Medium

**5** Prove that for any positive integer $n$, $(a-b)^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} a^{n-r}b^r$.

**6** Prove that for any positive integer $n$, $\frac{d}{dx}(\cos^n x) = -n\cos^{n-1} x \sin x$.

**7** Prove that for every positive integer $n$ that $\binom{n}{1} = n$.

You may assume $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ and $\binom{n}{0} = 1$.

**8** A plane is divided into regions by one or more intersecting circles. Prove that it is possible to colour the regions with only two colours, such that no two regions sharing an edge are the same colour.

**9** Prove that $\sin(n\pi + x) = (-1)^n \sin x$ for $n \geq 1$ by induction.

**10** Prove that the sum of the exterior angles of an $n$−sided plane convex polygon is $360°$.

**11** Prove $\frac{d^n}{dx^n}\left(xe^{2x}\right) = 2^{n-1}(2x+n)e^{2x}$ for $n \geq 1$ by induction.

**12** Prove for all positive integers $n$, that when $n$ circles on the Cartesian plane intersect each other at two distinct points, with no three circles intersecting at the same point, the circles divide the plane into $n^2 - n + 2$ regions.

**13a** Show that: $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$.

**b** Hence, use the principle of mathematical induction to establish the result:
$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$.

Challenging

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**14** Prove $\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^n\alpha = \dfrac{\sin(2^{n+1}\alpha)}{2^{n+1}\sin\alpha}$ for $n \geq 0, \alpha \neq m\pi$ for $m \in \mathbb{Z}$ by induction.

**15** Let $T(m,y) = \dfrac{{}^mC_0}{y} - \dfrac{{}^mC_1}{y+1} + \dfrac{{}^mC_2}{y+2} - \ldots + (-1)^m\dfrac{{}^mC_m}{y+m}$.

**a** If it is given that $T(k,x) = \dfrac{k!}{x(x+1)(x+2)\ldots(x+k)}$ for a particular value of $k$, $x > 0$, show that $T(k,x) - T(k,x+1) = T(k+1,x)$.

**b** Hence prove by induction that for $n \geq 1$

$$T(n,x) = \frac{{}^nC_0}{x} - \frac{{}^nC_1}{x+1} + \frac{{}^nC_2}{x+2} - \ldots + (-1)^n\frac{{}^nC_n}{x+n} = \frac{n!}{x(x+1)(x+2)\ldots(x+n)}.$$

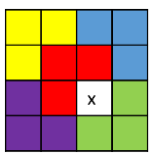
You may assume that ${}^{m+1}C_r = {}^mC_r + {}^mC_{r-1}$.

**c** Hence prove that $\dfrac{{}^nC_0}{1} - \dfrac{{}^nC_1}{3} + \dfrac{{}^nC_2}{5} - \ldots + (-1)^n\dfrac{{}^nC_n}{2n+1} = \dfrac{2^n n!}{1\times3\times5\times\ldots\times(2n+1)}$.

**16** A circle and a chord of that circle are drawn in a plane. Then a second circle and a chord of that circle, are added. Repeating this process, once there are $n$ circles with chords drawn, prove that the regions in the plane divided off by the circles and chords can be coloured with three colours in such a way that no two regions sharing a common boundary segment are the same colour.

**17** Chessboard Problem Prove that it is possible to cover a $2^n\times2^n$ grid with L tiles consisting of 3 squares if 1 square is removed.

For example, here we have a $2^2\times2^2$ grid with 1 square removed, with the remainder covered in 5 L tiles.



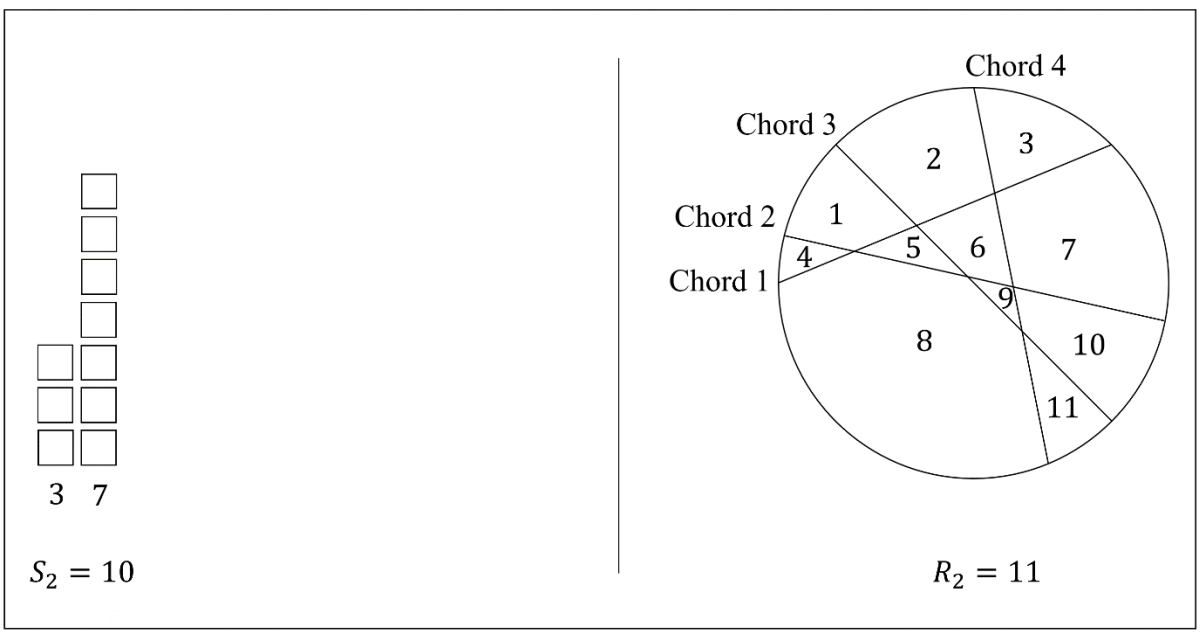

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**18** Tower of Hanoi You have three pegs and a collection of disks of different sizes. Initially all of the disks are stacked on top of each other according to size on the first peg − the largest disk being on the bottom and the smallest on top, as shown above. A move in this game consists of moving a disk from one peg to another, subject to the condition that a larger disk may never rest on a smaller one. The objective of the game is to find a number of permissible moves that will transfer all of the disks from the first peg to the third peg, making sure that the disks are assembled on the third peg according to size. The second peg is used as an intermediate peg. Prove that it takes at least $2^n - 1$ moves to move $n$ disks from the first peg to the third peg.



**19** Postage Stamp Problem Prove that any integer amount of postage in cents $n \geq 12$ can be paid for exactly using only $4$ cent and $5$ cent stamps.

**20** Two students are taking turns to draw on a classroom whiteboard. On each of his turns, Waylon draws squares on the left−hand side of the board. On his first turn he draws three squares, on his second turn he draws seven squares, and on each subsequent turn he draws four more squares than he did the turn before. After he has had $n$ turns, he has drawn $S_n$ squares. On each of her turns, Erin draws two chords in a circle on the right−hand side of the board. Each chord must be drawn so that it intersects every other chord that has already been drawn, and also in such a way that no three chords intersect at any point. After she has had $n$ turns, the circle has been divided into $R_n$ regions. The diagram shows the board after each student has had two turns. On the left−hand side there is a total of $3 + 7 = 10$ squares, while on the right−hand side there are $2 \times 2 = 4$ chords, dividing the circle into $11$ regions.




Prove by mathematical induction that $R_n = S_n + 1$ for $n \geq 1$.

# Chapter 4
# *Further integration*

Integration is a great Extension 2 topic, as the question often gives no hint as to how it should be approached, so you need to develop a deep understanding of the dozens of different techniques to know how to solve the integral. The solution is then relatively short, while often also relying on accurate algebra skills and using formulas from the reference sheet correctly, so students can show their ability without taking up too much exam time. Some of the integration skills we cover in this chapter are important in *Applications of calculus to mechanics* later in the course, which is largely applied integration.

**Lessons**

4.1 Completing the square & manipulating integrands
4.2 The reverse chain rule & $u$ substitutions
4.3 Other $u$ substitutions
4.4 Splitting the numerator
4.5 Partial fractions
4.6 Definite integrals
4.7 Trigonometric product to sum identities
4.8 Powers of trigonometric functions
4.9 Expressions and equations using the $t$-formulas
4.10 Trigonometric substitutions
4.11 Integration by parts
4.12 Recurrence relations
Chapter 4 Further integration - 100 revision questions

In Lesson 1 we cover:

- Integration in Extension 2
- Completing the square on a quadratic denominator
- Changing an integrand into an appropriate form using algebraic manipulation

This lesson mainly covers the 9th and 11th dot points from *Further integration*, with questions on the 15th and 16th dot points included in the exercises.

## Integration in Extension 2

Extension 2 integration involves a massive number of different techniques, some you have seen before in Advanced and Extension 1, and others which are new. As well as the new techniques you will learn, integration in Extension 2 is also harder as you will often be given a bare question with no hints as to how it should be solved – you generally won't be given the substitution if one is needed, or any other hint if a different technique is needed, and you may have to use more than one integration technique to complete the solution.

Integration questions occur throughout the HSC – most marks are easy to gain for those who have practised. There are occasionally harder questions scattered throughout the paper, not just in Q16. Focus on the small differences between integrands that require quite different approaches and learn the tricks that appear through the chapter.

There are online integral calculators that can help you if you are doing a question without worked solutions. Search for 'integral calculator' and end up at sites like integral-calculator.com – type in the integrand and it will give you the final solution plus working if you select it. Just keep in mind that it may use different techniques to what you would like, and the final answer may be equivalent to what you want but in a different form. This is especially true with trigonometric functions and inverse trigonometric functions.

At the end of this document is a summary of the topic, including many standard integrals that you would benefit from memorising.

One of the results of having so many techniques to learn is we often focus on how to do each type without really understanding what we are doing and why we are doing it.

The trick to successfully being able to solve the greatest variety of integration questions is to deeply understand that integration is the inverse operation of differentiation.

This might seem to be an obvious statement, but it is something we often forget. At its simplest it means that we can differentiate the primitive function to arrive back at the integrand.

Indefinite and definite integrals need to be understood in different ways.

Consider the indefinite integral, or primitive function, $\int x^2 dx$ – what does it mean? We are trying to find the family of functions whose derivative is $x^2$. The solution is $\frac{1}{3}x^3 + c$, which means that if we differentiate any of the infinite numbers of functions in the form $\frac{1}{3}x^3 + c$ the answer is $x^2$.

Now look at the definite integral $\int_1^3 x^2\, dx$ – how is this different? At its simplest we can say that it is the area bounded by the $x$–axis, the curve $y = x^2$ and the lines $x = 1$ and $x = 3$. If this curve was beneath the $x$–axis we would have to think of the definite integral as the negative of this area. We can find the definite integral by evaluating <u>any</u> primitive of $x^2$ at $x = 3$ and subtracting the value of the <u>same</u> primitive at $x = 1$. So, we find the difference between $\frac{1}{3}x^3 + c$ when $x = 3$ and $x = 1$. Since we can use any primitive, we usually let $c = 0$ for convenience without thinking about it, but we would get the same result if we used $\frac{1}{3}x^3 - 6$ twice, or any other primitive twice.

So indefinite integrals are a way of finding the family of functions that can be differentiated to arrive at the integrand, while definite integrals are most easily understood as the signed area between the curve and the $x$–axis between the upper and lower limits.

**Standard integrals**

Standard integrals are common primitive functions which form the basis of our integration. They all come from Advanced and Extension 1 – there are no new standard integrals in the Extension 2 course.

To find any standard integral we simply reverse the differentiation rules. For example, when $y = \sin f(x)$ is differentiated we find $y' = f'(x)\cos f(x)$, so the primitive of $f'(x)\cos f(x)$ is $\sin f(x) + c$. This gives us the standard integral

$$\int f'(x)\cos f(x)\, dx = \sin f(x) + c.$$

The reference sheet has a list of the only standard integrals we need to know for the HSC. They appear in two places on the reference sheet – there is a short list on the Extension 1 page and a longer list on the Advanced page.

**Advanced**

$$\int f'(x)\,[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c \quad \text{where } n \neq -1$$

This is one version of the reverse chain rule – a massively underrated tool for Extension 2 integration.

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

or use this version in terms of $f(x)$:
$$\int f'(x)\,a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

We will use the Advanced standard integrals throughout the chapter.

**Extension 1**

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{a}{a^2 + x^2}dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

Careful – in this version of the standard integral we need '$a$' in the numerator and there is no $\frac{1}{a}$ on the RHS.

or use these versions in terms of $f(x)$:
$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$
$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

Careful – in this version of the standard integral we don't have '$a$' in the numerator, so we need $\frac{1}{a}$ on the RHS.

When we use integrals whose primitive is the form $\tan^{-1}\left[\frac{f(x)}{a}\right]$ we will use the version shown in the box to the right as it is more useful, not the version from the reference sheet.

We will use the Extension 1 standard integrals later this lesson when we complete the square in the denominator of the integrand.

**Advanced**

$$\int f'(x)\,[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c \quad \text{where } n \neq -1$$

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$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

or use this version in terms of $f(x)$:

$$\int f'(x)\,a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

We will use the Advanced standard integrals throughout the chapter.

**Extension 1**

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{a}{a^2 + x^2}dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

Careful – in this version of the standard integral we need '$a$' in the numerator and there is no $\frac{1}{a}$ on the RHS.

or use these versions in terms of $f(x)$:

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

Careful – in this version of the standard integral we don't have '$a$' in the numerator, so we need $\frac{1}{a}$ on the RHS.

When we use integrals whose primitive is the form $\tan^{-1}\left[\frac{f(x)}{a}\right]$ we will use the version shown in the box to the right as it is more useful, not the version from the reference sheet.

We will use the Extension 1 standard integrals later this lesson when we complete the square in the denominator of the integrand.

There are other standard integrals that you might see in other resources, but they are NOT required in the HSC.

$$\int f'(x)\operatorname{cosec}^2 f(x)\ dx = -\cot f(x) + c$$

$$\int f'(x)\sec f(x)\tan f(x)\,dx = \sec f(x) + c$$

$$\int f'(x)\operatorname{cosec} f(x)\cot f(x)\,dx = -\operatorname{cosec} f(x) + c$$

$$\int f'(x)\sec f(x)\ dx = \ln\left|\sec f(x) + \tan f(x)\right| + c$$

$$\int f'(x)\operatorname{cosec} f(x)\,dx = -\ln\left|\operatorname{cosec} f(x) + \cot f(x)\right| + c$$

These are lesser used trig standard integrals. See the mnemonics at the end of the chapter.

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}\,dx = \ln\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + c$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}\,dx = \ln\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + c$$

These standard integrals come from a much earlier syllabus, and you shouldn't see them again.

You can safely ignore the seven standard integrals on this page for the HSC, although we will use them in some exercises in this textbook to increase our understanding.

We have just seen a lot of differentiation and integration rules for trig functions to remember – see the summary at the end of this chapter for some mnemonics to help you remember them.

## Completing the square on a quadratic denominator

Some integrands need to be manipulated into a different form before we can use the standard integrals. We will start by looking at some requiring completing the square in the denominator so that we can use the Extension 1 standard integrals. These are common HSC questions.

## Example 1

Find $\int \frac{dx}{x^2+10x+29}$.

$$= \int \frac{dx}{x^2+10x+25+4}$$

$$= \int \frac{dx}{(x+5)^2+2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+5}{2}\right)+c$$

Only perform one step of working on each line – it is very easy to make a simple mistake when completing the square within an integral.

When we use $\int \frac{f'(x)}{a^2+[f(x)]^2}dx$ $= \frac{1}{a}\tan^{-1}\frac{f(x)}{a}+c$, don't forget $\frac{1}{a}$ on the RHS – another common mistake.

## Example 2

Find $\int_{-3}^{2} \frac{1}{\sqrt{16-6x-x^2}}\,dx.$

$= \int_{-3}^{2} \frac{1}{\sqrt{-(x^2+6x-16)}}\,dx$

One way to avoid making a minus mistake is to take out a factor of minus one when the leading term is negative.

$= \int_{-3}^{2} \frac{1}{\sqrt{-(x^2+6x+9-25)}}\,dx$

$= \int_{-3}^{2} \frac{1}{\sqrt{5^2-(x+3)^2}}\,dx$

$= \left[\sin^{-1}\left(\frac{x+3}{5}\right)\right]_{-3}^{2}$

When we use $\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$, we do not need $\frac{1}{a}$ on the RHS.

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

Now let's have a look at some integrands where we need to complete the square using a function of $x$.

## Example 3

Find $\int_{0}^{\ln 4} \frac{3e^x}{e^{2x}-2e^x+10}\,dx.$

$= \int_{0}^{\ln 4} \frac{3e^x}{(e^x-1)^2+3^2}\,dx$

$= \left[\tan^{-1}\left(\frac{e^x-1}{3}\right)\right]_{0}^{\ln 4}$

Here we have $\int \frac{a\times f'(x)}{a^2+[f(x)]^2}\,dx$, so we do not need $\frac{1}{a}$ as $a$ was included in the numerator.

$= \tan^{-1}\left(\frac{4-1}{3}\right) - \tan^{-1}\left(\frac{1-1}{3}\right)$

$= \tan^{-1} 1 - \tan^{-1} 0$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$

## Example 4

Find $\int \frac{\cos 2x}{\sqrt{8 - 2\sin 2x - \sin^2 2x}}\,dx.$

$= \int \frac{\cos 2x}{\sqrt{-(\sin^2 2x + 2\sin 2x - 8)}}\,dx$

$= \int \frac{\cos 2x}{\sqrt{9 - (\sin^2 2x + 2\sin 2x + 1)}}\,dx$

$= \frac{1}{2}\int \frac{2\cos 2x}{\sqrt{3^2 - [\sin 2x + 1]^2}}\,dx$

$= \frac{1}{2}\sin^{-1}\left(\frac{\sin 2x + 1}{3}\right) + c$

Here we use $\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c.$

## Changing an integrand into an appropriate form using algebraic manipulation

Many integrands need to be manipulated into a different form before we can use out standard integrals. In the next example we start by rearranging the integrand using two successive trigonometric identities, then simplify before using standard integrals.

## Example 5

Find $\int \frac{1 + \sin 2x}{\cos x + \sin x}\,dx.$

$= \int \frac{1 + 2\sin x \cos x}{\sin x + \cos x}\,dx$

$= \int \frac{\sin^2 x + 2\sin x \cos x + \cos^2 x}{\sin x + \cos x}\,dx$

$= \int \frac{(\sin x + \cos x)^2}{\sin x + \cos x}\,dx$

$= \int (\sin x + \cos x)\,dx$

$= -\cos x + \sin x + c$

When we have different angles ($x$ and $2x$) our first step is usually to convert the trig ratios with the higher argument (angle) to the smaller argument.

At the moment we are at a dead end, but if we use a Pythagorean identity, we can simplify the integrand.

When we have a fraction with powers of $e$ in the numerator and denominator, we will generally try to rearrange the integrand so that the numerator is the derivative of the denominator. To do this, we must have any powers of $e$ in the numerator with the same powers as those in the denominator. The denominator can also have a constant term which disappears with differentiation. If one term in the denominator is the odd–one–out, then we need to divide the numerator and denominator by that power of $e$ so that that term becomes a constant. In the next example $e^x$ is the odd one out, so we start by dividing the numerator and denominator by $e^x$.

## Example 6

Find $\int \frac{e^{2x}}{e^{2x}-3e^x}\,dx$.

$$= \int \frac{e^x}{e^x-3}\,dx$$

$$= \ln\left|e^x-3\right| + c$$

## Basic

**1** Evaluate $\int_{-7}^{1} \frac{dx}{x^2+6x+25}$.

**2** Evaluate $\int_{0}^{\frac{3\sqrt{3}}{4}} \frac{1}{\sqrt{9-4x^2}}\,dx$.

**3** Find $\int \frac{1}{\sqrt{20-8x-x^2}}\,dx$.

**4** Find $\int \frac{dx}{12x^2+12x+78}$.

**5** Find $\int \frac{x^2+2x}{x^2+2x+5}\,dx$.

**6** Find $\int \frac{e^{-x}}{e^{-x}-1}\,dx$.

## Medium

**7** Find $\int \frac{1}{x^2-3x+3}\,dx$.

**8** Evaluate $\int_{-1}^{1} \frac{1}{5-2t+t^2}\,dt$.

**9** Find $\int \frac{2}{4x^2-4x+17}\,dx$.

**10** Find $\int \frac{2x}{x^4+2x^2+5}\,dx$.

**11** Find $\int \frac{2}{e^x+1}\,dx$.

**12** Find $-\int \frac{2x}{\sqrt{3-2x^2-x^4}}\,dx$.

**13** Find $\int \frac{2\cos x\,(\sin x+1)}{(\sin x+3)(\sin x-1)}\,dx$.

**14** Find $\int \frac{2}{x\ln 7}\,dx$.

**15** Find $\int \frac{dx}{(x^2+4)\tan^{-1}\left(\frac{x}{2}\right)}$.

**16** It is given that $\frac{dy}{dx}=\frac{\cos x}{1+\sin^2 x}$ and $y=1$ when $x=\frac{\pi}{2}$. Find $y$ as a function of $x$.

## Challenging

**17** Find $\int_{-\frac{\pi}{2}}^{0} \dfrac{\cos\theta}{\sin^2\theta + 2\sin\theta + 4}\,d\theta$.

**18** Find $\int \dfrac{e^{3x}}{\sqrt{7 + 12e^{3x} - 4e^{6x}}}\,dx$.

**19** Find $\int \dfrac{\cos x - \sin x}{2 + \sin 2x}\,dx$.

**20** Find $\int \dfrac{\sin^3 x}{\cos x - 1}\,dx$.

**21** Show that $\int \dfrac{e^{\log_2 x}}{x}\,dx = \ln 2\, x^{\frac{1}{\ln 2}} + c$.

**22** Find $\int \dfrac{\sin 2x + \cos 2x}{\cos x}\,dx$.

**23** Find $\int \dfrac{dx}{\sec x - 1}$.

**24** Find $\int (2x\,\operatorname{cosec} x - x^2 \cot x \operatorname{cosec} x)\,dx$.

**25** Given $\alpha, \beta$ are the roots of $ax^2 + bx + c = 0$, prove that:

$$\int \frac{\cos x\,(2a\sin x + b)}{(\sin x - \alpha)(\sin x - \beta)}\,dx = a\ln\left|a\sin^2 x + b\sin x + c\right| + k,$$

where $k$ is a constant.

# 4.2 The reverse chain rule & $u$ substitutions

In Lesson 2 we cover:

- The reverse chain rule, $u$ substitutions and $u$ squared substitutions

This lesson mainly covers the 6th dot point from *Further integration*, with questions on the 11th and 15th dot points included in the exercises.

## The reverse chain rule, $u$ substitutions and $u$ squared substitutions

Many integration questions in Extension 1 involve $u$ substitutions, and in Extension 1 you must be given the substitution to use. You may also be given the choice of whether you can use an alternative method such as the reverse chain rule. It depends whether the examiners are wanting to specifically check if you know how to solve integrals using substitution.

In Extension 2 we could get exactly the same integral, with either no guidance as to how we should solve it or being directed to 'use a suitable substitution' without being told what it should be. We either need to work out which substitution to use, or if the integrand is in the correct form, we can use the reverse chain rule if the wording of the question allows. If you learn to recognise integrands which are suitable for the reverse chain rule you can save yourself lots of effort and time, which benefit you later in the exam.

The reverse chain rule is generally much quicker than using substitution, involving less algebra and with less room for simple algebraic errors. It is a vastly underused technique in Extension 2.

One version of the reverse chain rule is the first standard integral on the Advanced page of the reference sheet, where we have a function of $x$ raised to a power other than $-1$.

$$\int f'(x)\,[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c \quad \text{where } n \neq -1$$

A more generalised form of the reverse chain rule is

$$\int f'(x)\,g\big(f(x)\big)dx = G\big(f(x)\big) + c$$

where $G(x)$ is the primitive function of $g(x)$.

Note that an integrand can be solved using the reverse chain rule if it involves the product of a composite function [a function of a function such as $g(f(x))$] and a multiple of the derivative of the inner function [$f'(x)$]. We may need to multiply the integral by a fudge factor to get it in the correct form, or rearrange the integrand, for example by rewriting it index form or using reciprocal trig ratios.

As well as being a much more efficient alternative to most integrals that would otherwise involve substitution, mastering the reverse chain rule also makes integration by parts and recurrence relations much easier when we get to them later in the chapter.

To help understand the reverse chain rule, consider the integral $\int x(2x^2+3)^3\,dx$.

Before we solve this integral using the reverse chain rule, let's differentiate the composite function $(2x^2+3)^4$, where we have the inner function of $2x^2+3$ raised to the power of $4$.

$$\frac{d}{dx}\left[(2x^2+3)^4\right] = 4(2x^2+3)^3(4x) = 16x(2x^2+3)^3$$

So, when we integrate $16x(2x^2+3)^3$ we will get $(2x^2+3)^4+c$, or more importantly for our use of the reverse chain rule, when we integrate $4x(2x^2+3)^3$ we will get $\frac{(2x^2+3)^4}{4}+c$.

- In this second form we have the product of the composite function, $(2x^2+3)^3$, and the derivative of its inner function, $4x$.
- To find the primitive of $4x(2x^2+3)^3$, we leave the inner function alone and take the primitive of the outer function – in this case the primitive of cubing a function is to increase the power by $1$ to become $4$ then divide by $4$.

The solution to the integral at the top of the page is shown below.

$$\int x(2x^2+3)^3\,dx$$

$$= \frac{1}{4}\int 4x(2x^2+3)^3\,dx$$

The derivative of $2x^2+3$ is $4x$, so change $x$ to $4x$, and place a fudge factor of $\frac{1}{4}$ at the front so the first two lines are equal.

$$= \frac{1}{4}\times\frac{(2x^2+3)^4}{4}+c$$

Leave the fudge factor at the front, take the primitive of the outer function, leaving the inner function alone.

$$= \frac{(2x^2+3)^4}{16}+c$$

Simplify the primitive.

We will now look at integrals that can be solved using either the reverse chain rule or by a suitable substitution – we will solve each question using both methods.

For most substitutions we will let $u$ equal the inner function of the composite function, then differentiate both sides to find an expression for $du$.

When the composite function is the square root of the inner function then we let $u^2$ equal the inner function (the radicand), as the algebra is easier because we work with integral powers rather than fractional powers

When we differentiate both sides, the LHS becomes $2u\,du$, which is an application of the chain rule.

Again, a reminder that you should master both methods – the reverse chain rule is more efficient in the questions where the integrand is in the correct form, but you may be required to solve the integral using a substitution, plus we will see integrals next lesson which can only be solved using substitution.

It is helpful to think of the functions we use in *Further integration* in five groups: Exponential functions, Trigonometric functions, Algebraic functions, Inverse trigonometric functions, and Logarithmic functions. We will use a mix of the five groups as the inner and outer functions.

In the examples below we solve the question using the reverse chain rule on the left and using a $u$ or $u^2$ substitution on the right of the dividing line.

## Example 1

Find $\int_0^1 x^3(x^4-2)^5\,dx$.

$$\int_0^1 x^3(x^4-2)^5\,dx$$

$$=\frac{1}{4}\int_0^1 4x^3(x^4-2)^5\,dx$$

$$=\frac{1}{4}\left[\frac{(x^4-2)^6}{6}\right]_0^1$$

$$=\frac{1}{4}\left(\frac{1}{6}-\left(\frac{64}{6}\right)\right)$$

$$=-\frac{63}{24}$$

$$\int_0^1 x^3(x^4-2)^5\,dx \qquad \boxed{\begin{aligned} u &= x^4-2 \\ du &= 4x^3dx \end{aligned}}$$

$$=\frac{1}{4}\int_0^1 (x^4-2)^5\times 4x^3dx$$

$$=\frac{1}{4}\int_{-2}^{-1} u^5\,du$$

Don't forget to change the limits when the variable changes

$$=\frac{1}{4}\left[\frac{u^6}{6}\right]_{-2}^{-1}$$

$$=\frac{1}{24}(1-64)$$

$$=-\frac{63}{24}$$

Note that the solution using the reverse chain rule required only four lines, while the solution using the $u$ substitution was seven lines long – two lines to find an equation for $du$ and $dx$, plus another five lines for the solution.

Common mistakes when using the substitution method for definite integrals include forgetting to change the limits so that they are in terms of $u$, or just replacing $dx$ with $du$, rather than differentiating the equation linking $u$ and $x$ to find the equation linking $du$ and $dx$.

We can see that using the reverse chain rule has many advantages, so always use it when the question allows.

## Example 2

Find $\int_1^2 \frac{x}{\sqrt{2x^2-1}}\,dx$.

$$\int_1^2 \frac{x}{\sqrt{2x^2-1}}\,dx$$

$$= \frac{1}{4}\int_1^2 4x\left(2x^2-1\right)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{4}\times 2\left[\left(2x^2-1\right)^{\frac{1}{2}}\right]_1^2$$

$$= \frac{1}{2}\left(\sqrt{7}-1\right)$$

$$\int_1^2 \frac{x}{\sqrt{2x^2-1}}\,dx$$

$$u^2 = 2x^2-1$$
$$2u\,du = 4x\,dx$$

$$= \frac{1}{4}\int_1^2 \frac{1}{\sqrt{2x^2-1}}\times 4x\,dx$$

$$= \frac{1}{4}\int_1^{\sqrt{7}} \frac{1}{u}\times 2u\,du$$

$$= \frac{1}{2}\int_1^{\sqrt{7}} du$$

$$= \frac{1}{2}\left[u\right]_1^{\sqrt{7}}$$

$$= \frac{1}{2}\left(\sqrt{7}-1\right)$$

## Example 3

Find $\int e^{3x}\sqrt{e^{3x}+1}\,dx$.

$$\int e^{3x}\sqrt{e^{3x}+1}\,dx$$

$$= \frac{1}{3}\int 3e^{3x}\left(e^{3x}+1\right)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\times\frac{2}{3}\left(e^{3x}+1\right)^{\frac{3}{2}}+c$$

$$= \frac{2\sqrt{\left(e^{3x}+1\right)^3}}{9}+c$$

$$\int e^{3x}\sqrt{e^{3x}+1}\,dx$$

$$u^2 = e^{3x}+1$$
$$2u\,du = 3e^{3x}dx$$

$$= \frac{1}{3}\int \sqrt{e^{3x}+1}\times 3e^{3x}dx$$

$$= \frac{1}{3}\int u\,2u\,du$$

$$= \frac{2}{3}\int u^2\,du$$

$$= \frac{2}{3}\times\frac{u^3}{3}+c$$

$$= \frac{2\sqrt{\left(e^{3x}+1\right)^3}}{9}+c$$

## Example 4

Find $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$.

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$$

$$= \frac{1}{2}\int_0^{\sqrt{\pi}} 2x\sin(x^2)\,dx$$

$$= -\frac{1}{2}\Big[\cos x^2\Big]_0^{\sqrt{\pi}}$$

$$= -\frac{1}{2}\Big(\cos\pi - \cos 0\Big)$$

$$= -\frac{1}{2}(-1-1)$$

$$= 1$$

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx \qquad \boxed{\begin{array}{c} u = x^2 \\ du = 2x\,dx \end{array}}$$

$$= \frac{1}{2}\int_0^{\sqrt{\pi}} \sin(x^2)\,2x\,dx$$

$$= \frac{1}{2}\int_0^{\pi} \sin u\,du$$

$$= -\frac{1}{2}\Big[\cos u\Big]_0^{\pi}$$

$$= -\frac{1}{2}\Big(\cos\pi - \cos 0\Big)$$

$$= -\frac{1}{2}(-1-1)$$

$$= 1$$

## Example 5

Find $\int \sqrt{\cos 2x}\sin 2x\,dx$.

$$\int \sqrt{\cos 2x}\sin 2x\,dx$$

$$= -\frac{1}{2}\int (-2\sin 2x)(\cos 2x)^{\frac{1}{2}}\,dx$$

$$= -\frac{1}{2}\times\frac{2}{3}(\cos 2x)^{\frac{3}{2}} + c$$

$$= -\frac{\sqrt{\cos^3 2x}}{3} + c$$

$$\int \sqrt{\cos 2x}\sin 2x\,dx \qquad \boxed{\begin{array}{c} u^2 = \cos 2x \\ 2u\,du = -2\sin 2x\,dx \end{array}}$$

$$= -\frac{1}{2}\int \sqrt{\cos 2x}\times(-2\sin 2x\,dx)$$

$$= -\frac{1}{2}\int u\times 2u\,du$$

$$= -\int u^2\,du$$

$$= -\frac{u^3}{3} + c$$

$$= -\frac{\sqrt{\cos^3 2x}}{3} + c$$

## Example 6

Find $\int e^x \sin(e^x)\,dx$.

$$\int e^x \sin(e^x)\,dx$$
$$= -\cos(e^x) + c$$

$$\int e^x \sin(e^x)\,dx \qquad \boxed{\begin{array}{c} u = e^x \\ du = e^x dx \end{array}}$$
$$= \int \sin(e^x) \times e^x dx$$
$$= \int \sin u\,du$$
$$= -\cos u + c$$
$$= -\cos e^x + c$$

## Example 7

Find $\int \dfrac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}\,dx$.

$$\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{1-x^2}} \times e^{\sin^{-1} x}\,dx$$
$$= e^{\sin^{-1} x} + c$$

$$\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}\,dx \qquad \boxed{\begin{array}{c} u = \sin^{-1} x \\ du = \dfrac{dx}{\sqrt{1-x^2}} \end{array}}$$
$$= \int e^{\sin^{-1} x} \times \frac{dx}{\sqrt{1-x^2}}$$
$$= \int e^u\,du$$
$$= e^u + c$$
$$= e^{\sin^{-1} x} + c$$

## Basic

**1** Find $\int x^2(x^3+4)^5\,dx$.

**2** Evaluate $\int_1^2 \frac{x^2}{\sqrt{2x^3-1}}\,dx$.

**3** Find $\int e^{2x}\sqrt{e^{2x}-1}\,dx$.

**4** Find $\int x\cos x^2\,dx$.

**5** Evaluate $\int_0^{\frac{\pi}{4}} \tan^3 x\sec^2 x\,dx$.

**6** Find $\int \frac{\cos x}{\sin^5 x}\,dx$.

## Medium

**7** Find $\int \sin^{\frac{3}{2}} 2x\cos 2x\,dx$.

**8** Find $\int \sqrt{\sin 2x}\cos 2x\,dx$.

**9** Find $\int e^x\cos e^x\,dx$.

**10** Find $\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx$.

**11** Evaluate $\int_1^{e^{\frac{\pi}{2}}} \frac{\cos(\ln x)}{x}\,dx$

**12** Find $\int (3x^2+2x)\sqrt{x^3+x^2}\,dx$.

**13** Find $\int x^2e^{x^3}\,dx$.

**14** Find $\int \frac{\sin(\tan^{-1}x)}{1+x^2}\,dx$.

**15** Find $\int (e^{t^2}+16)te^{t^2}\,dt$.

**16** Find $\int \frac{\operatorname{cosec} x\cot x}{1+\operatorname{cosec}^2 x}\,dx$.

**17** Find $\int e^xe^{e^x}\,dx$.

**18** Find $\int \frac{\ln^3 x}{x}\,dx$.

**19** Find $\int \sqrt{\dfrac{2+x}{2-x}}\,dx$.

**20** Find $\int \dfrac{\sin(\tan\theta)}{\cos^2\theta}\,d\theta$.

## Challenging

**21** Find $\int \dfrac{\cos^3 x}{\sqrt{\sin x}}\,dx$

**22** Find $\int \dfrac{x\sin\left(\sqrt{2x^2-1}\right)}{\sqrt{2x^2-1}}\,dx$.

**23** Find $\int \dfrac{e^{\sqrt{\sin x}}}{\sec x\sqrt{\sin x}}\,dx$.

**24** Find $\int \dfrac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$.

**25** Find $\int_1^{e^{\frac{\pi}{2}}} \dfrac{\sin(\ln|x|)}{x}\,dx$.

**26** Evaluate $\int_{\sqrt{2}}^{\sqrt{3}} \dfrac{2e^{\frac{2}{x^2}}}{x^3}\,dx$.

**27** Find $\int \dfrac{xe^{\sqrt{2x^2-1}}\sin\left(e^{\sqrt{2x^2-1}}\right)}{\sqrt{2x^2-1}}\,dx$.

**28** Indefinite integrals with trigonometric integrands can have many correct solutions. Prove that $\int \sin^3 2x\cos 2x\,dx$ is equal to each of the following expressions.

**a** $\dfrac{\sin^4 2x}{8}+c$

**b** $\dfrac{\cos^4 2x}{8}-\dfrac{\cos 4x}{8}+c$

**c** $\dfrac{\cos^4 2x}{8}+\dfrac{\sin^2 2x}{4}+c$

**d** $\dfrac{\cos^4 2x}{8}-\dfrac{\cos^2 2x}{4}+c$

**e** $-\dfrac{\cos 4x}{16}+\dfrac{\cos 8x}{64}+c$

**f** $-\dfrac{\cos 4x}{16}-\dfrac{\sin^2 4x}{16}+c$

# 4.3 Other $u$ substitutions

In Lesson 3 we cover:

- Other $u$ substitutions and $u$ squared substitutions

This lesson mainly covers the 6th dot point from *Further integration*, with questions on the 11th, 15th and 16th dot points included in the exercises.

## Other $u$ and $u^2$ substitutions

We now look at integrands which are not suitable for the reverse chain rule, so that we must use a $u$ or $u^2$ substitution. Again, the substitution may or may not be given.

In general, simpler integrands are easier to integrate, so we aim to choose a substitution that makes the integrand simpler. This will generally occur when we let $u$ or $u^2$ equal the inner function of the composite function, though in more complicated examples this will not be the case. The only way to determine the easiest method for unusual examples is lots of practice.

Later in the chapter we will look at trig substitutions, which are only needed in a small number of cases and often involve more lines of working.

In the next example we cannot use the reverse chain rule as we have $x^3$, rather than $x$, in front of the square root.

## Example 1

Find $\int x^3\sqrt{x^2-1}\,dx$.

$$\int x^3\sqrt{x^2-1}\,dx$$

$$\begin{aligned} u^2 &= x^2-1 \\ 2u\,du &= 2x\,dx \\ x^2 &= u^2+1 \end{aligned}$$

The outer function of the composite function is a square root, so we let $u^2$ equal the inner function. Making $x^2$ the subject is optional but helps simplify the integrand.

$$= \frac{1}{2}\int x^2\sqrt{x^2-1}\times 2x dx$$

$$= \frac{1}{2}\int (u^2+1)u \times 2u\,du$$

$$= \int (u^4+u^2)\,du$$

I always write brackets here, but the integral sign and $du$ act as grouping symbols, so the brackets are not needed.

$$= \frac{u^5}{5}+\frac{u^3}{3}+c$$

$$= \frac{(x^2-1)^{\frac{5}{2}}}{5}+\frac{(x^2-1)^{\frac{3}{2}}}{3}$$

I normally skip this step but notice how the denominator of the powers tell us to take the square roots in the next step, while the numerators tell us the power required inside the square root.

$$= \frac{\sqrt{(x^2-1)^5}}{5}+\frac{\sqrt{(x^2-1)^3}}{3}+c$$

## Example 2

Evaluate $\int_1^3 \frac{\sqrt{x}}{1+x}\,dx$.

$\int_1^3 \frac{\sqrt{x}}{1+x}\,dx$ $\qquad \boxed{\begin{array}{c} u^2 = x \\ 2u\,du = dx \end{array}}$

$= \int_1^{\sqrt{3}} \frac{u}{1+u^2} \times 2u\,du$

$= 2\int_1^{\sqrt{3}} \frac{u^2}{1+u^2}\,du$

$= 2\int_1^{\sqrt{3}} \frac{1+u^2-1}{1+u^2}\,du$

$= 2\int_1^{\sqrt{3}} \left(1 - \frac{1}{1+u^2}\right)du$

$= 2\left[u - \tan^{-1} u\right]_1^{\sqrt{3}}$

$= 2\left(\left(\sqrt{3} - \frac{\pi}{3}\right) - \left(1 - \frac{\pi}{4}\right)\right)$

$= 2\left(\sqrt{3} - \frac{\pi}{3} - 1 + \frac{\pi}{4}\right)$

$= 2\left(\sqrt{3} - 1 - \frac{\pi}{12}\right)$

$= 2\sqrt{3} - 2 - \frac{\pi}{6}$

## Example 3

Find $\int \frac{x^7}{\sqrt{1-x^4}}dx$.

$$\int \frac{x^7}{\sqrt{1-x^4}}dx$$

$$\boxed{\begin{array}{c} u^2 = 1-x^4 \\ 2u\,du = -4x^3dx \end{array}}$$

$$= -\frac{1}{4}\int \frac{x^4}{\sqrt{1-x^4}} \times (-4x^3)dx$$

$$= -\frac{1}{4}\int \frac{1-u^2}{u} \times 2u\,du$$

$$= -\frac{1}{2}\int (1-u^2)\,du$$

$$= -\frac{1}{2}\left(u - \frac{u^3}{3}\right) + c$$

$$= \frac{\sqrt{(1-x^4)^3}}{6} - \frac{\sqrt{1-x^4}}{2} + c$$

The next example is a classic integral. It is a great question to use in class, as we can solve it in so many ways. It can be a very poor exam question without some guidance, as many students would waste time trying a variety of substitutions that appear to lead to dead ends, or use a variety of other substitutions which could soak up lots of time getting to the answer. The final answer varies depending on the method used, although the answers are equivalent. If we graph them in GeoGebra or similar we can see that they only differ by the value of the constant.

Some relatively simple solutions involve letting $u^2 = 1 + x$ or $u^2 = 1 - x$, which still soak up lots of time. Later in the chapter we will solve this same integral using the substitution $u^2 = x + 1$ with a trig substitution, and separately just with a trig substitution, which is not as bad as the $u^2$ substitutions mentioned.

We do have a much simpler solution that does not involve substitutions but rather involves rearranging the integrand so that we can use standard integrals. Since the integrand involves the square root of a rational function, we can rationalise the numerator (not the denominator), allowing us to split the fraction in two, leading to two standard integrals.

## Example 4

Find $\int \sqrt{\dfrac{1-x}{1+x}}\,dx$.

$$= \int \sqrt{\frac{1-x}{1+x}} \times \sqrt{\frac{1-x}{1-x}}\,dx$$

← We will end up with the square root of a perfect square in the numerator.

$$= \int \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{dx}{\sqrt{1-x^2}}\,dx + \frac{1}{2}\int (-2x)\left(1-x^2\right)^{-\frac{1}{2}}\,dx$$

← We can now split the integral and deal with each part separately.

$$= \sin^{-1} x + \frac{1}{2} \times 2\left(1-x^2\right)^{\frac{1}{2}} + c$$

$$= \sin^{-1} x + \sqrt{1-x^2} + c$$

Some substitutions are much less obvious – the only way to be able to solve questions like these is lots of practice! This integral looks very similar to the last example, but we do need to start with a simple substitution. Since we are not replacing the whole radicand in the substitution, we use $u$ rather than $u^2$.

## Example 5

Find $\int x\sqrt{\dfrac{1-x^2}{1+x^2}}\,dx$.

$$\int x\sqrt{\frac{1-x^2}{1+x^2}}\,dx \qquad \boxed{\begin{array}{c} u = x^2 \\ du = 2x\,dx \end{array}}$$

$$= \frac{1}{2}\int \sqrt{\frac{1-x^2}{1+x^2}} \times 2x\,dx$$

$$= \frac{1}{2}\int \sqrt{\frac{1-u}{1+u}}\,du$$

$$= \frac{1}{2}\int \sqrt{\frac{1-u}{1+u}} \times \sqrt{\frac{1-u}{1-u}}\,du$$

$$= \frac{1}{2}\int \frac{1-u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du - \frac{1}{2}\int \frac{u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du + \frac{1}{4}\int (-2u)\left(1-u^2\right)^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\sin^{-1}u + \frac{1}{4} \times 2\left(1-u^2\right)^{\frac{1}{2}} + c$$

$$= \frac{1}{2}\sin^{-1}u + \frac{1}{2}\sqrt{1-u^2} + c$$

$$= \frac{\sin^{-1}x^2}{2} + \frac{\sqrt{1-x^4}}{2} + c$$

## Basic

**1** Find $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$.

**2** Find $\int x^3\sqrt{x^2+1}\,dx$.

**3** Find $\int x^5\sqrt{x^2+1}\,dx$.

**4** Evaluate $\int_0^1 x^3\sqrt{1-x^2}\,dx$.

**5** Evaluate $\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\,dx$.

**6** Evaluate $\int_0^3 x\sqrt{x+1}\,dx$.

## Medium

**7** Find $\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$.

**8** Evaluate $\int \frac{1}{2+\sqrt{x}}\,dx$.

**9** Find $\int x^5\sqrt{2-x^3}\,dx$.

**10** Find $\int \frac{2x^3}{(x^2-1)^2}\,dx$.

**11** Find $\int \frac{x^3}{(x^2+1)^2}\,dx$.

**12** Find $\int \frac{e^{2x}}{e^x+1}\,dx$.

**13** Find the area under the curve $y=\frac{1}{x\sqrt{2x-1}}$ from $x=1$ to $x=2$.

## Challenging

**14** Show that $\int \frac{x}{\sqrt{x+5}}\,dx = \frac{2}{3}(x-10)\sqrt{x+5}+c$

**15** The velocity of a particle, $t$ seconds after it starts moving along the $x-$axis, is given by $\dot{x}=t\sqrt{\frac{1+2t^2}{1-2t^2}}$ metres per second. Determine the distance of the particle from its initial position after $0.5$ seconds. Answer correct to the nearest millimetre.

**16** Evaluate $\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\,dx$.

**17** Show that $\int \frac{\sqrt{x}}{1-\sqrt{x}}dx = -x - 2\sqrt{x} - 2\ln\left|1-\sqrt{x}\right| + c$ using the following substitutions.

**a** $u^2 = x$

**b** $u = 1 - \sqrt{x}$

**18** Find $\int \frac{dx}{\sqrt{e^{2x}-1}}$ by letting $u = \frac{1}{e^x}$.

**19** Using the substitution $u = -x$, or otherwise, evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}dx$.

**20** Use integration to prove $\int \sqrt{\frac{x}{x+1}}dx = \sqrt{x^2+x} - \ln(\sqrt{x}+\sqrt{x+1}) + c$.

You may assume $\int \frac{f'(x)}{\sqrt{[f(x)]^2-a^2}}dx = \ln\left|f(x)+\sqrt{[f(x)]^2-a^2}\right|$.

# 4.4 Splitting the numerator

In Lesson 4 we cover:

- Splitting the numerator
- Partial fractions by inspection

This lesson mainly covers the 7th, 8th and 10th dot points from *Further integration*, with questions on the 9th, 11th, 15th and 16th dot points included in the exercises.

## Splitting the numerator

Rational functions are fractions where the numerator and denominator are both polynomials. Examples include:

$$\frac{x^2+2x+3}{x-1} \qquad \frac{x+2}{x+1} \qquad \frac{4x}{x^2+4x-6}$$

Splitting the numerator is a method of rewriting the numerator as multiples of either the denominator or the derivative of the denominator, so that we can then use standard integrals.

You may have used this method for dividing polynomials in Year 11, where it is a much better alternative to long division. We can also use this method for functions that would become rational functions after making a $u$ substitution.

The degree of the numerator and the degree of the denominator determine whether we rearrange the numerator as multiples of the denominator, or multiples of the derivative of the denominator.

In all cases we need to end up with a rational function where the degree of the numerator is less than the degree of the denominator before we can use any of the standard integrals.

**Case 1: The degree of the numerator is greater than or equal to that of the denominator**

If the numerator has degree that is greater than or equal to the degree of the denominator, then we rewrite the numerator as multiples of the denominator, leaving us with the sum of a polynomial and a rational fraction.

## Example 1

Find $\int \frac{x^2+2x+3}{x-1}dx$

$$= \int \frac{x(x-1)+3(x-1)+6}{x-1}dx$$

$$= \int \left(x+3+\frac{6}{x-1}\right)dx$$

$$= \frac{x^2}{2}+3x+6\ln\left|x-1\right|+c$$

Don't forget the absolute value signs – if the argument inside the absolute values could take a negative value, then you will lose marks in the HSC if it is missing.

## Example 2

Evaluate $\int_{-3}^{-2} \frac{x+2}{x+1}dx.$

The limits have been chosen here so that you cannot solve this definite integral if you don't use absolute values signs correctly.

$$= \int_{-3}^{-2} \frac{x+1+1}{x+1}dx$$

$$= \int_{-3}^{-2} \left(1+\frac{1}{x+1}\right)dx$$

$$= \Big[x+\ln\left|x+1\right|\Big]_{-3}^{-2}$$

Without the absolute values signs we could not evaluate the primitive functions here.

$$= \left(-2+\ln\left|-1\right|\right)-\left(-3+\ln\left|-2\right|\right)$$

$$= -2+0+3-\ln 2$$

$$= 1-\ln 2$$

## Example 3

Find $\int_1^{\sqrt{3}} \frac{x^2+5}{x^2+1}dx.$

$$= \int_1^{\sqrt{3}} \frac{x^2+1+4}{x^2+1}dx$$

$$= \int_1^{\sqrt{3}} \left(1+\frac{4}{x^2+1}\right)dx$$

$$= \Big[x+4\tan^{-1}x\Big]_1^{\sqrt{3}}$$

$$= \left(\sqrt{3}+4\left(\frac{\pi}{3}\right)\right)-\left(1+4\left(\frac{\pi}{4}\right)\right)$$

$$= \sqrt{3}-1+\frac{7\pi}{3}$$

The same techniques work for functions that would be rational functions after a $u$ substitution.

## Example 4

Find $\int \frac{e^{2x}+e^{x}}{e^{2x}-e^{x}}\,dx$.

$$= \int \frac{e^{2x}-e^{x}+2e^{x}}{e^{2x}-e^{x}}\,dx$$

$$= \int \left(1+\frac{2e^{x}}{e^{2x}-e^{x}}\right)dx$$

$$= \int \left(1+\frac{2e^{-x}}{1-e^{-x}}\right)dx$$

$$= x+2\ln\left|1-e^{-x}\right|+c$$

**Case 2: The degree of the numerator is one less than that of the denominator**

If the degree of the numerator is one less than the degree of the denominator, then we rewrite the numerator as multiples of the derivative of the denominator, leaving us with the sum of a polynomial and a rational function. The primitive will include a logarithmic function.

## Example 5

Find $\int \frac{4x}{x^2+4x+13}\,dx$.

$$= \int \frac{2(2x+4)-8}{x^2+4x+4+9}\,dx$$

$$= 2\int \frac{2x+4}{x^2+4x+13}\,dx - 8\int \frac{1}{(x+2)^2+3^2}\,dx$$

$$= 2\ln\left|x^2+4x+13\right| - \frac{8}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c$$

Since $x^2+4x+13$ is always positive we would not lose marks in the HSC if we did not include absolute values.

## Partial fractions by inspection

Partial fractions involve breaking a rational function into the sum or difference of two or more simpler fractions - the partial fractions - which we then solve using standard integrals. It only works for rational functions whose denominator can be factorised - think about cross multiplying to realise why.

We can create partial fractions by inspection for easier examples. Next lesson we will look at three formal methods for harder examples but never forget that it can be easier to break up the rational function by inspection.

## Example 6

Find $\int \frac{3}{x(x+1)}\,dx$.

$$= 3\int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= 3\ln|x| - 3\ln|x+1| + c$$

$$= 3\ln\left|\frac{x}{x+1}\right| + c$$

We will try putting 1 over each factor, and subtracting the two fractions, which will get rid of the $x$ in the numerator when we cross multiply. We then need to place a fudge factor out the front, so that when it multiplies by the new fractions, we get an answer equivalent to the original question. In this case $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}$ so, we need a fudge factor of 3 out the front.

## Example 7

Show that $\int_2^3 \frac{1}{x(x^2+1)}\,dx = \ln\left(\frac{3\sqrt{2}}{4}\right)$.

$$\text{LHS} = \int_2^3 \left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$$

We will try putting 1 over each factor, and subtracting the two fractions, but cross multiplying we would be left with an $x^2$ we don't want and be subtracting an $x$. We can get around these problems if we make the second denominator $x$. Cross multiplying in our heads we see that no fudge factor is needed.

$$= \int_2^3 \left(\frac{1}{x} - \frac{1}{2} \times \frac{2x}{x^2+1}\right)dx$$

$$= \left[\ln|x| - \frac{1}{2}\ln|x^2+1|\right]_2^3$$

$$= \left(\ln 3 - \frac{1}{2}\ln 10\right) - \left(\ln 2 - \frac{1}{2}\ln 5\right)$$

$$= \ln 3 - \ln\sqrt{10} - \ln 2 + \ln\sqrt{5}$$

$$= \ln\left(\frac{3\sqrt{5}}{2\sqrt{10}}\right)$$

$$= \ln\left(\frac{3}{2\sqrt{2}}\right)$$

$$= \ln\left(\frac{3\sqrt{2}}{4}\right)$$

## Basic

**1** Find $\int \frac{x^2 - 2x - 2}{x + 1} dx$.

**2** Evaluate $\int_1^3 \frac{2x - 3}{x + 1} dx$.

**3** Find $\int \frac{2x^2 + 19}{x^2 + 9} dx$.

**4** Find $\int \frac{3x + 2}{x^2 + 6x + 10} dx$.

**5** Find $\int \frac{2}{x(x - 1)} dx$.

**6** Find $\int \frac{2}{x(x^2 + 4)} dx$.

## Medium

**7** Evaluate $\int_{-2}^2 \frac{2x}{x^2 + 4x + 20} dx$.

**8** Find $\int \frac{e^{2x} + e^x}{e^{2x} + 1} dx$.

**9** Find $\int \frac{x^4 + 1}{x^2 + 2} dx$.

**10** Show that $\int \frac{x^3}{x^2 + 2} dx = 2 - \ln 3$.

**11** Find $\int \frac{x^6 + 3x^2 - 1}{x^3 + 1} dx$.

**12** Find $\int \frac{x^2 - 4x + 2}{(x - 2)^3} dx$.

## Challenging

**13** Show that $\int_0^1 \frac{e^{2x} - e^x}{e^x + 1} dx = \ln\left(\frac{4e^e}{[e(e + 1)]^2}\right)$.

**14** The area under the curve $y = \frac{e^{2x}}{\sqrt{e^{2x} + 1}}$ from $x = 0$ to $x = \ln 2$ is rotated about the $x-$axis. What is the volume of the solid of revolution formed? Answer correct to 2 decimal places.

**15** Find $\int \frac{\cos x}{\sin^2 x - 3 \sin x + 2} dx$.

**16** Find $\int \frac{8x}{1 + e^2 x} dx$.

**17** Find $\int \frac{\sec x \tan x}{\sec x + \sec^2 x}\,dx$.

**18** Show that $\int \frac{\tan^2 x + \tan x + \sec^2 x - 2}{\tan x - 1}\,dx = \ln\left|\frac{\tan x - 1}{\cos x}\right| + 2x + c$.

**19** Let $I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$.

**a** Show that $I = \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$.

**b** Hence, find the value of $I$.

**20** Find $\int \frac{\sin x}{\sin x + \cos x + e^x}\,dx$.

In Lesson 5 we cover:

- Partial fractions using formal methods
  - Equating coefficients
  - Elimination by substitution
  - The cover-up method

This lesson mainly covers the 7th and 8th dot points from *Further integration*, with questions on the 16th dot point included in the exercises.

## Partial fractions using formal methods

Last lesson we looked at simple examples where we could split the rational function up by inspection, while in this lesson we look at formal methods needed for harder examples. Never forget that breaking up the rational function by inspection can be a valid alternative to the formal methods - make sure you show enough working though if the question is worth more than one mark. We will work through examples, with important notes beside each solution.

There are three methods we will investigate. All students should learn the first method, as it will work for all questions, even if a little more slowly than the other methods. More capable students should also learn the second and third methods as they each allow more efficient solutions to some types of questions. In exams you can choose whichever method you like.

Most exam questions will tell you the original integrand (a rational function) and the form of the partial fractions to use. Use the hints in the following examples so that you can answer those questions where the form for the partial fractions is not given.

### Equating coefficients

The first method works for all questions but takes longer than the other two methods. We start by cross-multiplying the partial fractions on the RHS to create a single fraction, then equating the coefficients of $x^2, x$ and the constant in the numerators. In more complicated questions the other two methods need to use this method to find the value of one or more variables.

## Example 1

**a** Express $\dfrac{2x-7}{(x+1)(x-2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x-2}$ by equating coefficients.

$$\frac{2x-7}{(x+1)(x-2)}=\frac{a}{x+1}+\frac{b}{x-2}$$

$$\frac{2x-7}{(x+1)(x-2)}=\frac{a(x-2)+b(x+1)}{(x+1)(x-2)}$$

Cross-multiply on the RHS.

$$2x-7=(a+b)x-2a+b$$

Equate the numerators and separate the RHS into multiples of $x$ and the constant term.

$$\therefore 2=a+b \qquad (1)$$

$$-7=-2a+b \qquad (2)$$

Equate the coefficients of $x$ and the constant terms to create simultaneous equations, then solve them to find $a$ and $b$.

$(1)-(2)\quad 9=3a \quad\rightarrow\quad a=3$

sub in $(1)\quad 2=3+b \quad\rightarrow\quad b=-1$

$$\therefore \frac{2x-7}{(x+1)(x-2)}=\frac{3}{x+1}-\frac{1}{x-2}$$

Rewrite the original fraction in the required form as asked in the question.

**b** Hence find $\displaystyle\int \frac{2x-7}{(x+1)(x-2)}\,dx$.

$$=\int\left(\frac{3}{x+1}-\frac{1}{x-2}\right)dx$$

$$=3\ln|x+1|-\ln|x-2|+c$$

Since $x+1$ and $x-2$ could be negative we must use absolute values.

If the question didn't give the required form, we could work it out as follows:

- The rational function has the numerator with a degree one less than the denominator.
- The factors in the denominator are $x+1$ and $x-2$, which give us the denominators in the two partial fractions.
- Both denominators are linear functions, so we put a constant function in the numerator – each has a degree one less than the denominator, just like the original rational function.

**Elimination by substitution**

The second method works quickly to find variables in partial fractions with linear denominators, but for the variables in other partial fractions we need to equate one or more of the coefficients as we did in Method 1.

We start by cross-multiplying the partial fractions on the RHS, but do NOT expand the terms.

We then find the value of the variables one at a time by substituting values for $x$ that eliminate all but one of the variables.

If we used this method for Example 1 it would have saved us one line of working, so it is not that much more efficient.

Hints for the following example:

- When the original denominator has a squared factor the simplest form of the partial fractions involves a constant over a single power of that factor and a constant over the factor squared:

$$\frac{2x-3}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}$$

- Alternatively, we could expand the perfect square and use a linear numerator:

$$\frac{2x-3}{x^2(x-1)} = \frac{ax+b}{x^2} + \frac{c}{x-1}$$

- When there is a repeated factor then the substitution method cannot find all variables - equating coefficients will be needed for at least one variable – ‘a’ in this example as it is over the single $x$. This will also occur for the cover-up method we will use in a moment, which is an abbreviated form of the elimination by substitution method.

## Example 2

**a** Express $\dfrac{2x-3}{x^2(x-1)}$ in the form $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x-1}$ by elimination by substitution.

$$\frac{2x-3}{x^2(x-1)}=\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x-1}$$

$$(2x-3)=ax(x-1)+b(x-1)+cx^2$$

Cross-multiply on the RHS but do NOT expand.

Let $x=0$ $\therefore 2(0)-3=a(0)+b(0-1)+c(0)$

$$-3=-b$$

$$b=3$$

Let $x=1$ $\therefore 2(1)-3=a(0)+b(0)+c(1)$

$$-1=c$$

$$c=-1$$

Substitute values of $x$ that will eliminate other variables. We start by letting $x=0$ so that we can find $b$, then let $x=1$ so that we can find $c$.

equating coefficients of $x^2$: $0=a+c$

$$0=a-1$$

$$a=1$$

There is no substitution that leaves $a$ while eliminating $b$ and $c$, so we have to equate coefficients. Since $a$ is part of the coefficient of $x^2$ and of $x$ we could equate either one successfully.

$$\therefore \frac{2x-3}{x^2(x-1)}=\frac{1}{x}+\frac{3}{x^2}-\frac{1}{x-1}$$

**b** Hence evaluate $\displaystyle\int_2^3 \frac{2x-3}{x^2(x-1)}\,dx$.

$$\int_2^3 \frac{2x-3}{x^2(x-1)}\,dx=\int_2^3 \left(\frac{1}{x}+\frac{3}{x^2}-\frac{1}{x-1}\right)dx$$

$$=\left[\ln|x|-\frac{3}{x}-\ln|x-1|\right]_2^3$$

Since $x$ and $x-1$ cannot be negative for $2\le x\le 3$ we do NOT need to use absolute values.

$$=\left(\ln 3-1-\ln 2\right)-\left(\ln 2-\frac{3}{2}-0\right)$$

$$=\ln 3-2\ln 2+\frac{1}{2}$$

$$=\ln\left(\frac{3}{4}\right)+\frac{1}{2}$$

**The cover-up method**

The third method is a shortcut for elimination by substitution, allowing us to quickly find variables in partial fractions with linear denominators.

Just like elimination by substitution, it will not find variables in partial fractions with non-linear denominators. HSC questions are generally chosen so that this method cannot be used by itself, as it reduces these questions to mindless technique without understanding.

To find a variable over a linear denominator we do the following:

- Ignore (cover up) everything on the RHS except the variable we are trying to evaluate.
- Find the zero of the denominator under this variable.
- On the LHS ignore (cover up) the factor matching the denominator for the variable, then substitute the zero into what remains of the rational function and evaluate.

Let's redo part (a) of Example 1, then try a harder example.

## Example 3

Express $\dfrac{2x-7}{(x+1)(x-2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x-2}$ using the cover−up method.

$$a=\frac{2(-1)-7}{-1-2}=3$$

To find $a$, we find the zero of its denominator, which is $x=-1$, and substitute it into the LHS, ignoring the factor $x+1$.

$$b=\frac{2(2)-7}{2+1}=-1$$

To find $b$, we find the zero of its denominator, which is $x=2$, and substitute it into the LHS, ignoring the factor $x-2$.

$$\therefore \frac{2x-7}{(x+1)(x-2)}=\frac{3}{x+1}-\frac{1}{x-2}$$

Note that this took three lines of working, whereas equating coefficients in Example 1 took seven lines, making this a much more efficient method..

Let's look at standard HSC-style question where we can use the cover-up method to find some of the variables and need to equate coefficients to find the others.

## Example 4

Given $\frac{2x+1}{(x^2+1)(x+1)} = \frac{ax+b}{x^2+1} + \frac{c}{x+1}$, find the values of $a, b$ and $c$ using the $\text{cover} - \text{up}$ method and equating coefficients.

$$\frac{2x+1}{(x^2+1)(x+1)} \equiv \frac{ax+b}{x^2+1} + \frac{c}{x+1}$$

$$c = \frac{2(-1)+1}{(-1)^2+1} = -\frac{1}{2}$$ ← cover-up method

$$\frac{2x+1}{(x^2+1)(x+1)} \equiv \frac{(ax+b)(x+1) - \frac{1}{2}(x^2+1)}{(x^2+1)(x+1)}$$

$$\equiv \frac{\left(a-\frac{1}{2}\right)x^2 + (a+b)x + \left(b-\frac{1}{2}\right)}{(x^2+1)(x+1)}$$ ← equating coefficients

$$\therefore a - \frac{1}{2} = 0$$ ← equating coefficients of $x^2$

$$\therefore a = \frac{1}{2}$$

$$b - \frac{1}{2} = 1$$ ← equating constant terms

$$b = \frac{3}{2}$$

$\therefore a = \frac{1}{2}, b = \frac{3}{2}$ and $c = -\frac{1}{2}$.

In the last example one of the factors, $x^2+1$, is an irreducible quadratic – it cannot be factorised within the real number system.

We can use the cover-up method to find the values of $a$ and $b$ if we find and use the complex zero of $x^2+1$, which is $x=i$. We cover up everything on the RHS except for $ax+b$, and cover up $x^2+1$ on the LHS, then substitute $x=i$ into both sides. After rewriting the RHS in Cartesian form we find $a$ and $b$ by equating the real and imaginary components.

Note that we are using complex numbers as a tool to make our calculations easier, but the final answers must be purely real values for $a$ and $b$. We will still find $c$ in the normal manner.

## Example 5

Given $\dfrac{2x+1}{(x^2+1)(x+1)}$

$=\dfrac{ax+b}{x^2+1}+\dfrac{c}{x+1}$, find the values of $a, b$ and $c$ using the cover−up method.

$a(i)+b=\dfrac{2(i)+1}{i+1}\times\dfrac{1-i}{1-i}$ ← Substitute $x=i$ into both sides, then rationalise the denominator.

$b+ai=\dfrac{2i+2+1-i}{1^2+1^2}$

$\therefore b+ai=\dfrac{3}{2}+\dfrac{1}{2}i$

$\therefore a=\dfrac{1}{2}, b=\dfrac{3}{2}$ ← Equate the real and imaginary components.

$c=\dfrac{2(-1)+1}{(-1)^2+1}$ ← Use the cover-up method to find $c$.

$=-\dfrac{1}{2}$

$\therefore a=\dfrac{1}{2}, b=\dfrac{3}{2}$ and $c=-\dfrac{1}{2}$.

Basic

**1a** Express $\frac{2x+1}{(x+1)(x+2)}$ in the form $\frac{a}{x+1}+\frac{b}{x+2}$ by equating coefficients.

**b** Hence find $\int \frac{2x+1}{(x+1)(x+2)}dx$.

**2a** Express $\frac{2x+3}{x^2(x+1)}$ in the form $\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x+1}$ by elimination by substitution.

**b** Hence evaluate $\int_2^3 \frac{2x+3}{x^2(x+1)}dx$.

**3** Express $\frac{2x+1}{(x+1)(x+2)}$ in the form $\frac{a}{x+1}+\frac{b}{x+2}$ using the cover−up method.

**4** Given $\frac{2x+3}{x^2(x+1)}=\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x+1}$, find the values of $a, b$ and $c$ using the cover−up method and equating coefficients.

**5a** Find $a$ and $b$ given that $\frac{1-x}{(x+1)(x+2)} \equiv \frac{a}{x+1}+\frac{b}{x+2}$.

**b** Hence find $\int \frac{1-x}{(x+1)(x+2)}dx$.

**6** Rewrite $\frac{x+1}{x(x-1)^2}$ as the sum or difference of partial fractions.

**7** Find $\int_0^1 \frac{x+2}{x^2+7x+12}dx$.

Medium

**8** Express $\frac{5x-5}{(x+2)(x^2+1)}$ in the form $\frac{a}{x+2}+\frac{bx+c}{x^2+1}$ using the cover−up method and complex numbers, and hence evaluate $\int_3^4 \frac{5x-5}{(x+2)(x^2+1)}dx$.

**9** It can be shown that $\frac{2}{x^3+x^2+x+1}=\frac{1}{x+1}-\frac{x}{x^2+1}+\frac{1}{x^2+1}$ (Do NOT prove this.)
Use this result to evaluate $\int_{\frac{1}{2}}^2 \frac{2}{x^3+x^2+x+1}\,dx$.

**10** It can be shown that $\frac{8(1-x)}{(2-x^2)(2-2x+x^2)}=\frac{4-2x}{2-2x+x^2}-\frac{2x}{2-x^2}$ (Do NOT prove this.)
Use this result to evaluate $\int_0^1 \frac{8(1-x)}{(2-x^2)(2-2x+x^2)}dx$.

**11** Evaluate $\int_2^5 \frac{x-6}{x^2+3x-4}dx$.

**12a** Find $a, b$ and $c$ given that $\dfrac{-5x+2}{x^2(x-1)} \equiv \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x-1}$.

**b** Hence evaluate $\displaystyle\int_2^3 \frac{-5x+2}{x^2(x-1)}\,dx$.

**13a** Given that $\dfrac{16x-43}{(x-3)^2(x+2)}$ can be written as $\dfrac{16x-43}{(x-3)^2(x+2)} = \dfrac{a}{(x-3)^2} + \dfrac{b}{x-3} + \dfrac{c}{x+2}$.

where $a, b$ and $c$ are real numbers, find $a, b$ and $c$.

**b** Hence find $\displaystyle\int \frac{16x-43}{(x-3)^2(x+2)}\,dx$.

**14** Express $\dfrac{2x+3}{(x+1)(x^2+4)}$ in the form $\dfrac{a}{x+1} + \dfrac{bx+c}{x^2+4}$ and hence evaluate $\displaystyle\int_0^2 \frac{2x+3}{(x+1)(x^2+4)}\,dx$.

## Challenging

**15** Find $\displaystyle\int \frac{3x^2+8}{x(x^2+4)}\,dx$.

**16** Find $\displaystyle\int \frac{2x^3-x^2-8x-2}{x(x-2)}\,dx$.

**17** Find $\displaystyle\int \frac{e^x+1}{e^{2x}-5e^x+6}\,dx$.

**18** Given that $\dfrac{3x^3+x^2+12x-12}{(x^2+4)(x^2+2x+4)} = \dfrac{x-3}{x^2+2x+4} + \dfrac{2x}{x^2+4}$, show that

$$\int_0^1 \frac{3x^3+x^2+12x-12}{(x^2+4)(x^2+2x+4)}\,dx = \ln\frac{5\sqrt{7}}{8} + \frac{4}{\sqrt{3}}\left(\frac{\pi}{6} - \tan^{-1}\frac{2}{\sqrt{3}}\right).$$

**19** Use partial fractions to show that $\dfrac{3!}{x(x+1)(x+2)(x+3)} = \dfrac{1}{x} - \dfrac{3}{x+1} + \dfrac{3}{x+2} - \dfrac{1}{x+3}$.

**20** It is given that $t^4+4 = \left(t^2+2t+2\right)\left(t^2-2t+2\right)$.

**a** Find $A$ and $B$ so that $\dfrac{16}{t^4+4} = \dfrac{A+2t}{t^2+2t+2} + \dfrac{B-2t}{t^2-2t+2}$.

**b** A particle is initially at the origin and moves to the right along the $x$−axis. Its velocity $t$ seconds after it started moving is $\dfrac{16}{t^4+4}$ metres per second.
Show that its displacement after $t$ seconds is

$$x = \ln\left(\frac{t^2+2t+2}{t^2-2t+2}\right) + 2\tan^{-1}(t+1) + 2\tan^{-1}(t-1) \text{ metres.}$$

**c** Prove that the particle will approach but never reach $x = 2\pi$ metres.

In Lesson 6 we cover:

- Simplifying definite integrals using symmetry
- Variable of integration (dummy variables)
- Simplifying definite integrals using translations and dilations
- Simplifying definite integrals for odd and even functions
- Simplifying definite integrals using horizontal reflections

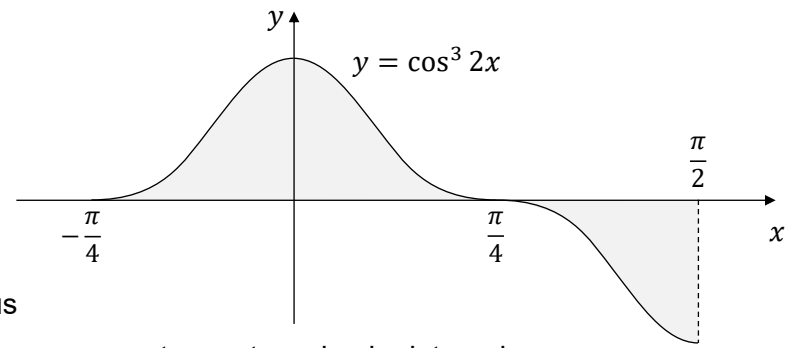
This lesson is massively important for harder questions but is not covered in the syllabus in any great detail. This content best fits with the 15th and 16th dot points from *Further integration*.

## Simplifying definite integrals using symmetry

An easy method to simplify definite integrals involves identifying parts of the integral that are:

- equal, where we simplify the integral by doubling one of the parts
- of the same magnitude but opposite sign, where we cancel out the two parts.

Consider the graph of $y = \cos^3 2x$ in the domain $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right]$.




There are three equal areas in this domain, which could lead us to a variety of obvious and not so obvious relationships between its parts, that we can use to create a simpler integral.

Some obvious results include:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^3 2x\, dx = 2\int_{0}^{\frac{\pi}{4}} \cos^3 2x\, dx \quad \text{and} \quad \int_{0}^{\frac{\pi}{2}} \cos^3 2x\, dx = 0.$$

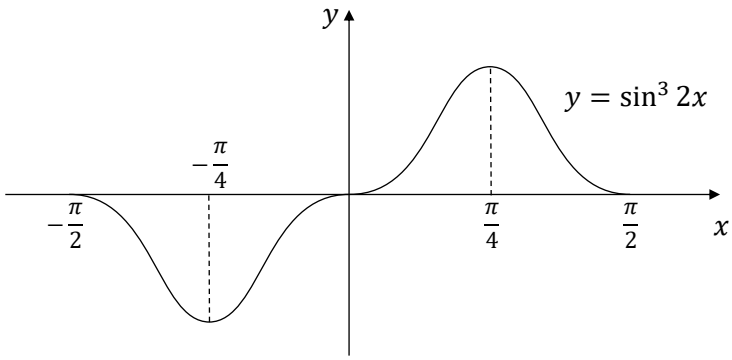
Less obvious results include:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 2x\, dx = \int_{0}^{\frac{\pi}{4}} \cos^3 2x\, dx \quad \text{and} \quad \int_{-\frac{\pi}{4}}^{0} \cos^3 2x\, dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^3 2x\, dx.$$

These types of relationships are particularly useful in simplifying definite integrals involving functions of sine and cosine. Since $\cos 2x$ has a period of $\pi$, we could add or subtract $\pi$ to the upper and lower limits of any integral above without changing its value.

## Example 1

Consider the graph of $y = \sin^3 2x$ for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.




**a** Use the graph to arrange these definite integrals in ascending order.

A: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^3 2x\,dx = \int_{-\frac{\pi}{4}}^{0} \sin^3 2x\,dx + \int_{0}^{\frac{\pi}{4}} \sin^3 2x\,dx = -\frac{1}{3} + \frac{1}{3} = 0$

B: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \sin^3 2x\,dx = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \sin^3 2x\,dx + \int_{-\frac{\pi}{4}}^{0} \sin^3 2x\,dx + \int_{0}^{\frac{\pi}{4}} \sin^3 2x\,dx = -\frac{1}{3} + 0 \text{ (from } a) = -\frac{1}{3}$

C: $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 2x\,dx = \int_{0}^{\frac{\pi}{4}} \sin^3 2x\,dx = \frac{1}{3}$ since $\sin^3 2x$ is symmetrical about $x = \frac{\pi}{4}$

In ascending order, we have: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \sin^3 2x\,dx, \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^3 2x\,dx, \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 2x\,dx.$

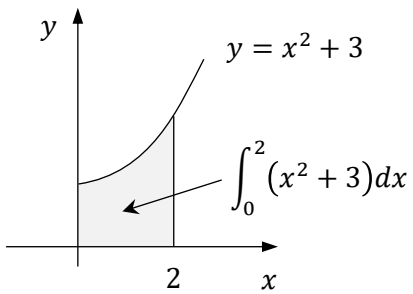
**b** What value of $a$ in the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ maximises the value of $\int_{a}^{\frac{\pi}{2}} \sin^3 2x\,dx$?

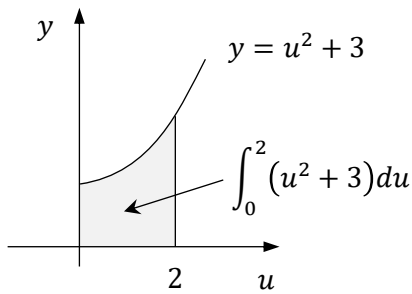
$\int_{a}^{\frac{\pi}{2}} \sin^3 2x\,dx$ is maximised when we exclude as much of the domain where the graph of $y = \sin^3 2x$ is below the $x$−axis, and include as much of the domain where the graph of $y = \sin^3 2x$ is above the $x$−axis.

This occurs when $a = 0$.

## Variable of integration (dummy variables)

The value of a definite integral is independent of the variable used − this is an important technique used in proofs, often called dummy variables. In the diagrams below we can easily see that $\int_0^2 (x^2+3)dx$ and $\int_0^2 (u^2+3)du$ must be identical.




This relationship only applies to definite integrals. If we compare $\int (x^2+3)\,dx$ and $\int (u^2+3)\,du$ we need to remember that not only do both sides end up with a function still in terms of the different variables, but they each include an unknown constant, which are not necessarily equal. For these indefinite integrals we end up with $\frac{x^3}{3}+3x+c_1$ and $\frac{u^3}{3}+3u+c_2$, which follow a similar pattern, but which are not equal - except by coincidence.

We will use symmetry and a $u$ substitution, together with a change of variable in the next example. Here we are comparing integrals under two different functions, rather than comparing integrals under different sections of the same function.

## Example 2

Show that $\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \cos^n 2x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx.$

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \cos^n 2x\, dx$$

$$= 2\int_{\pi}^{\frac{5\pi}{4}} \cos^n 2x\, dx \quad \text{since } \cos^n 2x \text{ is symmetrical about } x = \pi$$

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2dx \\ dx &= \frac{du}{2} \end{aligned}}$$

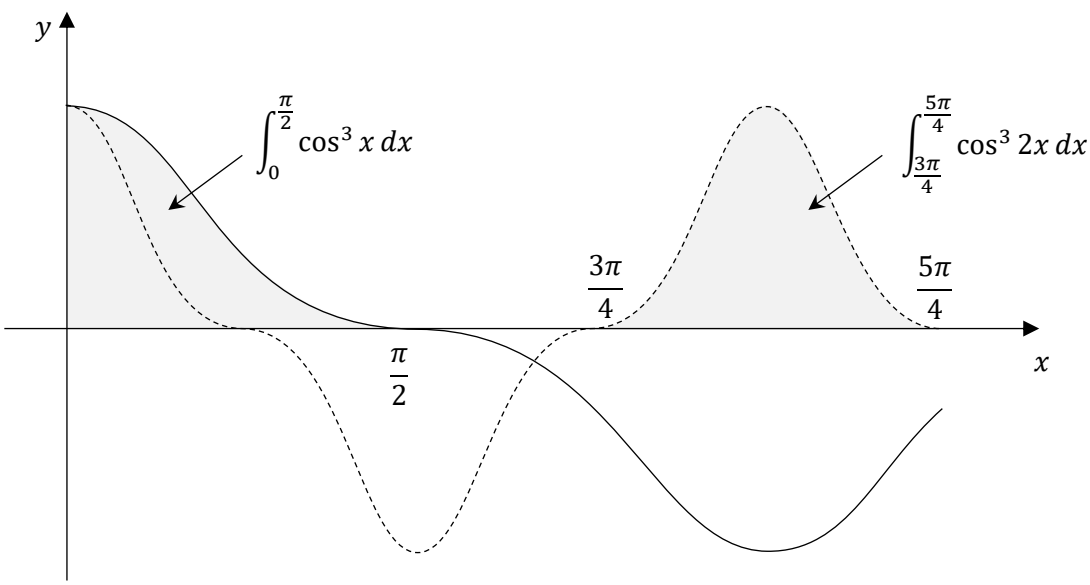
$$= 2\int_{2\pi}^{\frac{5\pi}{2}} \cos^n u \times \frac{du}{2}$$

$$= \int_{2\pi}^{\frac{5\pi}{2}} \cos^n u\, du$$

$$= \int_0^{\frac{\pi}{2}} \cos^n u\, du \quad \text{since } \cos u \text{ has a period of } 2\pi \text{ we can subtract } 2\pi \text{ from both limits}$$

$$= \int_0^{\frac{\pi}{2}} \cos^n x\, dx \quad \text{changing the variable}$$

If $n = 3$ we can see the relationship between the definite integrals as the areas below.



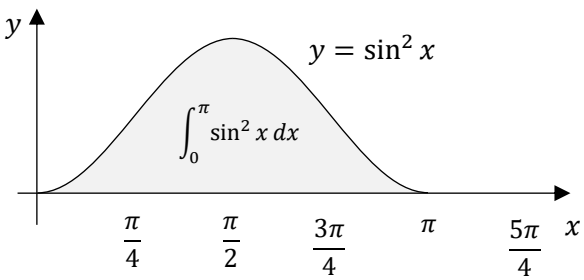
## Simplifying definite integrals using translations and dilations

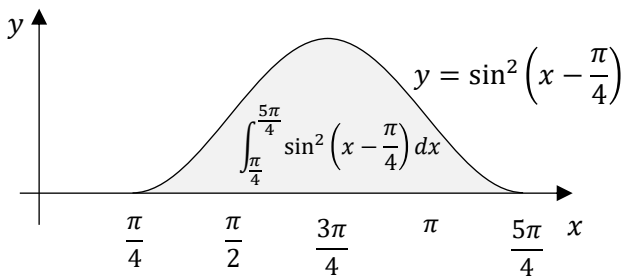
In Advanced we looked at how to quickly sketch transformations of a function, and we can use some of these skills to help simplify some definite integrals. We will look at horizontal reflections shortly, but let's consider horizontal translations and dilations. Again, we are comparing integrals under two different functions.

### Horizontal translations

When a function is translated horizontally, it is not only the function that moves, but also the upper and lower limits and the area between the curve and $x$-axis.

For example, the area under the curve $y = \sin^2 x$ from $x = 0$ to $\pi$, must be the same as the area under the curve $y = \sin^2\left(x - \frac{\pi}{4}\right)$ from $x = \frac{\pi}{4}$ to $\frac{5\pi}{4}$, shown below, where the curve, the limits and the area all move $\frac{\pi}{4}$ units to the right, preserving the area.




We will normally be trying to simplify the function in the definite integral, so moving from right to left we see that:

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin^2\left(x - \frac{\pi}{4}\right) dx = \int_0^{\pi} \sin^2 x \, dx.$$

In general, using translations to simplify the function in an integral, we have:

$$\int_a^b f(x-c)dx = \int_{a-c}^{b-c} f(x)dx\,.$$

We can prove this rule by letting $u = x - c$ and then changing the variable.

Note that we added $c$ units to the value inside the function, but subtracted $c$ units from the limits.

**Horizontal dilations**

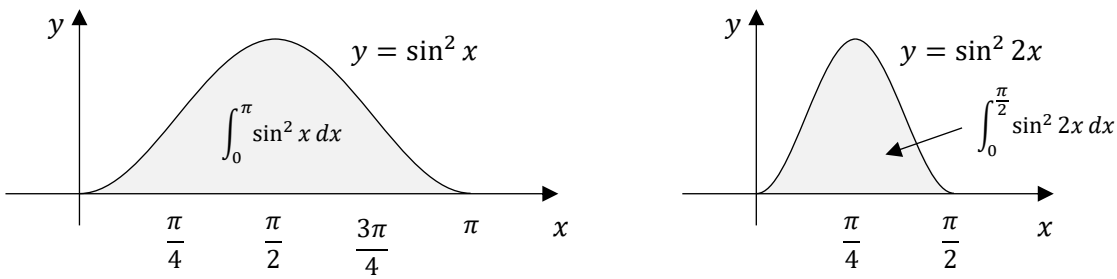
When the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is dilated horizontally by a factor of $k$:

- the new function is $y = f(kx)$
- the new limits are $x = \frac{a}{k}$ and $x = \frac{b}{k}$
- the new area under the curve is $\frac{1}{k}$ times the old area, since the width has been multiplied by $\frac{1}{k}$ while the height is unaffected.

Using definite integrals we see:

$$\int_a^b f(x)\,dx = k\int_{\frac{a}{k}}^{\frac{b}{k}} f(kx)\,dx$$

For example, the area under the curve $y = \sin^2 x$ from $x = 0$ to $\pi$, must be twice the area under the curve $y = \sin^2 2x$ from $x = 0$ to $\frac{\pi}{2}$, shown below, where the curve and the limits have been compressed towards the $y$-axis by a factor of $2$.




In general, using dilations to simplify the function in an integral, we have:

$$\int_a^b f(kx)dx = \frac{1}{k}\int_{ka}^{kb} f(x)dx$$

We can prove this rule by letting $u = kx$ and then changing the variable.

Note that we divided the value inside the function by $k$, but multiplied the limits by $k$. We also multiply the definite integral on the right by $\frac{1}{k}$ as its area increases by a factor of $k$.

## Example 3

Given that $f(x)$ is symmetrical on the line $x = a$, prove the following results.

**a** $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx.$

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$$

$$= \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx \quad \text{since } f(x) \text{ is symmetrical about } x = a$$

$$= 2\int_0^{a} f(x)\,dx$$

**b** $\int_{2a}^{3a} f(x)\,dx = \int_0^{a} f(x-a)\,dx.$

$$\int_{2a}^{3a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx \quad \text{since } f(x) \text{ is symmetrical about } x = a$$

$$= \int_0^{a} f(x-a)\,dx \quad \text{translating } a \text{ units to the right}$$

**c** $\int_0^{a} f(2x)\,dx = \int_0^{a} f(x)\,dx$

$$\int_0^{a} f(2x)\,dx = \frac{1}{2}\int_0^{2a} f(x)\,dx \quad \text{using a horizontal dilation of } f(x) \text{ by a factor of } 2$$

$$= \int_0^{a} f(x)\,dx \text{ from part (a)}$$

## Example 4

Show that the value of $\frac{1}{n}\int_0^{2na} e^{\frac{x}{2n}}\,dx + n\int_0^{\frac{a}{n}} e^{nx}\,dx$ for $a, n \in \mathbb{Z}^+$ is independent of the value of $n$.

$$\frac{1}{n}\int_0^{2na} e^{\frac{x}{2n}}\,dx + n\int_0^{\frac{a}{n}} e^{nx}\,dx = 2n\times\frac{1}{n}\int_0^{a} e^{x}\,dx + \frac{1}{n}\times n\int_0^{a} e^{x}\,dx$$

$$= 2\int_0^{a} e^{x}\,dx + \int_0^{a} e^{x}\,dx$$

$$= 3\int_0^{a} e^{x}\,dx$$

$$= 3\left[e^{x}\right]_0^{a}$$

$= 3(e^a - 1)$ which is independent of $n$.

## Simplifying definite integrals for odd and even functions

In Advanced we looked at odd and even functions, where the integrals that are equal or cancel out are everything to the left and right of the $y$-axis. We will see these rules used in Extension 2, often when we have a combination of two odd and/or even functions, where we first need to determine whether the total function is odd or even. To help determine the rules, rather than memorising them, we can use simple power functions as shown below. Realise that the rules apply to all odd or even functions - the power functions are simply a memory aid.

Sum or difference of two functions $f(x) \pm g(x)$

- If both functions are odd their sum and difference are both odd.
  - For example, $x^3$ and $x^5$ are odd, and we see that $x^3 + x^5$ and $x^3 - x^5$ are also odd since all powers are odd.
- If both functions are even their sum and difference are both even.
  - For example, $x^2$ and $x^4$ are even, and we see that $x^2 + x^4$ and $x^2 - x^4$ are also even as all powers are even.
- If one function is odd and the other even, then their sum and difference are neither odd nor even.
  - For example, $x^2$ is even and $x^3$ is odd, and we see that $x^2 + x^3$ and $x^2 - x^3$ are neither odd nor even, as they have a mix of odd and even powers.

Product or quotient of two functions $f(x) \times g(x)$ and $\dfrac{f(x)}{g(x)}$

- If both functions are odd, or both functions are even, their product and quotient are even.
  - For example, $x^4$ and $x^2$ are even, as are $x^4 \times x^2 = x^6$ and $x^4 \div x^2 = x^2$. Similarly, $x^5$ and $x^3$ are odd, while $x^5 \times x^3 = x^8$ is even as is $x^5 \div x^3 = x^2$.
- If one function is odd and the other even their product or quotient is odd.
  - For example, $x^4$ is even and $x^3$ is odd, while $x^4 \times x^3 = x^7$ is odd as is $x^4 \div x^3 = x^1$.

Composite functions $f\big(g(x)\big)$

- If both functions are odd, then the composite function is odd.
  - For example, $x^3$ and $x^5$ are odd, as is $\big(x^3\big)^5 = x^{15}$.
- If one or both functions are even, then the composite function is even.
  - For example, $x^4$ is even and $x^3$ is odd, while $(x^4)^3 = x^{12}$ is even.

**Equivalent definite integrals involving even functions**

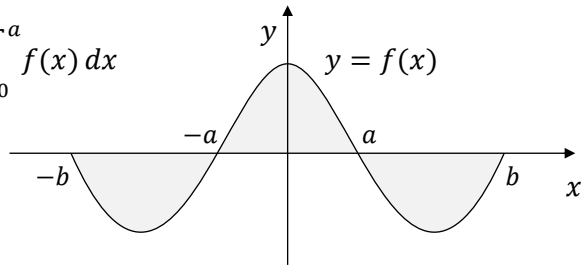
An even function is defined by $f(-x) = f(x)$, so the function has line symmetry on the $y$-axis.

Here are some of the rules that apply to even functions, with diagrams to help understanding.

1. Matching areas on opposite sides of the $y$-axis are equal.

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx \text{ and } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\int_{-b}^{-a} f(x)\,dx = \int_{a}^{b} f(x)\,dx$$

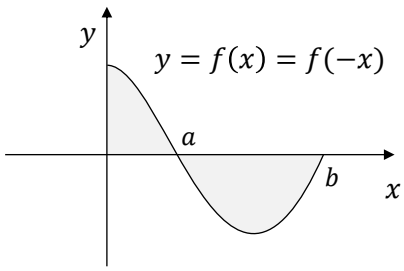



2. The function and its reflection on the $y$-axis coincide, so definite integrals between the same limits must be equal.

$$\int_{0}^{a} f(-x)\,dx = \int_{0}^{a} f(x)\,dx$$

$$\int_{a}^{b} f(-x)\,dx = \int_{a}^{b} f(x)\,dx$$

$$\int_{a}^{b} \big(f(x) + f(-x)\big)\,dx = 2\int_{a}^{b} f(x)\,dx$$




As an aside, the primitive of an even function is odd, so $F(-x) = -F(x)$.

**Equivalent definite integrals involving odd functions**

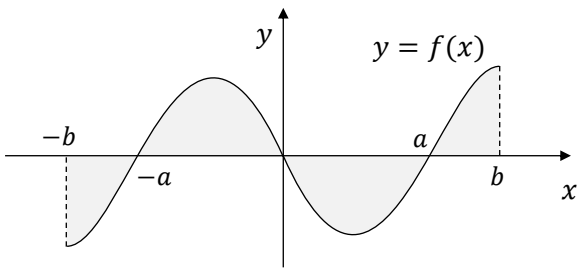
An odd function is defined by $f(-x) = -f(x)$, so the function has point symmetry at the origin.

Here are some of the rules that apply to odd functions, with diagrams to help understanding.

4. Matching areas on opposite sides of the $y$-axis are of equal magnitude but opposite sign.

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx \text{ and } \int_{-a}^{a} f(x)\,dx = 0$$

$$\int_{-b}^{-a} f(x)\,dx = -\int_{a}^{b} f(x)\,dx$$

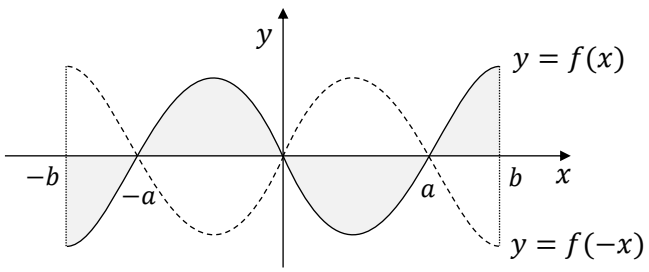



5. The function and its reflection on the $y$-axis are reflections on the $x$-axis, so definite integrals between the same limits must be multiplied by $-1$.

$$\int_{0}^{a} f(-x)\,dx = -\int_{0}^{a} f(x)\,dx$$

$$\int_{a}^{b} f(-x)\,dx = -\int_{a}^{b} f(x)\,dx$$

$$\int_{a}^{b} \big(f(x) + f(-x)\big)\,dx = 0$$




As an aside, the primitive of an odd function is even, so $F(-x) = F(x)$.

## Example 5

State whether each of the following definite integrals in the form $I = \int_{-a}^{a} f(x)\,dx$ is equal to zero or $2J$, where $J = \int_{0}^{a} f(x)\,dx$. Justify your answer.

**a** $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin x \tan x\,dx$

$\sin x$ and $\tan x$ are both odd, so their product is even, $\therefore I = 2J$.

**b** $I = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (x^3 - \sin x)\,dx$

$x^3$ and $\sin x$ are both odd, so their difference is odd, $\therefore I = 0$.

**c** $I = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{x^4}{\sin x}\,dx$

$x^4$ is even and $\sin x$ is odd, so their quotient is odd, $\therefore I = 0$.

**d** $I = \int_{-1}^{1} e^{x^2}\,dx$

$x^2$ is even so $e^{x^2}$ is even, $\therefore I = 2J$.

**e** $I = \int_{-3}^{3} \tan(x^3 + x)\,dx$

$\tan x$ and $x^3 + x$ are odd so the composite function is odd, $\therefore I = 0$.

## Simplifying definite integrals using horizontal reflections

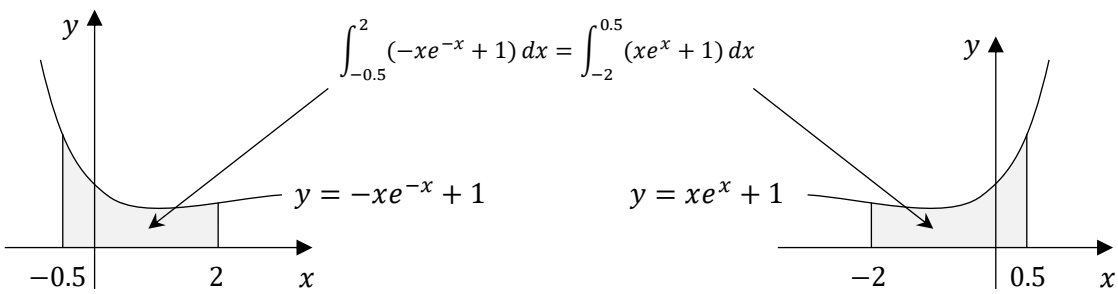
We now look at reflecting any function to create a second function and finding parts of the two different functions that are equal or cancel out. These results also apply to odd and even functions.

### Reflecting definite integrals on the $y$-axis

When we reflect a curve and the area under it about the $y$-axis, we replace $x$ with $-x$ in the function. Note that the upper and lower limits swap their relative order, so the negative of the old upper limit is now the lower limit, and vice versa.

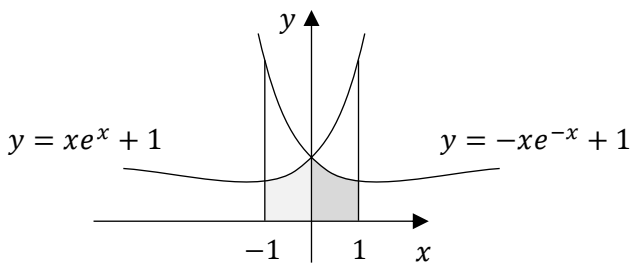
$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-x)\,dx$$

In the diagram below we have $f(x) = xe^x + 1, a = -2$ and $b = 0.5$.




If the lower limit is zero, we find a common application: $\int_0^a f(x)\,dx = \int_{-a}^{0} f(-x)\,dx.$

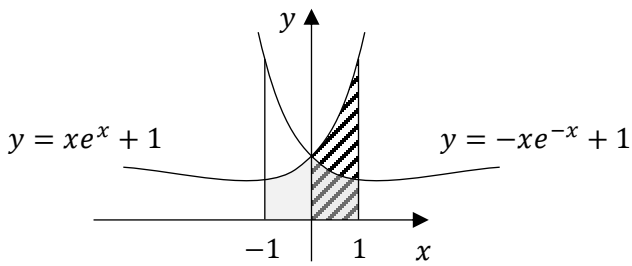
In the diagram at right, we have $f(x) = xe^x + 1, a = 1.$




We can extend this result to find:

$$\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$$

A little harder to see in the diagram, but the two overlapping areas under the two curves between $x = 0$ and $x = 1$ are equal to the area under $y = xe^x + 1$ between $x = -1$ and $x = 1$.



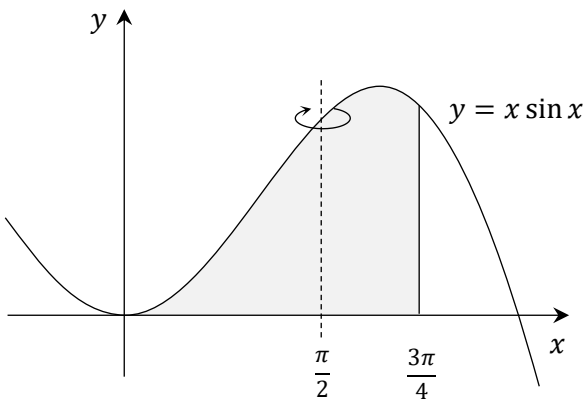
**Reflecting definite integrals on $x = k$**

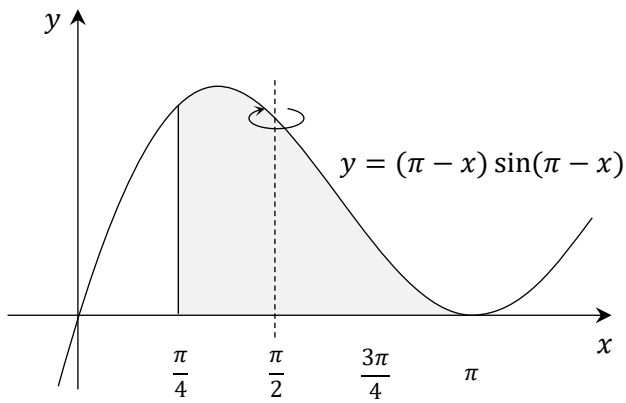
When we reflect the function $y = f(x)$ horizontally about the line $x = k$ the new function is $f(2k - x)$. You are NOT expected to know this, but it can come in handy. If it is needed in a question, you will be asked to prove it in part (a) then use it in part (b), or it will be given in the question.

If we also reflect the limits on $x = k$ then we create equivalent definite integrals:

$$\int_a^b f(x)\,dx = \int_{2k-b}^{2k-a} f(2k - x)\,dx$$

Consider the area between the curve $y = x\sin x$ and the $x$-axis between the lines $x = 0$ and $x = \frac{3\pi}{4}$, shown in the diagram below left. We will reflect it on the line $x = \frac{\pi}{2}$.




The equation of the new function is $y = \left(2\left(\frac{\pi}{2}\right) - x\right)\sin\left(2\left(\frac{\pi}{2}\right) - x\right) = (\pi - x)\sin(\pi - x)$.

The limits are also reflected about $x = \frac{\pi}{2}$.

- the new upper limit is $\frac{\pi}{2} + \left(\frac{\pi}{2} - \text{old lower limit}\right) = \frac{\pi}{2} + \left(\frac{\pi}{2} - 0\right) = 2\left(\frac{\pi}{2}\right) - 0 = \pi$
- the new lower limit is $\frac{\pi}{2} + \left(\frac{\pi}{2} - \text{old upper limit}\right) = \frac{\pi}{2} + \left(\frac{\pi}{2} - \frac{3\pi}{4}\right) = 2\left(\frac{\pi}{2}\right) - \frac{3\pi}{4} = \frac{\pi}{4}$.

From the diagram above we can see that $\int_{\frac{\pi}{4}}^{\pi} (\pi - x)\sin(\pi - x)\,dx = \int_0^{\frac{3\pi}{4}} x\sin x\,dx$.

We will see expressions like this when we look at integration by parts later in the chapter. We will normally use these rules in reverse, where we recognise a function in the form $f(2k - x)$ and change it to $f(x)$, adjusting the limits if needed.

Reflecting definite integrals on the midpoint of the limits, $\boldsymbol{x = \frac{a+b}{2}}$

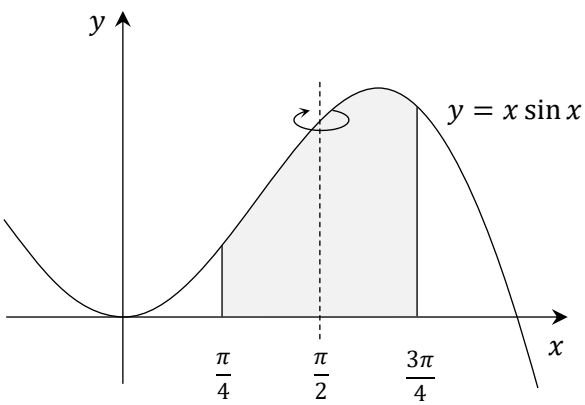
Generally, when we reflect a function on the line $x = k$ we choose a value of $k$ which is halfway between the two limits as it creates more useful results - we reflect on the line $x = \frac{a+b}{2}$. This leads us to the relationship which is informally known as King's rule, which is more useful when the lower limit is zero, as we will see over the page.

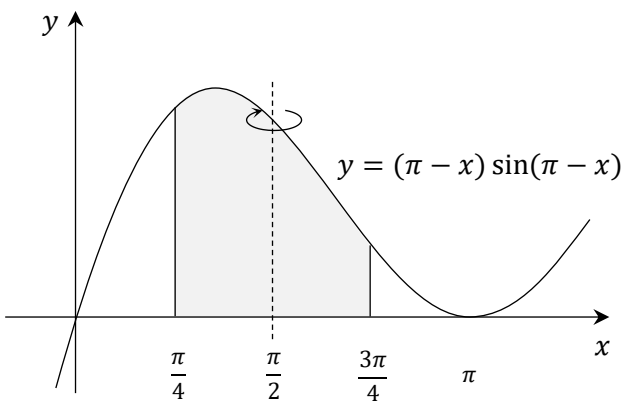
When we reflect the function $y = f(x)$ horizontally on the line $x = \frac{a+b}{2}$ the new function is $f\left(2\left(\frac{a+b}{2}\right) - x\right) = f(a + b - x)$. Again, you are NOT expected to know this, but it can come in handy. If it is needed in a question, you will be asked to prove it in part (a) then use it in part (b), or it will be given in the question.

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

Since the limits are equidistant from $x = \frac{a+b}{2}$ they do not appear to change, which can cause confusion.

Using similar definite integrals as before, but from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$ so that they are equidistant from $x = \frac{\pi}{2}$, we see the following diagrams. Note that the limits appear not to have changed, as they actually swapped.



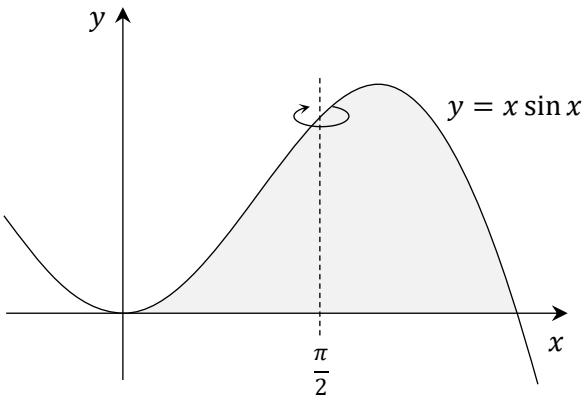

From the diagram above we can see that $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\pi - x)\sin(\pi - x)\,dx = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x \sin x\,dx.$

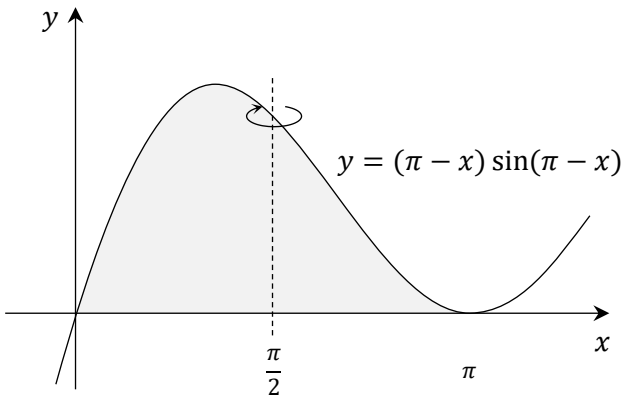
Letting the lower limit equal zero, we see:

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

This is the reflection of $f(x)$ on $x = \frac{a}{2}$.

Using similar definite integrals as before, but from $x = 0$ to $x = \pi$ so that they are equidistant from $x = \frac{\pi}{2}$ and with $x = 0$ as the lower limit, we see the following diagrams.




From the diagram above we can see that $\int_0^{\pi} (\pi - x)\sin(\pi - x)\,dx = \int_0^{\pi} x \sin x\,dx.$

The most common application of this version of King's rule involves trig functions between $0$ and $\frac{\pi}{2}$, where we let $a = \frac{\pi}{4}$. This leads to solutions using complementary angle properties, notably $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

In summary - keep the following rules in mind when you solve definite integrals. You will be asked to prove them if they are needed, but they are often an option in other questions.

$$\int_a^b f(x-c)dx = \int_{a-c}^{b-c} f(x)dx \qquad \int_a^b f(kx)dx = \frac{1}{k}\int_{ka}^{kb} f(x)dx$$

$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-x)\,dx \qquad \int_a^b f(x)\,dx = \int_{2k-b}^{2k-a} f(2k-x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \qquad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Let's use various reflection and translation properties in some examples.

## Example 6

**a** Prove $\int_0^a f(a-x)dx = \int_0^a f(x)dx$.

$$\int_0^a f(a-x)dx \qquad \boxed{\begin{array}{r} u = a-x \\ du = -dx \\ dx = -du \end{array}}$$

$$= \int_a^0 f(u)\times(-du)$$

$$= \int_0^a f(u)du$$

$$= \int_0^a f(x)dx$$

**b** Hence show that $\int_0^\pi x\sin x\,dx = \pi$.

$$\int_0^\pi x\sin x\,dx = \int_0^\pi (\pi-x)\sin(\pi-x)\,dx$$

$$\int_0^\pi x\sin x\,dx = \int_0^\pi \pi\sin(\pi-x)\,dx - \int_0^\pi x\sin(\pi-x)\,dx$$

$$2\int_0^\pi x\sin x\,dx = \pi\int_0^\pi \sin x\,dx$$

$$\int_0^\pi x\sin x\,dx = \frac{\pi}{2}\int_0^\pi \sin x\,dx$$

$$= \frac{\pi}{2}\Big[-\cos x\Big]_0^\pi$$

$$= \frac{\pi}{2}\big(-(-1)-(-1)\big)$$

$$= \pi$$

## Example 7

Given $\int_0^a f(a-x)dx = \int_0^a f(x)dx$, evaluate $\int_0^1 x(1-x)^7 x\,dx$.

$$\int_0^1 x(1-x)^7 x\,dx$$

$$= \int_0^1 (1-x)x^7\,dx$$

$$= \int_0^1 \left(x^7 - x^8\right)dx$$

$$= \left[\frac{x^8}{8} - \frac{x^9}{9}\right]_0^1$$

$$= \left(\frac{1}{8} - \frac{1}{9}\right) - \left(0 - 0\right)$$

$$= \frac{1}{72}$$

## Example 8

Evaluate $\int_0^{\frac{\pi}{2}} \tan\left(x - \frac{\pi}{4}\right)\sec\left(x - \frac{\pi}{4}\right)dx$.

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x \sec x\,dx \qquad \text{translating } \frac{\pi}{4} \text{ to the left}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x)\,dx \text{ where } f(x) \text{ is odd since } \tan x \text{ is odd and } \sec x \text{ is even}$$

$$= 0$$

## Example 9

The odd function $f(x)$ is defined for all real $x$.

**a** Show that $\int_0^a \big(f(a-x)-f(x-a)\big)\,dx = 2\int_0^a f(x)\,dx$ using reflections and translations.

$$\int_0^a \big(f(a-x)-f(x-a)\big)\,dx$$

$$= \int_0^a f(a-x)\,dx - \int_0^a f(x-a)\,dx$$

$$= \int_0^a f(x)\,dx - \int_{-a}^0 f(x)\,dx \quad \text{reflecting (1) about } x=\frac{a}{2} \text{ and translating (2) } a \text{ units to the left}$$

$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx \quad \text{since } f(x) \text{ is odd}$$

$$= 2\int_0^a f(x)\,dx$$

**b** Show that $\int_0^a \big(f(a-x)-f(x-a)\big)\,dx = 2\int_0^a f(x)\,dx$ using the substitution $u = x-a$.

$$\int_0^a \big(f(a-x)-f(x-a)\big)\,dx \qquad \boxed{\begin{array}{c} u = a-x \\ du = -dx \end{array}}$$

$$= \int_{-a}^0 \big(f(-u)-f(u)\big)\,du$$

$$= \int_0^a \big(f(u)-f(-u)\big)\,du$$

$$= \int_0^a \big(f(x)-f(-x)\big)\,dx \qquad \text{changing variables}$$

$$= \int_0^a \big(f(x)+f(x)\big)\,dx \qquad \text{since } f(x) \text{ is odd}$$

$$= 2\int_0^a f(x)\,dx$$

## Basic

**1** Prove $\int_0^a f(a-x)dx = \int_0^a f(x)dx$, and hence find $\int_0^{2\pi} x\cos x\,dx$.

**2** Given $\int_0^a f(a-x)dx = \int_0^a f(x)dx$, evaluate $\int_0^2 x\sqrt{2-x}\,dx$.

**3** Evaluate $\int_{-2}^{2}\left(x+x^3+x^5\right)\left(1+x^2+x^4\right)dx$.

**4** Evaluate $\int_{-\pi}^{0}\sin\left(x+\frac{\pi}{2}\right)\cos\left(x+\frac{\pi}{2}\right)dx$.

**5** Give a condition on a and b that guarantees $\int_a^b\left(x^3-x^2\right)dx$ is positive.

## Medium

**6** A function $f(x)$ has the property that $f(x)+f(a-x)=f(a)$.
Given $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ prove $\int_0^a f(x)\,dx = \frac{a}{2}f(a)$.

**7** Without evaluating the integrals, which one of the following integrals is greater than zero?

A $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x^2}\cos x\,dx$ B $\int_{-\pi}^{\pi} x^3\cos x\,dx$ C $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\sin^2 x-\cos^2 x)\,dx$ D $\int_{-1}^{1}\sin^{-1}\left(x^3\right)dx$

**8** Which integral is equal to $\int_{-a}^{a} f(x)\,dx$ for all $f(x)$?

A $\int_0^a\left(f(x-a)-f(-x)\right)dx$ B $\int_0^a\left(f(a+x)-f(x)\right)dx$
C $\int_0^a\left(f(x)+f(-x)\right)dx$ D $\int_0^a\left(f(x-a)-f(a-x)\right)dx$

**9** It is given that $f(x)$ is a non−zero even function and $g(x)$ is a non−zero odd function.
Which expression is equal to $\int_{-a}^{a} f(x)+g(x)\,dx$?

A $\int_0^a g(x)+g(-x)\,dx$ B $2\int_0^a g(x)+g(-x)\,dx$
C $\int_0^a f(x)+f(-x)\,dx$ D $2\int_0^a f(x)+f(-x)\,dx$

**10** Prove $\int_a^{3a} f(x-a)\,dx = \int_{-a}^{a} f(a-x)\,dx$.

**11** Which of these integrals has the smallest value?

A $\int_0^{\frac{\pi}{6}}\sin x\,dx$ B $\int_0^{\frac{\pi}{6}}\sin^2 x\,dx$ C $\int_0^{\frac{\pi}{6}}(1-\sin x)\,dx$ D $\int_0^{\frac{\pi}{6}}(1-\sin^2 x)\,dx$

**12** Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx$.

**13** Evaluate $\int_0^2 \left(1 + \sin\left(\pi(1-x)^3\right)\right) dx$.

**14** For what values of $a$ is $\int_0^2 \frac{dx}{1-\sqrt{x-a}}$ defined?

**15** Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2\left(x - \frac{\pi}{4}\right)}$.

**16** Given that $xy = x + 1$, evaluate $\int_3^5 x\, dy$.

**17** Evaluate $\int_{-1}^1 \frac{\tan^{-1} x}{1 + \sin^2 x}\, dx$.

**18** Find the smallest value of $\int_0^1 \left(x^2 - a\right)^2 dx$ as $a$ varies.

**19** Given that $f(a+x) = f(a-x)$, prove each of the following results.

**a** $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$

**b** $\int_0^a f(2x)\, dx = \int_0^a f(x)\, dx$

**c** $\int_0^a f(x+a)\, dx = \int_0^a f(x)\, dx$

**d** $\int_0^a f(x-a)\, dx = \int_{2a}^{3a} f(x)\, dx$

Challenging

**20** Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\, dx$.

**21** Given $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, which expression must be equal to $\int_0^a [f(a-x) + f(a+x)] dx$?

A $\int_0^a f(x)\, dx$ B $\int_0^{2a} f(x)\, dx$ C $2\int_0^a f(x)\, dx$ D $\int_{-a}^a f(x)\, dx$

**22** Evaluate $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$.

**23a** Prove $\int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}dx = \int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x}\right)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}dx.$

**b** Hence or otherwise evaluate $I = \int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x + \sin\frac{1}{x}\right)}dx.$

**24** Given that $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \big(f(x)+f(-x)\big)\,dx$, evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \sin^2 x}{1+e^x}dx.$

**25a** Prove $\int_{a}^{b} \frac{f(x)}{f(a+b-x)+f(x)}dx = \frac{b-a}{2}.$

**b** Hence evaluate $\int_{2}^{5} \frac{x^2}{\sqrt{29-x^2}+x}dx$, using the substitution $u = x^2$.

**26** Given $\int_{a}^{b} f(x)\,dx = \frac{1}{2}\int_{a}^{b} \big(f(x)+f(a+b-x)\big)\,dx,$ evaluate $\int_{3}^{7} \frac{\ln(x+2)}{\ln(24+10x-x^2)}dx.$

# 4.7 Trigonometric product to sum identities

In Lesson 7 we cover:

- Trigonometric product to sum and difference identities
- Solving equations using the product to sum and difference identities
- Sum and difference to product identities (optional)
- Product to sum identities and integration

This lesson covers the 1st, 2nd, 3rd and 4th dot points from *Further integration*.

**Background information**

In this lesson we look at the product to sum and difference identities, content that has been moved from Year 11 Extension 1. Expect to see this content used in harder questions than they have been in previous Extension 1 HSCs.

We also briefly look at the sum and difference to product identities as an alternative when we need to change a sum or difference into a product, although they generally offer little advantage over the product to sum and difference identities. They are not in the syllabus, so feel free to skip them.

Note that the first part of this lesson doesn't actually use integration at all – we will be working with expressions and equations.

## Trigonometric product to sum identities

When we are given the product of the sine or cosine of one argument and the sine or cosine of a different argument, we can convert the product into the sum or difference of sines or cosines using the product to sum and difference identities listed below. We can use them when the two arguments are the same, but it is usually easier to use the double angle formula.

We need to be able to convert back and forth between the two forms of expression as:

- The product of two trig ratios allows us to find zeros when solving equations or graphing more easily.
- The sum or difference of two trig ratios allows us to integrate and differentiate more easily.

These identities are applications of the double angle formula from the Extension 1 course.

The four product to sum rules are:

$$\cos A \cos B = \frac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big]$$

$$\sin A \sin B = \frac{1}{2}\Big[\cos(A-B) - \cos(A+B)\Big]$$

$$\sin A \cos B = \frac{1}{2}\Big[\sin(A+B) + \sin(A-B)\Big]$$

$$\cos A \sin B = \frac{1}{2}\Big[\sin(A+B) - \sin(A-B)\Big]$$

These identities ARE given on the reference sheet

Can we remember these identities without the reference sheet? Yes - but always check the reference sheet in an exam!

1. If the two ratios on the LHS are the same (both cos or both sin) then use cos on the RHS.
2. If the two ratios on the LHS are different (one cos and one sin) then use sin on the RHS.
3. If cos is on the RHS use the difference $(A-B)$ first and the sum $(A+B)$ second.
4. If sin is on the RHS use the sum $(A+B)$ first and the difference $(A-B)$ second.
5. If cos is the second ratio on the LHS then add the ratios on the RHS
6. If sin is the second ratio on the LHS then subtract the ratios on the RHS.

Now let's look at how one of the four rules is derived - the other three rules are proved in a similar way.

Prove $\cos A \cos B = \frac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big]$.

$$\text{RHS} = \frac{1}{2}\Big[\cos A \cos B + \sin A \sin B + \cos A \cos B - \sin A \sin B\Big]$$

$$= \frac{1}{2}\Big[2\cos A \cos B\Big]$$

$$= \cos A \cos B$$

$$\therefore \cos A \cos B = \frac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big]$$

## Example 1

Write $\sin 2x \sin 4x$ as the difference of cosines.

$$\sin 2x \sin 4x = \frac{1}{2}\Big[\cos(2x-4x) - \cos(2x+4x)\Big]$$

$$= \frac{1}{2}\Big(\cos(-2x) - \cos 6x\Big)$$

$$= \frac{1}{2}(\cos 2x - \cos 6x) \quad \text{cosine is even}$$

- Both ratios on the LHS are the same so use cos on the RHS.
- When we use cos, we use the difference first and the sum second.
- Since sin is second on the LHS we subtract the ratios on the RHS.

*Alternatively:*

$$\sin 2x \sin 4x = \sin 4x \sin 2x$$

$$= \frac{1}{2}\Big[\cos(4x-2x) - \cos(4x+2x)\Big]$$

$$= \frac{1}{2}(\cos 2x - \cos 6x)$$

Note that it is much easier to rearrange the terms so the ratio with the larger angle is first

## Example 2

Show that $\sin 52.5° \cos 7.5° = \dfrac{\sqrt{3}+\sqrt{2}}{4}$.

$$\text{LHS} = \frac{1}{2}\Big[\sin(52.5° + 7.5°) + \sin(52.5° - 7.5°)\Big]$$

$$= \frac{1}{2}(\sin 60° + \sin 45°)$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{6}+2}{2\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}+\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{3}+\sqrt{2}}{4}$$

- The ratios on the LHS are different so use sin on the RHS.
- When we use sin we use the sum first and the difference second.
- Since sin is second on the LHS we subtract the ratios on the RHS.

## Example 3

**a** Write $\cos\theta - \cos 3\theta$ as the product of trigonometric functions.

$$\cos\theta - \cos 3\theta = \cos(2\theta - \theta) - \cos(2\theta + \theta)$$

$$= 2\sin 2\theta \sin\theta$$

- We are using the identities backwards.
- When both ratios on the RHS are cos the two ratios on the LHS are the same.
- Since we subtract the ratios on the RHS the second one on the LHS must be cos, so they are both cos.
- The first angle is the difference, and the second angle is the sum, so we have $2\theta$ then $\theta$.
- We double it, as we don't have the $\frac{1}{2}$ from the identities.

**b** Hence show that $\dfrac{\cos\theta - \cos 3\theta}{2 - 2\cos^2\theta} = 2\cos\theta$.

$$\frac{\cos\theta - \cos 3\theta}{2 - 2\cos^2\theta} = \frac{2\sin 2\theta \sin\theta}{2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta\sin\theta}{\sin^2\theta}$$

$$= 2\cos\theta$$

## Solving equations using the product to sum and difference identities

Let's use the identities to solve trig equations.

## Example 4

Solve $\cos 3x - \cos 7x = \sin 5x$ for $0 \leq x \leq \frac{\pi}{2}$.

$$\begin{aligned}
\cos 3x - \cos 7x &= \cos(5x - 2x) - \cos(5x + 2x) \\
&= 2 \sin 5x \sin 2x \\
\therefore 2 \sin 5x \sin 2x &= \sin 5x \\
\sin 5x\,(2 \sin 2x - 1) &= 0
\end{aligned}$$

$\sin 5x = 0$ or $2 \sin 2x - 1 = 0$

$5x = 0, \pi, 2\pi$ $\qquad$ $2 \sin 2x = 1$

$x = 0, \frac{\pi}{5}, \frac{2\pi}{5}$ $\qquad$ $\sin 2x = \frac{1}{2}$

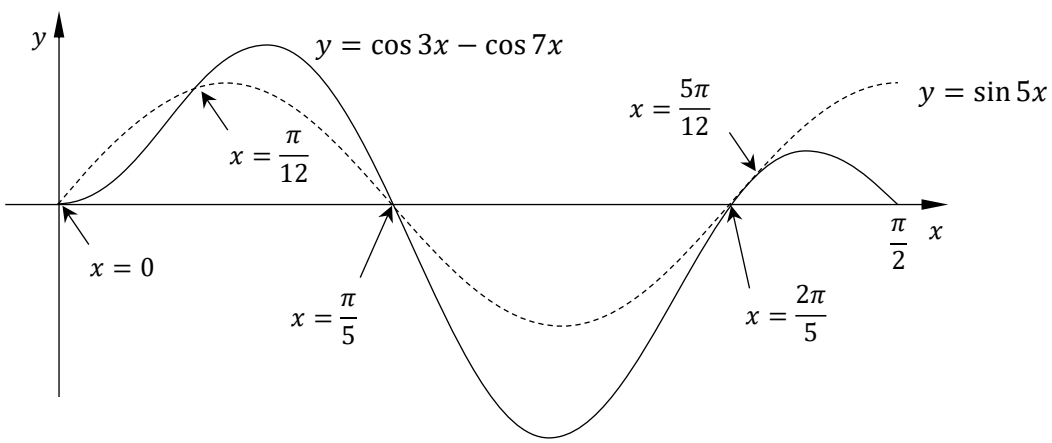
$$2x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

$\therefore x = 0, \frac{\pi}{12}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{5\pi}{12}$

Optional: It can help your understanding to graph both sides of the equation in your favourite graphing application (GeoGebra, Desmos etc) and see where they intersect, as shown below.

Note that since $\frac{2\pi}{5}$ and $\frac{5\pi}{12}$ are so close together, it would be easy to miss one of them if we just look at the graphs. Graphing $y = \cos 3x - \cos 7x - \sin 5x$ and looking for the $x$-intercepts can be safer.



## Sum and difference to product identities (optional)

Although not mentioned in the syllabus, the sum to product identities are applications of the product to sum identities. You do not need to learn these rules, as the product to sum and difference identities solve related questions just as efficiently – they are just provided here so that you can be assured that you do NOT need to know them.

The four sum and difference to product identities are:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

These identities ARE NOT given on the reference sheet.

The identities can be determined by using the product to sum and difference identities and replacing $A$ with $\frac{A+B}{2}$ and $B$ with $\frac{A-B}{2}$, simplifying and rearranging.

If you are very good at remembering formulas, then you can try to memorise these identities, but it is safer to use the product to sum and difference identities from earlier in the lesson.

In the following examples we will solve each question two ways – first using the product to sum and difference identities and then using the sum and difference to product identities. As you will see, there is no advantage in using this second set of identities.

## Example 5

**a** Prove that $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

$$\text{RHS} = 2\times\frac{1}{2}\left[\cos\left(\frac{A+B}{2}-\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}+\frac{A-B}{2}\right)\right]$$

$$= \cos\left(\frac{A+B-A+B}{2}\right)+\cos\left(\frac{A+B+A-B}{2}\right)$$

$$= \cos\left(\frac{2B}{2}\right)+\cos\left(\frac{2A}{2}\right)$$

$$= \cos A + \cos B$$

**b** Hence solve $\cos 4x + \cos 3x - \cos 2x - \cos x = 0$ for $0 \le x \le \pi$.

$$2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{x}{2}\right) - 2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) = 0$$

$$2\cos\left(\frac{x}{2}\right)\left[\cos\left(\frac{7x}{2}\right)-\cos\left(\frac{3x}{2}\right)\right] = 0$$

$$-2\cos\left(\frac{x}{2}\right)\left[\cos\left(\frac{5x}{2}-x\right)-\cos\left(\frac{5x}{2}+x\right)\right] = 0$$

$$-2\cos\left(\frac{x}{2}\right)\sin\left(\frac{5x}{2}\right)\sin\left(\frac{x}{2}\right) = 0$$

$$-\sin x \sin\left(\frac{5x}{2}\right) = 0$$

$$\therefore \sin x = 0 \quad or \quad \sin\left(\frac{5x}{2}\right) = 0$$

$$x = 0, \pi \qquad \frac{5x}{2} = 0, \pi, 2\pi$$

$$x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}$$

$$\therefore x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \pi$$

The pairs of angles $4x$ and $3x$, and $2x$ and $x$ each have a difference of $x$, so we can leave them as these pairs. We could also rearrange the LHS to use $4x$ and $2x$, and $3x$ and $x$.

## Example 6

Write $\cos 3x + \cos x$ as a function of $\cos x$ and show that $\cos 3x = 4\cos^3 x - 3\cos x$:

**a** using the product to sum and difference identities

$$\begin{aligned}\cos 3x + \cos x &= \cos(2x + x) + \cos(2x - x) \\ &= 2\cos 2x \cos x \\ &= 2\left(2\cos^2 x - 1\right)\cos x \\ &= 4\cos^3 x - 2\cos x \\ \therefore \cos 3x &= 4\cos^3 x - 3\cos x\end{aligned}$$

**b** using the sum and difference to product identities.

$$\begin{aligned}\cos 3x + \cos x &= 2\cos\left(\frac{3x + x}{2}\right)\cos\left(\frac{3x - x}{2}\right) \\ &= 2\cos 2x \cos x \\ &= 2\left(2\cos^2 x - 1\right)\cos x \\ &= 4\cos^3 x - 2\cos x \\ \therefore \cos 3x &= 4\cos^3 x - 3\cos x\end{aligned}$$

As we can see both methods took five lines, so there is little if any point learning the sum and difference to product identities.

## Product to sum identities and integration

The product to sum identities can also be used to solve definite and indefinite integrals. Once the identities have been used, the integration is fairly simple.

## Example 7

Show that $\int_0^{\frac{\pi}{6}} \cos 3x \sin 2x \, dx = \frac{3\sqrt{3}-4}{10}$.

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}\Big[\sin(3x+2x) - \sin(3x-2x)\Big]\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \Big[\sin 5x - \sin x\Big]\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos 5x}{5} + \cos x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\left(-\frac{\cos\frac{5\pi}{6}}{5} + \cos\frac{\pi}{6}\right) - \left(-\frac{\cos 0}{5} + \cos 0\right)\right)$$

$$= \frac{1}{2}\left(\frac{\cos\frac{\pi}{6}}{5} + \cos\frac{\pi}{6} + \frac{1}{5} - 1\right)$$

$$= \frac{1}{2}\left(\frac{6}{5} \times \frac{\sqrt{3}}{2} - \frac{4}{5}\right)$$

$$= \frac{3\sqrt{3}-4}{10}$$

## Example 8

Show that $\int \sin mx \sin nx \, dx = \dfrac{\sin(m-n)x}{2(m-n)} - \dfrac{\sin(m+n)x}{2(m+n)} + c$ for $m, n \in Z^+$ and $m \neq n$.

$$= \int \frac{1}{2}\Big[\cos(mx - nx) - \cos(mx + nx)\Big] dx$$

$$= \int \frac{1}{2}\Big[\cos(m-n)x - \cos(m+n)x\Big] dx$$

$$= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c$$

## Example 9

The curve $y = \sin 2x + \sin 4x$ between $x = 0$ and $x = \dfrac{\pi}{4}$ is rotated about the $x$−axis.

Find the volume of the solid of revolution created.

$$V = \pi \int_0^{\frac{\pi}{4}} (\sin 2x + \sin 4x)^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(\sin^2 2x + 2 \sin 4x \sin 2x + \sin^2 4x\right) dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}(1 - \cos 4x) + 2\left(\frac{1}{2}[\cos(4x - 2x) - \cos(4x + 2x)]\right) + \frac{1}{2}(1 - \cos 8x)\right) dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(1 - \frac{1}{2}\cos 4x + \cos 2x - \cos 6x - \frac{1}{2}\cos 8x\right) dx$$

$$= \pi \left[x - \frac{\sin 8x}{16} - \frac{\sin 4x}{8} - \frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \pi \left(\left(\frac{\pi}{4} - 0 - 0 + \frac{1}{6} + \frac{1}{2}\right) - (0 - 0 - 0 - 0 + 0)\right)$$

$$= \frac{\pi(3\pi + 8)}{12} \text{ units}^3$$

## Basic

**1** Write $\sin 4x \cos 3x$ as the sum of sines.

**2** Show that $\cos 75° \cos 15° = \frac{1}{4}$.

**3** Show that $\frac{\sin(A+B)+\sin(A-B)}{\cos(A-B)+\cos(A+B)} = \tan A$.

**4** Show that $\cos 130° + \cos 10° = \cos 70°$.

**5** Use the product to sum results to show that $\cos\frac{\pi}{12} - \cos\frac{5\pi}{12} = \cos\frac{\pi}{4}$.

**6** Prove that $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$.

**7a** Find $\int \cos 3x \cos 2x \, dx$.

**b** Find $\int \sin 8x \cos 5x \, dx$.

**8a** Evaluate $\int_0^{\frac{\pi}{4}} \sin 3x \sin 4x \, dx$.

**b** Evaluate $\int_{\frac{\pi}{2}}^{\pi} \cos 5x \sin 3x \, dx$.

## Medium

**9** Solve $\sin 4x - \sin 2x = 0$ for $0 \le x \le \pi$, using:

**a** the double angle results

**b** the product to sum identities

**10** Prove $\int \sin x \sin 2x \, dx = \frac{2\sin^3 x}{3} + c$:

**a** Using the double angle results and the reverse chain rule

**b** Using the product to sum identities.

**c** Which method is most efficient for this integral?

**11** Find the values of $A$ and $B$ such that $\cos 6x \cos 3x - \sin 7x \sin 4x = \cos Ax \cos Bx$.

**12** If $\sin(x-y) = M$ and $\sin(x+y) = N$ show that $\cos x \sin y = \frac{M+N}{2}$.

**13** Show that $\cot A \cot B = \frac{\cos(A-B)+\cos(A+B)}{\cos(A-B)-\cos(A+B)}$.

**14** Write $\sin 3x + \sin x$ as a function of $\sin x$.

**15** Prove that $\tan\left(\frac{A+B}{2}\right) = \frac{\sin A + \sin B}{\cos A + \cos B}$.

**16** Solve $\sin 3x + \cos 2x - \sin x = 0$ for $0 \leq x \leq 2\pi$.

**17** Evaluate $\int_0^{\pi} \sin(mx)\cos(nx)\,dx$ where $m$ and $n$ are positive and even, with $m \neq n$.

**18a** Show that $\sin 2\theta + \sin 4\theta - \sin 6\theta = 4\sin\theta \sin 2\theta \sin 3\theta$.

**b** Hence solve $\sin 6\theta = \sin 2\theta + \sin 4\theta$ for $[0, \pi]$.

**19** Solve $\sin 2x + \sin 3x + \cos 2x + \cos 3x = 0$ for $[0, 2\pi]$.

**20a** Show that $\dfrac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$ where $\cos 2x \neq -\dfrac{1}{2}$ and $\cos 3x \neq 0$.

**b** Hence solve $\sin x + \sin 3x + \sin 5x = \cos x + \cos 3x + \cos 5x$ for $0 \leq x \leq \pi$, given $\cos 2x \neq -\dfrac{1}{2}$.

## Challenging

**21** Solve $\cos\left(x + \dfrac{\pi}{6}\right)\sin\left(x - \dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{4}$ for $0 \leq x \leq 2\pi$.

**22a** Show that $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$.

**b** Hence solve $\sin^2 4x - \sin^2 2x = \sin 6x$ for $0 \leq x \leq \dfrac{\pi}{2}$.

**23** Express $4\cos 2x \sin x + \cos 3x - 2\sin x + 1$ in the form $R\sin(3x + \alpha) + c$.

**24a** Prove that $\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right) + \cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right) = \sin A$.

**b** Hence solve $\sin\left(x + \dfrac{\pi}{6}\right)\cos\left(x - \dfrac{\pi}{6}\right) + \cos\left(x + \dfrac{\pi}{6}\right)\sin\left(x - \dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$ for $-\pi \leq x \leq \pi$.

**25** Find $\int x\cos 4x^2 \cos 2x^2\,dx$.

In Lesson 8 we cover:

- Powers of trigonometric functions
  - Powers of sine and cosine
  - Powers of tangent and secant
  - Powers of cotangent and cosecant (briefly)

This lesson mainly covers the 11th, 15th and 16th dot points from *Further integration*.

## Powers of trigonometric functions

Trigonometric functions can occur on their own, or in pairs, often with the same function. For example, sine and cosine are commonly paired, as are tangent and secant, due to their links through their derivatives and in the Pythagorean identities.

Cotangent and cosecant are paired but are rarely used in exams. The relationships are identical to those between tangent and secant, with negative signs included. We will only cover them briefly, then use them in exercises.

### Powers of sine and cosine

To integrate powers of sine and cosine we use the following rules and standard integrals.

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

Memorise these identities. Notice that the only difference is the sign.

It is best to practise many examples until you are able to determine the technique to be used. Alternatively, consider these suggestions.

- If the integrand is in the form $\sin x \cos^n x$ or $\sin^m x \cos x$ then use the reverse chain rule
- If the integrand is in the form $\sin^m x$ or $\cos^n x$, then break off $\sin^2 x$ or $\cos^2 x$ respectively, expand, then use the reverse chain rule

- If the integrand is in the form $\sin^m x \cos^n x$ we have different techniques required depending on the values of $m$ and $n$.
  - if $m$ or $n$ is odd take out a factor of the trig function with the odd power - if they are both odd, it doesn't matter which one you choose - then use the Pythagorean identity to express the trig function that now has an even power in terms of the other trig function
  - if $m$ and $n$ are both even, then replace both functions using the half-angle identities: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

Personally, I prefer to do enough examples so that the correct method is obvious, rather than trying to remember more rules.

It is best to practise many examples until you are able to determine the technique to be used. Alternatively, consider these suggestions.

- If the integrand is in the form $\sin x \cos^n x$ or $\sin^m x \cos x$ then use the reverse chain rule
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  - if $m$ or $n$ is odd take out a factor of the trig function with the odd power - if they are both odd, it doesn't matter which one you choose - then use the Pythagorean identity to express the trig function that now has an even power in terms of the other trig function
  - if $m$ and $n$ are both even, then replace both functions using the half-angle identities: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

Personally, I prefer to do enough examples so that the correct method is obvious, rather than trying to remember more rules.

## Example 1

**a** Evaluate $\int_0^{\frac{\pi}{6}} \cos x \sin^3 x \, dx$

$$= \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left(\left(\frac{1}{2}\right)^4 - 0\right)$$

$$= \frac{1}{64}$$

**b** Find $\int \sin^3 x \ dx$.

$$= \int \sin^2 x \sin x \ dx$$

$$= \int (1 - \cos^2 x) \sin x \, dx$$

$$= \int \sin x \, dx - \int \sin x \cos^2 x \, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + c$$

## Example 2

**a** Evaluate $\int_0^{\frac{\pi}{3}} \cos^2 x \sin^3 x \, dx$

$$= \int_0^{\frac{\pi}{3}} \sin x \cos^2 x \sin^2 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \cos^2 x \, (1 - \cos^2 x) \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \ (\cos^2 x - \cos^4 x) \, dx$$

$$= -\int_0^{\frac{\pi}{3}} (-\sin x)\left(\cos^2 x - \cos^4 x\right) dx$$

$$= -\left[\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right]_0^{\frac{\pi}{3}}$$

$$= -\left(\left(\frac{1}{24} - \frac{1}{160}\right) - \left(\frac{1}{3} - \frac{1}{5}\right)\right)$$

$$= \frac{47}{480}$$

**b** Find $\int \cos^2 x \sin^2 x \ dx$.

$$= \int \frac{1}{2}(1 + \cos 2x)\frac{1}{2}(1 - \cos 2x) dx$$

$$= \frac{1}{4}\int \left(1 - \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2}(1 + \cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{x}{8} - \frac{1}{32}\sin 4x + c$$

**Powers of tangent and secant**

Integration involving $\tan x$ and $\sec x$ can be more involved, as the two functions are linked by many different rules, which can cause confusion. Not all integrands involving tangent and/or secant can be integrated using the simple techniques below - some rely on integration by parts as we will see later in the chapter.

This complexity makes integration using tangent and secant great for Extension 2. We need to solve the integrals using the following Pythagorean identities and standard integrals.

$$\tan^2 x + 1 = \sec^2 x$$
$$\int \tan x \, dx = -\ln|\cos x| + c$$
$$\int \sec x \, dx = \ln|\sec x + \tan x| + c$$
$$\int \sec^2 x \, dx = \tan x + c$$
$$\int \sec x \tan x \, dx = \sec x + c$$

Memorise these rules - see the mnemonics at the end of the chapter.

Again, it is best to practise many examples until you are able to determine the technique to be used. Alternatively, consider these suggestions.

- If the integrand is in the form $\tan^m x \sec^2 x$ or $\sec x \tan x \sec^n x$ then use the reverse chain rule
- If the integrand is in the form $\tan^m x$ or $\sec^n x$, then break off $\tan^2 x$ or $\sec^2 x$ respectively, expand, then use the reverse chain rule - if you can - it won't always work
- If the integrand is in the form $\tan^m x \sec^n x$ we have different techniques required depending on the values of $m$ and $n$.
  - if $m$ is odd take out a factor of $\tan x \sec x$, then use the Pythagorean identity to express the remaining function in terms of $\sec x$ then use the reverse chain rule
  - if $n$ is even then take out a factor of $\sec^2 x$, , then use the Pythagorean identity to express the remaining function in terms of $\tan x$ then use the reverse chain rule
  - for other cases there is no one method to follow, which makes these questions a challenge!

## Example 3

**a** Find $\int \sec^2 x \tan^4 x \ dx$

$$= \frac{\tan^5 x}{5} + c$$

**b** Find $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^4 x \, dx$.

$$= \int_0^{\frac{\pi}{4}} \tan^2 x \tan^2 x \, dx$$
$$= \int_0^{\frac{\pi}{4}} \tan^2 x \, (\sec^2 x - 1) \, dx$$
$$= \int_0^{\frac{\pi}{4}} (\sec^2 x \, (\tan x)^2 - \tan^2 x) \, dx$$
$$= \int_0^{\frac{\pi}{4}} (\sec^2 x \, (\tan x)^2 - (\sec^2 x - 1)$$
$$= \left[\frac{\tan^3 x}{3} - \tan x + x\right]_0^{\frac{\pi}{4}}$$
$$= \left(\frac{1}{3} - 1 + \frac{\pi}{4}\right) - \left(0 - 0 + 0\right)$$
$$= \frac{3\pi - 8}{12}$$

## Example 4

**a** Find $\int \tan^3 x \sec^3 x \, dx$

$$= \int \tan x \sec x \tan^2 x \sec^2 x \, dx$$
$$= \int \tan x \sec x \, (\sec^2 x - 1) \sec^2 x \, dx$$
$$= \int \tan x \sec x \, (\sec^4 x - \sec^2 x) \ dx$$
$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c$$

**b** Find $\int \tan^2 x \sec^4 x \, dx$.

$$= \int \tan^2 x \sec^4 x \, dx$$
$$= \int \sec^2 x \tan^2 x \, (1 + \tan^2 x) dx$$
$$= \int \sec^2 x \ (\tan^2 x + \tan^4 x) dx$$
$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + c$$

## Example 5

Prove $\int \sec x \, dx = \ln|\tan x + \sec x| + c$.

$$\int \sec x \, dx = \int \sec x \times \frac{\tan x + \sec x}{\sec x + \tan x} dx$$
$$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$$
$$= \ln|\tan x + \sec x| + c$$

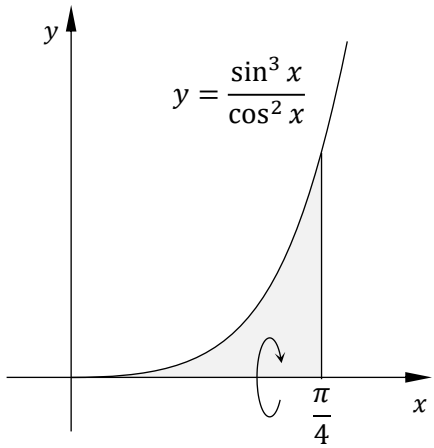
If this question was asked in an exam, it would be scaffolded so that students could see the method required. Alternatively, we work backwards from the primitive to see that we need to change the integrand to be

$$\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}.$$

The next example is interesting as it looks like it should be included with the integrals including powers of sine and cosine, but as we can see, after a slight manipulation it involves powers of tangent and secant.

## Example 6

The area between $y = \dfrac{\sin^2 x}{\cos^4 x}$ and the $x-$axis between $x = 0$ and $x = \dfrac{\pi}{4}$ is rotated about the $x-$axis. What is the volume of solid of revolution formed?




$$V = \pi \int_0^{\frac{\pi}{4}} \left(\frac{\sin^2 x}{\cos^4 x}\right)^2 dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos^8 x}\, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x\, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \tan^4 x \sec^2 x\, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \tan^4 x \left(1 + \tan^2 x\right) dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \left(\tan^4 x + \tan^6 x\right) dx$$

$$= \pi \left[\frac{\tan^5 x}{5} + \frac{\tan^7 x}{7}\right]_0^{\frac{\pi}{4}}$$

$$= \pi \left(\left(\frac{1}{5} + \frac{1}{7}\right) - (0 + 0)\right)$$

$$= \frac{12\pi}{35} \text{ units}^3$$

### Powers of cotangent and cosecant

There are many similarities between the pair of functions $\cot x$ and $\operatorname{cosec} x$ and the pair of functions $\tan x$ and $\sec x$ that we have just looked at. There are no separate examples for this pair – have a look at the previous slides for the techniques to be used, replacing $\sec x$ with $\operatorname{cosec} x$ and $\tan x$ with $\cot x$, remembering that there will also be a minus sign involved.

## Basic

**1a** Find $\int \sin x \cos^3 x \ dx$.

**b** Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 x \, dx$.

**2a** Find $\int \sec^4 x \tan^2 x \, dx$.

**b** Find $\int \sec^4 x \, dx$.

**3** Prove $\int \operatorname{cosec} x \, dx = -\ln|\cot x + \operatorname{cosec} x| + c$.

**4** Find $\int \tan x \sec^3 x \, dx$.

**5** Find $\int \cos^4 x \, dx$.

**6** Find $\int \tan^3 x \, dx$.

**7** Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot^3 x \operatorname{cosec}^2 x \, dx$.

**8** Find $\int \operatorname{cosec}^4 x \, dx$.

**9** Find $\int \sqrt{\cos x} \sin^3 x \, dx$.

**10** Find $\int \sin^2 x \cos^4 x \ dx$.

## Medium

**11** Find $\int \dfrac{\cos 2x}{\cos x} dx$.

**12** Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \dfrac{\tan x}{\cos^2 x} dx$.

**13** Find $\int (\cos^4 x - \sin^4 x) \, dx$.

**14** Find $\int (\tan^4 x - \sec^4 x) \, dx$.

**15** Find $\int \sec^6 x \, dx$.

**16** The area between $y = \sec^2 x \tan^3 x$ and the $x$−axis between $x = 0$ and $x = \dfrac{\pi}{4}$ is rotated about the $x$−axis. What is the volume of solid of revolution formed?

## Challenging

**17** Find $\int \frac{(\cos x + \sin x)^3}{\sin 2x + 1}\,dx$.

**18** Find $\int \frac{\sqrt{1+\sin x}}{\sec x}\,dx$.

**19** Find $\int \cot^2 x\,dx$.

**20** Find $\int \frac{\sin^5 x}{\cos^4 x}\,dx$.

**21** Find $\int \cot^3 x\,dx$.

**22** Find $\int \cot^3 x \operatorname{cosec}^3 x\,dx$.

**23** Show that $\int \sqrt{1+\sin x}\,dx = 2\sin\frac{x}{2} - 2\cos\frac{x}{2} + c$.

# 4.9 Expressions and equations using the $t$-formulas

In Lesson 9 we cover:

- $t-$formulas
- Using $t-$formulas to simplify expressions
- Using $t-$formulas to solve equations

This lesson covers the 5th dot point from *Further integration*.

## $t-$formulas

Some proofs, equations and integrals are difficult to solve using standard trig techniques, but we can use the double angle results to find expressions that allow us to use algebraic approaches.

If we let $t = \tan\frac{A}{2}$, then we can write each of the trigonometric ratios in terms of $t$ – these are called the $t$-formulas and are listed below.

$$\sin A = \frac{2t}{1+t^2}$$

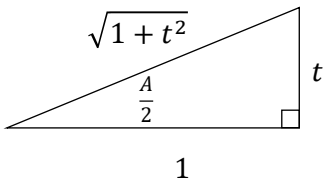
$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

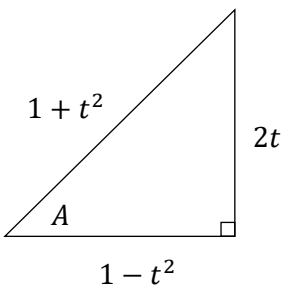
These identities ARE given on the reference sheet.

Note that tan is undefined for odd multiples of $\frac{\pi}{2}$ (odd multiples of $90°$), so $\frac{A}{2}$ cannot equal $\frac{\pi}{2}, \frac{3\pi}{2}$ etc (or $90°, 270°$). This means that $A$ cannot equal $\pi, 3\pi$ etc (or $180°, 540°$). This becomes important later in the lesson when we use $t$-formulas to solve equations.

You can think of the $t$-formulas as the Swiss army knife of trigonometry - not always the best tool for the job, but useful for all sorts of little odd jobs.




When $A$ is acute we can visualise $t = \tan\frac{A}{2}$ using the triangle above right and the $t-$formulas using the triangle below right. These diagrams become useful in solutions.



## Example 1

Prove $\sin A = \dfrac{2t}{1+t^2}$, where $t = \tan\dfrac{A}{2}$.

$$\begin{aligned}
\sin A &= \sin\left[2\left(\frac{A}{2}\right)\right] \\
&= 2\sin\frac{A}{2}\cos\frac{A}{2} && \text{using the double angle result} \\
&= \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}} && \text{since } \cos^2\theta+\sin^2\theta = 1 \\
&= \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} && \text{dividing the numerator and denominator by } \cos^2\frac{A}{2} \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

We can prove $\cos A = \frac{1-t^2}{1+t^2}$ in a similar manner, while the proof of $\tan A = \frac{2t}{1-t^2}$ is simpler as we don't need to use the Pythagorean identity.

## Using $t$−formulas to simplify expressions

In the next set of examples, we will use $t$−formulas to simplify expressions and find the exact value of trigonometric ratios.

## Example 2

**a** Use $t$−formulas to prove that $\dfrac{1-\cos A}{\sin A} = \tan\dfrac{A}{2}$.

$$\begin{aligned}
\frac{1-\cos A}{\sin A} &= \frac{1-\dfrac{1-t^2}{1+t^2}}{\dfrac{2t}{1+t^2}} \\
&= \frac{1+t^2-1+t^2}{2t} \\
&= \frac{2t^2}{2t} \\
&= t \\
&= \tan\frac{A}{2}
\end{aligned}$$

Be careful when you are working with fractions within fractions – this is a common cause of simple mistakes in the HSC. Make your handwriting large and use plenty of space on the page!

We multiplied the numerator and denominator by $1+t^2$ to remove the fractions within fractions correctly.

**b** Hence show that $\tan 15° = 2-\sqrt{3}$.

$$\begin{aligned}
\tan 15° &= \tan\left(\frac{30°}{2}\right) \\
&= \frac{1-\cos 30°}{\sin 30°} \qquad \text{from (a)} \\
&= \frac{1-\dfrac{\sqrt{3}}{2}}{\dfrac{1}{2}} \\
&= 2(1) - 2\left(\frac{\sqrt{3}}{2}\right) \\
&= 2-\sqrt{3}
\end{aligned}$$

## Example 3

Prove that $\dfrac{2}{1+\cos x} = \sec^2\left(\dfrac{x}{2}\right)$.

$$\begin{aligned}
\frac{2}{1+\cos x} &= \frac{2}{1+\dfrac{1-t^2}{1+t^2}} \\
&= \frac{2+2t^2}{1+t^2+1-t^2} \\
&= \frac{2\left(1+t^2\right)}{2} \\
&= 1+t^2 \\
&= 1+\tan^2\left(\frac{x}{2}\right) \\
&= \sec^2\left(\frac{x}{2}\right)
\end{aligned}$$

Let's prove that $\tan 15° = 2-\sqrt{3}$ directly from the $t-$formulas.

## Example 4

Use the formula $\sin\theta = \dfrac{2t}{1+t^2}$ to show that $\tan 15° = 2-\sqrt{3}$.

let $t = \tan 15°$

$$\sin 30° = \frac{2t}{1+t^2}$$

Note $\sin 150°$ is also $\frac{1}{2}$, so we will get two solutions to this equation, but only one of them answers the question.

$$\frac{1}{2} = \frac{2t}{1+t^2}$$

$$1+t^2 = 4t$$

$$t^2-4t+1 = 0$$

This quadratic has two solutions for $t$. The smaller one is the value of $\tan 15°$ and the larger one is the value of $\tan 75°$ since $\tan$ is an increasing function in the domain and $0° < 15° < 75° < 90°$, so we only use '$-$' rather than '$\pm$' in the quadratic formula.

$$t = \frac{4-\sqrt{(-4)^2-4(1)(1)}}{2\times 1}$$

$$= \frac{4-\sqrt{12}}{2}$$

$$= \frac{4-2\sqrt{3}}{2}$$

$$= 2-\sqrt{3}$$

## Using $t-$formulas to solve equations

In Extension 1 you will have used a variety of methods to solve trigonometric equations. In the next few examples, we will use $t-$formulas to solve trigonometric equations.

As we noted earlier in the lesson, when we use $t-$formulas, $A$ cannot equal any odd multiple of $\pi$ since $\tan\frac{A}{2}$ is undefined for these angles. This means that when we solve equations using $t-$formulas, we will only find solutions that are not odd multiples of $\pi$, and so we need to manually check whether any angles in the domain for which $\tan\frac{A}{2}$ is undefined also solve the equation.

> Expect to lose marks in the Extension 2 HSC unless you specifically check whether $\pi$ radians or $180°$ is also a solution or justify why it cannot be a solution.

We will generally look for solutions in the domain $[0, 2\pi]$, so we only need to manually check whether $\pi$ also solves the equation. If the equation is solved over a different domain, then you will need to manually check all odd multiples of $\pi$ within the domain.

In the next example we will solve the same equation using the auxiliary angle method from Extension 1 and then using $t-$formulas. The equation has been chosen since $x = \pi$ is a solution but it will not appear using $t-$formulas, only from the manual check.

While the auxiliary angle method shouldn't be required in Extension 2, it can still be used to solve some trig equations where $t-$formulas are used. I would expect that the standard expected in Extension 2 will be higher than in Extension 1, so we will follow the trend from recent Extension 1 HSCs and avoid using the shortcut method. I will use one method in the next example, and a different method in the exercises.

## Example 5

**a** Solve $\sin x + \cos x = -1$ in the domain $[0,2\pi]$ by first writing $\sin x + \cos x$ in the form $R\sin(x+\alpha)$ where $\alpha$ is acute.

$$\sin x + \cos x = R(\sin x \cos\alpha + \cos x \sin\alpha)$$
$$R\cos\alpha = 1 \quad (1) \quad (\cos\alpha > 0)$$
$$R\sin\alpha = 1 \quad (2) \quad (\sin\alpha > 0)$$
$\therefore \alpha$ is in the 1st quadrant

Get into the habit of finding the values of $R$ and $\alpha$ using simultaneous equations, rather than using the shortcuts.

$$(1)^2 + (2)^2: \quad R^2(\cos^2\alpha + \sin^2\alpha) = 1^2 + 1^2$$
$$R^2 = 2$$
$$R = \sqrt{2}$$

$$(2) \div (1): \quad \frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{1}$$
$$\tan\alpha = 1$$
$$\alpha = \frac{\pi}{4}$$
$$\therefore \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$
$$\therefore \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = -1$$
$$\sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$
$$x + \frac{\pi}{4} = \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$x = \pi, \frac{3\pi}{2}$$

**b** Solve $\sin x + \cos x = -1$ in the domain $[0,2\pi]$ using t formulas.

$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = -1$$

The solutions to this equation with $t$ will include all solutions to the original equation that are NOT odd multiples of $\pi$.

$$2t + 1 - t^2 = -1 - t^2$$
$$2t = -2$$
$$t = -1$$
$$\tan\frac{A}{2} = -1$$
$$\frac{A}{2} = \frac{3\pi}{4}$$

This is the only solution to the original equation that is NOT an odd multiple of $\pi$.

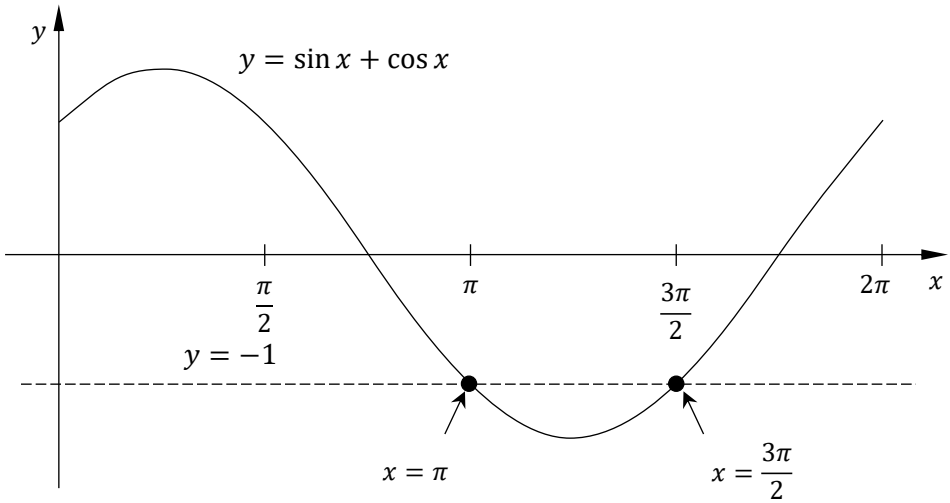
$$A = \frac{3\pi}{2}$$

Check $x = \pi$: $\sin\pi + \cos\pi = 0 + (-1) = -1$ ✓

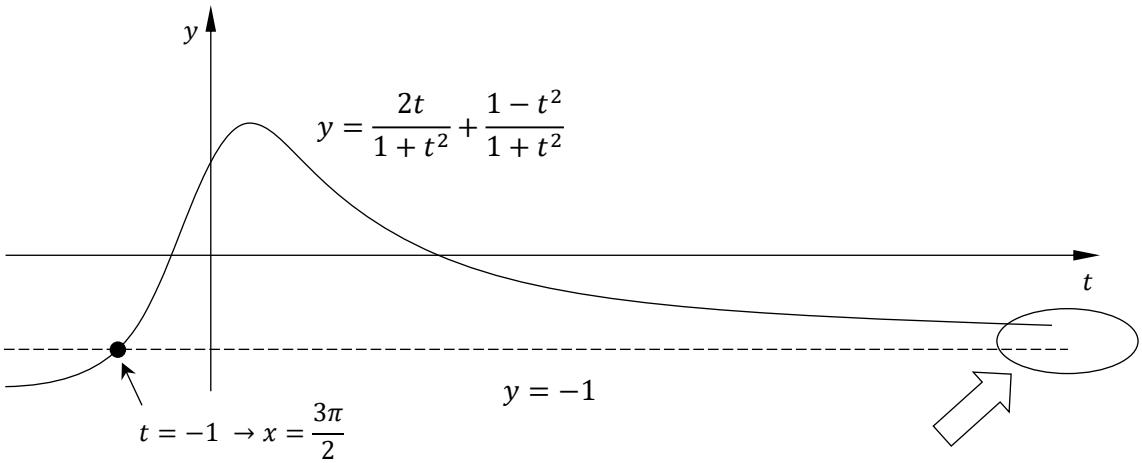
Always manually check whether any odd multiple of $\pi$ in the domain solves the equation when using $t$ – formulas.

$$\therefore x = \pi, \frac{3\pi}{2}$$

How can one solution just disappear? If we look at the graphs of $y = \sin x + \cos x$ and $y = -1$ within the domain $[0, 2\pi]$, shown below, we can see that they intersect when $x = \pi$ and $x = \frac{3\pi}{2}$, which matches the two solutions to the equation.




When we look at the graphs of $y = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$ and $y = -1$ with an unrestricted domain, shown below, we can see that there is only one point of intersection when $t = -1$, which leads to the solution $x = \frac{3\pi}{2}$.




What happened to the other solution? Even though there isn't a second point of intersection, we can see that as $y = -1$ is an asymptote to $y = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$, the curve and the line get closer as $t \to \infty$, in other words as $\tan\frac{A}{2} \to \infty$. This shows $\frac{A}{2} \to \frac{\pi}{2}$ and so $A \to \pi$. This is where the second solution of $x = \pi$ has 'disappeared'. If we extended the graph to the left, we would also see that the graph approaches the same asymptote as $t \to -\infty$, and if our domain included $-\pi$ then it would also be a solution to the equation.

Could we tell that $x = \pi$ is a solution without manually checking whether $x = \pi$ solves the equation? Yes, but it is much easier to just check whether $x = \pi$ solves the equation, plus what we look at below hasn't been in the last two syllabuses.

The alternative way to find if $x = \pi$ is a solution involves what happens to $y$ as $t \to \infty$ - if it approaches the RHS of the equation then $x = \pi$ is also a solution.

In this case:

$$y = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$$

$$= \frac{-t^2 + 2t + 1}{t^2 + 1}$$

Do NOT use this method in an exam – it wastes a massive amount of time and may not earn you full marks.

$$= \frac{-1 + \frac{2}{t} + \frac{1}{t^2}}{1 + \frac{1}{t^2}}$$ dividing the numerator and denominator by $t^2$

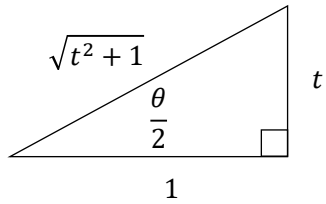
Now $\lim\limits_{t\to\infty}\left(\frac{-1 + \frac{2}{t} + \frac{1}{t^2}}{1 + \frac{1}{t^2}}\right) = \left(\frac{-1+0+0}{1+0}\right) = \frac{-1}{1} = -1.$

Since this matches the RHS of the equation that means all odd multiples of $\pi$ in the domain are also solutions – in this case it means that $x = \pi$ is also a solution to the equation.

## Example 6

Solve $\cos\theta - \sin\frac{\theta}{2} = 0$ for $-\pi \le \theta \le \pi$.

To find $\sin\frac{\theta}{2}$ in terms of $t$, draw a quick diagram where $\tan\frac{\theta}{2} = \frac{t}{1}$.




$$\frac{1-t^2}{1+t^2} - \frac{t}{\sqrt{t^2+1}} = 0$$

$$\frac{1-t^2}{1+t^2} = \frac{t}{\sqrt{t^2+1}}$$

$$1-t^2 = t\sqrt{1+t^2}$$

$$1-2t^2+t^4 = t^2+t^4$$

Since we have squared both sides of the equation, we must check that we have not introduced false solutions by checking that any solutions solve $1-t^2 = t\sqrt{1+t^2}$.

$$3t^2 = 1$$

$$t^2 = \frac{1}{3}$$

$$t = \frac{1}{\sqrt{3}} \quad \text{since } t = -\frac{1}{\sqrt{3}} \text{ does not solve } 1-t^2 = t\sqrt{1+t^2}$$

$$\tan\frac{\theta}{2} = \frac{1}{\sqrt{3}}$$

$$\frac{\theta}{2} = \frac{\pi}{6} \quad \text{since } -\pi \le \theta \le \pi \rightarrow -\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{\pi}{2}$$

$$\theta = \frac{\pi}{3}$$

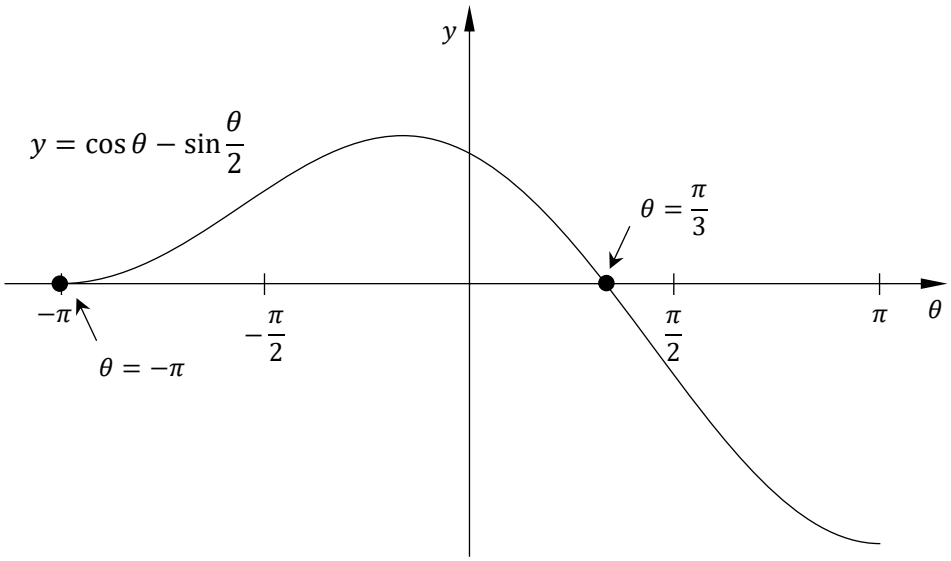
Check $\theta = -\pi$: $\cos(-\pi) - \sin\left(\frac{-\pi}{2}\right) = -1-(-1) = 0$ ✓

Check $\theta = \pi$: $\cos\pi - \sin\frac{\pi}{2} = -1-1 = -2 \neq 0$ ✗

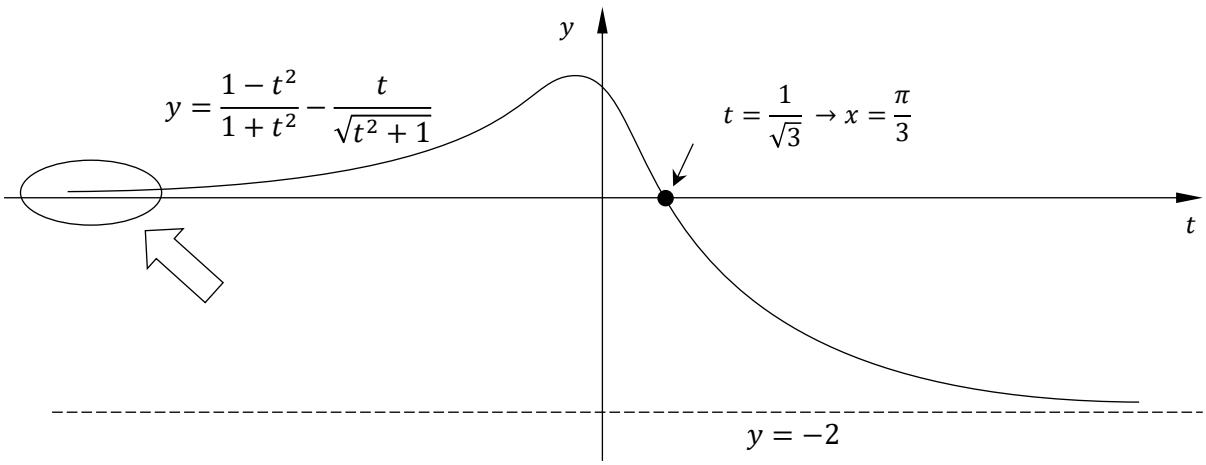
Always check all angles in the domain for which $t$ is undefined, even if it will not lead to a solution.

The solutions are $\theta = -\pi, \frac{\pi}{3}$.

If we look at the graph of $y = \cos\theta - \sin\frac{\theta}{2}$, we can see that it intersects the $\theta$-axis at two points in the domain, giving us the solutions of $x = -\pi, \frac{\pi}{3}$.




When we look at the graph of $y = \frac{1-t^2}{1+t^2} - \frac{t}{\sqrt{t^2+1}}$ within an unrestricted domain, shown below, we can see that there is one point of intersection when $t = \frac{1}{\sqrt{3}}$, which leads to the solutions of $x = \frac{\pi}{3}$.




There are two horizontal asymptotes - $y = 0$ as $t \rightarrow -\infty$ and $y = -2$ as $t \rightarrow \infty$. Since the RHS of the equation is $0$, this means that since $y = \frac{1-t^2}{1+t^2} - \frac{t}{\sqrt{t^2+1}}$ approaches $0$ as $t \rightarrow -\infty$, then $\frac{\theta}{2} \rightarrow -\frac{\pi}{2}$, so $\theta \rightarrow -\pi$. This gives us the second solution to the equation.

In these two examples, either $-\pi$ or $-\pi$ was a solution but that won't always be the case – these examples have been chosen to increase our understanding.

## Basic

**1** Prove $\cos A = \dfrac{1-t^2}{1+t^2}$, where $t = \tan\dfrac{A}{2}$.

**2a** Use $t$–formulas to prove that $\dfrac{1+\cos A}{\sin A} = \cot\dfrac{A}{2}$

**b** Hence show that $\cot 15° = 2+\sqrt{3}$.

**3** Solve $2\tan\dfrac{x}{2} = \sqrt{3}\left(1-\tan^2\dfrac{x}{2}\right)$ for $0 \le x \le 2\pi$.

**4** Prove the following results, where $t = \tan\dfrac{\theta}{2}$:

**a** $\tan\theta\sin\theta = \dfrac{4t^2}{1-t^4}$

**b** $\cos 2\theta = \dfrac{1-6t^2+t^4}{(1+t^2)^2}$

**5** Prove the following results:

**a** $\dfrac{2}{1-\cos x} = \operatorname{cosec}^2\left(\dfrac{x}{2}\right)$

**b** $\dfrac{1+\cos x+\sin x}{1-\cos x+\sin x} = \cot\left(\dfrac{x}{2}\right)$

**6a** Use the formula $\tan\theta = \dfrac{2t}{1-t^2}$ to show that $\tan\dfrac{\pi}{8} = \sqrt{2}-1$.

**b** Use the formula $\sin\theta = \dfrac{2t}{1+t^2}$ to show that $\tan\dfrac{\pi}{8} = \sqrt{2}-1$.

## Medium

**7** Prove $\cot^2\theta = \dfrac{1+\cos 2\theta}{1-\cos 2\theta}$.

**8a** Solve $\sin x - 2\cos x = 2$ in the domain $[0, 2\pi]$ by first writing $\sin x - \cos x$ in the form $R\sin(x-\alpha)$ where $\alpha$ is acute, giving answers to 2 decimal places if needed.

**b** Solve $\sin x - \cos x = 1$ in the domain $[0,2\pi]$ using t formulas, giving answers to 2 decimal places if needed.

**9** Solve $\cos x - 2\sin x = 1$ in the domain $[-\pi, \pi]$ using $t$–formulas, giving answers to 2 decimal places if needed.

**10** Prove that the only solutions to $\sqrt{3}\sin x - \cos x = 1$ for $[0, 2\pi]$ are $x = \dfrac{\pi}{3}$ and $x = \pi$.

**11** Prove that the only solutions to $\sin x + \cos x = 0.5$ for $[0, 2\pi]$ are $x = 2.00$ and $x = 5.86$, correct to 2 decimal places.

**12** Solve $\sin\theta - \sin\dfrac{\theta}{2} = 0$ for $-\pi \le \theta \le \pi$.

**13** Given $\sec x + \tan x = \tan(ax+b)$, use the substitution $t = \tan\left(\dfrac{x}{2}\right)$ to find $a$ and $b$ when $t \neq -1$.

## Challenging

**14** Use $t$–formulas to solve $3\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) = 3$ for $0 \leq x \leq 2\pi$.

**15** The equation $m\cos x - \sin x = 1$, where $m > 1$, has one acute solution.
(Do NOT prove this.)

**a** Show that $t = \frac{m-1}{m+1}$ where $t = \tan\frac{x}{2}$.

**b** Hence show that the single acute solution is $x = 2\tan^{-1} m - \frac{\pi}{2}$.

**16** The two solutions to the equation $2\sin x - \cos x = 0$ in the domain $[0, 2\pi]$ are $\alpha$ and $\beta$.
Prove that $\tan\frac{\alpha}{2}$ is the negative reciprocal of $\tan\frac{\beta}{2}$.

**17a** Show that $\sqrt{\frac{1-\cos x}{1+\sin x}} = \left|\frac{\sqrt{2}t}{t+1}\right|$ where $t = \tan\frac{x}{2}$.

**b** Hence solve $\sqrt{\frac{1-\cos x}{1+\sin x}} = \frac{1}{\sqrt{2}}$ for $[0, 2\pi]$.

In Lesson 10 we cover:

- Integrating with $t-$formulas
- Other trigonometric substitutions

This lesson mainly covers the 6th dot point from *Further integration*, with questions on the 15th dot point included in the exercises.

## Integrating with $t-$formulas

Integration using $t$-formulas is one of our methods of last resort, as many integrands involving trigonometric functions can be solved more easily using other methods.

Once you have substituted for $t$, the integrand can be rearranged into one of the standard integrals and solved. If you are solving an indefinite integral, remember to rewrite your answer in terms of the original variable so that you have answered the question.

Most questions end up using similar methods through the solution, so we will only go through a couple of examples. Take care during an exam, as simple errors often cost marks - incorrectly simplifying complex fractions (fractions within fractions) or not rewriting the final answer of an indefinite integral in terms of the original variable are common mistakes.

**A note on finding $dx$ or $d\theta$ as a function of $t$**

When we have an integrand as a function of $x$, and use the substitution $t = \tan\frac{x}{2}$, we need to find an expression for $dx$ in terms of $t$. This is always $dx = \frac{2dt}{1+t^2}$. Note that the RHS on the final line is almost the same as the $t$-result for $\sin A$ with $d$ inserted. The $t$-formula for $\sin A$ is on the Reference Sheet.

A common question that arises is - can we just use this result, or do we need to derive it each time? For most questions you can just memorise and use this result. See the marking guidelines for Q11(f) in 2022 and Q11(d) in 2024, where that is exactly what has been done. Other marking guidelines use the full derivation, but I would expect that you can achieve full marks using the shortcut.

There is always the possibility that part (i) of a question could ask you to derive the result - see Q1(e) in 2005.

See the full derivation on the right, which uses the Pythagorean identity $\sec^2\frac{x}{2} = 1 + \tan^2\frac{x}{2}$.

$$
\begin{aligned}
t &= \tan\frac{x}{2} \\
\frac{dt}{dx} &= \frac{1}{2}\sec^2\frac{x}{2} \\
&= \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) \\
&= \frac{1}{2}\left(1 + t^2\right) \\
dx &= \frac{2dt}{1+t^2}
\end{aligned}
$$

## Example 1

Find $\int \frac{dx}{1+\cos x}$ using $t = \tan\frac{x}{2}$.

$$
\begin{aligned}
&\int \frac{dx}{1+\cos x} \qquad \boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}} \\
&= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2} \\
&= 2\int \frac{1}{1+t^2+1-t^2}\,dt \\
&= 2\int \frac{1}{2}\,dt \\
&= t + c \\
&= \tan\frac{x}{2} + c
\end{aligned}
$$

Take care when simplifying nested fractions within integrals - this is a very common exam mistake.

## Example 2

Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\cos x}$ using $t = \tan\frac{x}{2}$.

$$
\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\cos x} \qquad \boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}} \\
&= \int_0^1 \frac{1}{4+5\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2} \\
&= 2\int_0^1 \frac{dt}{4+4t^2+5-5t^2} \\
&= 2\int_0^1 \frac{dt}{9-t^2} \\
&= \frac{1}{3}\int_0^1 \left(\frac{1}{3-t} + \frac{1}{3+t}\right) dt \\
&= \frac{1}{3}\Big[-\ln(3-t) + \ln(3+t)\Big]_0^1 \\
&= \frac{1}{3}\big((-\ln 2 + \ln 4) - (-\ln 3 + \ln 3)\big) \\
&= \frac{1}{3}\ln\frac{4}{2} \qquad \left(= \frac{1}{3}\ln 2\right)
\end{aligned}
$$

## Trigonometric substitutions

Another method we can use when nothing else works is to use a trig substitution, though they are rarely needed. Often, we can use a $u^2$ substitution instead and save a lot of work, as trig substitutions are often quite long.

Trig substitutions can be used whenever the integrand involves a square root (including in the denominator) where the radicand is the sum or difference of two squares. This might involve completing the square to make the appropriate substitution obvious.

In the following examples we will see how the trig substitution gives us an expression that we can then simplify using a Pythagorean identity.

## Example 3

Using the substitution $x = \sec\theta$, or otherwise, evaluate $\int x^3\sqrt{x^2-1}\,dx$ where $0 \le x \le \frac{\pi}{2}$.

$$\int x^3\sqrt{x^2-1}\,dx \qquad \boxed{\begin{aligned} x &= \sec\theta \\ dx &= \sec\theta\tan\theta\,d\theta \end{aligned}}$$

$$= \int \sec^3\theta\sqrt{\sec^2\theta-1}\sec\theta\tan\theta\,d\theta$$

$$= \int \sec^3\theta\sqrt{\tan^2\theta}\sec\theta\tan\theta\,d\theta$$

The trig substitution chosen is $x = \sec\theta$, as then the radicand becomes the LHS of the Pythagorean identity $\sec^2\theta - 1 = \tan^2\theta$, which simplifies to $\tan\theta$.

$$= \int \sec^4\theta\tan^2\theta\,d\theta$$

$$= \int \sec^2\theta\sec^2\theta\tan^2\theta\,d\theta$$

$$= \int \sec^2\theta(\tan^2\theta+1)\tan^2\theta\,d\theta$$

$$= \int \sec^2\theta\tan^4\theta\,d\theta + \int \sec^2\theta\tan^2\theta\,d\theta$$

$$= \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3} + c \qquad \boxed{\begin{aligned} \tan^2\theta &= \sec^2\theta - 1 \\ &= x^2 - 1 \\ \tan\theta &= \sqrt{x^2-1} \end{aligned}}$$

We take the positive root only, since $\tan x > 0$ in the domain

$$= \frac{\sqrt{(x^2-1)^5}}{5} + \frac{\sqrt{(x^2-1)^3}}{3} + c$$

## Example 4

Use a suitable trigonometric substitution to find $\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$, where $x > 0$.

$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx \qquad \boxed{\begin{array}{l} x = 2\tan u \\ dx = 2\sec^2 u\,du \end{array}}$$

The trig substitution chosen is $x = 2\tan u$, as then the radicand becomes the LHS of the Pythagorean identity $4(1+\tan^2 u) = 4\sec^2 u$, which then simplifies to $2\sec u$.

$$= \int \frac{1}{4\tan^2 u\sqrt{4+4\tan^2 u}} \times 2\sec^2 u\,du$$

$$= \frac{1}{2}\int \frac{\sec^2 u}{\tan^2 u\sqrt{4\sec^2 u}}\,du$$

$$= \frac{1}{2}\int \frac{\sec^2 u}{\tan^2 u \times 2\sec u}\,du$$

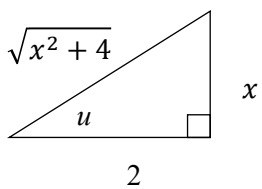
$$= \frac{1}{4}\int \frac{\sec u}{\tan^2 u}\,du$$

$$= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\,du$$

$$= \frac{1}{4}\int (\sin u)^{-2} \times \cos u\,du$$

$$= \frac{1}{4}\int \frac{d}{du}(-(\sin u)^{-1})\,du$$

$$= -\frac{1}{4\sin u} + c$$




$$= -\frac{1}{4\left(\frac{x}{\sqrt{x^2+4}}\right)} + c$$

$$= -\frac{\sqrt{x^2+4}}{4x} + c$$

Integrals in the form $\int \frac{1}{x\sqrt{x^n - c}}\,dx\ n = 1,2,3\ldots$ can be solved using $u^2 =$ radicand, avoiding trig substitutions. For example, in the integral $\int \frac{1}{x\sqrt{x-4}}\,dx$ we would let $u^2 = x - 4$.

## Trigonometric substitutions (continued)

We now return to an indefinite integral we first encountered in Lesson 3 Example 4, where we found the integral by rationalising the numerator. That is the simplest method for this integral, with 5 lines of working.

In the next example we will use a trig substitution which takes 8 lines. We will also solve this same integral in the exercises starting with a $u^2$ substitution - this takes 14 lines including a second substitution.

## Example 5

Using the substitution $x = \cos 2\theta$ , or otherwise, evaluate $\int \sqrt{\dfrac{1-x}{1+x}}\,dx$.

$$\int \sqrt{\frac{1-x}{1+x}}\,dx \qquad \boxed{\begin{array}{l} x = \cos 2\theta \\ dx = -2\sin 2\theta\, d\theta \end{array}}$$

$$= -2\int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\sin 2\theta\, d\theta$$

$$= -4\int \sqrt{\frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta-1)}}\sin\theta\cos\theta\, d\theta$$

$$= -4\int \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\sin\theta\cos\theta\, d\theta$$

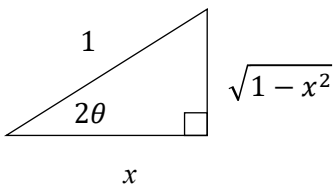
$$= -4\int \frac{\sin\theta}{\cos\theta}\times\sin\theta\cos\theta\, d\theta$$

$$= -4\int \sin^2\theta\, d\theta$$

$$= -4\int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= -2\theta + \sin 2\theta + c$$

$$= -\cos^{-1}(x) + \sqrt{1-x^2} + c$$



## Basic

**1** Find $\int \frac{dx}{1+\sin x}$ using the substitution $t = \tan\frac{x}{2}$.

**2** Using the substitution $t = \tan\frac{x}{2}$, or otherwise, find $\int \frac{5}{12+13\cos x}\,dx$.

**3** Using the substitution $x = \sin\theta$, or otherwise, find $\int x^3\sqrt{1-x^2}\,dx$.

**4** Using the substitution $x = 2\sec\theta$, or otherwise, find $\int \frac{1}{x\sqrt{x^2-4}}\,dx$.

**5** Let $t = \tan\frac{\theta}{2}$

**a** Show that $\frac{dt}{d\theta} = \frac{1}{2}(1+t^2)$

**b** Show that $\sin\theta = \frac{2t}{1+t^2}$

**c** Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \operatorname{cosec}\theta\ d\theta$.

**6** Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, find $\int \frac{1}{1-\cos x}\,dx$.

## Medium

**7** Using the substitution $t = \tan\frac{\theta}{2}$ evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$.

**8** Using the substitution $x = 4\tan^2 u$ find $\int \sqrt{x+4}\,dx$.

**9** Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx$.

**10** Using the substitution $x = \sin^2\theta$, or otherwise, evaluate $\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx$.

**11** Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, evaluate $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta}$.

**12** Prove $\int \operatorname{cosec} x\,dx = \ln\left|\tan\frac{x}{2}\right| + c$ using the substitution $t = \tan\frac{x}{2}$.

**13** Find $\int \frac{1}{\sin x + \cos x}\,dx$ using the substitution $t = \tan\frac{x}{2}$.

**14a** Use the substitution $t = \tan\frac{x}{2}$ to show that $\operatorname{cosec} x + \cot x = \cot\frac{x}{2}$.

**b** Hence show that $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{cosec} x + \cot x)\, dx = \ln 2$.

**15** Find $\int \frac{\sec^2 x}{1 - \tan^2 x}\, dx$.

Challenging

**16** Use a suitable trig substitution to find $\int \frac{x^2}{\sqrt{4 - x^2}}\, dx$.

**17** Use a suitable trig substitution to find $\int \frac{3 + \cos x}{2 - \cos x}\, dx$.

**18** Use a suitable trig substitution to find $\int \sqrt{2x - x^2}\, dx$.

**19** Use the substitution $u^2 = 1 + x$ to find $\int \sqrt{\frac{1 - x}{1 + x}}\, dx$.

**20** Use the fact that $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{(1 + \sin x)^3}$.

**21a** Show $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \frac{1}{2}\sin x} = \frac{2\pi}{3\sqrt{3}}$.

**b** Show $\int_0^{2a} f(x)\, dx = \int_0^a [f(x) + f(2a - x)]\, dx$.

**c** Hence evaluate $\int_0^{\pi} \frac{x\, dx}{1 + \frac{1}{2}\sin x}$.

In Lesson 11 we cover:

- Integration by parts
- Integration by parts using DETAIL
- Integration by parts using the reverse chain rule
- Other applications of integration by parts
- Tabular integration by parts (optional)

This lesson covers the 11th and 12th dot points from *Further integration*.

## Integration by parts

Integration by parts (IBP) is used to find or evaluate integrals when other methods don't work. You can use it on any integral - you will just waste your time and effort. We will mainly use IBP on integrals which are the product of two different types of functions, such as $e^x \sin x$, which is the product of an exponential function and a trigonometric function, though there are a small number of other integrands where it is required. IBP works for both indefinite and definite integrals.

Integration by parts comes about by taking the product rule from differentiation, integrating both sides and rearranging.

$$\frac{d}{dx}(u \times v) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\therefore \int \frac{d}{dx}(u \times v)dx = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$u \times v = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$\boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

In words:

1. split the integrand into two functions
2. integrate one factor to obtain $v$, which then appears in both terms on the right-hand side
3. the other function is $u$ and stays the same for the primitive function, $uv$, while is differentiated for the integrand, $v\frac{du}{dx}$.

This makes more sense when you see it used in the examples to follow.

We used integration to remove the derivative, but don't forget that when an indefinite integral 'disappears' it leaves a constant. The IBP formula should include $uv + c$ rather than just $uv$, but since there is still an indefinite integral in the formula which will have its own constant, we ignore it for the moment as it will be absorbed into the other constant at the end of the solution.

In some questions you will need to integrate by parts twice to get the answer. We will briefly look at tabular integration by parts, which is generally not recommended, but useful when we need to use IBP twice, though we can use it in any question.

## Integration by parts using DETAIL

Once you split the original expression into the correct functions for $u$ and $\frac{dv}{dx}$ then this work is straightforward. However, if you split the integrand incorrectly then the work becomes either much harder or impossible, and you have to start again.

**The trick to IBP lies in correctly splitting the integrand into one function that is easy to integrate and a second function that is simpler when it is differentiated.**

To split the integrand and make our solution work as easily as possible we have two choices.

(1) DETAIL − This rule of thumb helps us to choose which function is most easily integrated, which we make $\frac{dv}{dx}$, when we have two different types of functions. It works in the vast majority of questions in the HSC or from textbooks.

(2) Reverse chain rule − If one of the functions is a composite function we find $\frac{dv}{dx}$ as the product of the composite function and the derivative of the inner function. This can then be integrated using the reverse chain rule.

There are some other integrands where neither of these shortcuts work, but if we always try and find the most detailed function that can be integrated for $\frac{dv}{dx}$, we will split the function successfully.

**DETAIL**

Let's have a look at the DETAIL mnemonic, which focuses on choosing the function to be integrated.

D – Let $\frac{dv}{dx}$ equal the first function from this list to appear in the integrand

E – Exponential
T – Trigonometric } Easiest to Integrate, so these are the best functions for $\frac{dv}{dx}$

A – Algebraic } Easy to Integrate, so still good for $\frac{dv}{dx}$

I – Inverse Trig
L – Logarithmic } Hard to Integrate, so no good for $\frac{dv}{dx}$

For example, $\int x^2 \ln x \, dx$ has $x^2$ (an Algebraic function) and $\ln x$ (a Logarithmic function).

Since A comes before L in DETAIL we let $\frac{dv}{dx} = x^2$ while $u$ equals the rest of the integrand, in this case $u = \ln x$.

DETAIL focuses on the function that is best to integrate, but it is also interesting to consider how well each function works as $u$. We want $\frac{du}{dx}$ to be much simpler than $u$.

- Logarithmic functions differentiate to become simple algebraic functions, so are the best choice for $u$ if available.
- Inverse trig functions differentiate to become algebraic functions, so are also a good choice for $u$.
- Algebraic functions differentiate easily so are a good choice for $u$.
- Trig functions often cycle from sine to cosine, so while we can use them for $u$ they will require us to integrate twice (see Example 2)
- Exponential functions don't get simpler when they are differentiated, plus we need to know extra rules, so are no good as $u$.

If we only have one function in the integrand we often let $\frac{dv}{dx} = 1$ (as this is easy to integrate) and let $u$ equal the function.

In the exercises we will do one of the integrals using different choices for $u$ and $\frac{dv}{dx}$. In one case, we will use DETAIL; in the other, we will go against it.

- when we follow DETAIL, the answer is straightforward
- when we go against DETAIL, we will still get the correct answer, but it will be a harder solution.

Alternatives to **DETAIL** are **LIATE** or **ILATE**, which focus on choosing the function for $u$ rather than the function for $\frac{dv}{dx}$. In other countries **LIPET** is also used for choosing $u$, where P stands for Polynomials rather than Algebraic. Note that since we will probably never have an integrand with a logarithmic and an inverse trig function it doesn't matter whether LIATE or ILATE is used. Since exponential and trigonometric functions can both be integrated, we can swap the order of those terms and still achieve success.

## Example 1

Find $\int x \ln x \, dx$.

Using DETAIL $\frac{dv}{dx} = x$ and $u = \ln x$

$$\int x \ln x \, dx$$

$$u = \ln x \qquad \frac{dv}{dx} = x$$
$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^2}{2}$$

At the start it can help to draw these lines to see which pairs we need to multiply

$$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \times \frac{1}{x} dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

Using DETAIL $\frac{dv}{dx} = e^x$

## Example 2

Evaluate $\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$.

Using DETAIL $\frac{dv}{dx} = e^x$ and $u = \sin x$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$$u = \sin x \qquad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = \cos x \qquad v = e^x$$

$$= \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$u = \cos x \qquad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = -\sin x \qquad v = e^x$$

We have to use Integration by Parts a second time. Since the $u$ and $v$ don't form part of our actual solution we can use the same variables again. Using DETAIL $\frac{dv}{dx} = e^x$

$$= \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \left(\left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx\right)$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \left[e^x \cos x\right]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \frac{1}{2}\left(\left(e^{\frac{\pi}{2}} - 0\right) - \left(0 - 1\right)\right)$$

$$= \frac{e^{\frac{\pi}{2}} + 1}{2}$$

This is a very common integrand for IBP. $e^x$ stays the same while sine and cosine cycle, requiring us to use IBP twice.

We will come back and solve these two integrals using tabular IBP shortly.

## Integration by parts using the reverse chain rule

The vast majority of questions you come across will be solved using DETAIL, but there are more obscure questions where DETAIL will not work since both functions are of the same type, commonly with one of the functions being a composite function. Here we will use the reverse chain rule. While we could often simplify the integrand first using $u$ substitution it would cause more confusion (you would need a different variable as you couldn't use $u$ twice) and make for a longer solution. This method is also good for recurrence relations where we are not given the rule.

Let $\frac{dv}{dx}$ be the product of the composite function and the derivative of the inner function.

In Example 3 we will look at $\int x^3 e^{x^2}\,dx$.

- $e^{x^2}$ is the composite function.
- The derivative of the inner function, $x^2$, is $2x$
- Let $\frac{dv}{dx} = 2xe^{x^2}$
- That leaves us with $\frac{x^2}{2}$ from the integrand, which gives us $u$.

## Example 3

Find $\int x^3 e^{x^2}\,dx$.

Using RCR $\frac{dv}{dx} = 2xe^{x^2}$ and $u = \frac{x^2}{2}$

$$\int x^3 e^{x^2}\,dx$$

| | |
|---|---|
| $u = \frac{x^2}{2}$ | $\frac{dv}{dx} = 2xe^{x^2}$ |
| $\frac{du}{dx} = x$ | $v = e^{x^2}$ |

$$= \frac{x^2 e^{x^2}}{2} - \int xe^{x^2}\,dx$$
$$= \frac{x^2 e^{x^2}}{2} - \frac{1}{2}\int 2xe^{x^2}\,dx$$
$$= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + c$$

## Other applications of integration by parts

We can also use IBP with the reverse chain rule to integrate integrands with two similar functions, although quite often it is a hidden example of reversing the quotient rule as we will see in the next example.

## Example 4

Find $\int \frac{\ln x}{(1+\ln x)^2}dx$:

**a** Using IBP

$$\int \frac{\ln x}{(1+\ln x)^2}\,dx$$

| | |
|---|---|
| $u = x\ln x$ | $\frac{dv}{dx} = \frac{1}{x}(1+\ln x)^{-2}$ |
| $\frac{du}{dx} = \ln x + 1$ | $v = -(1+\ln x)^{-1}$ |
| | $= -\frac{1}{1+\ln x}$ |

$$= \int \frac{x\ln x}{x(1+\ln x)^2}\,dx$$

$$= -\frac{x\ln x}{1+\ln x} + \int \frac{\ln x + 1}{1+\ln x}\,dx$$

$$= -\frac{x\ln x}{1+\ln x} + x + c$$

$$= \frac{x}{1+\ln x} + c$$

**b** By first showing $\frac{\ln x}{(1+\ln x)^2} = \frac{(1+\ln x)(1) - (x)\left(\frac{1}{x}\right)}{(1+\ln x)^2}$.

$$\frac{\ln x}{(1+\ln x)^2} = \frac{1+\ln x - 1}{(1+\ln x)^2}$$

$$= \frac{(1+\ln x)(1) - (x)\left(\frac{1}{x}\right)}{(1+\ln x)^2}$$

$$\therefore \int \frac{\ln x}{(1+\ln x)^2}\,dx = \int \frac{(1+\ln x)(1) - (x)\left(\frac{1}{x}\right)}{(1+\ln x)^2}\,dx$$

$$= \int \frac{d}{dx}\left(\frac{x}{1+\ln x}\right)dx$$

$$= \frac{x}{1+\ln x} + c$$

There are some integrands that we would think should work using other methods, such as the next example, which is very similar to integrals in Lesson 8 where we used Pythagorean identities.

## Example 5

Given $\int \sec x \, dx = \ln \left| \sec x + \tan x \right|$, use integration by parts to show that

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln \left| \sec x + \tan x \right| + c.$$

$\int \sec^3 x \ dx$

$$\boxed{\begin{array}{ll} u = \sec x & \dfrac{dv}{dx} = \sec^2 x \\ \dfrac{du}{dx} = \sec x \tan x & v = \tan x \end{array}}$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \left( \sec^2 x - 1 \right) dx$$

$$= \sec x \tan x - \int \sec^3 x \ dx + \int \sec x \, dx$$

$$\therefore 2 \int \sec^3 x \ dx = \sec x \tan x + \ln \left| \sec x + \tan x \right| + c$$

$$\int \sec^3 x \ dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln \left| \sec x + \tan x \right| + c$$

## Tabular integration by parts (optional)

A little-known variation of integration by parts is tabular integration by parts. As with any unusual method, I would generally not recommend you use it in the HSC or assessments. The main problem occurs if you make an error - you may achieve no marks at all if the method is not covered by the marking scheme. This section is only included here as I often get asked about this method, and whether it is OK. Personally, I don't use it.

This method is mainly useful for quickly checking an answer you have found using the normal setting out of IBP. The only integrals where I would consider its use would be integrals that require using IBP twice, such as we saw in Example 2, where it is much more efficient.

We start with a table which has three columns.




- The first column is S for sign. We will place $+$ on the first row, then alternate between $-$ and $+$ down the column as we add new rows.
- The second column is D for derivative. We will place the function for $u$ in the first row and differentiate from row to row as we down the column.
- The third column is I for integrate. We will place the function for $\frac{dv}{dx}$ in the first row and integrate from row to row as we go down the column.

We choose the functions for $\frac{dv}{dx}$ and $u$ as normal. If the integrand requires use of the reverse chain rule this method will quickly reach a dead end, and it is much easier to use ordinary IBP.

We continue to add rows to the table until the product of the two terms on the same line is an integrand we can find or evaluate easily. This happens when either we:

- differentiate $u$ until it reaches $0$, so the remaining integral is zero
- create a simple integral we can solve
- create an integral that matches the original (so that we can swing it back with the original)

We will also look at using tabular IBP for a recurrence relation next lesson.

Let's start by solving $\int x \ln x \, dx$ from Example 1, using tabular IBP. As we will see, this integral is not one where tabular IBP is more useful than the normal setting out.

We used DETAIL to find that $\frac{dv}{dx} = x$ and $u = \ln x$, so here is the first row of the table.

| S | D | I |
|---|---|---|
| + | $\ln x$ | $x$ |

The next row is completed by placing $-$ in the S column, differentiating $\ln x$ and finding the primitive of $x$.

| S | D | I |
|---|---|---|
| + | $\ln x$ | $x$ |
| $-$ | $\frac{1}{x}$ | $\frac{x^2}{2}$ |

Here we see that $\frac{1}{x} \times \frac{x^2}{2} = \frac{x}{2}$, and we can easily find $\int \frac{x}{2} dx$.

So, what do we do with this table?

| S | D | I |
|---|---|---|
| + | $\ln x$ | $x$ |
| $-$ | $\frac{1}{x}$ | $\frac{x^2}{2}$ |

1. Green - Multiply S and D on the first row by I on the next row. In other examples we will need to do this more than once.
2. Red - Multiply all three cells on the last line and take their primitive.

So, $\int x \ln x \, dx = + \ln x \times \frac{x^2}{2} - \int \left( \frac{1}{x} \times \frac{x^2}{2} \right) dx$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \times \frac{x^2}{2} + c$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

As I mentioned, this integral is better done using the normal setting out.

Let's compare this example done both ways. We could skip a couple of steps in tabular IBP if we are careful.

$$\int x \ln x \, dx$$

$$\begin{array}{ll} u = \ln x & \dfrac{dv}{dx} = x \\ \dfrac{du}{dx} = \dfrac{1}{x} & v = \dfrac{x^2}{2} \end{array}$$

$$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \times \frac{1}{x} dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

| S | D | I |
|---|---|---|
| + | $\ln x$ | $x$ |
| − | $\dfrac{1}{x}$ | $\dfrac{x^2}{2}$ |

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$
$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

Now let's look at an example where we have to complete more rows of the table until we can find an integral that is easy to integrate.

## Example 6

Find $\int x^2 \sin x \, dx$ using tabular IBP.

Using DETAIL $\dfrac{dv}{dx} = \sin x$ and $u = x^2$

$$\int x^2 \sin x \, dx$$
$$= +(x^2)(-\cos x) - (2x)(-\sin x) + (2)(\cos x) + c$$
$$= (2 - x^2)\cos x + 2x \sin x + c$$

| S | D | I |
|---|---|---|
| + | $x^2$ | $\sin x$ |
| − | $2x$ | $-\cos x$ |
| + | $2$ | $-\sin x$ |
| − | $0$ | $\cos x$ |

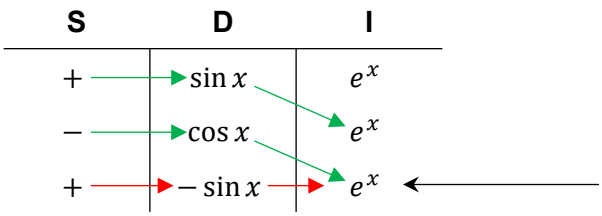
We could stop at the third row as we can easily find $\int (-2\sin x) dx$, but going one more row makes things easier as $\int 0 \, dx$ is even simpler.

Let's redo Example 2. This time we continue the table until we get an integral which is a multiple of the original integral, which we can then swing back to the other side.

## Example 7

Evaluate $\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$ using tabular IBP.

Using DETAIL $\frac{dv}{dx} = e^x$ and $u = \sin x$




Continue until the product of the two terms is a multiple of the original integral.

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \left[ e^x \sin x - e^x \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = e^{\frac{\pi}{2}}(1-0) - 1(0-1)$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \frac{e^{\frac{\pi}{2}} + 1}{2}$$

You can see that this is much shorter than the solution using the normal method.

This is why I would recommend that integrals which require using IBP twice are suitable for using tabular IBP.

## Basic

Use integration by parts to find or evaluate the following integrals.

**1** $\int x^2 \ln x \, dx$

**2** $\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$

**3** $\int x^3 \sin x^2 \, dx$

**4** $\int \ln x \, dx$. Hint: let $u = \ln x$ and $\frac{dv}{dx} = 1$. You may wish to memorise this result.

**5** $\int x e^{2x} \, dx$

**6** $\int_0^{\pi} x \cos x \, dx$

**7** $\int_0^2 te^{-t} \, dt$

**8** $\int_0^1 \tan^{-1} x \, dx$

## Medium

Use integration by parts to find or evaluate the following integrals.

**9** $\int \ln|1 + x| \, dx$

**10** $\int x 2^x \, dx$

**11** $\int \sin^{-1} x \, dx$

**12** $\int_4^{e+3} \ln|x - 3| \, dx$

**13** $\int \tan^{-1} x \, dx$

**14** $\int \ln|\cos x| \operatorname{cosec}^2 x \, dx$

**15** $\int 2^x \sin x \, dx$

## Challenging

**16** Find $\int x^2\sqrt{x-1}\,dx$:

**a** Using IBP

**b** Using a $u^2$ substitution

**17** Find $\int \dfrac{\ln x - 2}{(\ln x - 1)^2}dx$:

**a** Using IBP

**b** By using the quotient rule to find a function whose derivative equals the integrand.

**18** Using the result from Q4, repeat Q1 but ignore DETAIL and let $u = x^2$ and $\dfrac{dv}{dx} = \ln x$.

**19** Find $\int x \sin x \cos x\,dx$.

**20** Find $\int \dfrac{xe^x}{(1+x)^2}dx$.

**21** Prove $\int_0^\infty x^n e^{-x}\,dx = n!$ for $n \geq 0$ by induction.

In Lesson 12 we cover:

- Recurrence relations
- Recurrence relations using integration by parts
- Recurrence relations using other integration methods
- Finding recurrence relations when the relation is not given

This lesson covers the 14th dot point from *Further integration* , with questions on the 15th dot point included in the exercises.

## Recurrence relations

Recurrence relations express an integral (where the integrand includes a function to the power of $n$) as a function of another integral whose integrand includes the same function to a different power of $n$. For example, we can prove that

$$\int_0^1 x^n\sqrt{1-x^2}\,dx = \left(\frac{n-1}{n+2}\right)\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx$$

The first integral includes the term $x^n$ and is expressed as a relation of the integral including the term $x^{n-2}$. The power in the second integral is usually lower, as in this example, which gives us the common alternative name of reduction formula.

Typically, we use abbreviations such as $I_n$ for the integrals, and will find a relationship between $I_n$ and $I_{n-1}$, or $I_n$ and $I_{n-2}$, so, we would write the reduction formula in the above example in the much simpler form

$$I_n = \left(\frac{n-1}{n+2}\right)I_{n-2}.$$

We can use letters other than $I$ for the integral, such as $U$ or $A$, or leave the integral written in complete form.

We often use the recurrence relations to evaluate integrals with higher powers by breaking them down into functions of a lower-powered integral that we can integrate.

Recurrence relations are normally considered to be of medium to higher difficulty, as it is very easy to choose the wrong method and have to start again. Success has traditionally required much practice and a deep knowledge of differentiation and integration. Part of the problem is that while most recurrence relations are found using integration by parts, others can only be solved using standard integration techniques. While the 2024 syllabus only mentions using IBP for recurrence relations, other methods could easily be covered as an application of other dot points, so make sure you cover these types as well.

There is a very easy way to choose the correct method, which converts the topic to a simple mechanical process. This is great for quickly gaining marks in an exam, but our understanding suffers! We will also try to attain a deeper understanding of the work.

We can also use tabular IBP for those recurrence relations where IBP applies but is only an interesting aside - I recommend you use the traditional setting out.

As a rule of thumb, look at the coefficient of the lower-powered integral on the RHS ($I_{n-1}$ or $I_{n-2}$) and see if it involves $n$. Here are three examples:

A $I_n = \frac{e}{2} - nI_{n-1}$ B $I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$ C $I_n = \frac{1}{2n-1} - I_{n-1}$

**Integration by parts**

If the coefficient involves $n$ (examples A and B above) then use IBP. The numerator of the coefficient also gives us an important rule of thumb - use it (or the multiple closest to the power in the original integral) as the power to use for $u$. This rule of thumb requires that the integral with the higher power, when written in index form (usually $n$), is the subject of the reduction formula, so on the LHS. If the integrand involves a power in the denominator we need to rearrange to make the other integral the subject for this rule of thumb to work.

**Other methods**

If the coefficient does not involve $n$ (example C above) then we do not use integration by parts. It is normally easiest to move both integrals to the one side and simplify, then rearrange to find the given result.

Let's have a look at some integrals and recurrence relations to plan the easiest method.

**Original integral** **Recurrence relation to be proved**

$$I_n = \int_1^{e^2} (\log_e x)^n \, dx \qquad I_n = e^2 2^n - nI_{n-1}$$

- The coefficient of $I_{n-1}$ involves $n$ so use IBP.
- The numerator is $n$, so let $u = (\log_e x)^n$

$$I_n = \int_0^x \sec^n t \, dt \qquad I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

- The coefficient of $I_{n-2}$ involves $n$ so use IBP.
- The numerator is $n-2$, so let $u = \sec^{n-2} t$.

$$I_n = \int_0^1 x^{2n+1} e^{x^2} \, dx \qquad I_n = \frac{e}{2} - nI_{n-1}$$

- The coefficient of $I_{n-1}$ involves $n$ so use IBP.
- The numerator is $n$, and in this case we use a multiple of $n$, so $u = x^{2n}$.
- Where there are two functions in the integral, use the function that is already to the power of some function of $n$.

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{2n} \theta \, d\theta \qquad I_n = \frac{1}{2n-1} - I_{n-1}$$

- The coefficient of $I_{n-1}$ is $-1$. As it does not involve $n$ we don't use integration by parts.
- The easiest method is to rearrange the expression and first prove that

$$I_n + I_{n-1} = \frac{1}{2n-1}$$

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx \qquad I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$$

The coefficient of $I_{n-2}$ $n(n-1)$ is the product of two consecutive terms, which indicates that we will integrate by parts twice:

- the first time using $u = x^n$
- the second time using $u = x^{n-1}$

Note that we use the higher term for the first IBP. Since we use IBP twice, this is a situation where tabular IBP can be useful.

## Recurrence relations using integration by parts

Let's start with the most common questions you will encounter for recurrence relations, those where you do need to use IBP.

## Example 1

If $I_n = \int x^n e^x \, dx$ prove that $I_n = x^n e^x - nI_{n-1}$.

The coefficient of $I_{n-1}$ involves $n$, so use IBP. The numerator is $n$ which gives us the power we want, so we let $u = x^n$.

$$I_n = \int x^n e^x \, dx$$

$$\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = nx^{n-1} & v = e^x \end{array}$$

$$= x^n e^x - \int nx^{n-1} e^x \, dx$$

$$= x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - nI_{n-1}$$

We can also solve this using tabular IBP, but the normal method above is recommended.

$$I_n = \int x^n e^x \, dx$$

$$= +(x^n)(e^x) - \int (nx^{n-1})(e^x) \, dx$$

$$\therefore I_n = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - nI_{n-1}$$

| S | D | I |
|---|---|---|
| + | $x^n$ | $e^x$ |
| − | $nx^{n-1}$ | $e^x$ |

Add rows until the product involving the differentiated term produces the required lower-powered integrand.

This is the last time we will discuss tabular IBP, though we will use it as an alternative solution in exercises requiring IBP twice.

## Example 2

If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ prove that $I_n = \frac{n-1}{n} I_{n-2}$.

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

| | |
|---|---|
| $u = \cos^{n-1} x$ | $\frac{dv}{dx} = \cos x$ |
| $\frac{du}{dx} = -(n-1)\cos^{n-2} x \sin x$ | $v = \sin x$ |

$$= \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \left(1 - \cos^2 x\right) dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$\therefore I_n = (n-1)I_{n-2} - (n-1)I_n$

$nI_n = (n-1)I_{n-2}$

$$I_n = \frac{n-1}{n} I_{n-2}$$

The coefficient of $I_{n-2}$ involves $n$, so use IBP. Looking at the final answer, we see a numerator of $n-1$ which gives us the power we want, so we let $u = \cos^{n-1} x$.

## Example 3

In Example 1, we proved that if $I_n = \int x^n e^x \, dx$ then $I_n = x^n e^x - nI_{n-1}$.

Use this recurrence relation to evaluate $\int x^3 e^x \, dx$.

$\int x^3 e^x \, dx$

$= I_3$

$= x^3 e^x - 3I_2$ using the recurrence relation

$= x^3 e^x - 3\left(x^2 e^x - 2I_1\right)$

$= x^3 e^x - 3x^2 e^x + 6(x^1 e^x - 1I_0)$

$= x^3 e^x - 3x^2 e^x + 6xe^x - 6I_0$

$= x^3 e^x - 3x^2 e^x + 6xe^x - 6\int x^0 e^x \, dx$

$= x^3 e^x - 3x^2 e^x + 6xe^x - 6\int e^x \, dx$

$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$

1. We usually start by evaluating $I_0$, as it will be the easiest integral to evaluate. Sometimes $I_1$ or rarely $I_2$ will be easy to integrate.
2. Determine which integral you are trying to find.
3. Use the reduction formula to express the integral you are trying to find in terms of $I_0$.
4. Evaluate.

## Recurrence relations using other integration methods

We will also cover questions where we use methods other than IBP to find the recurrence relation. This could be included under the 15th and 16th dot points in the syllabus which relate to using multiple techniques of integration to solve theoretical and practical problems.

## Example 4

If $I_n = \int \tan^n x \, dx$ prove that $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$.

The coefficient of $I_{n-2}$ does not involve $n$, so move both integrals to the one side and simplify.

$$I_n + I_{n-2} = \int \tan^n x \, dx + \int \tan^{n-2} x \, dx$$

$$= \int \tan^{n-2} x \left(\tan^2 x + 1\right) dx$$

$$= \int \tan^{n-2} x \sec^2 x \; dx$$

$$= \int \frac{d}{dx}\left(\frac{\tan^{n-1} x}{n-1}\right) dx$$

$$= \frac{1}{n-1}\tan^{n-1} x$$

$$\therefore I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$$

## Finding recurrence relations when the relation is not given

In most HSC questions, the recurrence relation will be given in the question, so we can use the tricks from this lesson to quickly choose whether to use IBP, and if so the correct value of $u$.

But how do we proceed if the recurrence relation is not given? This might occur if the examiners want to create a much more difficult question for use late in the paper.

In our shortcut that we have used so far this lesson we have focused on $u$, but if the recurrence relation is not known we shift our focus back to $\frac{dv}{dx}$, choosing it so that it is the most complicated function that we can easily integrate. This often uses IBP with the reverse chain rule that we used last lesson or is sometimes equal to $1$. For trig functions we often need to split the powers so we can use the Pythagorean identities. It is important in all types that the $n$ stays as a power of $u$, not becoming part of $\frac{dv}{dx}$.

## Example 5

Find a recurrence relation for $I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx$, for $n = 0, 1, 2\ldots$, in terms of $I_{n-2}$.

$$I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx$$

$$\begin{array}{ll} u = \frac{1}{2}x^{n-1} & \frac{dv}{dx} = 2x(x^2+1)^{-2} \\ \frac{du}{dx} = \frac{1}{2}(n-1)x^{n-2} & v = -(x^2+1)^{-1} \end{array} \longleftarrow \text{RCR}$$

$$= -\frac{1}{2}\left[\frac{x^{n-1}}{x^2+1}\right]_0^1 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{x^2+1}\,dx$$

$$= -\frac{1}{4} - 0 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}(x^2+1)}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}\int_0^1 \frac{x^n}{(x^2+1)^2}\,dx + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}I_n + \frac{n-1}{2}I_{n-2}$$

$$\therefore \frac{3-n}{2}I_n = -\frac{1}{4} + \frac{n-1}{2}I_{n-2}$$

$$I_n = \frac{1}{2(n-3)} - \frac{n-1}{n-3}I_{n-2}$$

## Basic

**1** If $I_n = \int x^n e^{2x}\,dx$ prove that $I_n = \dfrac{x^n e^{2x}}{2} - \dfrac{n}{2} I_{n-1}$.

**2** If $I_n = \int \sin^n x\,dx$ prove that $I_n = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$.

**3** If $I_n = \int \cot^n x\,dx$ prove that $I_n = -\dfrac{1}{n-1}\cot^{n-1} x - I_{n-2}$.

**4** Use the result from Question 1 to find $\int x^2 e^{2x}\,dx$.

**5** It is given that when $I_n = \int_0^1 x^n\left(x^2-1\right)^5\,dx$ for $n \geq 0$, that $I_n = \dfrac{n-1}{n+11} I_{n-2}$ for $n \geq 2$.

Use this recurrence relation to evaluate $\int_0^1 x^3\left(x^2-1\right)^5\,dx$.

**6** It is given that $I_n = \int_0^{\frac{\pi}{4}} \tan^{2n}\theta\,d\theta$ for $n \geq 0$.

**a** Show that $I_n = \dfrac{1}{2n-1} - I_{n-1}$ for $n \geq 1$.

**b** Hence calculate $I_3$.

**7** It is given that $I_n = \int_1^{e^2} (\ln x)^n\,dx$ for $n \geq 0$. Show that $I_n = e^2 2^n - nI_{n-1}$.

## Medium

**8a** Let $I_n = \int_0^x \sec^n t\,dt$ where $0 \leq x \leq \dfrac{\pi}{2}$. Show that $I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$.

**b** Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sec^4 t\,dt$.

**9** Prove $\int x(\ln x)^n\,dx = \dfrac{x^2(\ln x)^n}{2} - \dfrac{n}{2}\int x(\ln x)^{n-1}\,dx$ and hence find $\int x \ln^2 x\,dx$.

**10** If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$ prove that $I_n = \left(\dfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$.

**11** If $I_n = \int x^n\sqrt{2x+1}\,dx$ prove that $I_n = \dfrac{x^n\sqrt{(2x+1)^3}}{2n+3} - \dfrac{n}{2n+3} I_{n-1}$.

**12** If $I_n = \int x^{-n} e^{2x}\,dx$ prove $I_n = -\dfrac{e^{2x}}{(n-1)x^{n-1}} + \dfrac{2}{n-1} I_{n-1}$ for $n \geq 2$.

**13a** If $I_{m,n} = \int x^m (\ln x)^n \, dx$ prove $I_{m,n} = \dfrac{x^{m+1}(\ln x)^n}{m+1} - \dfrac{n}{m+1} I_{m,n-1}$

**b** Hence find $\int x^2 (\ln x)^2 \, dx$.

**14** If $I_n = \int \dfrac{\sin 3x}{x^n} dx$ and $J_n = \int \dfrac{\cos 3x}{x^n} dx$ prove $I_n = -\dfrac{\sin 3x}{(n-1)x^{n-1}} + \dfrac{3}{n-1} J_{n-1}$ for $n \geq 1$.

Challenging

**15** Find a recurrence relationship for $I_n = \int_0^1 \dfrac{x^n}{(x^2+1)^2} dx$ in terms of $I_{n-2}$, for $n > 3$.

**16** Find a recurrence relationship for $I_m = \int_0^1 x^m (x^2 - 1)^5 \, dx$ in terms of $I_{m-2}$.

**17** Find a recurrence relationship for $I_n = \int_{-3}^0 x^n \sqrt{x+3} \, dx$ in terms of $I_{n-1}$, for $n \geq 1$.

**18** Find a recurrence relationship for $I_n = \int \dfrac{dx}{\sin^n x}$ in terms of $I_{n-2}$.

**19a** Given that $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2} \, dx$ where $n$ is a positive integer, find a recurrence relation for $I_{2n+1}$ in terms of $I_{2n-1}$.

**b** Hence evaluate $I_5$.

**20** Consider the integral $I_n = \int_0^1 \sqrt{x}(1-x)^n \, dx$, for $n = 0, 1, 2, 3, \ldots$
By finding a recurrence relation for $I_n$ in terms of $I_{n-1}$, evaluate $I_3$.

**21** Let $I_n = \int_0^1 (1 - x^2)^n \, dx$ and $J_n = \int_0^1 x^2 (1 - x^2)^n \, dx$.

**a** Show that $I_n = 2nJ_{n-1}$.

**b** Hence show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$.

**c** Show that $J_n = I_n - I_{n+1}$ and hence deduce that $J_n = \dfrac{1}{2n+3} I_n$.

**d** Hence write down a reduction formula for $J_n$ in terms of $J_{n-1}$.

**22a** Show that $\int \dfrac{dx}{5 - 4\cos x} = \dfrac{2}{3} \tan^{-1}\left(3 \tan \dfrac{x}{2}\right) + c$

**b** Given that $\int_0^{\pi} \dfrac{dx}{5 - 4\cos x} = \dfrac{\pi}{3}$ show that $\int_0^{\pi} \dfrac{\cos x}{5 - 4\cos x} dx = \dfrac{\pi}{6}$

**c** If $u_n = \int_0^{\pi} \dfrac{\cos nx}{5 - 4\cos x} dx$ show that $u_{n+1} + u_{n-1} - \dfrac{5}{2} u_n = 0$, for $n \geq 1$.

**23** Let $I_n = \int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^n}$, where $n$ is a positive integer.

**a** Find the value of $I_1$.

**b** Using integration by parts, show that $I_n = \frac{2nI_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)}$.

**c** Hence evaluate $I_3 = \int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^3}$.

**24** Let $I_n = \int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}} dt$ for $n = 0, 1, 2, \ldots$

**a** Show that $t^2\left(1-t^2\right)^{\frac{n-3}{2}} = \left(1-t^2\right)^{\frac{n-3}{2}} - \left(1-t^2\right)^{\frac{n-1}{2}}$.

**b** Using integration by parts, show that $nI_n = (n-1)I_{n-2}$ for $n = 2, 3, 4, \ldots$

**c** Let $J_n = nI_nI_{n-1}$ for $n = 1, 2, 3, \ldots$ By using the principle of mathematical induction, prove that $J_n = \frac{\pi}{2}$ for $n = 1, 2, 3, \ldots$

**d** Briefly explain why $0 < I_n < I_{n-1}$ for $n = 1, 2, 3, \ldots$

**e** Deduce that $\sqrt{\frac{\pi}{2(n+1)}} < I_n < \sqrt{\frac{\pi}{2n}}$ for $n = 1, 2, 3\ldots$

**25** Double factorials are defined as $n!! = \begin{cases} n(n-2)(n-4)\ldots 2 & \text{for even integers } n \\ n(n-2)(n-4)\ldots 1 & \text{for odd integers } n \end{cases}$

**a** Prove that for even integers $n = 2k$ that $(2k)!! = 2^k k!$

**b** Show $(2k+1)!!\, 2^k k! = (2k+1)!$ and thus for odd integers $n = 2k+1$ that $(2k+1)!! = \frac{(2k+1)!}{2^k k!}$.

**c** Prove $I_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$.

**26** Prove $I_{2n} = \int_{-1}^{1} \left(1-x^2\right)^n dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$.

Basic

**1** Find $\int \frac{dx}{x^2+6x+10}$.

**2** Find $\int \frac{1}{x^2+2x+2}dx$.

**3** Let $I_n = \int_1^e (\ln t)^n \, dt$ for $n = 1, 2, 3, \ldots$

**a** Show that $I_n = e - nI_{n-1}$ for $n = 1, 2, 3, \ldots$

**b** Hence, or otherwise, find the exact value $I_3$.

**4** Find $\int x \tan^{-1} x \, dx$.

**5** Find $\int \sin x \cos(\cos x) dx$.

**6** Which of the following is an expression for $\int \frac{1}{\sqrt{7-6x-x^2}}dx$?

A $\sin^{-1}\left(\frac{x-3}{2}\right)+c$ B $\sin^{-1}\left(\frac{x+3}{2}\right)+c$ C $\sin^{-1}\left(\frac{x-3}{4}\right)+c$ D $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

**7** Using the substitution $u = e^x + 1$ or otherwise, evaluate $\int_0^1 \frac{e^x}{(1+e^x)^2}dx$.

**8** Find $\int \frac{1}{x \ln x}dx$.

**9** What is the value of $\int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}}dx$?

A $\frac{\pi^2}{4}$ B $-\frac{\pi^2}{4}$ C $\frac{\pi^2}{8}$ D $-\frac{\pi^2}{8}$

**10** Evaluate $\int_0^{\frac{\pi}{3}} \sec^3 x \tan x \, dx$.

**11** If $n$ is a non−negative integer, then for what values of $n$ is $\int_0^1 x^n \, dx = \int_0^1 (1-x)^n \, dx$?

A no solution B non−zero $n$, only C $n$ even, only D all values of $n$

**12** Find $\int \frac{dx}{(x-1)(x+2)}$.

**13** Evaluate $\int_0^1 xe^{-x^2} \, dx$.

**14a** Find real numbers $a, b$ and $c$ such that $\frac{10}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$.

**b** Hence find $\int \frac{10}{(x+1)(x^2+4)}dx$.

**15** Find $\int \frac{t^2-1}{t^3}dt$.

**16** Find $\int \frac{dx}{\sqrt{6-x-x^2}}$.

**17** Find $\int x^2 \log_e(3x)\,dx$.

**18** Find $\int \frac{\cos\theta}{\sin^4\theta}d\theta$.

**19** Find $\int \frac{dx}{\sqrt{2-4x-2x^2}}$.

**20** Prove the following results:

**a** $\frac{\cos 2x-\cos 4x}{\sin 2x+\sin 4x}=\tan x$, for $\sin 3x \neq 0$.

**b** $\sin x+\sin 3x=4\sin x\cos^2 x$.

**c** $\cos x+\cos 3x=4\cos^3 x-2\cos x$.

**d** $2\tan x\cos 3x=\sec x\,(\sin 4x-\sin 2x)$.

**21** Find $\int x\cos x\,dx$.

**22** Find $\int \frac{x+1}{x-2}dx$.

**23** Which of the following is an expression for $\int \frac{x}{\sqrt{16-x^2}}dx$?

A $-2\sqrt{16-x^2}+c$ B $-\sqrt{16-x^2}+c$ C $\frac{1}{2}\sqrt{16-x^2}+c$ D $-\frac{1}{2}\sqrt{16-x^2}+c$

**24** Using $t=\tan\frac{x}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$.

**25** Use the substitution $u=e^x$, or otherwise, find $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$.

**26** Find $\int \frac{4x^3-2x^2+1}{2x-1}dx$.

**27** Find $\int \frac{dx}{\sqrt{3-4x-4x^2}}$.

**28** Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\sin^5 x\,dx$.

**29** Evaluate $\int_0^1 \tan^{-1} x\,dx$.

**30** Find $\int \frac{dx}{x^2-6x+13}$.

**31** Show that $\sin 130° + \sin 10° = \sin 70°$

**32** Find the amplitude and period of $f(x) = \sin 3x \cos x - \cos 2x \sin 2x.$

**33** Given the substitution $t = \tan\dfrac{\theta}{2}$, the expression $\dfrac{1-\cos\theta}{\sin\theta}$ can be written as which of the following.

A $2t$ B $t$ C $\dfrac{1}{t}$ D $0$

**34** Use the substitution $u = 1 + x^2$ to evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\,dx.$

**35** Find $\displaystyle\int x\sqrt{x}\left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right)dx.$

**36** Evaluate $\displaystyle\int_0^{\log_3 2} 3^{x\log_3 2}\,dx.$

Medium

**37** Show that $\cot\left(\dfrac{A+B}{2}\right) = \dfrac{\sin B - \sin A}{\cos A - \cos B}.$

**38** Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \frac{d\theta}{9 - 8\cos^2\theta}$ using the substitution $t = \tan\theta$.

**39** Find $\displaystyle\int \frac{e^{2x}}{e^x + 1}\,dx.$

**40a** Find real numbers $a, b$ and $c$ such that $\dfrac{8-2x}{(1+x)(4+x^2)} = \dfrac{a}{1+x} + \dfrac{bx+c}{4+x^2}.$

**b** Hence evaluate in simplest form $\displaystyle\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\,dx.$

**41** The algebraic fraction $\dfrac{x+1}{5(x+h)^2}$, where $h$ is a non−zero real number can be written in partial fraction form, where $A$ and $B$ are real numbers, as

A $\dfrac{A}{x+h} + \dfrac{B}{x+h}$ B $\dfrac{A}{5x+h} + \dfrac{B}{(x+h)^2}$ C $\dfrac{A}{x+h} + \dfrac{B}{(x+h)^2}$ D $\dfrac{A}{5(x+h)} + \dfrac{B}{x+h}$

**42** The value of $\displaystyle\int_{-1}^{1} \frac{1}{1+e^{-x}}\,dx$ is

A $\dfrac{1}{2}$ B $1$ C $\ln(1+e)$ D $2\ln(1+e)$

**43** Solve $2\cos\theta + 3\sin\theta = -2$ for $0 \le \theta \le 2\pi$ using the substitution $t = \tan\dfrac{\theta}{2}$.

**44** Consider the integral $I_n = \int_0^1 \sqrt{x}(1-x)^n \, dx$, for $n = 0, 1, 2, 3, \ldots$

**a** Show that $I_n = \left(\frac{2n}{2n+3}\right) I_{n-1}$, for $n = 1, 2, 3$.

**b** Hence evaluate $I_3 = \int_0^1 \sqrt{x}(1-x)^3 \, dx$.

**45** Show that $\sin 3x + \sin 2x + \sin x = \sin 2x \, (2\cos x + 1)$.

**46a** Find $a, b$ and $c$ such that $\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}$.

**b** Find $\int \frac{16}{(x^2+4)(2-x)} dx$.

**47** Evaluate $\int_0^1 \sin^{-1} x \, dx$.

**48** Evaluate $\int_0^1 \frac{x}{(x+1)(2x+1)} dx$.

**49** Which of the following equals $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$?

A $\frac{\pi}{4} - \frac{1}{2}\ln 2$ B $\frac{\pi}{4} + \frac{1}{2}\ln 2$ C $\frac{\pi}{4} + 2\ln\sqrt{2}$ D $\frac{\pi}{4} - 2\ln\sqrt{2}$

**50** Which of the following equals $\int (\tan^3 2x + \tan 2x) \, dx$?

A $\frac{1}{4}\tan^4 2x + \frac{1}{2}\sec^2 2x + c$ B $\tan^2 2x + c$ C $\frac{1}{4}\tan^2 2x + c$ D $\frac{1}{2}\tan^2 2x + c$

**51** Solve $\cos x - 3\sin x + 3 = 0$ for $0 \leq x \leq 2\pi$ using the substitution $t = \tan\frac{x}{2}$.

**52** Use the substitution $t = \tan\frac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin\theta} d\theta$.

**53** Find $\int \sqrt{\frac{5-x}{5+x}} dx$.

**54a** Find the real numbers $A, B$ and $C$ such that $\frac{10}{(3+x)(1+x^2)} = \frac{A}{3+x} + \frac{Bx+C}{1+x^2}$.

**b** Hence use the substitution $t = \tan\theta$ to find $\int \frac{10}{3+\tan\theta} d\theta$.

**55** Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}dx$, for $n \geq 0$.

**a** Show that $I_0 = 2\sqrt{2} - 2$.

**b** Given that $I_n + I_{n-1} = \int_0^1 x^{n-1}\sqrt{1+x}\,dx$, show that $I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$.

**c** Hence evaluate $I_2$ in exact form.

**56** Evaluate $\int_1^2 \frac{dx}{x(1+x^2)}$.

**57** Find $\int \frac{x}{\sqrt{1-x}}dx$.

**58** Find $\int e^{-2x}\cos x\,dx$.

**59** Consider the following two statements:

$$\textbf{I:}\quad \int_0^1 \frac{dx}{1+x^n} < \int_0^1 \frac{dx}{1+x^{n+1}} \quad (n > 1)$$

$$\textbf{II:}\quad \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx$$

Which of these statements is correct?

A Neither statement B Statement I only C Statement II only D Both statements

**60** Evaluate $\int_0^1 \cos^{-1} x\,dx$.

**61** Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}}dx$.

**a** Show that $I_1 = \frac{4}{3}$.

**b** Show that $I_{n-1} - I_n = \int_0^1 x^{n-1}\sqrt{1-x}\,dx$.

**c** Use integration by parts on the result of part (b) to show that $I_n = \frac{2n}{2n+1}I_{n-1}$.

**62** Use the substitution $t = \tan\frac{x}{2}$ to find which of the following are equal to $\int \sec x\,dx$.

A $\ln|(t+1)(t-1)| + c$ B $\ln\left|\frac{1+t}{1-t}\right| + c$ C $\ln|(1+t^2)(1-t^2)| + c$ D $\ln\left|\frac{t^2+1}{t^2-1}\right| + c$

**63** Evaluate $\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx$.

**64** Which of the following is equal to $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}dx$?

A $\sin(\sqrt{x}) + c$ B $2\sin(\sqrt{x}) + c$ C $\frac{1}{\sin(\sqrt{x})} + c$ D $\frac{2}{\sin(\sqrt{x})} + c$

**65a** Find real numbers $A$ and $B$ such that $\dfrac{3x^2+3x-2}{(x-1)(x+1)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{1}{(x+1)^2}$.

**b** Hence find $\displaystyle\int \frac{3x^2+3x-2}{(x-1)(x+1)^2}\,dx$.

**66** Use the substitution $x=2\cos\theta$ to evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$.

**67a** Find constants $A, B$ and $C$ such that $\dfrac{x^2-x+1}{(x+1)^2}=A+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$.

**b** Hence find $\displaystyle\int \frac{x^2-x+1}{(x+1)^2}\,dx$.

**68** Show, using integration by parts, that $\displaystyle\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx=\frac{\pi\sqrt{3}}{3}-\ln 2$.

**69** Let $\displaystyle I_n=\int_0^{\pi} x^n\sin x\,dx$, where $n=0,1,2,\ldots$

**a** Use integration by parts to show that $I_n=\pi^n-n(n-1)I_{n-2}$ for $n=2,3,4,..$

**b** Hence evaluate $\displaystyle\int_0^{\pi} x^4\sin x\,dx$.

**70** If $\displaystyle I=\int_0^{\ln 2}\frac{e^x}{e^x+e^{-x}}\,dx$ and $\displaystyle J=\int_0^{\ln 2}\frac{e^{-x}}{e^x+e^{-x}}\,dx$, then the exact value of $I-J$ is:

A $\ln\left(\dfrac{5}{2}\right)$ B $\ln 2$ C $\ln 5$ D $\ln\left(\dfrac{5}{4}\right)$

**71** Given $f(t)=3\sin(4t)$ and $g(t)=3\sin\left(4t+\dfrac{\pi}{3}\right)$, find $f(t)+g(t)$ in the form $R\sin(4t+\alpha)$.

**72a** If $\displaystyle I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$ show that $I_n=\dfrac{n-1}{n}I_{n-2}$.

**b** Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}}\sin^5 x\,dx$.

**73** Find $\displaystyle\int\sin^3 x\,dx$

A $\dfrac{1}{4}\sin^4 x+c$ B $-\cos x+\dfrac{1}{3}\cos^3 x+c$ C $-\dfrac{1}{4}\sin^4 x+c$ D $-\cos x-\dfrac{1}{3}\cos^3 x+c$

**74** Solve $\sin 4x+\sin 3x+\sin 2x=0$ for $0\le x\le\pi$.

**75** Show that $\displaystyle\int x\sqrt{x-1}\,dx=\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$.

**76** Find $\displaystyle\int\frac{x}{\sqrt{2-x^2}}\,dx$ using the substitution $x=\sqrt{2}\sin\theta$.

**77** Which of the following is an expression for $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x}\,dx$? You are given that $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

A $x + \frac{1}{4}\cos 2x + c$ B $x - \frac{1}{4}\cos 2x + c$ C $x + \frac{1}{2}\sin 2x + c$ D $x - \frac{1}{2}\sin 2x + c$

**78** Find $\int \frac{\sin 2x + \sin x}{\cos^2 x}\,dx$.

**79** Find $\int e^x \sin x\,dx$.

**80** Prove $\int \sin 2x \sin 4x\,dx = \frac{\sin 2x}{4} - \frac{\sin 6x}{12} + c$.

**81** Prove $\int \frac{\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)}{\cos\left(x - \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)}\,dx = -\ln|\cos x| + c$.

## Challenging

**82** $\int \frac{x}{\sqrt{x+5}}\,dx =$

A $2\sqrt{x+5} + c$ B $\frac{2}{3}\sqrt{(x+5)^3} + c$

C $\frac{2}{3}\left\{\sqrt{(x+5)^3} - 10\sqrt{x+5}\right\} + c$ D $\frac{2}{3}(x-10)\sqrt{x+5} + c$

**83** By using the fact that $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x + \sin x} = \ln 2$, evaluate $\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1 + \cos x + \sin x}$.

**84** Prove $\int \frac{dx}{x^3\sqrt{x^2-4}} = \frac{1}{16}\cos^{-1}\left(\frac{2}{x}\right) + \frac{\sqrt{x^2-4}}{8x^2} + c$.

**85** Using the substitution $u = \pi - x$:

**a** Show that $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\,dx$.

**b** Hence deduce that $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx = \frac{\pi^2}{4}$.

**86** The value of $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$ is equal to:

A $0$ B $\pi$ C $\frac{\pi}{2}$ D $\frac{\pi}{4}$

**87a** Use the substitution $u = \frac{\pi}{4} - x$, to show $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan x}\right)dx$.

**b** Hence find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\,dx$.

**88** Find $\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\,dx$.

**89** Let $I_n = \int_0^{\frac{\pi}{6}} \sin^{2n}\theta \sec\theta \, d\theta$, for $n \geq 1$.

**a** Show that $I_n - I_{n-1} = -\dfrac{1}{2^{2n-1}(2n-1)}$.

**b** Hence show that $I_n = \dfrac{1}{2}\ln 3 - \sum_{k=1}^{n} \dfrac{1}{2^{2k-1}(2k-1)}$.

**90** Given that $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x \, dx$, for $n = 1, 2, \ldots$

**a** Show that $I_1 = \dfrac{1}{2}\ln 2$.

**b** Show that $I_{n-2} + I_n = \dfrac{1}{n-1}\left(3^{\frac{1}{2}(n-1)} - 1\right)$, for $n = 2, 3, 4, \ldots$

**c** Find $I_5$.

**91** Use the substitution $t = \tan\dfrac{\theta}{2}$ to show that $\int_0^{\frac{\pi}{2}} \dfrac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \dfrac{1}{2}\tan^{-1}\left(\dfrac{1}{2}\right)$.

**92** Show that $\int_0^1 \dfrac{\ln(1+x)}{1+x^2} dx = \dfrac{\pi}{8}\ln 2$, using the substitution $x = \dfrac{1-u}{1+u}$.

**93a** Show that $\int_0^1 \dfrac{5-5x^2}{(1+2x)(1+x^2)} dx = \dfrac{1}{2}\left(\pi + \ln\dfrac{27}{16}\right)$.

**b** Hence evaluate $\int_0^{\frac{\pi}{4}} \dfrac{\cos 2x}{1 + 2\sin 2x + \cos 2x} dx$ using the substitution $t = \tan x$.

**94a** If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$, show that for $n > 1$, $I_n = \left(\dfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$.

**b** Hence find the area of the finite region bounded by the curve $y = x^4 \cos x$ and the $x$−axis for $0 \leq x \leq \dfrac{\pi}{2}$.

**95** Given $I_n = \int_0^1 \dfrac{dx}{(1+x^2)^n}$ for $n \geq 1$ show that $I_n = \dfrac{1}{2^n(1-2n)} + \dfrac{2n}{2n-1} I_{n+1}$.

**96** Let $I_n = \int_0^1 x^n \sqrt{1-x^3} \, dx$, for $n \geq 2$.

**a** Show that $I_n = \dfrac{2n-4}{2n+5} I_{n-3}$ for $n \geq 5$.

**b** Hence find $I_8$.

**97a** Show that $2\sin x \sin\big((2k+1)x\big) = \cos(2kx) - \cos(2(k+1)x)$.

**b** Hence, use mathematical induction to prove for all integers $n \geq 1$, that $\sin x + \sin 3x + \sin 5x + \ldots + \sin\big((2n-1)x\big) = \dfrac{1 - \cos(2nx)}{2\sin x}$.

**98a** Show that $(1+t^2)^{n-1} + t^2(1+t^2)^{n-1} = (1+t^2)^n$.

**b** Given $I_n = \int_0^x (1+t^2)^n \, dt$ for $n = 1, 2, 3, \ldots$, use integration by parts, and part (a) to show $I_n = \frac{1}{2n+1}(1+x^2)^n x + \frac{2n}{2n+1} I_{n-1}$.

**99** For $n = 0, 1, 2, 3, \ldots$, let $I_n = \int_{e^{-1}}^{1} (1+\log_e x)^n \, dx$ and $J_n = \int_{e^{-1}}^{1} (\log_e x)(1+\log_e x)^n \, dx$.

**a** Show that $I_n = 1 - nI_{n-1}$ for $n = 1, 2, 3, \ldots$

**b** Show that $J_n = 1 - (n+2)I_n$ for $n = 0, 1, 2, 3, \ldots$

**c** Hence find the value of $J_3$ in simplest exact form.

**100** By first multiplying the numerator and denominator of the integrand by $\sin 3x$, show that $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx = -\frac{1}{2}\sin 2x - \sin x + c$.

### Standard integrals - trigonometry

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\tan f(x)\,dx = \int \frac{f'(x)\sin f(x)}{\cos f(x)}\,dx = -\ln|\cos f(x)| + c$$

$$\int f'(x)\operatorname{cosec} f(x)\,dx = -\ln|\operatorname{cosec} f(x) + \cot f(x)| + c$$

$$\int f'(x)\sec f(x)\,dx = \ln|\sec f(x) + \tan f(x)| + c$$

$$\int f'(x)\cot f(x)\,dx = \int \frac{f'(x)\cos f(x)}{\sin f(x)}\,dx = \ln|\sin f(x)| + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)\operatorname{cosec}^2 f(x)\,dx = -\cot f(x) + c$$

$$\int f'(x)\sec f(x)\tan f(x)\,dx = \sec f(x) + c$$

$$\int f'(x)\operatorname{cosec} f(x)\cot f(x)\,dx = -\operatorname{cosec} f(x) + c$$

### Standard integrals - other

$$\int f'(x)\,[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq 1$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$$

$$\int f'(x)\,a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}\,dx = \ln\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + c$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}\,dx = \ln\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + c$$

### Trig identities

$$\sin^2 x + \cos^2 x = 1$$
$$\sin^2 x = 1 - \cos^2 x$$
$$\cos^2 x = 1 - \sin^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$
$$\tan^2 x = \sec^2 x - 1$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$
$$\cot^2 x = \operatorname{cosec}^2 x - 1$$

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

### Product to sum identities

$$\cos A\cos B = \frac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big]$$

$$\sin A\sin B = \frac{1}{2}\Big[\cos(A-B) - \cos(A+B)\Big]$$

$$\sin A\cos B = \frac{1}{2}\Big[\sin(A+B) + \sin(A-B)\Big]$$

$$\cos A\sin B = \frac{1}{2}\Big[\sin(A+B) - \sin(A-B)\Big]$$

### *t*-formulas

$$t = \tan\frac{x}{2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$
$$= \frac{1}{2}(1 + t^2)$$

$$dx = \frac{2dt}{1 + t^2}$$

$$\sin x = \frac{2t}{1 + t^2}$$

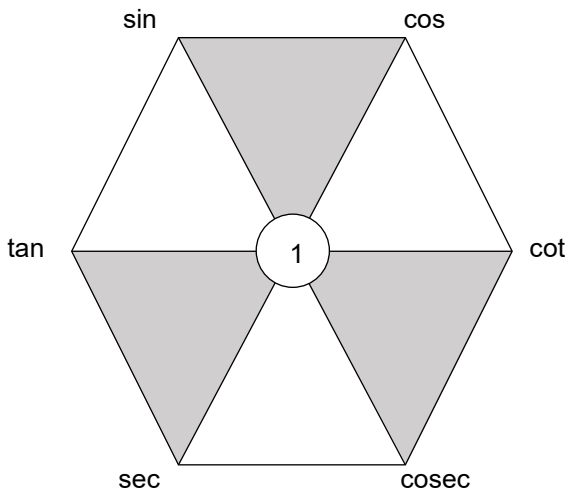
$$\cos x = \frac{1 - t^2}{1 + t^2}$$

$$\tan x = \frac{2t}{1 - t^2}$$

## Hexagon mnemonic

**Co-function relations:** The trigonometric functions cosine, cotangent, and cosecant on the right of the hexagon are co-functions of sine, tangent, and secant on the left respectively.

$\sin(90-\theta) = \cos\theta\,;\tan(90-\theta) = \cot\theta\,;\sec(90-\theta) = \operatorname{cosec}\theta$




**Reciprocal identities:** The two trigonometric functions of any diagonal are reciprocals of each other.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}\,;\sec\theta = \frac{1}{\cos\theta}\,;\cot\theta = \frac{1}{tan\,\theta}$$

**Product identities:** Along the outside edges of the hexagon any trigonometric function equals the product of the functions of the adjacent vertices.

$\tan\theta \times \cos\theta = \sin\theta\,;\sec\theta \times \cot\theta = \operatorname{cosec}\theta$ etc

**Quotient identities:** along the outside edges of the hexagon, any trigonometric function equals the quotient of the next two trigonometric functions going either clockwise or counter clock wise.

$$\sin\theta = \tan\theta \div \sec\theta\,;\cot\theta \div \operatorname{cosec}\theta = \cos\theta \;\; etc$$

**Pythagorean identities:** For each shaded triangle, the upper left function squared plus the upper right function squared, equals the bottom function squared.

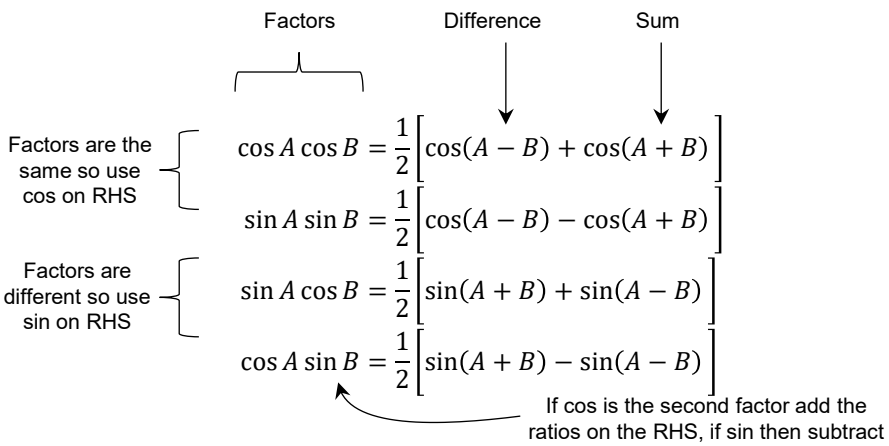
$$\sin^2\theta + \cos^2\theta = 1\,;\tan^2\theta + 1 = \sec^2\theta\,;1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

To rearrange any identity, drop any term from the top line to the right end of the bottom line with a minus in front.

$$\sin^2\theta = 1 - \cos^2\theta\,;\tan^2\theta = \sec^2\theta - 1\,;\operatorname{cosec}^2\theta - \cot^2\theta = 1$$

## Product to sum mnemonic

The fours Product to Sum Identities are very similar, so to save us looking at the Reference Sheet each time, try:




Sin or Cos on the RHS?

- If the factors use the same ratio then use cos on the RHS, if the factors use different ratios then use sin

Sum or Difference of Angles first on RHS?

- If using cos then difference of angles is used first, if using sin then sum of angles is used first

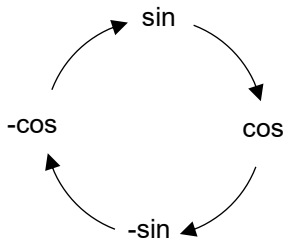
Add or Subtract the Ratios on RHS?

- If cos is the second factor then add the ratios on RHS, if sin is the second factor then subtract

The arrows show differentiation - go against the arrows for primitives/integration. The bottom two triangles show rules that are more often used in integration.

When differentiating:

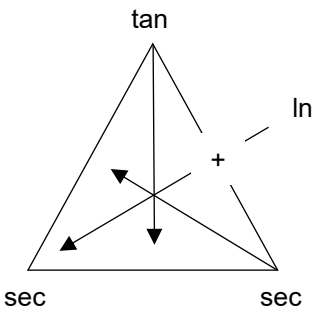
- any of the complementary ratios ($\cos\theta$, $\cot\theta$ or $\operatorname{cosec}\theta$) the sign changes.
- any ratio ending in 'c' ($\sec\theta$, $\operatorname{cosec}\theta$) gives two terms.



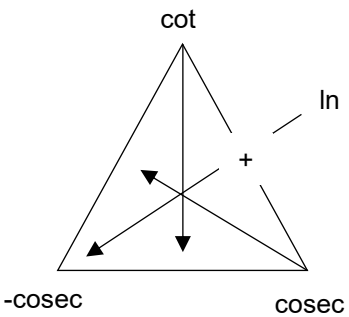

| Function | Derivative |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $-\sin x$ | $-\cos x$ |
| $-\cos x$ | $\sin x$ |

In the diagrams below:

- an arrow pointing at a side means the derivative is the product of the vertices, so the derivative of $\sec x$ is $\sec x \tan x$
- the plus sign on the RH side means to take the ln of the sum of the vertices, so the derivative of $\ln|\tan x + \sec x|$ is $\sec x$




| Function | Derivative |
|---|---|
| $\tan x$ | $\sec^2 x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\ln(\tan x + \sec x)$ | $\sec x$ |




| Function | Derivative |
|---|---|
| $\cot x$ | $-\operatorname{cosec}^2 x$ |
| $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cot x$ |
| $\ln(\cot x + \operatorname{cosec} x)$ | $-\operatorname{cosec} x$ |

# Solutions

## Exercise 1.1 solutions

**Question 1a**

$z^2 + 25$

$= z^2 - 25i^2$

$= z^2 - (5i)^2$

$= (z + 5i)(z - 5i)$

**Question 1b**

Let $z$ be a square root of $-25$

$$z^2 = -25$$

$$z^2 + 25 = 0$$

$\therefore (z + 5i)(z - 5i) = 0$ from (a)

$$z = \pm 5i$$

**Question 2a**

$z^2 + 2z + 5$

$= z^2 + 2z + 1 + 4$

$= (z + 1)^2 + 2^2$

$= (z + 1)^2 - (2i)^2$

$= (z + 1 + 2i)(z + 1 - 2i)$

**Question 2b**

$z^2 + 2z + 5 = 0$

$(z + 1 + 2i)(z + 1 - 2i) = 0$

$z = -1 \pm 2i$

**Question 2c**

$$z = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

**Question 3a**

$z^2 + 6iz + 7$

$= z^2 + 6iz + (3i)^2 - (3i)^2 + 7$

$= (z + 3i)^2 + 9 + 7$

$= (z + 3i)^2 + 4^2$

$= (z + 3i)^2 - (4i)^2$

$= (z + 3i + 4i)(z + 3i - 4i)$

$= (z + 7i)(z - i)$

**Question 3b**

$z^2 + 6iz + 7$

$= z^2 + 6iz - 7i^2$

$= (z + 7i)(z - i)$

**Question 3c**

$z^2 + 6iz + 7 = 0$

$(z + 7i)(z - i) = 0$

$z = -7i, i$

**Question 4a**

$i^6$

$= i^4 \times i^2$

$= 1 \times -1$

$= -1$

**Question 4b**

$i^{400}$

$= (i^4)^{100}$

$= 1^{100}$

$= 1$

**Question 4c**

$i^{63}$

$= (i^4)^{15} \times i^3$

$= 1^{15} \times (-i)$

$= -i$

**Question 4d**

$i^{-3}$

$= i^{-4} \times i$

$= (i^4)^{-1} \times i$

$= 1^{-1} \times i$

$= i$

**Question 4e**

$i^{-10}$

$= i^{-12} \times i^2$

$= (i^4)^{-3} \times i^2$

$= 1^{-3} \times (-1)$

$= -1$

**Question 5**

$i^9$

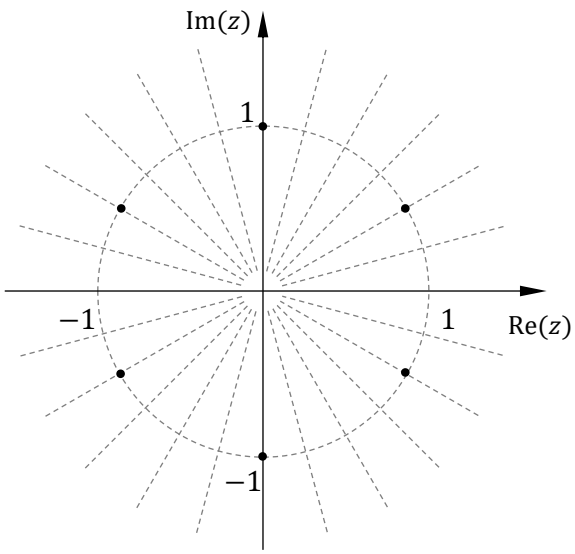
$= i^8 \times i$

$= 1 \times i$

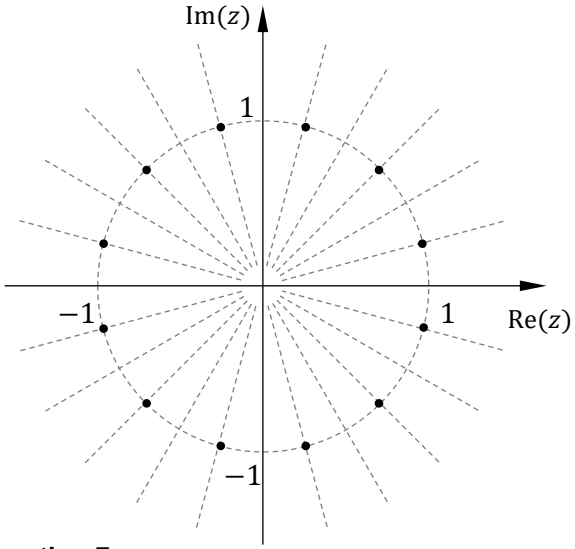
$= i$

$= 0 + 1i$

**Question 6a**



**Question 6b**




**Question 7a**

$i^5 + i^7$
$= i^4(i + i^3)$
$= 1(i - i)$
$= 0$

**Question 7b**

$i^6(i^2 + i^3)$
$= i^8(1 + i)$
$= (i^4)^2(1 + i)$
$= 1(1 + i)$
$= 1 + i$

**Question 8**

$z^2 = -36$
$z^2 = 36i^2$
$z^2 = (\pm 6i)^2$
$z = \pm 6i$

**Alternatively:**

Let $z$ be a square root of $-36$

$\therefore z^2 = -36$
$z^2 + 36 = 0$
$\therefore (z + 6i)(z - 6i) = 0$
$z = \pm 6i$

**Question 9**

Let $z$ be a cube root of $8$

$\therefore z^3 = 8$
$\therefore z^3 - 8 = 0$
$(z - 2)(z^2 + 2z + 4) = 0$
$z - 2 = 0$ or $z^2 + 2z + 4 = 0$
$z = 2$
$z = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$
$= \dfrac{-2 \pm \sqrt{-12}}{2}$
$= \dfrac{-2 \pm 2\sqrt{3}i}{2}$
$= -1 \pm \sqrt{3}i$

The three cube roots of $8$ are $2$ and $-1 \pm \sqrt{3}i$.

**Question 10a**

$z^4 + z^2 - 12$
$= (z^2 - 3)(z^2 + 4)$

**Question 10b**

$z^4 + z^2 - 12$
$= (z^2 - 3)(z^2 + 4)$
$= (z + \sqrt{3})(z - \sqrt{3})(z^2 + 4)$

**Question 10c**

$z^4 + z^2 - 12$
$= (z + \sqrt{3})(z - \sqrt{3})(z^2 + 4)$ from (b)
$= (z + \sqrt{3})(z - \sqrt{3})(z + 2i)(z - 2i)$

**Question 11a**

$z^2 + 2z + 5 = 0$
$z^2 + 2z + 1 + 4 = 0$
$(z + 1)^2 + 4 = 0$
$(z + 1)^2 = -4$
$(z + 1)^2 = 4i^2$
$z + 1 = \pm 2i$
$z = -1 \pm 2i$

**Question 11b**

$\Delta = 2^2 - 4(1)(5)$
$= -16$
$= 16i^2$
$= (4i)^2$
$z = \dfrac{-b \pm \lambda}{2a}$
$= \dfrac{-2 \pm 4i}{2 \times 1}$
$= -1 \pm 2i$

**Question 12**

This question is easier if you notice that the sum of $2 + 3i$ is one more than the product $1 + 3i$, so $1$ and $1 + 3i$ are the numbers needed.

Alternatively use the quadratic formula.

$z^2 + (2 + 3i)z + (1 + 3i) = 0$
$(z + 1)(z + 1 + 3i) = 0$
$\therefore z = -1, -1 - 3i$

**Question 13**

$4z^2 + 4z + 5 = 0$
$(2z + 1)^2 + 2^2 = 0$
$(2z + 1)^2 - (2i)^2 = 0$
$(2z + 1 + 2i)(2z + 1 - 2i) = 0$
$2z = -1 \pm 2i$
$z = -\dfrac{1}{2} \pm i$

**Question 14**

$z^2 + (z + 1)^2 = 0$
$2z^2 + 2z + 1 = 0$
$z = \dfrac{-2 \pm \sqrt{2^2 - 4(2)(1)}}{2(2)}$
$= \dfrac{-2 \pm \sqrt{-4}}{4}$
$= \dfrac{-2 \pm 2i}{4}$
$= \dfrac{-1 \pm i}{2}$

**Question 15**

$$\sum_{r=2}^{6} i^{3r}$$
$$= i^6 + i^9 + i^{12} + i^{15} + i^{18}$$
$$= i^4 \times i^2 + (i^4)^2 \times i + (i^4)^3 + (i^4)^3 \times i^3 + (i^4)^4 \times i^2$$
$$= 1(-1) + 1(i) + 1 + 1(-i) + 1(-1)$$
$$= -1$$

**Question 16**

$$z^3 - i$$
$$= z^3 + i^3$$
$$= (z + i)\left(z^2 - zi + i^2\right)$$
$$= (z + i)(z^2 - iz - 1)$$

**Question 17**

$$(a - ib)^3$$
$$= a^3 + 3a^2(-ib) + 3a(-ib)^2 + (-ib)^3$$
$$= a^3 - 3a^2bi - 3ab^2 + b^3 i$$
$$= \left(a^3 - 3ab^2\right) + \left(b^3 - 3a^2b\right)i$$

**Question 18**

$$z^3 - z^2 + 9z - 9 = 0$$
$$z^2(z - 1) + 9(z - 1) = 0$$
$$\left(z^2 + 9\right)(z - 1) = 0$$
$$(z + 3i)(z - 3i)(z - 1) = 0$$
$$z = 1, \pm 3i$$

The three roots are $1, -3i$ and $3i$.

**Question 19**

$$f(x) = (x - 1)(x - 3)$$
$$= x^2 - 4x + 3$$
$$= x^2 - 4x + 4 - 1$$
$$= (x - 2)^2 - 1$$
$$f(x + m) + n = (x - 2i)(x + 2i)$$
$$= x^2 + 4$$
$$= \left[\left((x + 2) - 2\right)^2 - 1\right] + 5$$
$$= f(x + 2) + 5$$

So $m = 2$ and $n = 5$.

**Question 1a**

$7+3i$

**Question 1b**

$-1+i$

**Question 2a**

$6+12i$

**Question 2b**

$-8+6i$

**Question 2c**

$i(3+i)$
$=3i+i^2$
$=-1+3i$

**Question 2d**

$-2i(3-4i)$
$=-6i+8i^2$
$=-8-6i$

**Question 3a**

$(3+i)(4-2i)$
$=12-6i+4i-2i^2$
$=12-2i+2$
$=14-2i$

**Question 3b**

$(2-i)^2$
$=4-4i+i^2$
$=4-4i-1$
$=3-4i$

**Question 3c**

$(2+i)^3$
$=(2)^3+3(2)^2(i)^1+3(2)(i)^2+i^3$
$=8+12i-6-i$
$=2+11i$

**Question 4a**

$\overline{2-2i}$
$=2+2i$

**Question 4b**

$\overline{-6i}$
$=6i$

**Question 5a**

$\dfrac{2-3i}{2i}\times\dfrac{i}{i}$
$=\dfrac{2i-3i^2}{-2}$
$=\dfrac{2i+3}{-2}$
$=\dfrac{-3-2i}{2}$

**Question 5b**

$\dfrac{3-2i}{1-i}\times\dfrac{1+i}{1+i}$
$=\dfrac{3+3i-2i-2i^2}{1^2+1^2}$
$=\dfrac{3+3i-2i+2}{1^2+1^2}$
$=\dfrac{5+i}{2}$

**Question 6**

$\bar{z}-w$
$=(3+i)-(2i-1)$
$=4-i$

**Question 7**

$z_1\bar{z_2}$
$=(5-2i)(3-4i)$
$=15-20i-6i-8$
$=7-26i$

**Question 8**

$\mathrm{Re}(2z+iz)$
$=\mathrm{Re}\big(2(3+2i)+i(3+2i)\big)$
$=\mathrm{Re}\big(6+4i+3i+2i^2\big)$
$=\mathrm{Re}(6+4i+3i-2)$
$=\mathrm{Re}(4+7i)$
$=4$

**Question 9**

$3x+5y=7 \quad (1)$
$2y-x=5 \quad (2)$
$(1)+3(2)$: $11y=22$
$y=2$
sub in (2): $2(2)-x=5$
$x=-1$
$\therefore x=-1, y=2.$

**Question 10**

$1-3i+2i-6i^2$
$=7-i$
$\therefore a=7, b=-1$

**Question 11**

$z-2i\bar{z}=3+4i$
$a+ib-2i(a-ib)=3+4i$
$a+ib-2ai-2b=3+4i$
$(a-2b)+(b-2a)i=3+4i$
$\therefore a-2b=3 \quad (1)$
$b-2a=4 \quad (2)$
$(1)+2(2)$: $-3a=11$
$a=-\dfrac{11}{3}$
sub in (1): $-\dfrac{11}{3}-2b=3$
$b=-\dfrac{10}{3}$
$\therefore z=-\dfrac{11}{3}-\dfrac{10}{3}i$

**Question 12a**

$z+iw$
$=1+i+i(1-i)$
$=1+i+i+1$
$=2+2i$

**Question 12b**

$z^2\bar{w}$
$=(1+i)^2(1+i)$
$=(1+i)^3$
$=1^3+3(1)^2i+3(1)i^2+i^3$
$=1+3i-3-i$
$=-2+2i$

**Question 12c**

$\dfrac{z}{w}$
$=\dfrac{1+i}{1-i}\times\dfrac{1+i}{1+i}$
$=\dfrac{1+2i-1}{1^2+1^2}$
$=i$

**Question 13a**

$z + w$
$= (3 + i) + (-2 + 2i)$
$= 1 + 3i$

**Question 13b**

$z - w$
$= (3 + i) - (-2 + 2i)$
$= 5 - i$

**Question 13c**

$2z$
$= 2(3 + i)$
$= 6 + 2i$

**Question 13d**

$-3w$
$= -3(-2 + 2i)$
$= 6 - 6i$

**Question 13e**

$iz =$
$i(3 + i)$
$= 3i + i^2$
$= -1 + 3i$

**Question 13f**

$zw$
$= (3 + i)(-2 + 2i)$
$= -6 + 6i - 2i \pm 2i^2$
$= -6 + 6i - 2i - 2$
$= -8 + 4i$

**Question 13g**

$$\frac{w}{3i}$$
$$= \frac{-2 + 2i}{3i} \times \frac{i}{i}$$
$$= \frac{-2i - 2}{-3}$$
$$= \frac{-2i + 2i^2}{-3}$$
$$= \frac{2}{3} + \frac{2}{3}i$$

**Question 13h**

$$\frac{z}{w}$$
$$= \frac{3 + i}{-2 + 2i} \times \frac{-2 - 2i}{-2 - 2i}$$
$$= \frac{-6 - 6i - 2i - 2i^2}{(-2)^2 + 2^2}$$
$$= \frac{-6 - 6i - 2i + 2}{4 + 4}$$
$$= \frac{-4 - 8i}{8}$$
$$= \frac{-1 - 2i}{2}$$

**Question 14a**

$z + \bar{z} = 2\text{Re}(z)$
$= 2(-5)$
$= -10$

**Question 14b**

$$\frac{z + \bar{z}}{2} = \text{Re}(z)$$
$$= 6$$

**Question 15a**

$z - \bar{z} = 2i\text{Im}(z)$
$= 2i(-2)$
$= -4i$

**Question 15b**

$b = \text{Im}(z)$
$$= \frac{z - \bar{z}}{2i}$$
$$= \frac{6i}{2i}$$
$= 3$

**Question 16**

$$\frac{a}{1 + i} \times \frac{1 - i}{1 - i} - \frac{b}{2i} \times \frac{i}{i} = 2$$
$$\frac{a - ai}{1^2 + 1^2} - \frac{bi}{-2} = 2$$
$$\frac{a - ai + bi}{2} = 2$$
$$a + (b - a)i = 4$$
$$\therefore a = 4$$
$$b - a = 0$$
$$b = 4$$

**Question 17a**

$(1 - 2i)^2$

$= 1 - 4i + 4i^2$

$= 1 - 4i - 4$

$= -3 - 4i$

**Alt:**

$(1 - 2i)^2$

$= 1^2 - 2^2 + 2(1)(-2)i$

$= -3 - 4i$

**Question 17b**

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(7 + i)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25 - 28 - 4i}}{2}$$

$$= \frac{5 \pm \sqrt{-3 - 4i}}{2}$$

$$= \frac{5 \pm (1 - 2i)}{2}$$

$$= \frac{4 + 2i}{2}, \frac{6 - 2i}{2}$$

$= 2 + i, 3 - i$

**Question 18**

$$\frac{x}{i}\times\frac{i}{i}-\frac{y}{1+i}\times\frac{1-i}{1-i}=-1-3i$$

$$\frac{xi}{i^2}-\frac{y-iy}{1^2+1^2}=-1-3i$$

$$-xi-\frac{y}{2}+\frac{y}{2}i=-1-3i$$

$$-\frac{y}{2}+\left(\frac{y}{2}-x\right)=-1-3i$$

$$-\frac{y}{2}=-1 \qquad \frac{y}{2}-x=-3$$

$$y=2 \qquad \frac{2}{2}-x=-3$$

$$-x=-4$$

$$x=4$$

$$\therefore x=4, y=2$$

**Question 19**

$$z_1^2\div\overline{z_2}$$

$$=\frac{(1+2i)^2}{3+i}$$

$$=\frac{1+4i+4i^2}{3+i}$$

$$=\frac{1+4i-4}{3+i}\times\frac{3-i}{3-i}$$

$$=\frac{(-3+4i)(3-i)}{3^2+1^2}$$

$$=\frac{-9+3i+12i-4i^2}{10}$$

$$=\frac{-9+3i+12i+4}{10}$$

$$=\frac{-5+15i}{10}$$

$$=\frac{-1+3i}{2}$$

**Question 20**

$$4(a+ib)-3(a-ib)=\frac{1-18i}{2-i}$$

$$4a+4bi-3a+3bi=\frac{1-18i}{2-i}\times\frac{2+i}{2+i}$$

$$a+7bi=\frac{2+i-36i-18i^2}{2^2+1^2}$$

$$=\frac{2+i-36i+18}{5}$$

$$=4-7i$$

$$\therefore a=4$$

$$7b=-7$$

$$b=-1$$

$$\therefore a=4, b=-1$$

**Question 21a**

$$z=\frac{\lambda+4i}{1+\lambda i}\times\frac{1-\lambda i}{1-\lambda i}$$

$$=\frac{\lambda-\lambda^2 i+4i-4\lambda i^2}{1^2+\lambda^2}$$

$$=\frac{\lambda-\lambda^2 i+4i+4\lambda}{1+\lambda^2}$$

$$=\frac{5\lambda+(4-\lambda^2)i}{1+\lambda^2}$$

$\mathrm{Im}(z)=0$ since $z$ is purely real

$$\therefore 4-\lambda^2=0$$

$$\lambda=\pm 2$$

**Question 21b**

Let $\lambda=-2$

$$z=\frac{-2+4i}{1-2i}$$

$$=-\frac{2(1-2i)}{1-2i}$$

$=-2$ which is purely real

Let $\lambda=2$

$$z=\frac{2+4i}{1+2i}$$

$$=\frac{2(1+2i)}{1+2i}$$

$=2$ which is purely real

**Question 1a**

$|z| = \sqrt{12^2 + 5^2}$
$= 13$

**Question 1b**

$|\omega| = \sqrt{6^2 + (-3)^2}$
$= \sqrt{45}$
$= 3\sqrt{5}$

**Question 2**

$\left|\frac{z^3}{\bar{z}}\right|$
$= \frac{|z|^3}{|z|}$
$= |z|^2$

**ANSWER B**

**Question 3**

$|z^2| + |w^2| - |2wz|$
$= |z|^2 - 2|w||z| + |w|^2$
$= (|z| + |w|)^2$

**Question 4a**

$a^2 - b^2 = 3 \quad (1)$
$2ab = 4 \quad (2)$
rearranging (2): $b = \frac{2}{a} \quad (3)$
sub (3) into (1): $a^2 - \frac{4}{a^2} = 3$
$a^4 - 3a^2 - 4 = 0$
$(a^2 - 4)(a^2 + 1) = 0$
$a^2 - 4 = 0$ or $a^2 + 1 = 0$
$a = \pm 2$ no real solution
$b = \frac{2}{\pm 2}$
$= \pm 1$

The square roots of $3 + 4i$ are $\pm(2 + i)$.

**Question 4b**

$a = \pm\sqrt{\frac{\sqrt{3^2 + 4^2} + 3}{2}} = \pm 2$

$b = \pm(+)\sqrt{\frac{\sqrt{3^2 + 4^2} - 3}{2}} = \pm 1$

The square roots of $3 + 4i$ are $\pm(2 + i)$.

**Question 5**

$a^2 - b^2 = 5 \quad (1)$
$2ab = -12$
$ab = -6 \quad (2)$

By inspection, the square roots of $5 - 12i$ are $\pm(3 - 2i)$.

*Check:* $\pm 2$ *and* $\mp 3$*, or* $\pm 1$ *and* $\mp 6$*, to see which pair has a difference of their squares being* $+5$*.* $3^2 - 2^2 = 9 - 4 = 5$*, so the square roots are* $\pm(3 - 2i)$*.*

**Question 6**

$a^2 - b^2 = 0 \quad (1)$
$2ab = 18$
$ab = 9 \quad (2)$

By inspection, the square roots of $18i$ are $\pm(3 + 3i)$.

*Check* $\pm 1$ *and* $\pm 9$*,* $\pm 3$ *and* $\pm 3$*. Now the real part is zero.* $3^2 - 3^2 = 0$*, so the square roots are* $\pm(3 + 3i)$*.*

**Question 7**

$a^2 - b^2 = 2 \quad (1)$
$2ab = -2\sqrt{3}$
$ab = -\sqrt{3} \quad (2)$

By inspection, the square roots of $2 - 2\sqrt{3}i$ are $\pm(\sqrt{3} - i)$.

*Check* $\pm 1$ *and* $\mp\sqrt{3}$*. Now the real part is* $2$*.* $\sqrt{3}^2 - (-1)^2 = 2$*, so the square roots are* $\pm(\sqrt{3} - i)$*.*

**Question 8a**

$a^2 - b^2 = -6 \quad (1)$
$2ab = 8$
$ab = 4$
$= \sqrt{16} \quad (2)$

By inspection, the square roots of $-6 + 8i$ are $\pm(\sqrt{2} + 2\sqrt{2}i)$.

*Check:* $\pm\sqrt{2}$ *and* $\pm\sqrt{8}$*. Now the real part is* $-6$*.* $(\sqrt{2})^2 - (\sqrt{8})^2 = -6$*, so the square roots are* $\pm(\sqrt{2} + \sqrt{8}i) = \pm(\sqrt{2} + 2\sqrt{2}i)$*.*

**Question 8b**

$z = \frac{\sqrt{2}i \pm \sqrt{(-\sqrt{2}i)^2 - 4(1)(1 - 2i)}}{2(1)}$
$= \frac{\sqrt{2}i \pm \sqrt{-2 - 4 + 8i}}{2}$
$= \frac{\sqrt{2}i \pm \sqrt{-6 + 8i}}{2}$
$= \frac{\sqrt{2}i \pm (\sqrt{2} + 2\sqrt{2}i)}{2}$
$= \frac{\sqrt{2}i - \sqrt{2} - 2\sqrt{2}i}{2}, \frac{\sqrt{2}i + \sqrt{2} + 2\sqrt{2}i}{2}$
$= -\frac{\sqrt{2} + \sqrt{2}i}{2}, \frac{\sqrt{2} + 3\sqrt{2}i}{2}$
$= -\frac{1 + i}{\sqrt{2}}, \frac{1 + 3i}{\sqrt{2}}$

**Question 9**

$|z| = \sqrt{z\bar{z}}$

$|z|^2 = z\bar{z}$

$\frac{|z|^2}{z} = \bar{z}$

$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

**Question 10**

$a^2 - b^2 = 21 \qquad (1)$

$2ab = -20$

$ab = -10 \qquad (2)$

By inspection, the square roots of $21 - 20i$ are $\pm(5 - 2i)$.

**Question 11a**

$a^2 - b^2 = 0 \qquad (1)$

$2ab = 2$

$ab = 1 \qquad (2)$

By inspection, the square roots of $2i$ are $\pm(1 + i)$.

**Question 11b**

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)\left(1 - \frac{i}{2}\right)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 - 4 + 2i}}{2}$$

$$= \frac{-2 \pm (1 + i)}{2}$$

$$= \frac{-3 - i}{2}, \frac{-1 + i}{2}$$

**Question 12a**

$|z| - z - 3 - 4i = 0$

$|z| - a - ib - 3 - 4i = 0$

$|z| - a - 3 - (b + 4)i = 0 + 0i$

$\therefore b + 4 = 0$

$b = -4$

**Question 12b**

$|z| - a - 3 = 0$

$\sqrt{a^2 + (-4)^2} = a + 3$

$a^2 + 16 = a^2 + 6a + 9$

$6a = 7$

$a = \frac{7}{6}$

**Question 13**

We have a problem here – the coefficient of $i$ isn't even so we cannot use our shortcut yet. We will double it and put it over 2 so we can use our shortcut.

$$4 - 3i = \frac{8 - 6i}{2}$$

Taking the coefficient of $i$ in the numerator and halving it we get 3, so we need to find a pair of numbers that have a product of 3. We have $\pm 1$ and $\mp 3$. $3^2 - 1^2 = 8$, so the square roots of $8 - 6i$ are $\pm(3 - i)$. So, we have square rooted the numerator, but we must also square root the denominator.

So, the square roots of $4 - 3i$ are $\pm\frac{3-i}{\sqrt{2}}$.

**Question 14**

$$z = \frac{-4 \pm \sqrt{4^2 - 4(1)(4 - i)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{4i}}{2}$$

To find the square roots of $4i$:

$a^2 - b^2 = 0 \qquad (1)$

$2ab = 4$

$ab = 2 \qquad (2)$

By inspection, the square roots of $4i$ are $\pm(\sqrt{2} + \sqrt{2}i)$.

$$z = \frac{-4 \pm (\sqrt{2} + \sqrt{2}i)}{2}$$

$$= \frac{-4 - \sqrt{2} - \sqrt{2}i}{2}, \frac{-4 + \sqrt{2} + \sqrt{2}i}{2}$$

**Question 15**

$$z = \frac{(4i - 1) \pm \sqrt{(1 - 4i)^2 - 4(1)(-5 + i)}}{2(1)}$$

$$= \frac{4i - 1 \pm \sqrt{1 - 8i - 16 + 20 - 4i}}{2}$$

$$= \frac{4i - 1 \pm \sqrt{5 - 12i}}{2}$$

To find the square roots of $5 - 12i$:

$a^2 - b^2 = 5 \qquad (1)$

$2ab = -12$

$ab = -6 \qquad (2)$

By inspection, the square roots of $5 - 12i$ are $\pm(3 - 2i)$.

$$\therefore z = \frac{4i - 1 \pm (3 - 2i)}{2}$$

$$= \frac{4i - 1 - (3 - 2i)}{2}, \frac{4i - 1 + (3 - 2i)}{2}$$

$$= -2 + 3i, 1 + i$$

**Question 16**

$$\frac{3 + i}{1 + 2i} + \lambda$$

$$= \frac{3 + i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i} + \lambda$$

$$= \frac{3 - 6i + i - 2i^2}{1^2 + 2^2} + \lambda$$

$$= \frac{3 - 6i + i + 2}{5} + \lambda$$

$$= 1 - i + \lambda$$

$$= (1 + \lambda) - i$$

$\therefore |(1 + \lambda) - i| = \sqrt{\lambda + 2}$

$\sqrt{(1 + \lambda)^2 + (-1)^2} = \sqrt{\lambda + 2}$

$1 + 2\lambda + \lambda^2 + 1 = \lambda + 2$

$\lambda^2 + \lambda = 0$

$\lambda(\lambda + 1) = 0$

$\lambda = 0, -1$

Check − always a good idea when you square a square root:

$|(1 + 0) - i| = |1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$= \sqrt{0 + 2}$ ✓

$|(1 - 1) - i| = |-i| = 1 = \sqrt{1} = \sqrt{-1 + 2}$ ✓

$\therefore \lambda = 0, -1$

**Question 17a**

$$z = \frac{-\sqrt{3} \pm \sqrt{\left(\sqrt{3}\right)^2 - 4(i)(-1)}}{2(i)}$$
$$= \frac{-\sqrt{3} \pm \sqrt{3 + 4i}}{2i}$$

To find the square roots of $3 + 4i$:

$$a^2 - b^2 = 3 \qquad (1)$$
$$2ab = 4$$
$$ab = 2 \qquad (2)$$

By inspection, the square roots of $3 + 4i$ are $\pm(2 + i)$.

$$\alpha, \beta = \frac{-\sqrt{3} \pm (2 + i)}{2i}$$
$$= \frac{-\sqrt{3} + 2 + i}{2i}, \frac{-\sqrt{3} - 2 - i}{2i}$$
$$= \frac{1 + \left(\sqrt{3} - 2\right)i}{2}, \frac{-1 + \left(\sqrt{3} + 2\right)i}{2}$$

in either order

**Question 17b**

$$\alpha\beta = \frac{c}{a}$$
$$= -\frac{1}{i} \times \frac{i}{i}$$
$$= -\frac{i}{-1}$$
$$= i$$

$\therefore\ \alpha^2\beta^2 + 1$
$= i^2 + 1$
$= -1 + 1$
$= 0$

**Question 18**

$$\mathrm{Re}(z) = a^2 - b^2 \qquad (1)$$
$$|z| = a^2 + b^2 \qquad (2)$$

$(1) + (2)$: $|z| + \mathrm{Re}(z) = 2a^2$

$$a^2 = \frac{|z| + \mathrm{Re}(z)}{2} \qquad (3)$$

$(2) - (1)$: $|z| - \mathrm{Re}(z) = 2b^2$

$$b^2 = \frac{|z| - \mathrm{Re}(z)}{2} \qquad (4)$$

Now $\mathrm{Im}(x + iy) = \mathrm{Im}\left((a + ib)^2\right)$
$y = 2ab$

$\therefore$ if $y$ is positive then $a$ and $b$ have the same sign, while if $y$ is negative then $a$ and $b$ have opposite sign.

The solutions to (3) and (4) are:

$$a = \pm\sqrt{\frac{|z| + \mathrm{Re}(z)}{2}} \quad \text{and}$$

$$b = \pm\mathrm{sgn}(y)\sqrt{\frac{|z| - \mathrm{Re}(z)}{2}}$$

**Question 19**

$$(a + ib) + 2(a - ib) = |a + ib + 2|$$
$$3a - ib = |(a + 2) + ib|$$

equating real and imaginary parts

$$3a = |(a + 2) + ib| \qquad b = 0$$
$$9a^2 = a^2 + 4a + 4$$
$$8a^2 - 4a - 4 = 0$$
$$2a^2 - a - 1 = 0$$
$$(2a + 1)(a - 1) = 0$$
$$a = -\frac{1}{2}, 1$$

Check whether they solve the equation:

$$\mathrm{z} = -\frac{1}{2} + 0i = -\frac{1}{2}$$

$$\text{LHS} = z + 2\bar{z} = -\frac{1}{2} + 2\left(-\frac{1}{2}\right) = -\frac{3}{2}$$

$$\text{RHS} = |z + 2| = \left|-\frac{1}{2} + 2\right| = \frac{3}{2}$$ ✗

$$\mathrm{z} = 1 + 0i = 1$$
$$\text{LHS} = z + 2\bar{z} = 1 + 2(1) = 3$$
$$\text{RHS} = |z + 2| = |1 + 2| = 3$$ ✓

$\therefore\ z = 1$ is the only solution

**Question 20**

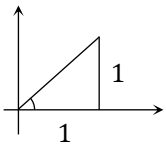
The inequalities are symmetrical in $x$ and $y$, so the maximum occurs when $y = x$.

$$|z| = \sqrt{x^2 + x^2}$$
$$= \sqrt{2x^2}$$
$$\therefore \sqrt{2x^2} = x + 1$$
$$2x^2 = x^2 + 2x + 1$$
$$x^2 - 2x - 1 = 0$$
$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{2 \pm 2\sqrt{2}}{2}$$
$$= 1 \pm \sqrt{2}$$
$$y = 1 \pm \sqrt{2}$$

$|z|_{\max}$ when $x = y = 1 + \sqrt{2}$ since $\left|1 + \sqrt{2}\right| > \left|1 - \sqrt{2}\right|$

$|z|_{\max}$
$= \sqrt{2}\left(1 + \sqrt{2}\right)$ using the $45°$ exact triangle
$= \sqrt{2} + 2$

**Question 1a**




$\therefore 1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ exact triangles

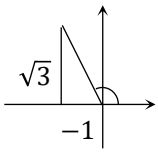
Alt:

$|z|=\sqrt{1^2+1^2}=\sqrt{2}$

$\arg(z)=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4}$

$\therefore 1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

**Question 1b**




$\therefore -1+\sqrt{3}i=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

exact triangles

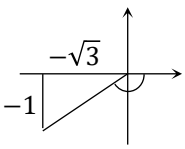
Alt:

$|z|=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=2$

$\arg(z)=\pi-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\frac{2\pi}{3}$

$\therefore -1+\sqrt{3}i=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

**Question 1c**




$\therefore -\sqrt{3}-i=2\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)$

exact triangles

Alt:

$|z|=\sqrt{(-\sqrt{3})^2+(-1)^2}=2$

$\arg(z)=-\pi+\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=-\frac{5\pi}{6}$

$\therefore -\sqrt{3}-i=2\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)$

**Question 2a**

$4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$=4\times\left(\frac{\sqrt{3}}{2}+i\times\frac{1}{2}\right)$

$=2\sqrt{3}+2i$

**Question 2b**

$\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$

$=\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)$

$=\frac{1}{2}-i\times\frac{\sqrt{3}}{2}$

$=\frac{1}{2}-\frac{\sqrt{3}}{2}i$

**Question 3**

$w=uv$

$=4\operatorname{cis}\frac{\pi}{6}\times\operatorname{cis}\left(-\frac{\pi}{3}\right)$

$=4\operatorname{cis}\left(\frac{\pi}{6}-\frac{\pi}{3}\right)$

$=4\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$

**Question 4**

$w=\frac{u}{v}$

$=4\operatorname{cis}\frac{\pi}{6}\div\operatorname{cis}\left(-\frac{\pi}{3}\right)$

$=(4\div 1)\operatorname{cis}\left(\frac{\pi}{6}--\frac{\pi}{3}\right)$

$=4\operatorname{cis}\frac{\pi}{2}$

$=4i$

**Question 5a**

$\overline{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$

$=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$

**Question 5b**

$\overline{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)}$

$=\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)$

**Question 6**

The argument in B has doubled, so the answer involves $z^2$, A or D. $|z|=1$, so $\left|z^2\right|=1$, so we need to use $2z^2$ to get a modulus greater than 1.

**Answer D**

**Question 7**

$\frac{z^2}{|z|}=\frac{z^2}{1}$ since $z$ lies on the unit circle

$=\operatorname{cis}2\theta$ since $\arg\left(z^2\right)=2\arg(z)$

We are looking for the point that has twice the argument of $z$, which is point *D*.

**Question 8**

$z-\bar{z}$

$=r(\cos\theta+i\sin\theta)-r(\cos(-\theta)+i\sin(-\theta))$

$=r(\cos\theta+i\sin\theta-\cos(-\theta)-i\sin(-\theta))$

$=r\left(\cos\theta+i\sin\theta-\cos\theta+i\sin\theta\right)$

since cosine is even and sine is odd

$=2ri\sin\theta$

$=2i(r\sin\theta)$

$=2i\mathrm{Im}(z)$

**Question 9a**

$r = \sqrt{9} = 3$

$\theta_1 = \frac{3\pi}{4} \div 2 = \frac{3\pi}{8}$

$\theta_2 = \theta_1 - \pi = \frac{3\pi}{8} - \pi = -\frac{5\pi}{8}$

The square roots are $3\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$ and $3\left(\cos\left(-\frac{5\pi}{8}\right) + i\sin\left(-\frac{5\pi}{8}\right)\right)$.

**Question 9b**

$r = \sqrt{8} = 2\sqrt{2}$

$-\frac{11\pi}{4} \div 2 = -\frac{11\pi}{8} \rightarrow \theta_1 = -\frac{11\pi}{8} + 2\pi = \frac{5\pi}{8}$

$\theta_2 = \theta_1 - \pi = \frac{5\pi}{8} - \pi = -\frac{3\pi}{8}$

The square roots are $2\sqrt{2}\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$ and $2\sqrt{2}\left(\cos\left(-\frac{3\pi}{8}\right) + i\sin\left(-\frac{3\pi}{8}\right)\right)$.

**Question 10**

$(\cos\pi + i\sin\pi) - \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= (-1 + 0i) - (0 + i)$

$= -1 - i$

$= \sqrt{2}\,\mathrm{cis}\left(-\frac{3\pi}{4}\right)$ (exact triangles)

$\therefore r = \sqrt{2}, \theta = -\frac{3\pi}{4}$

**Question 11**

$\sin\frac{\pi}{6}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\frac{3\pi}{4}\right)$

$= \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

**Question 12a**

$|w| = 2, \arg(w) = -\frac{\pi}{3}$ by inspection

**Question 12b**

$\bar{w} = 1 + i\sqrt{3}$

$w^2 = \left(1 - i\sqrt{3}\right)^2 = 1 - 2\sqrt{3}i - 3 = -2 - 2\sqrt{3}i$

$\frac{1}{w} = \frac{1}{1 - \sqrt{3}i} \times \frac{1 + \sqrt{3}i}{1 + \sqrt{3}i} = \frac{1 + \sqrt{3}i}{1 + 3} = \frac{1}{4} + \frac{\sqrt{3}}{4}$

**Question 13**

$z = \mathrm{cis}\frac{3\pi}{4}$

A: $\bar{z} = \mathrm{cis}\left(-\frac{3\pi}{4}\right) \neq i$

B: $-z = -\mathrm{cis}\frac{3\pi}{4} = \mathrm{cis}(-\pi) \times \mathrm{cis}\frac{3\pi}{4} = \mathrm{cis}\left(-\frac{\pi}{4}\right) \neq i$

C: $z^2 = \mathrm{cis}\left(2 \times \frac{3\pi}{4}\right) = \mathrm{cis}\frac{3\pi}{2} = -i \neq i$

D: $-z^2 = i$

**Answer D**

**Question 14a**

$\frac{\sqrt{3} + i}{\sqrt{3} - i} \times \frac{\sqrt{3} + i}{\sqrt{3} + i}$

$= \frac{3 + 2\sqrt{3}i + 1}{3 + 1}$

$= \frac{4 + 2\sqrt{3}i}{4}$

$= 1 + \frac{\sqrt{3}}{2}i$

**Question 14b**

$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\sqrt{3} - i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$\frac{\sqrt{3} + i}{\sqrt{3} - i} = \frac{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}{2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)}$

$= \frac{2}{2}\left(\cos\left(\frac{\pi}{6} - -\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6} - -\frac{\pi}{6}\right)\right)$

$= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

**Question 14c**

Equating imaginary parts of **a** and **b**

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

**Question 15a**

$i \times 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= \mathrm{cis}\frac{\pi}{2} \times 2 \times \mathrm{cis}\frac{\pi}{6}$

$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

**Question 15b**

$-3i \times \frac{1}{\sqrt{3}}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$= 3 \times \mathrm{cis}\left(-\frac{\pi}{2}\right) \times \frac{1}{\sqrt{3}} \times \mathrm{cis}\left(-\frac{\pi}{3}\right)$

$= \sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$

**Question 16a**

$2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \div (2i)$

$= 2\sqrt{2}\,\mathrm{cis}\frac{3\pi}{4} \div \left(2\,\mathrm{cis}\frac{\pi}{2}\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

**Question 16b**

$i \div \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \mathrm{cis}\frac{\pi}{2} \div \mathrm{cis}\frac{\pi}{4}$

$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

**Question 17**

$$\frac{1-i}{-1+i}=\frac{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)}{\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)}$$

$$=\cos(-\pi)+i\sin(-\pi)=-1$$

The rotation is $\pi$ radians (180°).

**Question 18**

$$(\bar{z})^{-1}$$

$$=\frac{1}{2\ \mathrm{cis}\left(-\frac{\pi}{6}\right)}\times\frac{\mathrm{cis}\frac{\pi}{6}}{\mathrm{cis}\frac{\pi}{6}}$$

$$=\frac{\mathrm{cis}\frac{\pi}{6}}{2}$$

$$=\frac{1}{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

**Question 19**

$$-3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$=\mathrm{cis}(-\pi)\times3\times\mathrm{cis}\frac{\pi}{4}$$

$$=3\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)$$

**Question 20**

$$|z|=\sqrt{\left(\sqrt{6}\right)^2+\left(-\sqrt{2}\right)^2}$$

$$=\sqrt{8}$$

$$=2\sqrt{2}$$

$$\arg z=\tan^{-1}\left(-\frac{\sqrt{2}}{\sqrt{6}}\right)$$

$$=-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$=-\frac{\pi}{6}$$

$$\therefore\sqrt{6}-\sqrt{2}i=2\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$$

**Question 21a**

by inspection (exact triangles):

$$1+i\sqrt{3}=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

**Question 21b**

$$\left(1+i\sqrt{3}\right)(1+i)$$

$$=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\times\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$=2\sqrt{2}\left(\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)$$

$$=2\sqrt{2}\left(\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right)$$

**Question 21c**

$$\left(1+i\sqrt{3}\right)(1+i)=1+i+\sqrt{3}i-\sqrt{3}$$

$$=\left(1-\sqrt{3}\right)+\left(1+\sqrt{3}\right)i$$

$$\tan\left(\frac{7\pi}{12}\right)=\frac{2\sqrt{2}\sin\left(\frac{7\pi}{12}\right)}{2\sqrt{2}\cos\left(\frac{7\pi}{12}\right)}$$

$$=\frac{1+\sqrt{3}}{1-\sqrt{3}}$$

**Question 22**

The principal argument of $z$ is between $-\frac{3\pi}{4}$ and $-\pi$, so when it is squared we get an argument between $-\frac{3\pi}{2}$ and $-2\pi$, which places it in the first quadrant, where $a$ and $b$ are both positive.

**Answer D**

Alternative solution I:

$$z^2=(x+iy)^2$$

$$=x^2-y^2+2xyi$$

$$\therefore\ a=x^2-y^2 \text{ and } b=2xy$$

Now $x^2>y^2$ since $x<y$ and they are both negative, so $a>0$.

$2xy>0$ since $x$ and $y$ are both negative, so $b>0$.

**Question 23**

$$\frac{z}{w}=\pm2$$

$$z=\pm2w$$

$$w=\pm\frac{1}{2}z$$

$$=\pm\frac{1}{2}\left(4\ \mathrm{cis}\frac{\pi}{3}\right)$$

$$=\pm2\ \mathrm{cis}\frac{\pi}{3}$$

$$=2\ \mathrm{cis}\left(\frac{\pi}{3}-\pi\right),2\ \mathrm{cis}\frac{\pi}{3}$$

$$w=2\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right),$$

$$2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

**Question 24a**

$$zw=(1-3i)(2+i)=2+i-6i+3=5-5i$$

**Question 24b**

$$|zw|=\sqrt{5^2+5^2}=5\sqrt{2}$$

$$\arg(zw)=-\tan^{-1}\left(\frac{5}{5}\right)=-\frac{\pi}{4}$$

$$\therefore zw=5\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$$

**Question 24c**

$$\frac{\sqrt{2}(\cos x-i\sin x)}{2+i}=\frac{1-3i}{5}$$

$$5\sqrt{2}(\cos x-i\sin x)=(1-3i)(2+i)$$

$$\therefore\cos x-i\sin x=\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)$$

from (i), (ii)

$$=\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)$$

$$\therefore x=\frac{\pi}{4}$$

**Question 25a**

$$z = \frac{4+3i}{2+4i} \times \frac{2-4i}{2-4i}$$
$$= \frac{8-16i+6i+12}{2^2+4^2}$$
$$= \frac{20-10i}{20}$$
$$= 1-\frac{1}{2}i$$
$$\therefore |z| = \sqrt{1^2+\left(-\frac{1}{2}\right)^2}$$
$$= \frac{\sqrt{5}}{2}$$

**Question 256b**

$$z = \frac{a+3i}{2+ai} \times \frac{2-ai}{2-ai}$$
$$= \frac{2a-a^2i+6i+3a}{2^2+a^2}$$
$$= \frac{5a+(6-a^2)i}{a^2+4}$$
$$= \frac{5a}{a^2+4}+\frac{6-a^2}{a^2+4}i$$
$$\arg z = \tan^{-1}\frac{6-a^2}{5a}$$
$$\therefore \frac{6-a^2}{5a} = \tan\frac{\pi}{4}$$
$$6-a^2 = 5a$$
$$a^2+5a-6 = 0$$
$$(a+6)(a-1) = 0$$
$$\therefore a = -6,1$$

$\therefore a = 1$ is the only solution, since $z$ is in the first quadrant.

**Question 26a**

$$\overline{1-z} = \overline{1-3\cos\theta-3i\sin\theta}$$
$$= 1-3\cos\theta+3i\sin\theta$$

**Question 26b**

$$\frac{1}{1-z} = \frac{1}{1-3(\cos\theta+i\sin\theta)}$$
$$= \frac{1}{(1-3\cos\theta)-3i\sin\theta} \times \frac{(1-3\cos\theta)+3i\sin\theta}{(1-3\cos\theta)+3i\sin\theta}$$
$$= \frac{(1-3\cos\theta)+3i\sin\theta}{(1-3\cos\theta)^2+(3\sin\theta)^2}$$
$$= \frac{(1-3\cos\theta)+3i\sin\theta}{1-6\cos\theta+9\cos^2\theta+9\sin^2\theta}$$
$$= \frac{(1-3\cos\theta)+3i\sin\theta}{10-6\cos\theta}$$
$$\therefore \operatorname{Im}\left(\frac{1}{1-z}\right) = \frac{3\sin\theta}{10-6\cos\theta}$$

**Question 27a**

LHS $= a(\cos\alpha + i\sin\alpha) + b(\cos\beta + i\sin\beta)$

$= (a\cos\alpha + b\cos\beta) + i(a\sin\alpha + b\sin\beta)$

RHS $= r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta$

Equating real parts: $r\cos\theta = a\cos\alpha + b\cos\beta$

$$\cos\theta = \frac{a\cos\alpha + b\cos\beta}{r}$$

Equating imaginary parts: $r\sin\theta = a\sin\alpha + b\sin\beta$

$$\sin\theta = \frac{a\sin\alpha + b\sin\beta}{r}$$

**Question 27b**

$r^2 = r(\cos\theta + i\sin\theta)\times r(\cos(-\theta) + i\sin(-\theta))$

$= (a\,\text{cis}\,\alpha + b\,\text{cis}\,\beta)(a\,\text{cis}(-\alpha) + b\,\text{cis}(-\beta))$

$= a^2 + ab\,\text{cis}(\alpha-\beta) + ab\,\text{cis}(\beta-\alpha) + b^2$

$= a^2 + b^2 + ab\big(\cos(\alpha-\beta) + i\sin(\alpha-\beta) + \cos\big(-(\alpha-\beta)\big) + i\sin\big(-(\alpha-\beta)\big)\big)$

$= a^2 + b^2 + ab(\cos(\alpha-\beta) + i\sin(\alpha-\beta) + \cos(\alpha-\beta) - i\sin(\alpha-\beta))$

$= a^2 + b^2 + ab(2\cos(\alpha-\beta))$

$= a^2 + b^2 + 2ab\cos(\alpha-\beta)$

$\therefore r = \sqrt{a^2 + b^2 + 2ab\cos(\alpha-\beta)}$

**Question 27c**

Let $z_1 = a\,\text{cis}\,\alpha$, $z_2 = b\,\text{cis}\,\beta$

$|z_1| + |z_2| = a + b \quad (1)$

$|z_1 + z_2| = \sqrt{a^2 + b^2 + 2ab\cos(\alpha-\beta)}$ from (ii)

$\leq \sqrt{a^2 + b^2 + 2ab}$ since $\cos\theta \leq 1$

$\leq \sqrt{(a+b)^2}$

$\leq a + b \quad (2)$

$\therefore$ from (1) and (2) $|z_1| + |z_2| \geq |z_1 + z_2|$

**Question 28**

$\Delta AOC$ is equilateral since $z_3 = \text{cis}\frac{\pi}{3}z_1$ and so $\angle AOC = \frac{\pi}{3}$.

$\Delta OAB$ is right-angled isosceles since $z_2 - z_1 = iz_1$

and so $\angle AOB = \frac{\pi}{4}$ and $|z_2| = \sqrt{2}|z_1|$.

$\therefore |z_1| = |z_3| = |z_2 - z_1|$ since $OA$ is common to both triangles.

$\angle BOC = \angle AOC - \angle AOB$

$= \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$

$z_2 - z_1 = iz_1 \therefore z_2 = (1+i)z_1 \quad (1)$

$z_3 = \text{cis}\frac{\pi}{3}z_1$

$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)z_1$

$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)z_1 \quad (2)$

$\therefore z_2 - z_3 = \left[\left(1+i\right) - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right]z_1$ from (1) and (2)

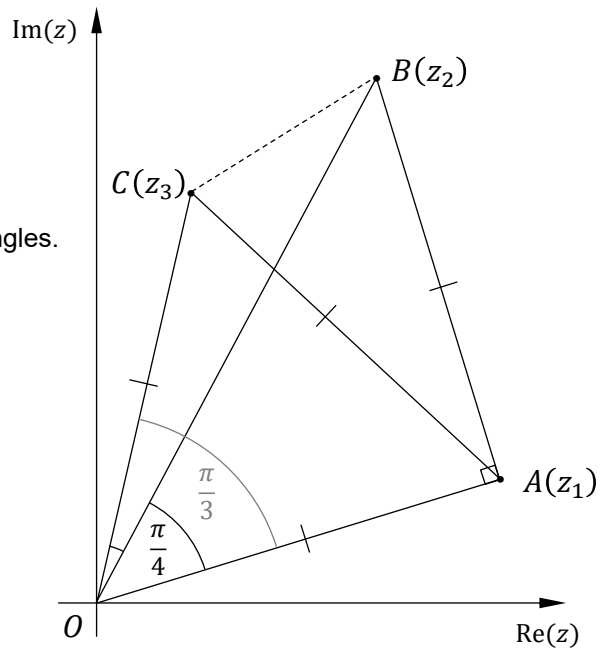
$= \left[\frac{1}{2} + \left(1 - \frac{\sqrt{3}}{2}\right)i\right]z_1$

$= \frac{1 + (2-\sqrt{3})i}{2}z_1$

$|z_2 - z_3| = \left|\frac{1 + (2-\sqrt{3})i}{2}z_1\right|$

$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{2-\sqrt{3}}{2}\right)^2}\times|z_1|$

$= \sqrt{\frac{1+4-4\sqrt{3}+3}{4}}\times|z_1|$




$= \sqrt{2-\sqrt{3}}\times|z_1|$

$\cos\angle BOC = \frac{OC^2 + OB^2 - BC^2}{2\times OC\times OB}$

$\cos\frac{\pi}{12} = \frac{|z_3|^2 + |z_2|^2 - |z_2 - z_3|^2}{2\times|z_3|\times|z_2|}$

$= \frac{|z_1|^2 + \left(\sqrt{2}|z_1|\right)^2 - \left(\sqrt{2-\sqrt{3}}|z_1|\right)^2}{2\times|z_1|\times\sqrt{2}|z_1|}$

$= \frac{1 + 2 - (2-\sqrt{3})}{2\times1\times\sqrt{2}}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

**Question 1ai**

$$\begin{aligned} z+\bar{z} &= 2\text{Re}(z) \\ &= 2(2) \\ &= 4 \end{aligned}$$

**Question 1aii**

$$\begin{aligned} z+\bar{z} &= 2\text{Re}(z) \\ &= 2\times 2\cos\frac{\pi}{6} \\ &= 4\times\frac{\sqrt{3}}{2} \\ &= 2\sqrt{3} \end{aligned}$$

**Question 1bi**

$$\begin{aligned} z-\bar{z} &= 2i\text{Im}(z) \\ &= 2i(-3) \\ &= -6i \end{aligned}$$

**Question 1bii**

$$\begin{aligned} z-\bar{z} &= 2i\text{Im}(z) \\ &= 2i\left(2\sin\frac{\pi}{6}\right) \\ &= 4i\times\frac{1}{2} \\ &= 2i \end{aligned}$$

**Question 1ci**

$$\begin{aligned} z\bar{z} &= |z|^2 \\ &= a^2+b^2 \\ &= 2^2+(-3)^2 \\ &= 13 \end{aligned}$$

**Question 1cii**

$$\begin{aligned} z\bar{z} &= |z|^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

**Question 1di**

$$\begin{aligned} \frac{1}{z} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{2+3i}{2^2+(-3)^2} \\ &= \frac{2}{13}+\frac{3}{13}i \end{aligned}$$

**Question 1dii**

$$\begin{aligned} \frac{1}{z} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)}{2^2} \\ &= \frac{1}{2}\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right) \end{aligned}$$

**Question 2**

$$\begin{aligned} |z| &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+\left(-\frac{1}{\sqrt{2}}\right)^2} \\ &= 1 \\ \therefore \frac{1}{z} &= \bar{z} \\ &= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i \end{aligned}$$

**Question 3a**

$$\begin{aligned} \bar{p}+\bar{q} &= \overline{p+q} \\ &= \overline{-\frac{b}{a}} \\ &= -\left(\frac{2-i}{3}\right) \\ &= -\frac{2+i}{3} \end{aligned}$$

**Question 3b**

$$\begin{aligned} \bar{p}\times\bar{q} &= \overline{p\times q} \\ &= \overline{\left(\frac{c}{a}\right)} \\ &= \overline{\left(\frac{6i}{3}\right)} \\ &= \overline{(2i)} \\ &= -2i \end{aligned}$$

**Question 4a**

$$\begin{aligned} &(1+i)^3-2(1+i)+4 \\ &= 1^3+3(1)^2i+3(1)(i)^2+i^3-2(1+i)+4 \\ &= 1+3i-3-i-2-2i+4 \\ &= 0 \end{aligned}$$

$\therefore 1+i$ is a root.

**Question 4b**

$\therefore 1-i$ is a root.

$$\therefore \sum\alpha = -\frac{b}{a}$$

$$1+i+1-i+\gamma = -\frac{0}{1}$$

$$2+\gamma = 0$$

$$\therefore \gamma = -2$$

$\therefore$ The roots of $z^3-2z+4=0$ are $-2, 1\pm i$.

**Question 5**

$$\begin{aligned} P(z) &= \big(z-(2+i)\big)\big(z-(2-i)\big)(z-\gamma) \\ &= \big(z^2-2\text{Re}(2+i)z+|2+i|^2\big)(z-\gamma) \\ &= \big(z^2-4z+2^2+1^2\big)(z-\gamma) \\ &= \big(z^2-4z+5\big)(z-\gamma) \end{aligned}$$

$Let\ \gamma = 0$

$$\begin{aligned} \therefore P(z) &= \big(z^2-4z+5\big)(z) \\ &= z^3-4z^2+5z \end{aligned}$$

**Answer A**

**Question 6**

The roots are $3, 2+i$ and $2-i$ using the conjugate root theorem.

$$\begin{aligned} \sum\alpha &= 3+2+i+2-i \\ &= 7 \end{aligned}$$

$\therefore b = -7$ so eliminate C and D

$$\begin{aligned} \prod\alpha &= 3(2+i)(2-i) \\ &= 3\big(2^2+1^2\big) \\ &= 15 \end{aligned}$$

$\therefore d = -15$

**Answer B**

**Question 7**

$z+w=0 \ \rightarrow \ w=-z$ or $z=-w$

For A: $\frac{w}{z}=\frac{-z}{z}=-1$ so true.

For B: $z-w=z-(-z)=2z$ so true.

For C: $w-z=w-(-w)=2w$ so true.

For D: $z+\overline{w}=z+\overline{-z}=z-\bar{z}=2i\text{Im}(z)\neq 0$ since $z$ is a non-real root, so FALSE

**Question 8**

Let $z_1=a_1+ib_1, z_2=a_2+ib_2$ etc

$$\begin{aligned}\text{LHS} &= \overline{a_1+ib_1+a_2+ib_2+\ldots+a_n+ib_n}\\ &= \overline{(a_1+a_2+\ldots a_n)+i(b_1+b_2+\ldots+b_n)}\\ &= (a_1+a_2+\ldots a_n)-i(b_1+b_2+\ldots+b_n)\\ &= a_1-ib_1+a_2-ib_2+\ldots+a_n-ib_n\\ &= \overline{z_1}+\overline{z_2}+\ldots+\overline{z_n}\\ &= \text{RHS}\end{aligned}$$

**Question 9**

Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$,
$z_2=r_2(\cos\theta_2+i\sin\theta_2)$ etc.

$$\begin{aligned}\text{LHS} &= \overline{r_1\,\text{cis}\,\theta_1\times r_2\,\text{cis}\,\theta_2\times\ldots\times r_n\,\text{cis}\,\theta_n}\\ &= \overline{(r_1r_2\ldots r_n)\,\text{cis}(\theta_1+\theta_2+\ldots\theta_n)}\\ &= (r_1r_2\ldots r_n)\,\text{cis}\big(-(\theta_1+\theta_2+\ldots\theta_n)\big)\\ &= r_1\,\text{cis}(-\theta_1)\times r_2\,\text{cis}(-\theta_2)\times\ldots\times r_n\,\text{cis}(-\theta_n)\\ &= \overline{r_1\,\text{cis}\,\theta_1}\times\overline{r_2\,\text{cis}\,\theta_2}\times\ldots\times\overline{r_n\,\text{cis}\,\theta_n}\\ &= \overline{z_1}\times\overline{z_2}\times\ldots\times\overline{z_n}\\ &= \text{RHS}\end{aligned}$$

**Question 10**

$2-3i$ must also be a root

$\therefore$ the following quadratic must be a factor.

$$\begin{aligned}&\big(z-(2+3i)\big)\big(z-(2-3i)\big)\\ &=(z-2-3i)(z-2+3i)\\ &=(z-2)^2-(3i)^2\\ &=z^2-4z+4+9\\ &=z^2-4z+13\end{aligned}$$

**Question 11**

$1-i$ is also a root (conjugate root theorem).

$$\begin{aligned}1+i+1-i+\alpha+\beta &= -\frac{-2}{1}\\ 2+\alpha+\beta &= 2\\ \alpha+\beta &= 0 \qquad (1)\\ (1+i)(1-i)\alpha\beta &= \frac{-2}{1}\\ \big(1^2+1^2\big)\alpha\beta &= -2\\ \alpha\beta &= -1 \qquad (2)\\ \alpha,\beta &= \pm 1 \text{ by inspection}\end{aligned}$$

The other roots are $1-i, -1$ and $1$.

Alternative solution I:

$1-i$ is also a root (conjugate root theorem).

$\big(z-(1+i)\big)\big(z-(1-i)\big)=z^2-2z+2$ is a factor of $P(z)$.

$$\begin{aligned}&z^4-2z^3+z^2+2z-2\\ &=z^2\big(z^2-2z+2\big)-\big(z^2-2z+2\big)\\ &=\big(z^2-1\big)\big(z^2-2z+2\big)\\ &=(z+1)(z-1)\big(z-(1+i)\big)\big(z-(1-i)\big)\end{aligned}$$

The other roots are $1-i, -1$ and $1$.

**Question 12**

$$\begin{aligned}z\bar{z} &= |z|^2\\ \therefore 25 &= a^2+b^2 \qquad (1)\\ z+\bar{z} &= 2\text{Re}(z)\\ 8 &= 2a\\ a &= 4\\ \text{sub in } (1):\ 25 &= 4^2+b^2\\ b^2 &= 9\\ b &= \pm 3\\ z &= 4\pm 3i\end{aligned}$$

Alternative solution I:

Let $z$ and $\bar{z}$ be the solutions of the quadratic $P(\omega)=0$.

$$\begin{aligned}\therefore\ &(\omega-z)(\omega-\bar{z})=0\\ &\omega^2-(z+\bar{z})\omega+z\bar{z}=0\\ &\omega^2-8\omega+25=0\\ &\omega^2-8\omega+16=16-25\\ &(\omega-4)^2=-9\\ &\omega-4=\pm 3i\\ &\omega=4\pm 3i\\ &\therefore z=4\pm 3i\end{aligned}$$

**Question 13**

The graph shows one real root, a pair of non-real conjugate roots and a single non-real root. Since not all non-real roots occur in conjugate pairs then we cannot have a quartic with all real coefficients, so eliminate A.

B is a distractor, as there is no rule stating that if at least one coefficient is real anything can be known about the roots, so eliminate B.

If a polynomial has all real coefficients then any non-real roots would occur in conjugate pairs, so the conjugate of $\beta$ would also be a root, so $P(x)$ must have a degree of $5$ or higher.

D is a distractor, as we do not know the arguments of the roots, so cannot determine of they are n$^{\text{th}}$ roots of $-1$.

**Answer C**

**Question 14**

Since the coefficients are real then the conjugate of non-real zeros are also zeros.

The conjugate of $\alpha$ is $\bar{\alpha}$.

The conjugate of $-i\alpha$ is $\overline{-i\alpha}=\overline{-i}\times\bar{\alpha}=i\bar{\alpha}$

$\therefore\ \bar{\alpha}$ and $i\bar{\alpha}$ are roots of $P(z)$

**Question 15a**

$$\begin{aligned}&a\bar{z}^3+b\bar{z}^2+c\bar{z}+d\\ &=\overline{az^3}+\overline{bz^2}+c\bar{z}+d\\ &=\overline{az^3}+\overline{bz^2}+\overline{cz}+d\\ &=\overline{az^3+bz^2+cz+d}\\ &=\overline{0}\\ &=0\end{aligned}$$

$\therefore\ \bar{z}$ is also a root.

**Question 15b**

$1-2i$ is also a root.

$$\begin{aligned}\alpha+\beta+\gamma &= -\frac{b}{a}\\ 1+2i+1-2i+\gamma &= 6\\ \gamma &= 4\end{aligned}$$

The roots of the equation are $1\pm 2i$ and $4$.

**Question 16**

A is true if $\arg(z) = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$

B cannot be true, as $\bar{z}$ must be non-real since $z$ is non-real, while $|z|^2$ must be real

C cannot be true as $\text{Re}(iz) = -\text{Im}(z) \neq \text{Im}(z)$ since $z$ is non-real.

D cannot be true as $\arg(z^3) = 3\arg(z)$, which is impossible since $\arg(z) \neq 0$ since $z$ is non-real.

**Answer A**

**Question 17**

$z + \bar{z} = a + ib + a - ib = 2a$ which is real

$\frac{z}{\bar{z}} = \frac{a+ib}{a-ib} \times \frac{a+ib}{a+ib} = \frac{a^2-b^2+2abi}{a^2+b^2}$ which is non-real

$z^2 - (\bar{z})^2 = (a+ib)^2 - (a-ib)^2$

$= (a^2 - b^2 + 2abi) - (a^2 - b^2$

$- 2abi) = 4abi$ which is non−real

$z\bar{z} = (a+ib)(a-ib) = a^2 + b^2$ which is real and positive

**Answer C**

**Question 18**

$2z^3 = z + \bar{z}$

$2z^3 = 2\text{Re}(z)$

$z^3 = \text{Re}(z)$

$z = 0, \pm 1$ are the three solutions since their cube equals their real component.

**Question 19**

$\bar{z} = 3 - i$ is also a root.

$$p = -\sum \alpha$$

$= -(3 + i + 3 - i)$

$= -6$

$$q = \prod \alpha$$

$= (3 + i)(3 - i)$

$= 3^2 + 1^2$

$= 10$

**Question 20**

$(z_1 + z_2 + z_3)^2$

$= z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_1z_3 + z_2z_3)$

$\therefore\ z_1^2 + z_2^2 + z_3^2$

$= (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_1z_3 + z_2z_3)$

$= 0 - 2z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$

$= -2z_1z_2z_3(\overline{z_1} + \overline{z_2} + \overline{z_3})$

$= -2z_1z_2z_3(z_1 + z_2 + z_3)$ since $|z_n| = 1$

$= -2z_1z_2z_3(\overline{0})$

$= 0$

**Question 21**

$z_1z_2$ is a complex number

$\therefore |z_1z_2|^2 = (z_1z_2)\overline{z_1z_2}$

$= z_1 \times z_2 \times \overline{z_1} \times \overline{z_2}$

$= z_1 \times \overline{z_1} \times z_2 \times \overline{z_2}$

$= |z_1|^2 \times |z_2|^2$

$\therefore |z_1z_2| = |z_1| \times |z_2|$

since moduli are non−negative

**Question 22**

$$w = \frac{z^2 - 1}{z^2 + 1}$$

$$= \frac{z\left(z - \frac{1}{z}\right)}{z\left(z + \frac{1}{z}\right)}$$

$$= \frac{z - \frac{1}{z}}{z + \frac{1}{z}}$$

$$= \frac{z - \bar{z}}{z + \bar{z}}$$

$$= \frac{2i\text{Im}(z)}{2\text{Re}(z)}$$

$$= i \times \frac{\sin\theta}{\cos\theta}$$

$= i\tan\theta$ which is purely imaginary

**Question 23**

$|z_1| = |z_2| = |z_3| \quad (1)$

$z_1 + z_2 + z_3 = 1 \quad (2)$

$z_1z_2z_3 = 1 \quad (3)$

From (1) and (3) $|z_n| = 1$ for $n = 1, 2, 3$

$$\therefore \frac{1}{z_n} = \overline{z_n} \quad (4)$$

From (3) and (4) $z_1z_2 = \frac{1}{z_3} = \overline{z_3}$.

Similarly $z_1z_3 = \overline{z_2}$ and $z_2z_3 = \overline{z_1}$.

$\therefore\ z_1z_2 + z_1z_3 + z_2z_3 = \overline{z_3} + \overline{z_2} + \overline{z_1}$

$= \overline{z_1 + z_2 + z_3}$

$= \overline{1}$

$= 1 \quad (5)$

The complex numbers $z_1, z_2$ and $z_3$ are the roots of the cubic:

$z^3 - (z_1 + z_2 + z_3)z^2 + (z_1z_2 + z_1z_3 + z_2z_3)z$

$- z_1z_2z_3 = 0$

$z^3 - z^2 + z - 1 = 0$ from $(2), (3)$ and $(5)$

$z^2(z - 1) + (z - 1) = 0$

$(z^2 + 1)(z - 1) = 0$

$(z + i)(z - i)(z - 1) = 0$

$\therefore z = 1, \pm i$

$z_1, z_2, z_3 = 1, i, -i$ in any order.

**Question 24a**

Since the cubic equation is not $z^3 = 0$ at least one of the roots $\alpha, \beta, \gamma$ is non-zero.
The coefficients are real, so any non-real roots must occur in conjugate pairs.
In each equation the sum of squares is zero, and since at least one of the roots is non-zero this means that either: $\alpha, \gamma$ are real and $\beta$ is non-real, or $\beta$ is real and $\alpha, \gamma$ are non-real.
Since by the conjugate root theorem any non-real roots must occur in conjugate pairs, $\beta$ must be real while $\alpha$ and $\gamma$ are non-real.

**Question 24b**

Let $\alpha = x + iy, \gamma = x - iy$

$\alpha^2 = \gamma^2 = -\beta^2$ from the given equations

$\therefore (x \pm iy)^2$

$= (\beta i)^2$ where $\beta$ is purely real from (a)

$x + iy = \pm\beta i$

$\therefore x = 0, y = \pm\beta$

$\therefore \alpha, \gamma$ are purely imaginary.

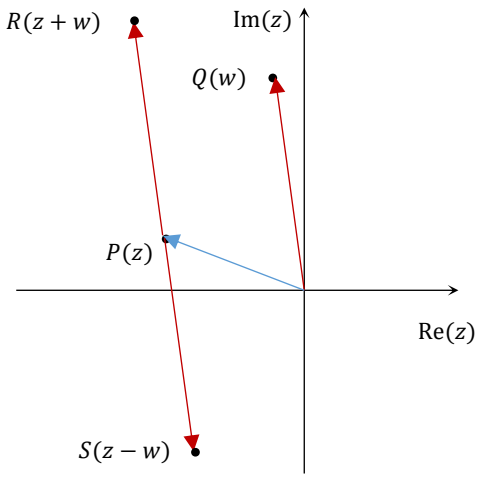
**Question 24c**

$$\alpha\beta\gamma = -8$$
$$(-\beta \times \beta) \times \beta = -8$$
$$\beta^3 = 8$$
$$\beta = 2$$
$$\therefore \alpha = 2i, \gamma = -2i \text{ (or vice versa)}$$
$$\alpha + \beta + \gamma = -b$$
$$b = -(2i + 2 - 2i)$$
$$= -2$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = c$$
$$c = (2i)(2) + (2i)(-2i) + (2)(-2i)$$
$$= 4$$
$$\therefore b = -2, c = 4$$

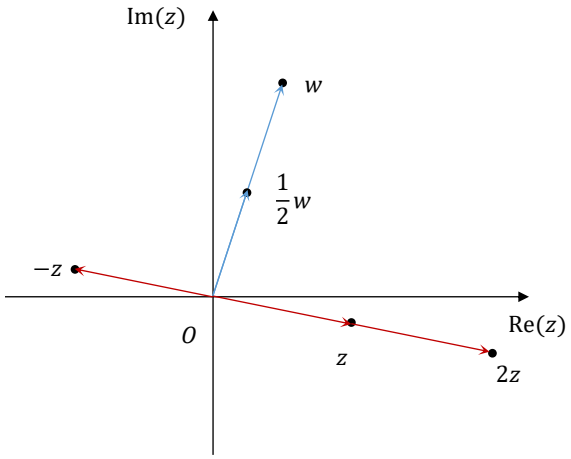
**Question 25**

$$32\sin^4\theta\cos^2\theta$$
$$= 32\sin^4\theta\left(1 - \sin^2\theta\right)$$
$$= 32\sin^4\theta - 32\sin^6\theta$$
$$= 32\left(\frac{z - z^{-1}}{2i}\right)^4 - 32\left(\frac{z - z^{-1}}{2i}\right)^6$$
$$= 2\left(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}\right)$$
$$+\frac{1}{2}\left(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}\right)$$
$$= \frac{1}{2}\left(z^6 + z^{-6}\right) - \left(z^4 + z^{-4}\right) - \frac{1}{2}\left(z^2 + z^{-2}\right) + 2$$
$$= \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$

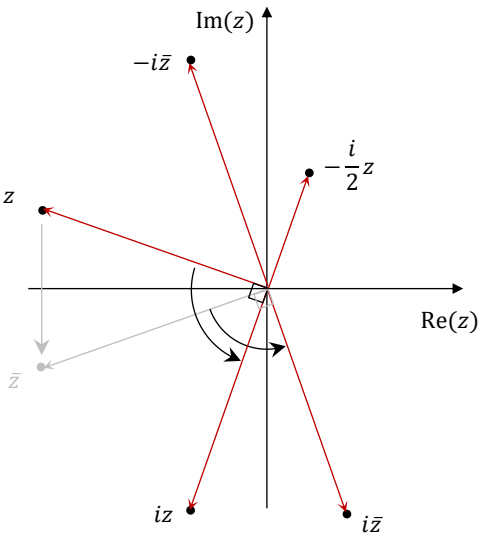
**Question 1**




**Question 2**




**Question 3**




**Question 4**

$$\frac{-1+5}{2}+\frac{3-1}{2}i=2+i$$

**Question 5**

$\angle VOU=\frac{\pi}{3}$ angle in an equilateral triangle

Let $v=u\ \mathrm{cis}\frac{\pi}{3}=\mathrm{cis}\frac{\pi}{6}\mathrm{cis}\frac{\pi}{3}=\mathrm{cis}\frac{\pi}{2}$

$$\begin{aligned}
&u^2+v^2\\
&=\left(\mathrm{cis}\frac{\pi}{6}\right)^2+\left(\mathrm{cis}\frac{\pi}{2}\right)^2\\
&=\mathrm{cis}\frac{\pi}{3}+\mathrm{cis}\,\pi\\
&=\mathrm{cis}\frac{2\pi}{3}\left(\mathrm{cis}\left(-\frac{\pi}{3}\right)+\mathrm{cis}\frac{\pi}{3}\right)\\
&=\mathrm{cis}\frac{2\pi}{3}\left(2\mathrm{Re}\left(\mathrm{cis}\frac{\pi}{3}\right)\right)\\
&=\mathrm{cis}\frac{2\pi}{3}\left(2\cos\frac{\pi}{3}\right)\\
&=\mathrm{cis}\frac{2\pi}{3}\left(2\times\frac{1}{2}\right)\\
&=\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\\
&=-\frac{1}{2}+\frac{\sqrt{3}}{2}i
\end{aligned}$$

**Question 6a**

$$\begin{aligned}
&M=\frac{1}{2}(\omega z+\bar{\omega}z)\\
&=\frac{z}{2}(\omega+\bar{\omega})\\
&=\frac{z}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}+\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)\\
&=\frac{z}{2}\left(2\cos\frac{\pi}{3}\right)\\
&=\left(\cos\frac{\pi}{3}\right)z\\
&=\frac{z}{2}
\end{aligned}$$

**Question 6b**

$M$ is also the midpoint of SP

$$\begin{aligned}
&\frac{s+p}{2}=\frac{z}{2}\\
&s+z=z\\
&s=0
\end{aligned}$$

**Question 7**

$$\begin{aligned}
&\left|(1-i)z^2+\sqrt{2}iz\right|\\
&\leq\left|(1-i)z^2\right|+\left|\sqrt{2}iz\right| && \text{triangle inequality}\\
&=|1-i||z|^2+\left|\sqrt{2}\right||i||z| && \text{modulus rules}\\
&=\sqrt{1^2+(-1)^2}|z|^2+\sqrt{2}|z|\\
&<\sqrt{2}(1)^2+\sqrt{2} && \text{since } |z|<1\\
&=2\sqrt{2}\\
&<2\times1.5 && \text{since } \sqrt{2}<1.5\\
&=3
\end{aligned}$$

**Question 8**

Diagonals of a square are of equal length and at right angles, so

$$\overrightarrow{QS} = i\overrightarrow{PR}$$
$$\therefore q - s = i(p - r)$$

**Answer A**

**Question 9**

$z_2 - z_1$ is the vector from $P$ to $Q$, so $\overrightarrow{PQ}$.

**Answer B**

**Question 10**

$$z + 2iz = -4 - 3i$$
$$x + iy + 2ix - 2y = -4 - 3i$$
$$x - 2y = -4 \quad (1)$$
$$2x + y = -3 \quad (2)$$
$$(1) + 2(2): \quad 5x = -10$$
$$x = -2$$

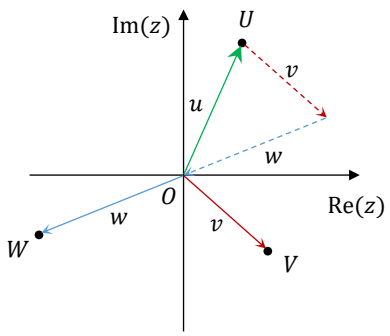
**Answer C**

**Alt:**

$$z + 2iz = -4 - 3i$$
$$z(1 + 2i) = -(4 + 3i)$$
$$z = -\frac{4 + 3i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i}$$
$$= -\frac{4 - 8i + 3i - 6i^2}{1^2 + 2^2}$$
$$= -\frac{4 - 5i + 6}{5}$$
$$= -2 + i$$

**Answer C**

**Question 11**




**Question 12**

$$\overrightarrow{OC} = 3i \times \overrightarrow{OA}$$
$$= 3i(1 + i)$$
$$= -3 + 3i$$

**Question 13**

$$\overrightarrow{AC} = -i\overrightarrow{AB}$$
$$c - a = -i(b - a)$$
$$c = -ib + ia + a$$
$$c = (1 + i)a - ib$$

**Question 14**

Let $W$ represent $a + ib$

$$-i\overrightarrow{WV} = \overrightarrow{OW}$$
$$-i(4 + i - a - ib) = a + ib$$
$$-4i + 1 + ai - b = a + bi$$
$$-b + 1 + i(a - 4) = a + bi$$
$$\therefore -b + 1 = a \qquad b = 1 - a$$
$$a - 4 = b$$
$$a - 4 = 1 - a$$
$$2a = 5$$
$$a = \frac{5}{2}$$
$$b = 1 - \frac{5}{2} = -\frac{3}{2}$$

$W$ represents $\dfrac{5 - 3i}{2}$.

**Question 15**

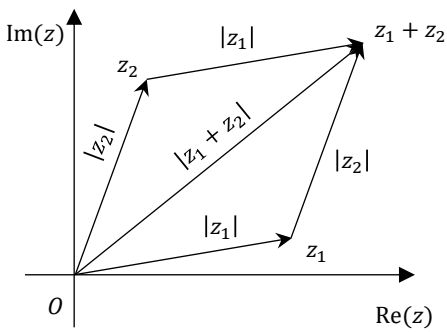
$$\angle POR = \angle ROQ = \frac{\pi}{4}, |r| = \sqrt{2}|p| = \sqrt{2}|q|$$
$$\therefore q = \tfrac{1}{\sqrt{2}} \operatorname{cis} \tfrac{\pi}{4} \times r \text{ and } p = \tfrac{1}{\sqrt{2}} \operatorname{cis}\left(-\tfrac{\pi}{4}\right) \times r$$
$$pq = \frac{r}{\sqrt{2}} \operatorname{cis} \frac{\pi}{4} \times \frac{r}{\sqrt{2}} \operatorname{cis}\left(-\frac{\pi}{4}\right)$$
$$= \frac{r^2}{2} \operatorname{cis}\left(\frac{\pi}{4} - \frac{\pi}{4}\right)$$
$$= \frac{r^2}{2} \operatorname{cis} 0$$
$$= \frac{r^2}{2}$$

Alternatively (thanks to Chris B):

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$$
$$= \overrightarrow{OP} + \overrightarrow{OQ}$$
$$(\overrightarrow{PR} = \overrightarrow{OQ} \text{ opposite sides of a square})$$
$$\therefore r = p + q$$
$$r^2 = (p + q)^2$$
$$= p^2 + 2pq + q^2$$
$$= p^2 + 2pq + (ip)^2 \qquad \text{since } q = ip$$
$$= p^2 + 2pq - p^2$$
$$= 2pq$$
$$\therefore pq = \frac{r^2}{2}$$

**Question 16**

In the diagram we can see that $O, z_1, z_2$ and $z_1 + z_2$ form a parallelogram.




The diagonal from zero to $z_1 + z_2$ splits the parallelogram into two congruent triangles with lengths $|z_1|, |z_2|$ and $|z_1 + z_2|$.

$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$ (any side of a triangle is less than the sum of the other two sides).

**Question 17a**

$\overrightarrow{OC}$ is $\overrightarrow{OA}$ rotated $\frac{\pi}{3}$ anticlockwise, since the sides of a rhombus are equal, and $\angle COA = \frac{\pi}{3}$

$\therefore \overrightarrow{OC} = \overrightarrow{OA} \times \operatorname{cis}\frac{\pi}{3}$

$\therefore \omega_2 = \omega_1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

**Question 17b**

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OC} + \overrightarrow{OA} = \omega_2 + \omega_1$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \omega_2 - \omega_1$

**Question 17c**

$$
\begin{aligned}
&i(\omega_1 + \omega_2) \\
&= i\left(\omega_1 + \omega_1 \operatorname{cis}\frac{\pi}{3}\right) \\
&= \omega_1\left(\operatorname{cis}\frac{\pi}{2}\right)\left(1 + \operatorname{cis}\frac{\pi}{3}\right) \\
&= \omega_1\left(\operatorname{cis}\frac{\pi}{2} + \operatorname{cis}\frac{5\pi}{6}\right) \\
&= \omega_1\left(i + \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \\
&= \omega_1\left(i - \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\
&= \omega_1\left(-\frac{\sqrt{3}}{2} + \frac{3}{2}i\right) \\
&= \sqrt{3}\omega_1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= \sqrt{3}\omega_1\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i - 1\right) \\
&= \sqrt{3}\omega_1\left(\operatorname{cis}\frac{\pi}{3} - 1\right) \\
&= \sqrt{3}\left(\omega_1 \operatorname{cis}\frac{\pi}{3} - \omega_1\right) \\
&= \sqrt{3}(\omega_2 - \omega_1)
\end{aligned}
$$

$\therefore \overrightarrow{AC}$ is at right angles to $\overrightarrow{OB}$ (and $\sqrt{3}$ times its magnitude).

**Question 18**

$C$ can be in two positions, $C_1$ and $C_2$, forming a square with diagonal $AB$.

The diagonals of a square are equal and bisect each other at right angles.

Let $M = \frac{1}{2}(2 - i + 8 + i) = 5$ be the midpoint of $AB$.

$\overrightarrow{MB} = 8 + i - 5 = 3 + i$

$\overrightarrow{MC_1} = i(3 + i) = 3i - 1 \ \therefore C_1 = 5 + 3i - 1$
$= 4 + 3i$

$\overrightarrow{MC_2} = -i(3 + i) = -3i + 1 \ \therefore C_2 = 5 - 3i + 1$
$= 6 - 3i$

**Question 19**

$\Delta ODA$ is right angled (diagonals of a kite are perpendicular)

$\therefore OD = \frac{\sqrt{3}}{2}|w|$ (exact triangles)

$OB = 2 \times OD = \sqrt{3}|w|$

$\angle DCB = \angle DCO = \frac{\pi}{6}$ (angles of a kite are bisected by the diagonals)

$\therefore \Delta OBC$ is equilateral (isosceles triangle with apex angle $\frac{\pi}{3}$)

$\therefore \angle COB = \frac{\pi}{3}$

$\angle AOC = \frac{\pi}{2}$ (angle sum $\Delta AOC$)

$\angle AOB = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

$$
\begin{aligned}
\therefore \overrightarrow{OB} &= \sqrt{3}w \operatorname{cis}\frac{\pi}{6} \\
&= \sqrt{3}w\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \\
&= \frac{w(3 + \sqrt{3}i)}{2} \\
\overrightarrow{OC} &= \overrightarrow{OB} \operatorname{cis}\frac{\pi}{3} \\
&= \sqrt{3}w \operatorname{cis}\frac{\pi}{6}\operatorname{cis}\frac{\pi}{3} \\
&= \sqrt{3}w \operatorname{cis}\frac{\pi}{2} \\
&= \sqrt{3}wi
\end{aligned}
$$

**Question 20**

$\therefore \phi = \omega \operatorname{cis}\frac{2\pi}{3}$

$$
\begin{aligned}
(\omega + \phi)^2 &= \left(\omega + \omega \operatorname{cis}\frac{2\pi}{3}\right)^2 \\
&= \omega^2\left(1 + \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^2 \\
&= \omega^2\left(1 - \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2 \\
&= \omega^2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 \\
&= \omega^2\left(\frac{1}{4} + \frac{\sqrt{3}}{2}i - \frac{3}{4}\right) \\
&= \omega^2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= \omega^2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \\
&= \omega \cdot \omega \operatorname{cis}\frac{2\pi}{3} \\
&= \omega\phi
\end{aligned}
$$

**Question 21**

Splitting $\Delta OKL$ in half to create two right angled triangles, we can see that $|OL| = \sqrt{3}r$, so $|OM| = \sqrt{3}r$ as $\Delta OLM$ is equilateral.

$\angle KOL = \frac{\pi}{6}$ (angle sum of $\Delta OKL$)

$\angle LOM = \frac{\pi}{3}$ (angle in an equilateral triangle)

$\therefore\ \angle KOM = \frac{\pi}{2}$

$\therefore OM$ is $OK$ rotated anticlockwise by $\frac{\pi}{2}$ and stretched by $\sqrt{3}$.

$\therefore \beta = \alpha\sqrt{3}i$

$$\begin{aligned}\therefore 3\alpha^2 + \beta^2 &= 3\alpha^2 + \left(\alpha\sqrt{3}i\right)^2 \\ &= 3\alpha^2 - 3\alpha^2 \\ &= 0\end{aligned}$$

**Question 22**

$$\begin{aligned}|z| &\le \left|z - \frac{4}{z}\right| + \left|\frac{4}{z}\right| \\ |z| &\le 2 + \frac{4}{|z|} \\ |z|^2 &\le 2|z| + 4 \\ |z|^2 - 2|z| - 4 &\le 0 \\ |z| &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)} \\ &= \frac{2 \pm 2\sqrt{5}}{2} \\ &= 1 \pm \sqrt{5} \\ \therefore 0 \le |z| &\le 1 + \sqrt{5} \\ \therefore |z| &\le 1 + \sqrt{5}\end{aligned}$$

**Question 23**

$$\begin{aligned}\arg\left(\frac{z}{w}\right) &= -\frac{\pi}{2} \\ \therefore \frac{z}{w} &= ki \quad k \in \mathbb{R} \\ \therefore \left|\frac{z-w}{z+w}\right| &= \left|\frac{\frac{z}{w} - 1}{\frac{z}{w} + 1}\right| \\ &= \frac{|ki - 1|}{|ki + 1|} \\ &= \frac{\sqrt{k^2+1}}{\sqrt{k^2+1}} \\ &= 1\end{aligned}$$

Alternatively:

$z + w$ and $z - w$ are the diagonals of a parallelogram.

Since $\arg\left(\frac{z}{w}\right) = -\frac{\pi}{2}$ they are the diagonals of a rectangle (a parallelogram with one right angle must be a rectangle).

The diagonals of a rectangle are equal, so $|z - w| = |z + w|$.

$$\begin{aligned}\left|\frac{z-w}{z+w}\right| &= \frac{|z-w|}{|z+w|} \\ &= 1\end{aligned}$$

**Question 24a**

$$\begin{aligned}&|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 \\ &= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha - \beta)\left(\overline{\alpha - \beta}\right) \\ &= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha - \beta)\left(\bar{\alpha}\right. \\ &= \alpha\bar{\alpha} + \beta\bar{\beta} - \alpha\bar{\alpha} + \alpha\bar{\beta} \\ &= \bar{\alpha}\beta + \overline{\bar{\alpha}\beta} \\ &= 2\mathrm{Re}(\bar{\alpha}\beta)\end{aligned}$$

**Question 24b**

Using the cosine rule:

$$\begin{aligned}|\alpha - \beta|^2 &= |\alpha|^2 + |\beta|^2 - 2|\alpha||\beta|\cos\theta \\ 2|\alpha||\beta|\cos\theta &= |\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 \\ &= 2\mathrm{Re}(\bar{\alpha}\beta) \quad \text{from (i)} \\ \therefore |\alpha||\beta|\cos\theta &= \mathrm{Re}\left(\alpha\bar{\beta}\right)\end{aligned}$$

**Question 25ai**

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(z_2 - z_1)$$

**Question 25aii**

$$\begin{aligned}\tan\left(\frac{\alpha}{2}\right) &= \frac{|AM|}{|MP|} \\ |MP| &= \frac{|AM|}{\tan\left(\frac{\alpha}{2}\right)} = |AM|\cot\left(\frac{\alpha}{2}\right) \\ \overrightarrow{MP} &= \frac{1}{2}(z_2 - z_1)\cot\left(\frac{\alpha}{2}\right)i\end{aligned}$$

**Question 25b**

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AM} + \overrightarrow{(MP)} \\ &= z_1 + \frac{1}{2}(z_2 - z_1) + \frac{1}{2}(z_2 - z_1)\cot\left(\frac{\alpha}{2}\right)i \\ &= \frac{1}{2}\left(1 - i\cot\frac{\alpha}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)z_2\end{aligned}$$

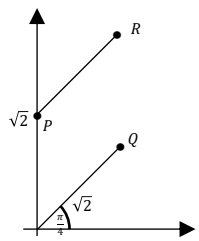
**Question 26a**

$$\begin{aligned}z_1 &= i\sqrt{2} = \sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\ z_2 &= \frac{2}{1-i} = \frac{2}{\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\end{aligned}$$

**Question 26b**

$$\begin{aligned}z_1 &= wz_2 \\ \therefore w &= \frac{z_1}{z_2} \\ &= \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{2}\right) \div \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right) \\ &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\end{aligned}$$

**Question 26c**



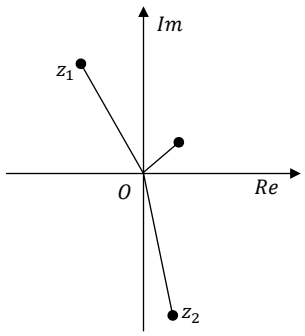
**Question 26d**

$|z_1| = |z_2| = \sqrt{2}$ so $OPRQ$ is a rhombus.

$\therefore OR$ bisects $\angle POQ$

$$\therefore \arg(z_1 + z_2) = \arg z_2 + \frac{1}{2}(\arg z_1 - \arg z_2)$$
$$= \frac{1}{2}(\arg z_1 + \arg z_2)$$
$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$$
$$= \frac{3\pi}{8}$$
$$z_1 + z_2 = i\sqrt{2} + \frac{2}{1-i} \times \frac{1+i}{1+i}$$
$$= i\sqrt{2} + \frac{2(1+i)}{1^2 + 1^2}$$
$$= 1 + \left(1 + \sqrt{2}\right)i$$
$$\therefore \tan\left(\frac{3\pi}{8}\right) = \frac{1+\sqrt{2}}{1} = 1 + \sqrt{2}$$

**Question 27a**




The sum of vectors must add to zero, so when laid tip to tail they should start and end at zero.

**Question 27b**

$z_1 = \cos\alpha + i\sin\alpha$

$z_2 = k(\cos\beta + i\sin\beta)$

$z_3 = (2-k)(\cos\gamma + i\sin\gamma)$

**Question 27c**

$z_1 + z_2 + z_3 = 0$

$$\therefore \cos\alpha + i\sin\alpha + k(\cos\beta + i\sin\beta) + (2-k)(\cos\gamma + i\sin\gamma) = 0$$
$$(\cos\alpha + k\cos\beta + (2-k)\cos\gamma) + i(\sin\alpha + k\sin\beta + (2-k)\sin\gamma) = 0$$

Equating real and imaginary components:

$$\cos\alpha + k\cos\beta + (2-k)\cos\gamma = 0$$
$$\sin\alpha + k\sin\beta + (2-k)\sin\gamma = 0$$

**Question 27d**

$$\sin\alpha = -(k\sin\beta + (2-k)\sin\gamma)$$
$$\sin^2\alpha = k^2\sin^2\beta + 2k(2-k)\sin\beta\sin\gamma + (2-k)^2\sin^2\gamma$$
$$\cos\alpha = -(k\cos\beta + (2-k)\cos\gamma)$$
$$\cos^2\alpha = k^2\cos^2\beta + 2k(2-k)\cos\beta\cos\gamma + (2-k)^2\cos^2\gamma$$
$$\sin^2\alpha + \cos^2\alpha = 1$$
$$k^2\left(\sin^2\beta + \cos^2\beta\right) + 2k(2-k)(\cos\beta\cos\gamma + \sin\beta\sin\gamma) + (2-k)^2\left(\sin^2\gamma + \cos^2\gamma\right) = 1$$
$$k^2 + 2k(2-k)\cos(\beta-\gamma) + (2-k)^2 = 1$$

**Question 27e**

$$2k(2-k)\cos(\beta-\gamma) = 1 - (2-k)^2 - k^2$$
$$\cos(\beta-\gamma) = \frac{1 - 4 + 4k - k^2 - k^2}{2k(2-k)}$$
$$= \frac{2k^2 - 4k + 3}{2k^2 - 4k}$$
$$= 1 + \frac{3}{2k^2 - 4k}$$
$$\therefore \left|1 + \frac{3}{2k^2 - 4k}\right| \leq 1$$
$$-2 \leq \frac{3}{2k^2 - 4k} \leq 0$$

$$-2 \leq \frac{3}{2k^2 - 4k}$$
$$-4k^2 + 8k \geq 3 \quad \text{since } 2k^2 - 4k < 0$$
$$4k^2 - 8k + 3 \leq 0$$
$$(2k-1)(2k-3) \leq 0$$
$$\frac{1}{2} \leq k \leq \frac{3}{2}$$

and

$$\frac{3}{2k^2 - 4k} \leq 0$$
$$\therefore 2k^2 - 4k \leq 0$$
$$2k(k-2) \leq 0$$
$$0 \leq k \leq 2$$

$$\therefore \frac{1}{2} \leq k \leq \frac{3}{2}$$

**Question 28**

If $4\alpha - \beta = \bar{\beta} + 1 - \alpha$, then $\overrightarrow{OC} - \overrightarrow{OB} = \overrightarrow{OD} - \overrightarrow{OA} \rightarrow \overrightarrow{BC} = \overrightarrow{AD}$, so $ABCD$ is a parallelogram.

$\alpha + \beta + 4\alpha + \bar{\beta} + 1 = -\dfrac{b}{a}$ sum of the roots one at a time

$5\alpha + \beta + \bar{\beta} + 1 = 10$

$5\alpha + 2\text{Re}(\beta) = 9$

$2\text{Re}(\beta) = 9 - 5\alpha \quad (1)$

$\dfrac{\alpha + 4\alpha}{2} = \dfrac{\beta + \bar{\beta} + 1}{2}$ the diagonals of a parallelogram bisect

$\dfrac{5\alpha}{2} = \dfrac{2\text{Re}(\beta) + 1}{2}$

$2\text{Re}(\beta) = 5\alpha - 1 \quad (2)$

$9 - 5\alpha = 5\alpha - 1$ from (1) and (2)

$10\alpha = 10$

$\alpha = 1$

$\therefore 4\alpha = 4$

sub in (1): $2\text{Re}(\beta) = 9 - 5(1)$

$= 4$

$\text{Re}(\beta) = 2$

Let $\beta = 2 + ki$

$\therefore \bar{\beta} + 1 = 3 - ki$

$\therefore 1 \times 4 \times (2 + ki)(3 - ki) = 4(7 + i)$ product of the roots

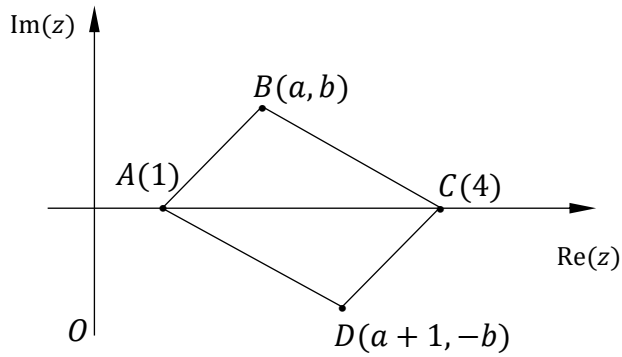
$6 - 2ki + 3ki + k^2 = 7 + i$

$6 + k^2 = 7$ and $k = 1$

$\therefore k = 1$

$\therefore \beta = 2 + i, \bar{\beta} + 1 = 3 - i$

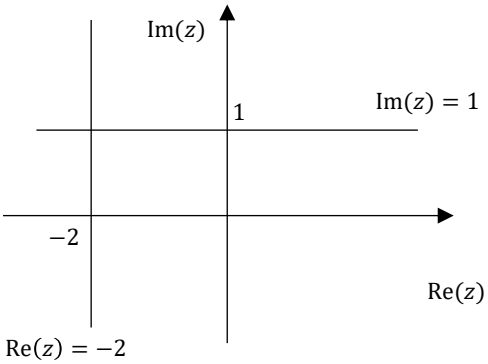
The four roots are $1, 4, 2 + i$ and $3 - i$.



**Question 1**

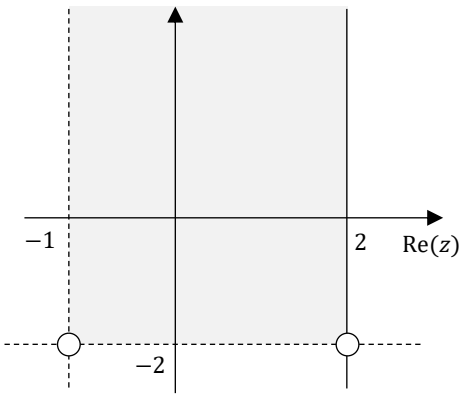
**a** A vertical line at $\text{Re}(z) = -2$

**b** A horizontal line at $\text{Im}(z) = 1$



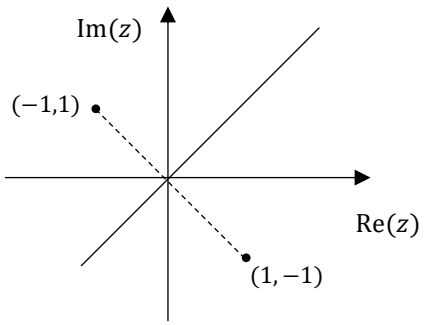

**Question 2**

The region between the vertical lines at $\text{Re}(z) = -1$ (dotted) and $\text{Re}(z) = 2$ (solid), and above the horizontal line at $\text{Im}(z) = -2$ (dotted).




**Question 3**

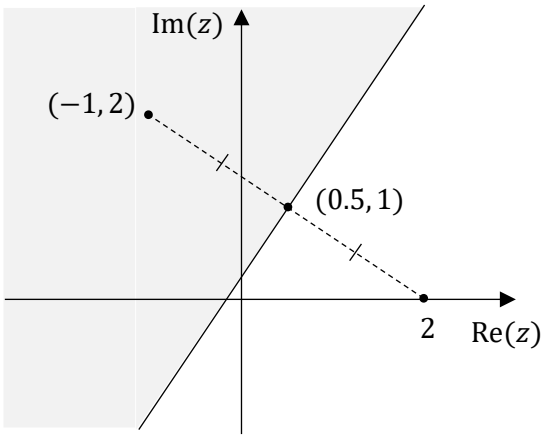
This is the perpendicular bisector of the interval joining $(-1,1)$ and $(1,-1)$.




**Question 4**

This is the perpendicular bisector of the interval joining $(-1,2)$ and $(2,0)$ and the region including $(-1,2)$. The midpoint is $(0.5,1)$ and the gradient is $-\frac{2}{3}$, so draw a solid line through $(0.5,1)$ with $m = \frac{3}{2}$, and shade upper left-hand side.

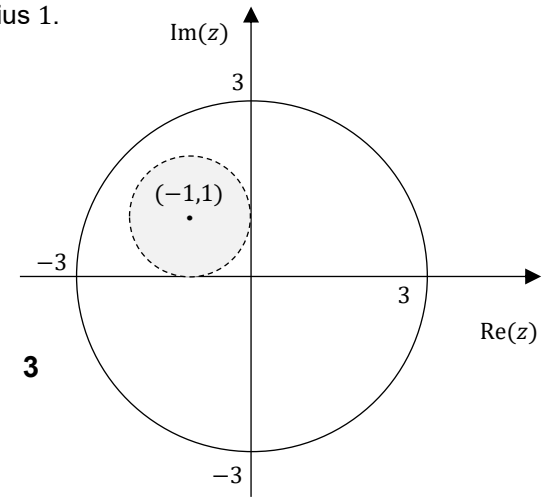
**Question 4 (continued)**




**Question 5**

**a** This is a circle centred at zero with radius $3$

**b** This is the region inside but not including the circumference of a circle centred at $(-1,1)$ with radius $1$.

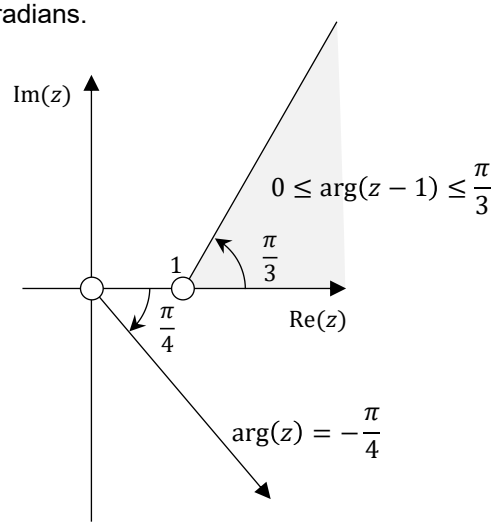



**3**

**Question 6**

**a** A ray from the point $(-2,0)$ at an angle of $\frac{\pi}{3}$.

**b** A sector centred at zero, with rays at 0 radians and $\frac{\pi}{4}$ radians.



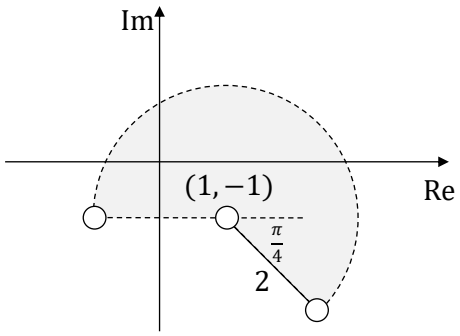
**Question 7**

The perpendicular bisector of $(0,-2)$ and the origin, so $\text{Im}(z) = -1$

**Answer B**

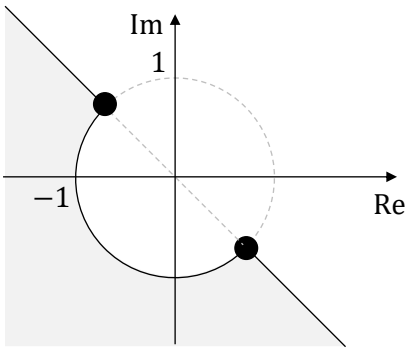
**Question 8**

This is intersection of the inside of the circle centred at $(1,-1)$ with radius 2, with the sector with the same centre, with argument clockwise from $-\frac{\pi}{4}$ (solid) to $\pi$ (dotted).




**Question 9**

This is the intersection of the region on or to the lower-left of the perpendicular bisector of $(-1,0)$ and $(0,1)$ and the area on and outside the unit circle.




**Question 10**

The interval joining $(-2,0)$ and $(2,0)$ subtends an angle at the circumference of $\frac{\pi}{4}$, so the angle subtended at the centre is $\frac{\pi}{2}$, so we want three-quarters of the circumference.

**Answer C**

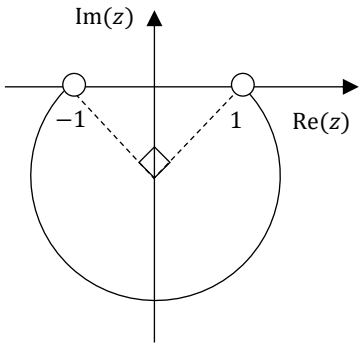
**Question 11**

$$\sqrt{(x+1)^2+y^2} = 3\sqrt{x^2+y^2}$$
$$x^2+2x+1+y^2 = 9(x^2+y^2)$$
$$x^2+2x+1+y^2 = 9x^2+9y^2$$
$$8x^2-2x+8y^2-1=0$$
$$x^2-\frac{1}{4}x+y^2=\frac{1}{8}$$
$$x^2-\frac{1}{4}x+\frac{1}{64}+y^2=\frac{1}{8}+\frac{1}{64}$$
$$\left(x-\frac{1}{8}\right)^2+y^2=\frac{9}{64}$$

Which is a circle with centre $\left(\frac{1}{8},0\right)$ with radius $\frac{3}{8}$.

**Question 12**

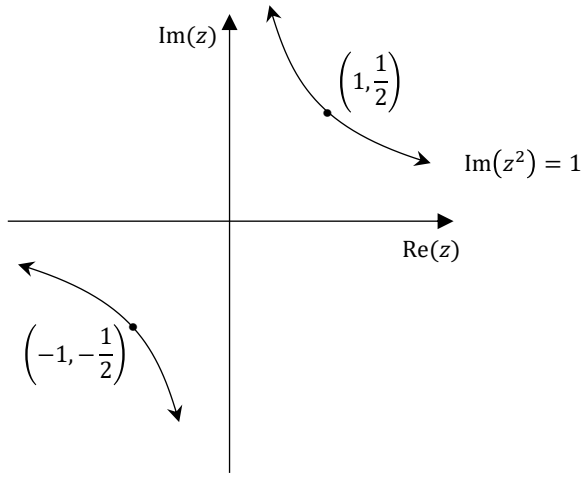
The angle at $z$ (formed by the vectors from $(-1,0)$ and $(1,0)$ is equal to $\frac{\pi}{4}$, so an arc with ends at $(-1,0)$ and $(1,0)$. Standing at $(1,0)$ and facing $(-1,0)$ we want the arc on the left (since the angle is positive). We double the $\frac{\pi}{4}$ angle at the circumference to get $\frac{\pi}{2}$, which is the apex angle of our isosceles triangle. Since our angle is less than $\frac{\pi}{2}$ put the centre to the left (below) the chord, and draw our arc.




**Question 13**

$$\text{Im}(z^2) = 1$$
$$\text{Im}((x+iy)^2) = 1$$
$$\text{Im}(x^2-y^2+2xyi) = 1$$
$$2xy = 1$$
$$y = \frac{1}{2x}$$

A hyperbola with asymptotes the $x$- and $y$-axes passing through $\left(1,\frac{1}{2}\right)$.

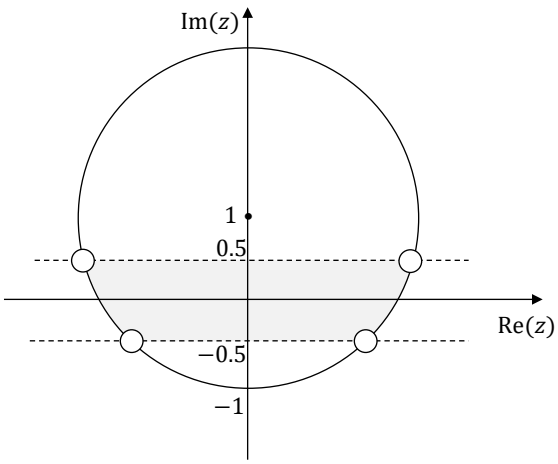


**Question 14**

$$|z-\bar{z}| = |2i\text{Im}(z)|$$
$$= 2|\text{Im}(z)|$$
$$|z-\bar{z}| < 1$$
$$2|\text{Im}(z)| < 1$$
$$|\text{Im}(z)| < \frac{1}{2}$$
$$-0.5 < \text{Im}(z) < 0.5$$

This is everything between the lines $\text{Im}(z) = 0.5$ and $\text{Im}(z) = -0.5$. $|z-i| \leq 2$ is everything on or inside a circle centred at (0,1) with radius 2.

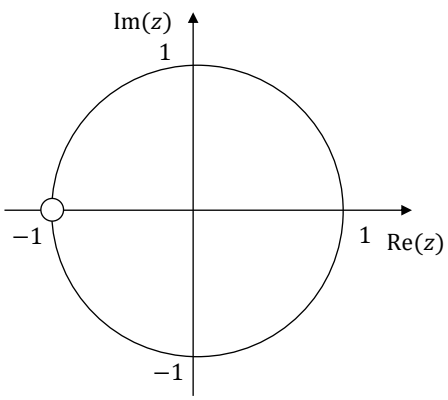
Find the intersection of the two regions.




**Question 15**

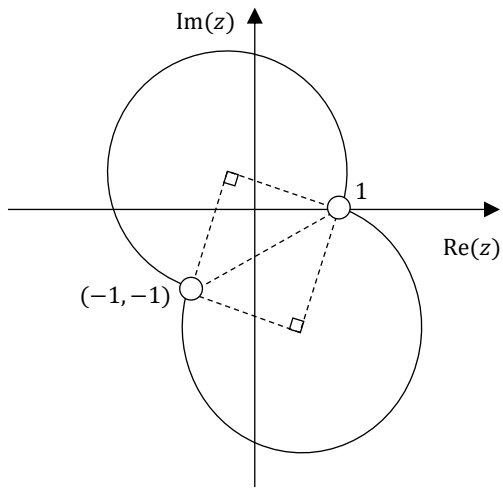
$$\frac{1}{z+1} + \frac{1}{\bar{z}+1} = 1 \qquad (z \neq -1)$$
$$\bar{z}+1+z+1 = (z+1)(\bar{z}+1)$$
$$2\text{Re}(z)+2 = z\bar{z}+z+\bar{z}+1$$
$$= |z|^2 + 2\text{Re}(z) + 1$$
$$|z|^2 = 1$$
$$|z| = 1 \text{ since } |z| \geq 0$$

So the curve is the unit circle, excluding $z = -1$.



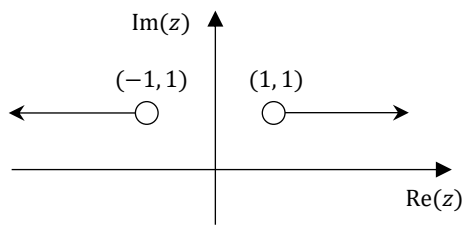

**Question 16**

The angle at $z$ (formed by the vectors from $(-1,-1)$ and $(1,0)$ is equal to $\pm\frac{\pi}{4}$, so arcs with ends at $(-1,-1)$ and $(1,0)$. We want isosceles triangles drawn to both sides since the angle can be positive or negative, with apex angles of $2\times\frac{\pi}{4} = \frac{\pi}{2}$ and can then draw both arcs to form a snowman.



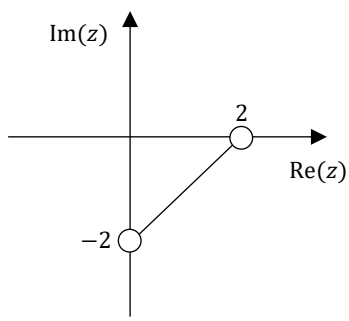

**Question 17**

The angle at $z$ formed by the vectors from $(1,1)$ and $(-1,1)$ is equal to $0$. The line passing through $(1,1)$ and $(-1,1)$, but outside those points.



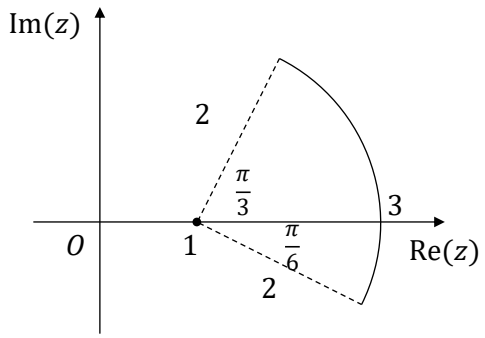

**Question 18**

The angle at z formed by the vectors from $(2,0)$ and $(0.-2)$ is equal to $\pi$. The line between $(2,0)$ and $(0,-2)$ not including the points.



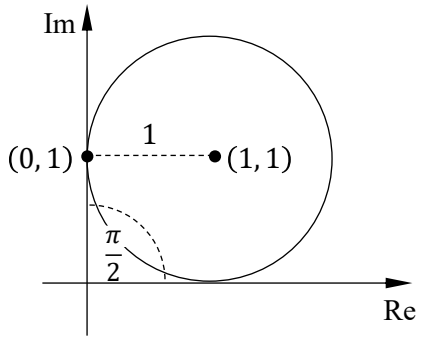
**Question 19**

$z = 2(\cos\theta + i\sin\theta)$ is the circle centred at $z = 0$ with radius 2. The restriction $-\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$ gives the arc between arguments $-\frac{\pi}{6}$ and $\frac{\pi}{3}$. Adding 1 moves the arc 1 unit to the right.




**Question 20**

$|z - 1 - i| = 1$ is a circle of radius one unit centred at $(1,1)$. The maximum value of $\text{Arg}(z)$ occurs when $z$ is directly above zero at $(0,1)$, where it takes a value of $\frac{\pi}{2}$ as shown.




**Question 21**

$x^2 - 2x + y^2 - 2y - 2 = 0 \ \ (y \neq 1)$

$x^2 - 2x + 1 + y^2 - 2y + 1 - 4 = 0$

$(x-1)^2 + (y-1)^2 = 4$

This is the circle centred at $(1,1)$ with radius 2, excluding the points $(3,1)$ and $(-1,1)$. The angle in a semicircle is $\frac{\pi}{2}$, so:

$z_1 = (3,1), z_2 = (-1,1)$ (the two can be swapped) and $\theta = \frac{\pi}{2}$.

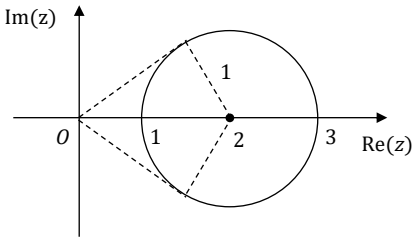
**Question 22**

Let $w = a + ib, z = x + iy$

$$w = \frac{z+3}{z} \times \frac{\bar{z}}{\bar{z}}$$
$$= \frac{z\bar{z} + 3\bar{z}}{z\bar{z}}$$
$$= \frac{|z|^2 + 3\bar{z}}{|z|^2}$$
$$= \frac{4 + 3\bar{z}}{4}$$
$$= 1 + \frac{3}{4}\bar{z}$$

Which is a circle centred at $(1,0)$ with radius $\frac{3}{4} \times 2 = \frac{3}{2}$

**Question 23a**




**Question 23b**

$1 \le |z| \le 3$ (closest and furthest points on the circle are the real intercepts).

$$-\sin^{-1}\left(\frac{1}{2}\right) \le \arg z \le \sin^{-1}\left(\frac{1}{2}\right)$$
$$-\frac{\pi}{6} \le \arg z \le \frac{\pi}{6}$$

**Question 232c**

$|P_1| = |P_2| = \sqrt{3}$ exact triangles

$\therefore z_1 = \sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and

$z_2 = \sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

**Question 23d**

$z_1{}^2 + z_2{}^2$

$$= \left(\sqrt{3}\,\text{cis}\frac{\pi}{6}\right)^2 + \left(\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^2$$
$$= 3\left[\text{cis}\frac{\pi}{3} + \text{cis}\left(-\frac{\pi}{3}\right)\right]$$
$$= 3\left[2\text{Re}\left(\text{cis}\frac{\pi}{3}\right)\right]$$
$$= 3\left[2\cos\frac{\pi}{3}\right]$$
$$= 3\left(2\left(\frac{1}{2}\right)\right)$$
$$= 3$$

**Question 24a**

$z = 2i + 1(1 + 3i - 2i) = 1 + 3i$

$z$ is the point $(1,3)$

**Question 24b**

When $k = 0$ $z = z_1 = 2i$

For $0 < k < 1$ we have the interval between (but not including) $z_1$ and $z_2$, so between $(0,2)$ and $(1,3)$.

**Question 24c**

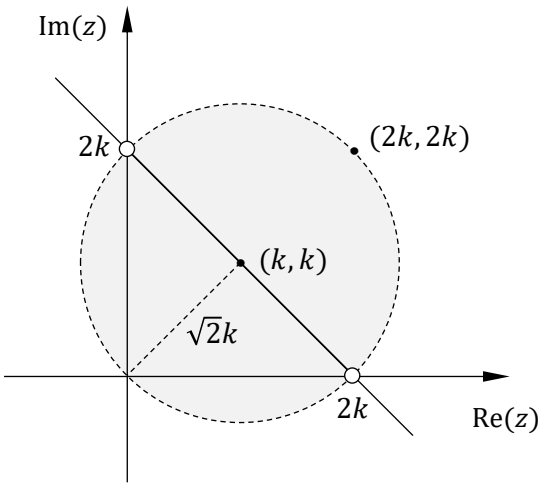
The line through $z_1$ and $z_2$, $\text{Im}(z) = \text{Re}(z) + 2$.

**Question 24d**

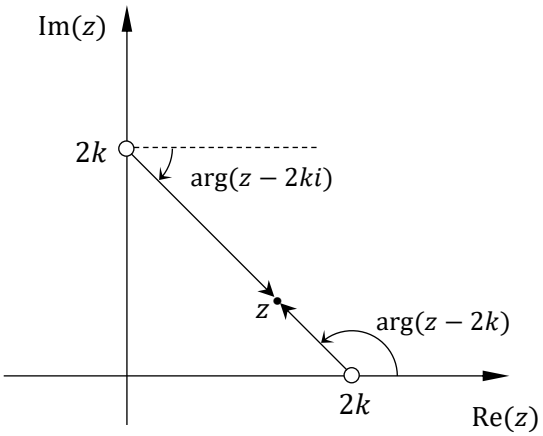
$\alpha = 2i, \beta = 1 + 3i$ (the two can be swapped) and $\theta = \pm\pi$.

**Question 25**

$|z| = |z - 2k - 2ki|$ is the perpendicular bisector of $0$ and $2k + 2ki$, which is a line passing through $2k$ and $2ki$. $|z - k - ki| < \sqrt{2}k$ is the region inside the circle centred at $k + ki$ with radius $\sqrt{2}k$. The intersection of these is the interval between (but not including) $2k$ and $2ki$ (shown below).




For every point on this line the argument from $2k$ and the argument taken from $2ki$ are cointerior angles to parallel lines.




So $\arg(z - 2k) - \arg(z - 2ki) = \pi$
$\therefore\ \arg(z - 2k) = \arg(z - 2ki) + \pi.$
**Answer D**

**Question 26**

$\dfrac{\text{Im}(z)}{\text{Re}(z)}$ is undefined for $\text{Re}(z) = 0$, so there are no purely imaginary numbers for which it is true

$\text{Arg}(z) = \tan^{-1}\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)$ for $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, so there are no points in the first or fourth quadrants for which it is true

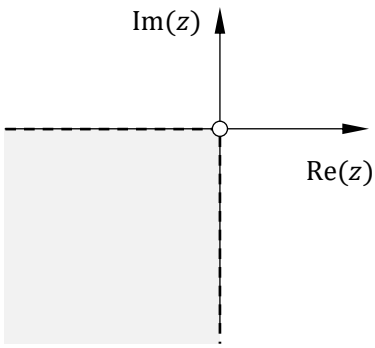
In the second quadrant

$\text{Arg}(z) = \pi + \tan^{-1}\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)$, so there are no points in the second quadrant for which it is true

In the third quadrant

$\text{Arg}(z) = -\pi + \tan^{-1}\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)$, so all points in the third quadrant are in the region

The region is the third quadrant excluding the axes and zero.



**Question 1a**

$|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\arg(1-i) = \tan^{-1}\dfrac{-1}{1} = -\dfrac{\pi}{4}$

$1+i = \sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right)$

**Question 1b**

$(1-i)^5$

$= \left(\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right)+i\sin\left(-\dfrac{\pi}{4}\right)\right)\right)^5$

$= \left(\sqrt{2}\right)^5\left(\cos\left(-\dfrac{5\pi}{4}\right)+i\sin\left(-\dfrac{5\pi}{4}\right)\right)$

$= 4\sqrt{2}\left(\cos\left(-\dfrac{5\pi}{4}\right)+i\sin\left(-\dfrac{5\pi}{4}\right)\right)$

**Question 1c**

$(1-i)^5 = 4\sqrt{2}\left(-\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$

$= 4\sqrt{2}\left(-\dfrac{1}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}\right)$

$= -4+4i$

**Question 2**

$1-\sqrt{3}i = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$

$1-i = \sqrt{2}\,\text{cis}\left(\dfrac{\pi}{4}\right)$

$\therefore \left(1+\sqrt{3}i\right)^4(1-i)^2$

$= \left(2\,\text{cis}\left(-\dfrac{\pi}{3}\right)\right)^4\left(\sqrt{2}\,\text{cis}\left(\dfrac{\pi}{4}\right)\right)^2$

$= 2^4\times\text{cis}\left(-\dfrac{4\pi}{3}\right)\times\sqrt{2}^2\times\text{cis}\left(\dfrac{2\pi}{4}\right)$

$= 32\,\text{cis}\left(-\dfrac{5\pi}{6}\right)$

$= 32\left(-\cos\dfrac{\pi}{6}-i\sin\dfrac{\pi}{6}\right)$

$= 32\left(-\dfrac{\sqrt{3}}{2}-i\dfrac{1}{2}\right)$

$= -16\sqrt{3}-16i$

**Question 3a**

$\sqrt{3}+i = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ by inspection (exact triangles)

**Question 3b**

$\left(\sqrt{3}+i\right)^6 = 2^6\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)^6$

$= 64(\cos\pi+i\sin\pi)$

$= 64(-1)$

$= -64$

**Question 4**

$\left(1-\sqrt{3}i\right)^{21}$

$= \left(2\,\text{cis}\left(-\dfrac{\pi}{3}\right)\right)^{21}$

$= 2^{21}\,\text{cis}(-7\pi)$

$= 2^{21}\,\text{cis}(\pi)$

$= 2^{21}(-1)$

$= -2^{21}$

**Question 5**

$\left(\sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)\right)^3$

$= 2\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

$= 2\sqrt{2}\left(-\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$

$= 2\sqrt{2}\left(-\dfrac{1}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}\right)$

$= -2+2i$

**Answer B**

Alternatively:

$\left(\sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)\right)^3$

$= \left[\sqrt{2}\left(\dfrac{1}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}\right)\right]^3$

$= (1+i)^3$

$= 1+3i+3i^2+i^3$

$= 1+3i-3-i$

$= -2+2i$

**Answer B**

**Question 6**

$(3+i)^n+(3-i)^n$

$= (3+i)^n+\overline{(3+i)}^n$

$= (3+i)^n+\overline{(3+i)^n}$

$= 2\text{Re}[(3+i)^n]$

which is purely real.

**Question 7**

Let $z = \cos\theta+i\sin\theta$

$\therefore \bar{z} = \cos(-\theta)+i\sin(-\theta)$

$= (\cos\theta+i\sin\theta)^{-1}$

$= z^{-1}$

$= \dfrac{1}{z}$

**Question 8**

$1+(1+i)+(1+i)^2+\ldots+(1+i)^{99}$

$= \dfrac{1\left((1+i)^{100}-1\right)}{1+i-1}$

$= \dfrac{(1+i)^{100}-1}{i}$

$= \dfrac{\left(\sqrt{2}\,\text{cis}\dfrac{\pi}{4}\right)^{100}-1}{i}$

$= \dfrac{2^{50}\,\text{cis}\,25\pi-1}{i}$

$= -\dfrac{2^{50}+1}{i}\times\dfrac{i}{i}$

$= \left(2^{50}+1\right)i$

**Question 9a**

Let $c = \cos\theta$ and $is = i\sin\theta$.

$(c + is)^5$

$= c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$

$= c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$

$= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

**Question 9b**

From de Moivre's Theorem $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$.

Equating imaginary parts with part (a) we get:

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1 - \sin^2\theta)^2\sin\theta - 10\left(1 - \sin^2\theta\right)\sin^3\theta + \sin^5\theta$

$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10\left(1 - \sin^2\theta\right)\sin^3\theta + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ as required

**Question 10**

$(1 - i)^n = \left(\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)\right)^n$

$= 2^{\frac{n}{2}}\text{cis}\left(-\frac{n\pi}{4}\right)$

The real part is $2^{\frac{n}{2}}\cos\left(-\frac{n\pi}{4}\right)$, which we let equal 0.

$\cos\left(-\frac{n\pi}{4}\right) = 0$

$\therefore \cos\left(\frac{n\pi}{4}\right) = 0$ since cosine is even

$\therefore \frac{n\pi}{4} = \frac{(2k+1)\pi}{2}$ where $k$ is an integer

$n = 4k + 2$ so $n$ is two more than a multiple of 4

$\therefore (1 + i)^n$ is purely imaginary when $n$ is two more than a multiple of 4.

**Question 11**

Let $P(n)$ represent the proposition.

$P(0)$ is true since LHS $= [r(\cos\theta + i\sin\theta)]^0 = 1$

RHS $= r^0(\cos 0\theta + i\sin 0\theta) = 1(1 + 0i) = 1$

$P(k)$: let $k \geq 0$ be an arbitrary integer for which $P(k)$ is true, that is

$[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$

RTP $P(k+1)$: $[r(\cos\theta + i\sin\theta)]^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$

LHS $= [r(\cos\theta + i\sin\theta)]^k \times r(\cos\theta + i\sin\theta)$

$= r^k(\cos k\theta + i\sin k\theta) \times r(\cos\theta + i\sin\theta)$

$= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= r^{k+1}[\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta]$

$= r^{k+1}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i\,(\sin k\theta\cos\theta + \cos k\theta\sin\theta)]$

$= r^{k+1}[\cos(k\theta + \theta) + i\sin(k\theta + \theta)]$

$= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$

$\therefore P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for all $n \geq 0$ by induction.

**Question 12**

Let $n = -k$ where $k \in \mathbb{Z}^+$

$[r\,\text{cis}\,\theta]^n$

$= [r\,\text{cis}\,\theta]^{-k}$

$= r^{-k} \times \left((\text{cis}\,\theta)^{-1}\right)^k$

$= r^{-k}\left(\overline{\text{cis}\,\theta}\right)^k$ since $|\text{cis}\,\theta| = 1$

$= r^{-k}(\text{cis}(-\theta))^k$

$= r^{-k}(\text{cis}(-k\theta))$ since $k \geq 0$

$= r^n(\text{cis}(n\theta))$

$\therefore [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ for negative integers.

**Question 13a**

Let $c = \cos\theta$ and $is = i\sin\theta$.

$$(c+is)^4$$
$$= c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$$
$$= c^4 + i4c^3s - 6c^2s^2 - i4cs^3 + s^4$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i\left(4\cos^3\theta - 4\cos\theta\sin^3\theta\right)$$

**Question 13b**

$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

Equating real parts with (a)

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta\left(1-\cos^2\theta\right) + \left(1-\cos^2\theta\right)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

**Question 13c**

$$8\cos^4\theta - 8\cos^2\theta = -1$$
$$8\cos^4\theta - 8\cos^2\theta + 1 = 0$$
$$\cos 4\theta = 0$$
$$4\theta = \frac{(2k+1)\pi}{2} \quad \text{for } k = 0,1,\ldots$$
$$\theta = \frac{(2k+1)\pi}{8}$$
$$= \frac{\pi}{8}, \frac{3\pi}{8}\ldots$$

The smallest positive solution is $\frac{\pi}{8}$

**Question 14a**

$$z^n + z^{-n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \quad \text{since cosine is even and sine is odd}$$
$$= 2\cos n\theta$$

Alternatively:

$$z^n + z^{-n} = z^n + \overline{z^n} \quad \text{since } |z| = 1$$
$$= 2\text{Re}(z^n)$$
$$= 2\text{Re}\left((\cos\theta + i\sin\theta)^n\right)$$
$$= 2\text{Re}(\cos n\theta + i\sin n\theta)$$
$$= 2\cos n\theta$$

**Question 14b**

$$8\cos^4\theta - 10\cos^2\theta + 2 = 8\left[\frac{1}{2}(z+z^{-1})\right]^4 - 10\left[\frac{1}{2}(z+z^{-1})\right]^2 + 2$$
$$= \frac{1}{2}\left[(z^4+z^{-4}) + 4\left(z^2+z^{-2}\right) + 6\right] - \frac{5}{2}\left[\left(z^2+z^{-2}\right) + 2\right] + 2$$
$$= \frac{1}{2}(z^4+z^{-4}) - \frac{1}{2}\left(z^2+z^{-2}\right)$$
$$= \frac{1}{2}(2\cos 4\theta) - \frac{1}{2}(2\cos 2\theta)$$
$$= \cos 4\theta - \cos 2\theta$$

Alternatively:

$$\cos 4\theta - \cos 2\theta = \frac{1}{2}(z^4+z^{-4}) - \frac{1}{2}\left(z^2+z^{-2}\right)$$
$$= \frac{1}{2}z^4 + \frac{1}{2}z^{-4} - \frac{1}{2}z^2 - \frac{1}{2}z^{-2}$$
$$= \left[\frac{1}{2}z^4 + 2z^2 + 3 + 2z^{-2} + \frac{1}{2}z^{-4}\right] - \left[\frac{5}{2}z^2 + 5 + \frac{5}{2}z^{-2}\right] + 2$$
$$= \frac{1}{2}(z+z^{-1})^4 - \frac{5}{2}(z+z^{-1})^2 + 2$$
$$= \frac{1}{2}(2\cos\theta)^4 - \frac{5}{2}(2\cos\theta)^2 + 2$$
$$= 8\cos^4\theta - 10\cos^2\theta + 2$$

**Question 15a**

$$\begin{aligned} z^n - z^{-n} &= (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta) \\ &= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta \quad \text{since cosine is even and sine is odd} \\ &= 2i\sin n\theta \end{aligned}$$

Alternatively:

$$\begin{aligned} z^n - z^{-n} &= z^n - \overline{z^n} \quad \text{since } |z| = 1 \\ &= 2i\text{Im}(z^n) \\ &= 2i\text{Im}\left((\cos\theta + i\sin\theta)^n\right) \\ &= 2i\text{Im}(\cos n\theta + i\sin n\theta) \\ &= 2i\sin n\theta \end{aligned}$$

**Question 15b**

$$\begin{aligned} &(z - z^{-1})^3 = z^3 + 3z^2(-z^{-1}) + 3(z)(-z^{-1})^2 + (-z^{-1})^3 \\ &= z^3 - 3z + 3z^{-1} - z^{-3} \\ &= \left(z^3 - z^{-3}\right) - 3(z - z^{-1}) \\ &= 2i\sin 3\theta - 6i\sin\theta \qquad (1) \end{aligned}$$

From (i):

$$\begin{aligned} &(z - z^{-1})^3 \\ &= (2i\sin\theta)^3 \\ &= -8i\sin^3\theta \qquad (2) \end{aligned}$$

From (1) and (2):

$$-8\sin^3\theta = 2\sin 3\theta - 6\sin\theta$$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

**Question 16**

$$\begin{aligned} &\left(1 - \sqrt{3}i\right)^6 + \left(1 + \sqrt{3}i\right)^6 \\ &= \left(1 - \sqrt{3}i\right)^6 + \overline{\left(1 - \sqrt{3}i\right)^6} \\ &= \left(1 - \sqrt{3}i\right)^6 + \overline{\left(1 - \sqrt{3}i\right)^6} \\ &= 2\text{Re}\left(\left(1 - \sqrt{3}i\right)^6\right) \\ &= 2\text{Re}\left(\left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^6\right) \\ &= 2\text{Re}\left(2^6\,\text{cis}(-2\pi)\right) \\ &= 2\text{Re}\left(2^6\right) \\ &= 2\times 2^6 \\ &= 128 \end{aligned}$$

**Question 17a**

$z = \sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ by inspection

**Question 17b**

$$\begin{aligned} &z^7 + 64z \\ &= \left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^7 + 64\left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right) \\ &= 128\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) + 128\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\ &= 128\left[\left(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) + \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right] \\ &= 0 \end{aligned}$$

**Question 18a**

$|z| = \sqrt{\left(-3\sqrt{3}\right)^2 + 3^2} = 6$

$\arg z = \pi - \tan^{-1}\left(\dfrac{-3}{3\sqrt{3}}\right) = \pi - \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{5\pi}{6}$

$\therefore z = 6\left(\cos\left(\dfrac{5\pi}{6}\right) + i\sin\left(\dfrac{5\pi}{6}\right)\right)$

**Question 18b**

$$z^n = 6^n\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)^n$$
$$= 6^n\left(\cos\left(\frac{5n\pi}{6}\right) + i\sin\left(\frac{5n\pi}{6}\right)\right)$$

For $z^n$ to be real $\frac{5n\pi}{6} = k\pi$ for integral $k$

$\therefore \frac{5n}{6}$ must be integral. $n = 6$ is the smallest positive value for which this occurs.

**Question 19a**

$|-2 + 2i| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\arg(-2 + 2i) = \pi - \tan^{-1}\left(\dfrac{2}{2}\right) = \dfrac{3\pi}{4}$

**Question 19b**

$$(-2 + 2i)^{8k} = \left(2\sqrt{2}\,\mathrm{cis}\,\frac{3\pi}{4}\right)^{8k}$$
$$= 2^{12k}\,\mathrm{cis}\,6k\pi$$
$$= 2^{12k} \text{ since } k \in \mathbb{Z}$$

**Question 20a**

$1 + i = \sqrt{2}\mathrm{cis}\dfrac{\pi}{4}$ and $1 - i = \sqrt{2}\ \mathrm{cis}\ \left(-\dfrac{\pi}{4}\right)$

$\therefore (1 + i)^n - (1 - i)^n$

$$= \left(\sqrt{2}\,\mathrm{cis}\frac{\pi}{4}\right)^n - \left(\sqrt{2}\ \mathrm{cis}\ \left(-\frac{\pi}{4}\right)\right)^n$$
$$= 2^{\frac{n}{2}}\,\mathrm{cis}\frac{n\pi}{4} - 2^{\frac{n}{2}}\,\mathrm{cis}\left(-\frac{n\pi}{4}\right)$$
$$= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} - \cos\left(-\frac{n\pi}{4}\right) - i\sin\left(-\frac{n\pi}{4}\right)\right)$$
$$= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} - \cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right)$$
$$= 2^{\frac{n}{2}}\left(2\sin\frac{n\pi}{4}\right)i$$
$= 2\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}i$ as required

**Question 20b**

$$(1 + i)^n - (1 - i)^n$$
$$= \binom{n}{0} + \binom{n}{1}i + \binom{n}{2}i^2 + \binom{n}{3}i^3 + \ldots + \binom{n}{n}i^n$$
$$-\binom{n}{0} + \binom{n}{1}i - \binom{n}{2}i^2 + \binom{n}{3}i^3 - \ldots - \binom{n}{n}i^n$$
$$= 2\left(\binom{n}{1}i + \binom{n}{3}i^3 + \binom{n}{5}i^5 + \ldots + \binom{n}{n-1}i^{n-1}\right)$$
$$= 2\left(\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \ldots - \binom{n}{n-1}\right)i$$

$\therefore 2\left(\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \ldots - \binom{n}{n-1}\right) = 2\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$

$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \ldots - \binom{n}{n-1} = \left(\sqrt{2}\right)^n \sin\dfrac{n\pi}{4}$

but since $n$ is a multiple of 4, $\frac{n\pi}{4}$ is equal to 0 or $\pi$, and thus $\sin\frac{n\pi}{4}$ is equal to 0.

$\therefore \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \ldots - \binom{n}{n-1} = 0$ as required

**Question 21a**

$$(1+i\tan\theta)^n-(1-i\tan\theta)^n$$
$$=\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n-\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$$
$$=\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n-\left(\frac{\cos(-\theta)+i\sin(-\theta)}{\cos\theta}\right)^n$$
$$=\frac{\cos n\theta+i\sin n\theta-\cos(-n\theta)-i\sin(-n\theta)}{\cos^{\mathrm{n}}\theta}$$
$$=\frac{\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta}{\cos^{\mathrm{n}}\theta}$$
$$=\frac{2\sin n\theta}{\cos^n\theta}i$$

**Question 21b**

Note that when we expand and collect like terms this is a cubic polynomial, which is why we only get three solutions.

$$(1+z)^4-(1-z)^4=0$$
Let $z=i\tan\theta,\theta\neq\frac{k\pi}{2}$ for $k\in\mathbb{Z}$
$$\therefore\left((1+i\tan\theta)^4-(1-i\tan\theta)^4\right)=0$$
$$\therefore\frac{2\sin 4\theta}{\cos^4\theta}=0 \text{ from (a)}$$
$$\sin 4\theta=0$$
$$4\theta=k\pi \quad \text{for } k=-1,0,1,2$$
$$\theta=\frac{k\pi}{4}$$
$$=-\frac{\pi}{4},0,\frac{\pi}{4},\frac{\pi}{2}$$

$$\therefore z=i\tan\left(-\frac{\pi}{4}\right),i\tan(0),i\tan\left(\frac{\pi}{4}\right),\tan\left(\frac{\pi}{2}\right)$$
$$=0,\pm i \quad \text{since}\tan\frac{\pi}{2}\text{ is undefined}$$

**Question 22a**

$$\frac{1}{1-z\cos\alpha}$$
$$=\frac{1}{1-(\cos\alpha+i\sin\alpha)\cos\alpha}$$
$$=\frac{1}{1-\cos^2\alpha-i\sin\alpha\cos\alpha}$$
$$=\frac{1}{\sin^2\alpha-i\sin\alpha\cos\alpha}$$
$$=\frac{1}{\sin\alpha(\sin\alpha-i\cos\alpha)}\times\frac{\sin\alpha+i\cos\alpha}{\sin\alpha+i\cos\alpha}$$
$$=\frac{\sin\alpha+i\cos\alpha}{\sin\alpha(\sin^2\alpha+\cos^2\alpha)}$$
$$=\frac{\sin\alpha+i\cos\alpha}{\sin\alpha}$$
$$=1+i\cot\alpha$$

**Question 22b**

$$\sum_{k=0}^{\infty}(z\cos\alpha)^k=\frac{a}{1-r}=\frac{1}{1-z\cos\alpha}=1+i\cot\alpha \quad (1)$$

$$\sum_{k=0}^{\infty}(z\cos\alpha)^k=\sum_{k=0}^{\infty}\left((\cos\alpha+i\sin\alpha)\cos\alpha\right)^k$$
$$=\sum_{k=0}^{\infty}(\cos k\alpha+i\sin k\alpha)\cos^k\alpha$$
$$=\sum_{k=0}^{\infty}\cos k\alpha\cos^k\alpha+i\sum_{k=0}^{\infty}\sin k\alpha\cos^k\alpha \quad (2)$$

Equating the imaginary components of (1) and (2):

$$\sum_{k=0}^{\infty}\sin k\alpha\cos^k\alpha=\cot\alpha$$

**Question 23**

$5z^4 - z^3 + 6z^2 - z + 5 = 0$

$5z^2 - z + 6 - z^{-1} + 5z^{-2} = 0$

$5\left(z^2 + z^{-2}\right) - \left(z^1 + z^{-1}\right) + 6 = 0$

$5(2\cos 2\theta) - (2\cos\theta) + 6 = 0$

$10\left(2\cos^2\theta - 1\right) - 2\cos\theta + 6 = 0$

$20\cos^2\theta - 2\cos\theta - 4 = 0$

$10\cos^2\theta - \cos\theta - 2 = 0$

$$\cos\theta = \frac{1 \pm \sqrt{(-1)^2 - 4(10)(-2)}}{2(10)}$$

$$= \frac{1 \pm 9}{20}$$

$$= -\frac{2}{5}, \frac{1}{2}$$

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

$$= \pm\frac{\sqrt{21}}{5}, \pm\frac{\sqrt{3}}{2} \quad \text{respectively}$$

$$\therefore z = -\frac{2}{5} \pm \frac{\sqrt{21}}{5}i, \qquad \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

**Question 24a**

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$= \cos^3\theta + 3\cos^2\theta\sin\theta\, i - 3\cos\theta\sin^2\theta - \sin^3\theta\, i$$

Equating real components:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= \cos^3\theta - 3\cos\theta\left(1 - \cos^2\theta\right)$$

$$= 4\cos^3\theta - 3\cos\theta$$

**Question 24b**

$8x^3 - 6x - 1 = 0$

$8x^3 - 6x = 1$

$$4x^3 - 3x = \frac{1}{2}$$

Let $x = \cos\theta$

$$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$$

$$\therefore \cos 3\theta = = \frac{1}{2} \quad \text{from (i)}$$

$\therefore 8x^3 - 6x - 1 = 0$ has solutions in the form $x = \cos\theta$ where $\cos 3\theta = \frac{1}{2}$.

**Question 24c**

$$3\theta = \cos^{-1}\left(\frac{1}{2}\right), 2\pi - \cos^{-1}\left(\frac{1}{2}\right), 2\pi + \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

$$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$

$$\therefore x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$$

**Question 24d**

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)$$

$$= \cos\left(\frac{\pi}{9}\right)\left(-\cos\left(\frac{7\pi}{9}\right)\right)\left(-\cos\left(\frac{5\pi}{9}\right)\right)$$

$$= \cos\frac{\pi}{9} \times \cos\frac{5\pi}{9} \times \cos\frac{7\pi}{9}$$

$$= -\frac{d}{a}$$

$$= \frac{1}{8}$$

**Question 25a**

$$\begin{aligned}
&\left(z^{2n}+z^{2n-2}+z^{2n-4}+\ldots+z^{-2n+2}+z^{-2n}\right)\sin\theta \\
&=\left[\left(z^{2n}+z^{-2n}\right)+\left(z^{2n-2}+z^{-2n+2}\right)+\ldots+\left(z^{2}+z^{-2}\right)+1\right]\sin\theta \\
&=\left[2\cos(2n\theta)+2\cos\left((2n-2)\theta\right)+\ldots+2\cos 2\theta+1\right]\sin\theta \\
&=2\cos(2n\theta)\sin\theta+2\cos\left((2n-2)\theta\right)\sin\theta+\ldots+2\cos 2\theta\sin\theta+\sin\theta \\
&=(\sin(2n+1)\theta-\sin(2n-1)\theta)+(\sin(2n-1)\theta-\sin(2n-3)\theta)+\ldots+(\sin 3\theta-\sin\theta)+\sin\theta \\
&=\sin(2n+1)\theta
\end{aligned}$$

**Question 25b**

Let $n=3$ in (i):

$$\begin{aligned}
\left(z^6+z^4+z^2+1+z^{-2}+z^{-4}+z^{-6}\right)\sin\theta &= \sin 7\theta \\
z^6+z^{-6}+z^4+z^{-4}+z^2+z^{-2}+1 &= \frac{\sin 7\theta}{\sin\theta} \\
2\cos 6\theta+2\cos 4\theta+2\cos 2\theta+1 &= \frac{\sin 7\theta}{\sin\theta} \\
2\left(4\cos^3 2\theta-3\cos 2\theta\right)+2\left(2\cos^2 2\theta-1\right)+2\cos 2\theta+1 &= \frac{\sin 7\theta}{\sin\theta} \\
8\cos^3 2\theta+4\cos^2 2\theta-4\cos 2\theta-1 &= \frac{\sin 7\theta}{\sin\theta}
\end{aligned}$$

**Question 25c**

$8x^3+4x^2-4x-1=0$

Let $x=\cos 2\theta$

$$8\cos^3 2\theta+4\cos^2 2\theta-4\cos 2\theta-1=0$$

$$\begin{aligned}
\therefore \frac{\sin 7\theta}{\sin\theta} &= 0 \\
\therefore \sin 7\theta &= 0, \qquad \sin\theta\neq 0 \\
7\theta &= 0,\pi,2\pi,3\pi,\ldots \\
\theta &= \cancel{0},\frac{\pi}{7},\frac{2\pi}{7},\ldots
\end{aligned}$$

$\therefore\ \cos\dfrac{2\pi}{7}$ is one root of the equation.

**Question 26a**

$$\begin{aligned}
u^3 &= \left(z+\frac{1}{z}\right)^3 \\
&= z^3+3z^2\left(\frac{1}{z}\right)+3z\left(\frac{1}{z^2}\right)+\left(\frac{1}{z}\right)^3 \\
&= z^3+3z+3\left(\frac{1}{z}\right)+\frac{1}{z^3} \\
&= z^3+\frac{1}{z^3}+3u
\end{aligned}$$

$$z^3+\frac{1}{z^3}=u^3-3u$$

**Question 26b**

$$\begin{aligned}
1+\cos 10\theta &= 1+\cos\left(2(5\theta)\right) \\
&= 1+2\cos^2 5\theta-1 \\
&= 2\cos^2 5\theta \\
&= \frac{1}{2}(2\cos 5\theta)^2 \\
&= \frac{1}{2}\left(z^5+z^{-5}\right)^2 \\
&= \frac{1}{2}\left(u^5-5u^3+5u\right)^2 \\
&= \frac{1}{2}\left((2\cos\theta)^5-5(2\cos\theta)^3+5(2\cos\theta)\right)^2 \\
&= \frac{1}{2}\left(32\cos^5\theta-40\cos^3\theta+10\cos\theta\right)^2 \\
&= 2\left(16\cos^5\theta-20\cos^3\theta+5\cos\theta\right)^2
\end{aligned}$$

**Question 27a**

$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\sin\theta\, i - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin^3\theta\, i + \sin^4\theta$

Equating real components:

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta\left(1-\cos^2\theta\right) + \left(1-\cos^2\theta\right)^2 \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

**Question 27b**

$$\begin{aligned}16x^4 - 16x^2 + 1 &= 0 \\ 2\left(8x^4 - 8x^2 + 1\right) - 1 &= 0 \\ 8x^4 - 8x^2 + 1 &= \frac{1}{2}\end{aligned}$$

Let $x = \cos\theta$

$$\begin{aligned}8\cos^4\theta - 8\cos^2\theta + 1 &= \frac{1}{2} \\ \cos 4\theta &= \frac{1}{2} \\ 4\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ \theta &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12} \\ \therefore x &= \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12} \\ &= \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, -\cos\frac{5\pi}{12}, -\cos\frac{\pi}{12}\end{aligned}$$

$\therefore$ the roots are $x = \cos\frac{\pi}{12}, x = -\cos\frac{\pi}{12}, x = \cos\frac{5\pi}{12}$ and $x = -\cos\frac{5\pi}{12}$.

**Question 27c**

$$\begin{aligned}x^2 &= \frac{16 \pm \sqrt{16^2 - 4(16)(1)}}{2(16)} \\ &= \frac{16 \pm \sqrt{192}}{32} \\ &= \frac{16 \pm 8\sqrt{3}}{32} \\ &= \frac{2 \pm \sqrt{3}}{4}\end{aligned}$$

Since $\cos\theta$ is positive and decreasing in the first quadrant:

$$\begin{aligned}\cos\frac{5\pi}{12} &= \sqrt{\frac{2-\sqrt{3}}{4}} \\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

**Question 28**

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos 3\theta + i\sin 3\theta \\ &= \cos^3\theta + 3\cos^2\theta\sin\theta\, i - 3\cos\theta\sin^2\theta - \sin^3\theta\, i \\ &= \cos^3\theta - 3\cos\theta\left(1-\cos^2\theta\right) + i\left[3\left(1-\sin^2\theta\right)\sin\theta - \sin^3\theta\right] \\ &= 4\cos^3\theta - 3\cos\theta + i\left(3\sin\theta - 4\sin^3\theta\right)\end{aligned}$$

Equating real components:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$8x^3 - 6x - \sqrt{2} = 0$$

$$8x^3 - 6x = \sqrt{2}$$

$$4x^3 - 3x = \frac{1}{\sqrt{2}}$$

Let $x = \cos\theta$

$$4\cos^3\theta - 3\cos\theta = \frac{1}{\sqrt{2}}$$

$$\cos 3\theta = \frac{1}{\sqrt{2}}$$

$$3\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$$

$$\theta = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$$

$$\therefore x = \cos\frac{\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{3\pi}{4}$$

$$= 0.966, -0.259, -0.707$$

Alternatively, equating imaginary components:

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$8x^3 - 6x - \sqrt{2} = 0$$

$$6x - 8x^3 = -\sqrt{2}$$

$$3x - 4x^3 = -\frac{1}{\sqrt{2}}$$

Let $x = \sin\theta$

$$3\sin\theta - 4\sin^3\theta = -\frac{1}{\sqrt{2}}$$

$$\sin 3\theta = -\frac{1}{\sqrt{2}}$$

$$3\theta = -\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$$

$$\theta = -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$$

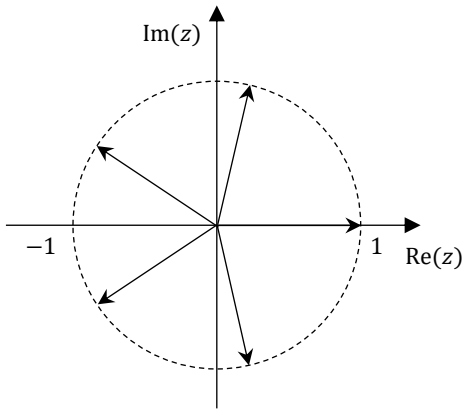
$$x = \sin\left(-\frac{\pi}{12}\right), \sin\frac{5\pi}{12}, \sin\frac{7\pi}{12}, \sin\frac{3\pi}{4}$$

$$= -0.259, 0.966, 0.966, 0.707$$

The three solutions are $x = -0.259, 0.966, 0.707$.

We are solving a cubic equation, so are looking for three solutions for $x$ - when we use another variable we can get the same solution more than once, but this does not mean that it is a multiple root.

**Question 1**




1 is a fifth root of 1, and the other roots must be spaced by $\frac{2\pi}{5}$ as shown. The fifth roots of 1 are $\text{cis}\left(-\frac{4\pi}{5}\right), \text{cis}\left(-\frac{2\pi}{5}\right) 1, \text{cis}\frac{2\pi}{5}$ and $\text{cis}\frac{4\pi}{5}$.

**Question 2**

Let one of the fifth roots be $z = r(\cos\theta + i\sin\theta)$

$$\therefore z^5 = 1$$

$$r^5(\cos\theta + i\sin\theta)^5 = \cos 2k\pi + i\sin 2k\pi \quad \text{for } k$$

$$= -2, -1, 0, 1, 2$$

$$r^5(\cos 5\theta + i\sin 5\theta) = \cos 2k\pi + i\sin 2k\pi$$

$$\therefore r^5 = 1 \quad \text{and } 5\theta = 2k\pi$$

$$\therefore r = 1 \quad \text{and } \theta = \frac{2k\pi}{5}$$

$$\therefore z$$

$$= \text{cis}\left(-\frac{4\pi}{5}\right), \text{cis}\left(-\frac{2\pi}{5}\right), \text{cis}\,0, \text{cis}\left(\frac{2\pi}{5}\right), \text{cis}\left(\frac{4\pi}{5}\right)$$

$$= 1, \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$$

**Question 3**

$$1 + \omega + \omega^2 = 0$$

$$\therefore \left(1 - \omega + w^2\right)^4$$

$$= \left(1 + \omega + \omega^2 - 2\omega\right)^4$$

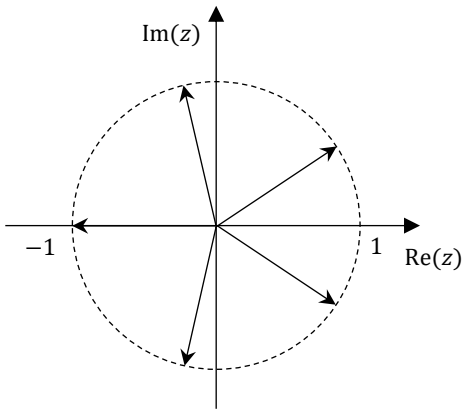
$$= (-2\omega)^4$$

$$= 16\omega^4$$

$$= 16\omega^3\omega$$

$$= 16\omega$$

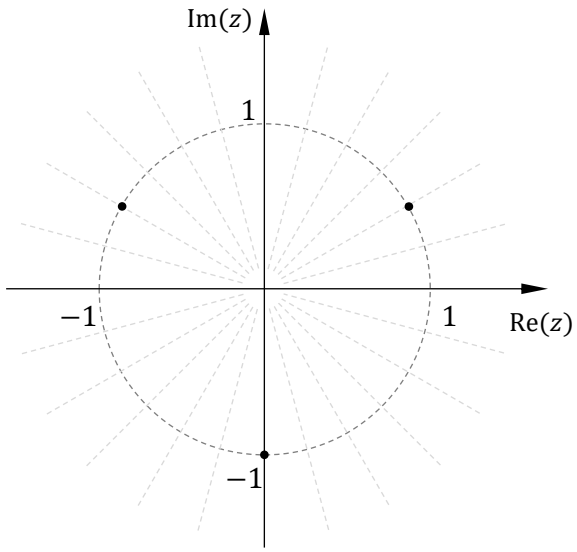
**Question 4**




$-1$ is one of the roots, and the others are spread evenly around the unit circle, so there is $\frac{2\pi}{5}$ between each fifth root of $-1$.

$\therefore$ the roots are $-1, \text{cis}\left(\pm\frac{3\pi}{5}\right), \text{cis}\left(\pm\frac{\pi}{5}\right)$.

**Question 5**



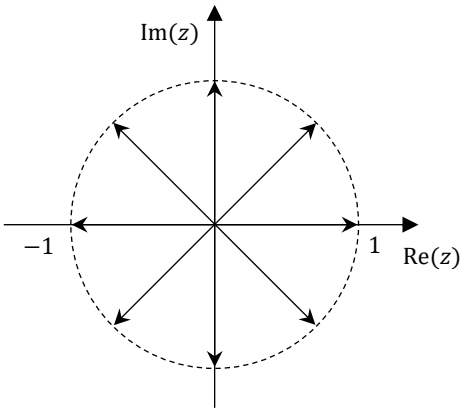

$$i = 1\,\text{cis}\frac{\pi}{2}$$

One of the cube roots of $i$ is $\left(\text{cis}\frac{\pi}{2}\right)^{\frac{1}{3}} = \text{cis}\left(\frac{\pi}{6}\right)$

The other roots have arguments which differ by $\frac{2\pi}{3} = \frac{4\pi}{6}$. The cube roots of $i$ are $-i, \text{cis}\frac{\pi}{6}$ and $\text{cis}\frac{5\pi}{6}$.

**Question 6**

1 is an eighth root of 1, and the other roots must be spaced by $\frac{2\pi}{8} = \frac{\pi}{4}$ as shown.




The eighth roots of 1 are $\pm 1, \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, \pm i,$

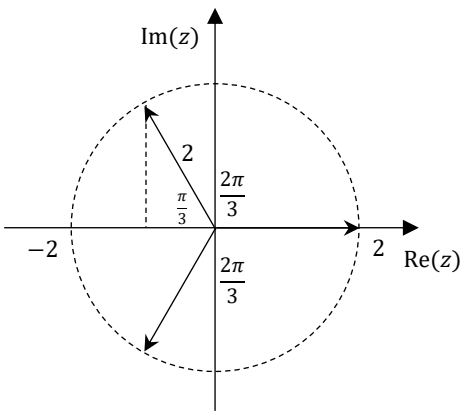
$-\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i.$

**Question 7**

$$\frac{1}{\omega} + \frac{1}{\omega^2} + 1$$
$$= \frac{\omega^2 + \omega + \omega^3}{\omega^3}$$
$$= \frac{\omega^2 + \omega + 1}{1}$$
$$= 0$$

**Question 8**

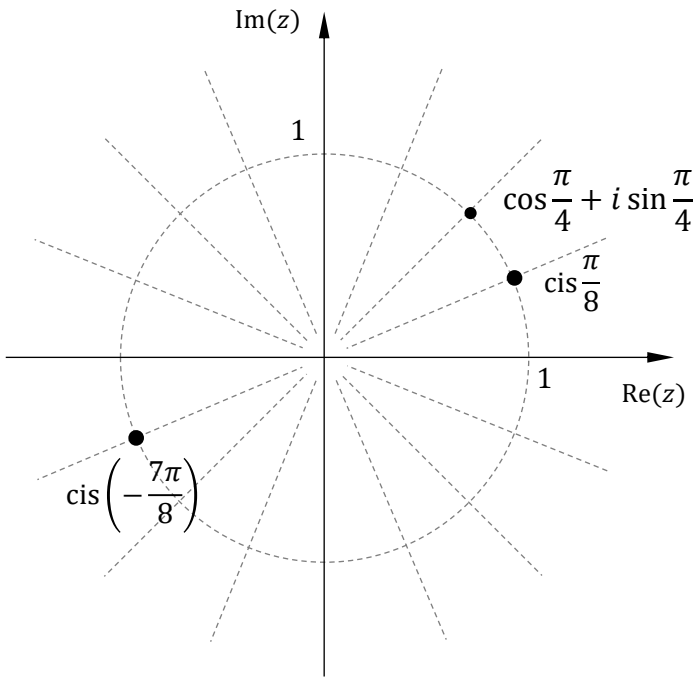
Sketching the roots we see:




Using exact triangles and reflection we can see the roots are $2, -1 + \sqrt{3}i, -1 - \sqrt{3}i$

**Answer C**

**Question 9a**




**Question 9b**

The square roots of $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ are $\text{cis}\frac{\pi}{8}$ and $\text{cis}\left(-\frac{7\pi}{8}\right)$.

$$\left(\text{cis}\frac{\pi}{8}\right)^4 = \text{cis}\frac{\pi}{2}$$
$$= i$$
$$\left(\text{cis}\left(-\frac{7\pi}{8}\right)\right)^4 = \text{cis}\left(-\frac{7\pi}{2}\right)$$
$$= \text{cis}\left(\frac{\pi}{2}\right)$$
$$= i$$

$\therefore$ both the square roots of $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ are also fourth roots of $i$.

**Question 9c**

The fourth root of $i$ in the second quadrant is $\text{cis}\frac{5\pi}{8}$.

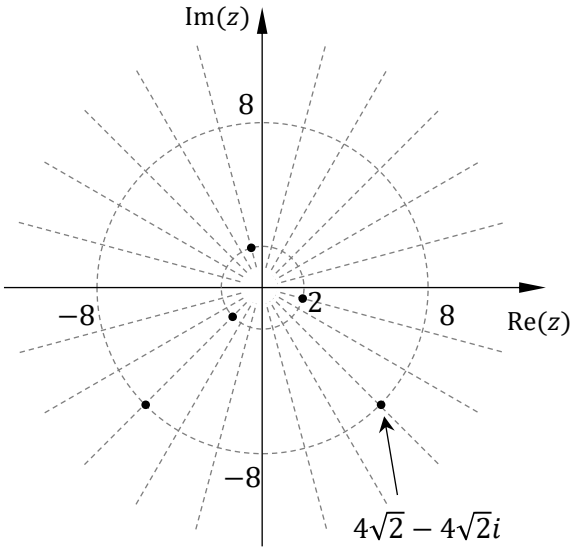
$$\therefore \left(\text{cis}\frac{5\pi}{8}\right)^n = \text{cis}\left(2k\pi + \frac{\pi}{4}\right) \text{ for } k = 0, 1, 2, \ldots$$
$$\text{cis}\frac{5n\pi}{8} = \text{cis}\left(\frac{(8k+1)\pi}{4}\right)$$
$$\frac{5n\pi}{8} = \frac{8k+1)\pi}{4}$$
$$n = \frac{2(8k+1)}{5}$$

For $n$ to be a positive integer, $8k + 1$ must be a multiple of 5, which first occurs when $k = 3$. The smallest positive value of $n$ is $\frac{2(8(3)+1)}{5} = 10$.

**Question 10**

For $4\sqrt{2} - 4\sqrt{2}i$ $r = \sqrt{\left(4\sqrt{2}\right)^2 + \left(-4\sqrt{2}\right)^2}$ $= 8$ and $\theta = -\frac{\pi}{4}$.

$4\sqrt{2} - 4\sqrt{2}i = 8\,\text{cis}\left(-\frac{\pi}{4}\right)$




One of the cube roots of $4\sqrt{2}$ $- 4\sqrt{2}i$ is $\left(8\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^{\frac{1}{3}} = 2\,\text{cis}\left(-\frac{\pi}{12}\right)$

The other roots have arguments which differ by $\frac{2\pi}{3} = \frac{8\pi}{12}$.

The cube roots of $4\sqrt{2}$ $- 4\sqrt{2}i$ are $2\,\text{cis}\left(-\frac{3\pi}{4}\right), 2\,\text{cis}\left(-\frac{\pi}{12}\right)$ and $2\,\text{cis}\frac{7\pi}{12}$.

**Question 11**

Note: $1 + w + w^2 = 0$

$$\frac{1}{1+w} + \frac{1}{1+w^2} = \frac{1 + w^2 + 1 + w}{(1+w)(1+w^2)}$$
$$= \frac{0+1}{1 + w^2 + w + w^3}$$
$$= \frac{1}{0+1}$$
$$= 1$$

**Answer C**

**Question 12**

$$\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$\omega^2 + \frac{1}{\omega^2} = \omega^2 + \overline{\omega^2}$

since $\left|\omega^2\right| = |\omega|^2 = |1|^2 = 1$

$$= 2\text{Re}\left(\omega^2\right)$$
$$= 2\text{Re}\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right) \quad \text{using DMT}$$
$$= 2\cos\frac{4\pi}{5}$$

$\therefore \left(2\cos\frac{4\pi}{5}\right)^2 + 2\cos\frac{4\pi}{5} - 1 = 0$

$4\cos^2\frac{4\pi}{5} + 2\cos\frac{4\pi}{5} - 1 = 0$

$\cos\frac{4\pi}{5} = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$

$= \frac{-2 \pm \sqrt{20}}{8}$

$= \frac{-2 \pm 2\sqrt{5}}{8}$

$= \frac{-1 \pm \sqrt{5}}{4}$

Now $\frac{4\pi}{5}$ is obtuse, so $\cos\frac{4\pi}{5} < 0$.

$\therefore \cos\frac{4\pi}{5} = \frac{-1 - \sqrt{5}}{4}$

$= -\frac{\sqrt{5} + 1}{4}$.

**Question 13**

Let $z$ be a cube root of $8$

$\therefore z^3 = 8$

$\therefore z^3 - 8 = 0$

$(z-2)\left(z^2 + 2z + 4\right) = 0$

$z - 2 = 0$ or $z^2 + 2z + 4 = 0$

$z = 2$ $\qquad z = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$

$= \frac{-2 \pm \sqrt{-12}}{2}$

$= -1 \pm \sqrt{3}i$

The three cube roots of $8$ are $2$ and $-1 \pm \sqrt{3}i$.

**Question 14**

We are looking for the fourth roots of $\left(\text{cis}\frac{\pi}{6}\right)^3 = \text{cis}\frac{\pi}{2} = i$.

One of the fourth roots of $\text{cis}\frac{\pi}{2}$ is $\left(\text{cis}\frac{\pi}{2}\right)^{\frac{1}{4}} = \text{cis}\frac{\pi}{8}$.

The other roots have arguments which differ by $\frac{2\pi}{4} = \frac{\pi}{2}$.

The solutions to $\left( \left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^{\frac{3}{4}}\right.$ are $\text{cis}\left(-\frac{7\pi}{8}\right), \text{cis}\left(-\frac{3\pi}{8}\right)$, $\text{cis}\left(\frac{\pi}{8}\right)$ and $\text{cis}\frac{5\pi}{8}$.

**Question 15**

$$\frac{1}{1+w} - \frac{1}{1+w^2}$$
$$= \frac{1+w^2-1-w}{1+w^2+1+w}$$
$$= \frac{w^2-w}{1+w+w^2+1}$$
$$= \frac{1+w+w^2-1-2w}{0+1}$$
$$= -(1+2w)$$

**Question 16**

$$(a-b)(a-\omega b)\left(a-\omega^2 b\right)$$
$$= (a-b)\left(a^2 - ab\omega^2 - ab\omega + b^2\omega^3\right)$$
$$= (a-b)\left(a^2 - ab\left(1+\omega+\omega^2-1\right) + b^2\right)$$
$$= (a-b)\left(a^2 - ab(0-1) + b^2\right)$$
$$= (a-b)\left(a^2 + ab + b^2\right)$$
$$= a^3 - b^3$$

**Question 17a**

$$w^5 - 1 = 0$$
$$\therefore (w-1)\left(w^4 + w^3 + w^2 + w + 1\right) = 0$$
$$\therefore w^4 + w^3 + w^2 + w + 1 = 0 \qquad \text{since } w \neq -1$$
$$\therefore 1 + w^2 + w^4 = -(w + w^3)$$

**Question 17b**

Let $z = r\,\text{cis}\,\theta$

$$z^5 - 1 = 0$$
$$z^5 = 1$$
$$\therefore r^5\text{cis}^5\theta = \text{cis}\,2k\pi$$
$$r^5(\cos 5\theta + i\sin 5\theta) = \cos(2k\pi) + i\sin(2k\pi)$$
$$\therefore r = 1 \text{ and } 5\theta = (2k\pi)$$
$$\theta = \frac{2k\pi}{5}$$
$$= 0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}$$

Let $w = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$$\therefore 1 + \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^2 + \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^4 = -\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^3\right)$$
$$1 + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = -\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$$
$$1 - \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = -\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} - \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$$
$$1 - \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = -\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} + \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$
$$\therefore 1 - 2\cos\frac{\pi}{5} + 2\cos\frac{2\pi}{5} = 0$$
$$\therefore \cos\frac{2\pi}{5} - \cos\frac{\pi}{5} = -\frac{1}{2}$$

**Question 18a**

In polar form the roots are $z_1 = 1, z_2 = \text{cis}\frac{2\pi}{3}$ and $z_3 = \text{cis}\frac{4\pi}{3}$.

$$\left(z_2\right)^2 = \left(\text{cis}\frac{2\pi}{3}\right)^2 = \text{cis}\frac{4\pi}{3} = z_3.$$

$$\left(z_3\right)^2 = \left(\text{cis}\frac{4\pi}{3}\right)^2 = \text{cis}\frac{8\pi}{3} = \text{cis}\frac{2\pi}{3} = z_2.$$

**Question 18b**

The second square root of $z_3$ is the first square root multiplied by $-1$, so $\omega = -z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

**Question 18c**

$\omega$ is the square root of a cube root of unity, so is a sixth root of unity, $\therefore n = 6$.

**Question 19a**

$$z^5 + 1 = 0$$
$$\therefore (z+1)\left(z^4 - z^3 + z^2 - z + 1\right) = 0$$

**Question 19b**

$z = -1 = \cos\pi + i\sin\pi$ is one root, and the other roots are spaced by $\frac{2\pi}{5}$, so the roots are:

$$-1, \text{cis}\left(\pm\frac{\pi}{5}\right), \text{cis}\left(\pm\frac{3\pi}{5}\right)$$

**Question 19c**

$$\therefore \sum\alpha = -\frac{b}{a}$$
$$-1 + \text{cis}\frac{\pi}{5} + \text{cis}\left(-\frac{\pi}{5}\right) + \text{cis}\frac{3\pi}{5} + \text{cis}\left(-\frac{3\pi}{5}\right) = 0$$
$$1 + \text{cis}\frac{\pi}{5} + \overline{\text{cis}\frac{\pi}{5}} + \text{cis}\frac{3\pi}{5} + \overline{\text{cis}\frac{3\pi}{5}} = 0$$
$$1 + 2\text{Re}\left(\text{cis}\frac{\pi}{5}\right) + 2\text{Re}\left(\text{cis}\frac{3\pi}{5}\right) = 0$$
$$2\cos\left(\frac{\pi}{5}\right) + 2\cos\left(\frac{3\pi}{5}\right) - 1 = 0$$

**Question 20**

Let $n = 3k + j$ for $k \in \mathbb{Z}$ and $j = 0,1,2$

$$\begin{aligned} 1 + \omega^n + \omega^{2n} &= 1 + \omega^{3k+j} + \omega^{6k+2j} \\ &= 1 + \left(\omega^3\right)^k \times \omega^j + \left(\omega^3\right)^{2k} \times \omega^{2j} \\ &= 1 + \omega^j + \omega^{2j} \end{aligned}$$

If $j = 0$ then $1 + \omega^n + \omega^{2n} = 1 + 1 + 1 = 3$

$\therefore$ if $n$ is a multiple of 3 then $1 + \omega^n + \omega^{2n} = 3$

If $j = 1$ then $1 + \omega^n + \omega^{2n} = 1 + \omega + \omega^2 = 0$

If $j = 2$ then $1 + \omega^n + \omega^{2n} = 1 + \omega^2 + \omega^4 = 1 + \omega^2 + \omega = 0$

$\therefore$ if $n$ is not a multiple of 3 then $1 + \omega^n + \omega^{2n} = 0$

**Question 21a**

$$\begin{aligned} \frac{1}{\omega^2} + \frac{1}{\omega} &= \frac{\omega + \omega^2}{\omega^3} \\ &= \omega + \omega^2 \\ &= 1 + \omega + \omega^2 - 1 \\ &= -1 \end{aligned}$$

**Question 21b**

$$\begin{aligned} &\frac{1 + 2\omega + 3\omega^2}{2 + 3\omega + \omega^2} \times \frac{\omega}{\omega} + \frac{1 + 2\omega + 3\omega^2}{3 + \omega + 2\omega^2} \times \frac{\omega^2}{\omega^2} \\ &= \frac{1 + 2\omega + 3\omega^2}{2\omega + 3\omega^2 + \omega^3} \times \omega + \frac{1 + 2\omega + 3\omega^2}{3\omega^2 + \omega^3 + 2\omega^4} \times \omega^2 \\ &= \frac{1 + 2\omega + 3\omega^2}{2\omega + 3\omega^2 + 1} \times \omega + \frac{1 + 2\omega + 3\omega^2}{3\omega^2 + 1 + 2\omega} \times \omega^2 \\ &= \omega + \omega^2 \\ &= 1 + \omega + \omega^2 - 1 \\ &= -1 \end{aligned}$$

**Question 22a**

$$z^5 = 1$$
$$z^5 - 1 = 0$$
$$(z-1)\left(z^4 + z^3 + z^2 + z + 1\right) = 0$$
$$\therefore (\omega - 1)\left(\omega^4 + \omega^3 + \omega^2 + \omega + 1\right) = 0$$
$$\therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0 \quad \text{since } \omega \text{ is non-real}$$

**Question 22b**

$$\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1$$
$$= \omega^2 + 2 + \frac{1}{\omega^2} + \omega + \frac{1}{\omega} - 1$$
$$= \omega^2 + 1 + \frac{1}{\omega^2} + \omega + \frac{1}{\omega}$$
$$= \frac{\omega^4 + \omega^2 + 1 + \omega^3 + \omega}{\omega^2}$$
$$= \frac{0}{\omega^2}$$
$$= 0$$

**Question 22c**

$$\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$
$$\frac{1}{\omega} = \bar{\omega} = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right) = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} \text{since } |\omega| = 1$$
$$\omega + \bar{\omega} = 2\cos\frac{2\pi}{5}$$
$$\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$$
$$\left(2\cos\frac{2\pi}{5}\right)^2 + \left(2\cos\frac{2\pi}{5}\right) - 1 = 0$$
$$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0$$
$$\cos\frac{2\pi}{5} = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$$
$$= \frac{-2 \pm \sqrt{20}}{8}$$
$$= \frac{-1 \pm \sqrt{5}}{4}$$

Now $0 < \frac{2\pi}{5} < \frac{\pi}{2} \therefore \cos\frac{2\pi}{5} > 0$

$$\therefore \cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$$

**Question 23**

$\Delta OA_1A_2$ is isosceles, with $\angle OA_1A_2 = \frac{2\pi}{3}$ and $|OA_1| = |OA_2| = 1$

$\Delta OA_1A_3$ is congruent to $\Delta OA_1A_2$, so $A_1A_2 \times A_1A_3 = (A_1A_2)^2$

Using the Cosine Rule in $\Delta OA_1A_2$:

$$(A_1A_2)^2 = |OA_1|^2 + |OA_2|^2 - 2|OA_1| \times |OA_2| \cos\angle OA_1A_2$$
$$= 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\left(\frac{2\pi}{3}\right)$$
$$= 2 - 2\left(-\cos\frac{\pi}{3}\right)$$
$$= 2 + 2\left(\frac{1}{2}\right)$$
$$= 3$$

**Question 24a**

$\alpha^7 = 1$ since $\alpha$ is a root of $z^7 = 1$

$$z^7 = 1$$
$$z^7 - 1 = 0$$
$$(z-1)\left(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1\right) = 0$$
$$\therefore (\alpha - 1)\left(\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1\right) = 0$$
$$\therefore 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0 \quad \text{since } \alpha \neq 1 \text{ as it is non-real.}$$

**Question 24b**

$$\begin{aligned}\theta + \delta &= \alpha + \alpha^2 + \alpha^4 + \alpha^3 + \alpha^5 + \alpha^6 \\ &= \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 \\ &= 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 - 1 \\ &= 0 - 1 \\ &= -1\end{aligned}$$

$$\begin{aligned}\theta\delta &= \left(\alpha + \alpha^2 + \alpha^4\right)\left(\alpha^3 + \alpha^5 + \alpha^6\right) \\ &= \alpha^4 + \alpha^6 + \alpha^7 + \alpha^5 + \alpha^7 + \alpha^8 + \alpha^7 + \alpha^9 + \alpha^{10} \\ &= \alpha^4 + \alpha^6 + 1 + \alpha^5 + 1 + \alpha + 1 + \alpha^2 + \alpha^3 \\ &= 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + 2 \\ &= 0 + 2 \\ &= 2\end{aligned}$$

$$(z - \theta)(z - \delta) = 0$$
$$z^2 - (\theta + \delta)z + \theta\delta = 0$$
$$z^2 + z + 2 = 0$$

**Question 24c**

$$z = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm i\sqrt{7}}{2}$$

$$\therefore \theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2} \text{ and } \delta = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$$

**Question 24d**

$$\alpha = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$$

$$\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$$
$$\alpha + \alpha^2 + \alpha^4 = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$$
$$\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} + \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7} = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$$

Equating real and imaginary components:

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2} \rightarrow \therefore \cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$$
$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = -\frac{\sqrt{7}}{2} \rightarrow \therefore \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$$

**Question 25a**

$$(r\,\mathrm{cis}\,\theta)^5 = \mathrm{cis}\,2k\pi \quad \text{for } 4k = -2, -1, 0, 1, 2$$
$$r^5\,\mathrm{cis}\,5\theta = \mathrm{cis}\,2k\pi$$
$$r = 1 \quad 5\theta = 2\pi k$$
$$\theta = \frac{2\pi k}{5}$$
$$= -\frac{4\pi}{5}, -\frac{2\pi}{5}, 0, \frac{2\pi}{5}, \frac{4\pi}{5}$$
$$\therefore z = 1, \cos\left(\pm\frac{4\pi}{5}\right) + i\sin\left(\pm\frac{4\pi}{5}\right), \cos\left(\pm\frac{2\pi}{5}\right) + i\sin\left(\pm\frac{2\pi}{5}\right)$$

**Question 25b**

$$\begin{aligned}
z^5-1 &= (z-\alpha)(z-\beta)(z-\gamma)(z-\delta)(z-\epsilon)\\
&= (z-1)\left(z-\operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)\left(z-\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)\left(z-\operatorname{cis}\left(-\frac{4\pi}{5}\right)\right)\left(z-\operatorname{cis}\left(\frac{4\pi}{5}\right)\right)\\
&= (z-1)\left(z^2-z\left(\operatorname{cis}\left(-\frac{2\pi}{5}\right)+\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)+\operatorname{cis}0\right)\left(z^2-z\left(\operatorname{cis}\left(-\frac{4\pi}{5}\right)+\operatorname{cis}\left(\frac{4\pi}{5}\right)\right)+\operatorname{cis}0\right)\\
&= (z-1)\left(z^2-z\left(2\operatorname{Re}\left(\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)\right)+1\right)\left(z^2-z\left(2\operatorname{Re}\left(\operatorname{cis}\left(\frac{4\pi}{5}\right)\right)\right)+1\right)\\
&= (z-1)\left(z^2-2z\cos\left(\frac{2\pi}{5}\right)+1\right)\left(z^2-2z\cos\left(\frac{4\pi}{5}\right)+1\right)
\end{aligned}$$

**Question 25c**

$$\alpha+\beta+\gamma+\delta+\epsilon=-\frac{b}{a}$$

$$1+\operatorname{cis}\left(\frac{2\pi}{5}\right)+\operatorname{cis}\left(-\frac{2\pi}{5}\right)+\operatorname{cis}\left(\frac{4\pi}{5}\right)+\operatorname{cis}\left(-\frac{4\pi}{5}\right)=0$$

$$1+2\operatorname{Re}\left(\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)+2\operatorname{Re}\left(\operatorname{cis}\left(\frac{4\pi}{5}\right)\right)=0$$

$$2\left[\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)\right]=-1$$

$$\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)=-\frac{1}{2}$$

**Question 26a**

$$z_1=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$$

$$\begin{aligned}
z_1^r &= \left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)^r \quad \text{for } r=1,2,\ldots,n\\
&= \cos\frac{2\pi r}{n}+i\sin\frac{2\pi r}{n} \quad \text{for } r=1,2,\ldots,n
\end{aligned}$$

Since $z_1^r$ covers all possible roots of $z^n=1$ it is a primitive root.

**Question 26b**

$$\begin{aligned}
z_5 &= \cos\frac{2\pi\times5}{6}+i\sin\frac{2\pi\times5}{6}\\
&= \cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}
\end{aligned}$$

$$z_1^r=\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)^r \quad \text{for } r=1,2,..,6$$

$$\begin{aligned}
\theta &= \frac{5\pi}{3},\frac{10\pi}{3},\frac{15\pi}{3},\frac{20\pi}{3},\frac{25\pi}{3},\frac{30\pi}{3}\\
&= \frac{5\pi}{3},\frac{4\pi}{3},\frac{3\pi}{3},\frac{2\pi}{3},\frac{\pi}{3},\frac{6\pi}{3}\\
&= \frac{2\pi k}{6} \quad \text{for } k=1,2,\ldots,6
\end{aligned}$$

Since $z_5^r$ covers all possible roots of $z^6=1$ it is a primitive root.

**Question 27a**

$$\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1}$$

$$= \frac{\frac{1-t^2}{1+t^2} + \frac{2ti}{1+t^2} - 1}{\frac{1-t^2}{1+t^2} + \frac{2ti}{1+t^2} + 1}$$

$$= \frac{1 - t^2 + 2ti - 1 - t^2}{1 - t^2 + 2ti + 1 + t^2}$$

$$= \frac{2ti - 2t^2}{2 + 2ti}$$

$$= \frac{t(i-t)}{(1+ti)} \times \frac{1-ti}{1-ti}$$

$$= \frac{ti + t^2 - t^2 + t^3 i}{1 + t^2}$$

$$= \frac{it\left(1 + t^2\right)}{1 + t^2}$$

$$= it$$

$$= i\tan\frac{\theta}{2}$$

**Question 27b**

$$\omega = 1, \operatorname{cis}\left(\pm\frac{2\pi}{5}\right), \operatorname{cis}\left(\pm\frac{4\pi}{5}\right)$$

**Question 27c**

Let $\omega = \frac{2+z}{2-z}$

$$2\omega - \omega z = 2 + z$$

$$z(\omega + 1) = 2(\omega - 1)$$

$$z = 2\left(\frac{\omega - 1}{\omega + 1}\right)$$

$$= 2i\tan\frac{\theta}{2} \quad \text{for } \omega = \cos\theta + i\sin\theta$$

$$= 2i\tan\left(\frac{k\pi}{5}\right) \quad \text{for } k = 0, \pm 1, \pm 2$$

**Question 27d**

$$\left(\frac{2+z}{2-z}\right)^5 = 1$$

$$(2+z)^5 = (2-z)^5$$

$$32 + 5(2^4)z + 10\left(2^3\right)z^2 + 10\left(2^2\right)z^3 + 5(2)z^4 + z^5 = 32 - 5(2^4)z + 10\left(2^3\right)z^2 - 10\left(2^2\right)z^3 + 5(2)z^4 - z^5$$

$$10(2^4)z + 20\left(2^2\right)z^3 + 2z^5 = 0$$

$$80z + 40z^3 + z^5 = 0$$

$$z^5 + 40z^3 + 80z = 0$$

$$z\left(z^4 + 40z^2 + 80\right) = 0$$

The product of the roots excluding $z = 0$ is 80

$$\therefore 2i\tan\left(-\frac{\pi}{5}\right) \times 2i\tan\left(-\frac{2\pi}{5}\right) \times 2i\tan\left(\frac{\pi}{5}\right) \times 2i\tan\left(\frac{\pi}{5}\right) = 80$$

$$-2\tan\left(\frac{\pi}{5}\right) \times \left(-2\tan\left(\frac{2\pi}{5}\right)\right) \times 2\tan\left(\frac{\pi}{5}\right) \times 2\tan\left(\frac{\pi}{5}\right) = 80$$

$$\left(\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\right)^2 = 5$$

$$\therefore \tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5} \text{ since } \tan\frac{\pi}{5}, \tan\frac{2\pi}{5} > 0$$

**Question 28a**

$$\begin{aligned}\frac{1+w}{1-w} &= \frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}\\ &= \frac{1+\dfrac{1-t^2}{1+t^2}+i\dfrac{2t}{1+t^2}}{1-\dfrac{1-t^2}{1+t^2}-i\dfrac{2t}{1+t^2}}\\ &= \frac{1+t^2+1-t^2+2ti}{1+t^2-1+t^2-2ti}\\ &= \frac{2+2ti}{2t^2-2ti}\\ &= \frac{1+ti}{t(t-i)}\\ &= \frac{1+ti}{t(t-i)}\times\frac{t+i}{t+i}\\ &= \frac{t+i+t^2i-t}{t(t^2+1)}\\ &= \frac{i\left(1+t^2\right)}{t(1+t^2)}\\ &= \frac{i}{t}\\ &= i\cot\frac{\theta}{2}\end{aligned}$$

Alternative solution I:

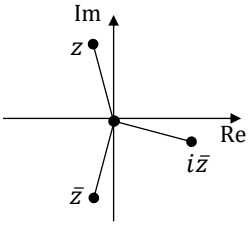
$$\begin{aligned}\frac{1+w}{1-w} &= \frac{1+\operatorname{cis}\theta}{1-\operatorname{cis}\theta}\\ &= \frac{\operatorname{cis}\left(-\frac{\theta}{2}\right)+\operatorname{cis}\left(\frac{\theta}{2}\right)}{\operatorname{cis}\left(-\frac{\theta}{2}\right)-\operatorname{cis}\left(\frac{\theta}{2}\right)}\\ &= \frac{2\operatorname{Re}\left(\operatorname{cis}\left(-\frac{\theta}{2}\right)\right)}{2i\operatorname{Im}\left(\operatorname{cis}\left(-\frac{\theta}{2}\right)\right)}\\ &= \frac{\cos\left(-\frac{\theta}{2}\right)}{i\sin\left(-\frac{\theta}{2}\right)}\\ &= \frac{\cos\left(-\frac{\theta}{2}\right)}{i\sin\left(-\frac{\theta}{2}\right)}\times\frac{i}{i}\\ &= \frac{i\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \quad \text{since cosine is even, sine is odd and } i^2=-1\\ &= i\cot\frac{\theta}{2}\end{aligned}$$

**Question 28b**

$$\begin{aligned}\left(\frac{z-1}{z+1}\right)^8 &= -1\\ &= \operatorname{cis}(2k+1)\pi\\ \therefore \frac{z-1}{z+1} &= \operatorname{cis}\frac{(2k+1)\pi}{8} \quad \text{for } k=-4,-3,\ldots,3\\ \therefore z-1 &= z\operatorname{cis}\frac{(2k+1)\pi}{8}+\operatorname{cis}\frac{(2k+1)\pi}{8}\\ z\left(1-\operatorname{cis}\frac{(2k+1)\pi}{8}\right) &= 1+\operatorname{cis}\frac{(2k+1)\pi}{8}\\ z &= \frac{1+\operatorname{cis}\frac{(2k+1)\pi}{8}}{1-\operatorname{cis}\frac{(2k+1)\pi}{8}}\\ &= i\cot\frac{(2k+1)\pi}{16} \quad \text{with } w=\operatorname{cis}\frac{(2k+1)\pi}{8} \text{ in part (a)}\\ &= i\cot\left(\pm\frac{\pi}{16}\right), i\cot\left(\pm\frac{3\pi}{16}\right),\ldots,i\cot\left(\pm\frac{7\pi}{16}\right)\\ &= \pm i\cot\left(\frac{\pi}{16}\right), \pm i\cot\left(\frac{3\pi}{16}\right),\ldots,\pm i\cot\left(\frac{7\pi}{16}\right)\end{aligned}$$

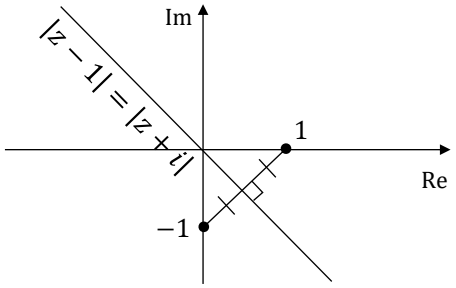
**Question 1**

$\bar{z}$ is a reflection of $z$ in the real axis, $\bar{z} \times i$ is an anticlockwise rotation of 90°




**Answer C**

**Question 2**




**Question 3**

$$\left|-\sqrt{3}+i\right| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2}$$
$$= 2$$
$$\text{Arg}(-\sqrt{3}+i) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{5\pi}{6}$$
$$-\sqrt{3}+i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

**Answer D**

**Question 4**

$$iz = i(3-i)$$
$$= 3i - i^2$$
$$= 1 + 3i$$
$$\therefore \bar{iz} = 1 - 3i$$

Alternatively:

$$\bar{iz} = \bar{i} \times \bar{z}$$
$$= -i \times (3+i)$$
$$= -3i - i^2$$
$$= 1 - 3i$$

**Answer C**

**Question 5a**

$$|z| = \sqrt{(1)^2 + (2)^2}$$
$$= \sqrt{5}$$

**Question 5b**

$$zw = (1+2i)(-2+i)$$
$$= -2 + i - 4i - 2$$
$$= -4 - 3i$$

**Question 5c**

$$\frac{5}{iw} = \frac{5}{i(-2+i)} \times \frac{-1+2i}{-1+2i}$$
$$= \frac{-5+10i}{(-1)^2 + (2)^2}$$
$$= -1 + 2i$$

**Question 6**

The region is on and in the circle centred at $(1,0)$ with radius 2. Its equation is

$$|z - (1+0i)| \leq 2$$
$$\therefore |z-1| \leq 2$$

**Answer A**

**Question 7**

$$\frac{2-i}{-2-i} \times \frac{-2+i}{-2+i}$$
$$= \frac{-4+2i+2i+1}{4+1}$$
$$= -\frac{3}{5} + \frac{4}{5}i$$

**Answer A**

**Question 8**

Let $(a+ib)^2 = -8+6i$

$\therefore a^2 - b^2 = -8$

and $2ab = 6$

by inspection $a = 1, b = 3$ or $a = -1, b = -3$

The square roots are $\pm(1+3i)$

Alternatively:

$$a = \pm\sqrt{\frac{|z| + \text{Re}(z)}{2}}$$
$$= \pm\sqrt{\frac{\sqrt{(-8)^2 + (6)^2} - 8}{2}}$$
$$= \pm 1$$
$$b = \text{sgn}(6) \pm \sqrt{\frac{|z| - \text{Re}(z)}{2}}$$
$$= (+) \pm \sqrt{\frac{\sqrt{(-8)^2 + (6)^2} + 8}{2}}$$
$$= \pm 3$$

The square roots are $\pm(1+3i)$

**Question 9**

$$\cos\pi + i\sin\pi + 1$$
$$= -1 + i(0) + 1$$
$$= 0$$

**Question 10**

The shaded area, when measured from $(1,0)$, has an argument between 0 and $\frac{3\pi}{4}$ inclusive (since the x and y intercepts are both 1 the gradient of the left hand ray is -1, so an angle of $\frac{3\pi}{4}$. So $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$.

The shaded area is on or inside the circle of radius 2 centred at $(0,1)$, so $|z-i| \leq 2$

**Answer A**

**Question 11a**

$$wz^2 = -\sqrt{2}\left(\sqrt{2} - \sqrt{2}i\right)^2$$
$$= -\sqrt{2}(2 - 4i - 2)$$
$$= 4\sqrt{2}i$$

**Question 11b**

$$\text{Arg}\, z = -\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$
$$= -\frac{\pi}{4}$$

**Question 11c**

$$\begin{aligned}\frac{z}{z+w} &= \frac{\sqrt{2}-\sqrt{2}i}{\sqrt{2}-\sqrt{2}i-\sqrt{2}} \\ &= \frac{\sqrt{2}-\sqrt{2}i}{-\sqrt{2}i} \times \frac{\sqrt{2}i}{\sqrt{2}i} \\ &= \frac{2+2i}{2} \\ &= 1+i\end{aligned}$$

**Question 11d**

$$\begin{aligned}|z| &= \sqrt{\left(\sqrt{2}\right)^2+\left(-\sqrt{2}\right)^2} \\ &= 2\end{aligned}$$

**Question 11e**

$$\begin{aligned}z^{10} &= \left(2\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^{10} \\ &= 2^{10}\text{cis}\left(-\frac{5\pi}{2}\right) \\ &= 1024\,\text{cis}\left(-\frac{\pi}{2}\right) \\ &= -1024i\end{aligned}$$

**Question 12**

Let $(a+ib)^2 = 5-12i$

$\therefore a^2-b^2=5$

$and \quad 2ab=-12$

by inspection $a=3, b=-2$ or $a=-3, b=2$

The square roots are $\pm(3-2i)$

Alternatively:

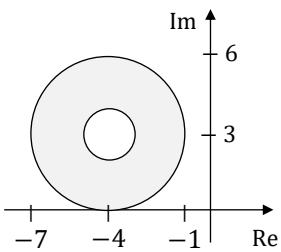
$$\begin{aligned}a &= \pm\sqrt{\frac{|z|+\text{Re}(z)}{2}} \\ &= \pm\sqrt{\frac{\sqrt{(5)^2+(-12)^2}+5}{2}} \\ &= \pm 3\end{aligned}$$

$$\begin{aligned}b &= \text{sgn}(-12)\pm\sqrt{\frac{|z|+\text{Re}(z)}{2}} \\ &= (-)\pm\sqrt{\frac{\sqrt{(5)^2+(-12)^2}-5}{2}} \\ &= \mp 2\end{aligned}$$

The square roots are $\pm(3-2i)$

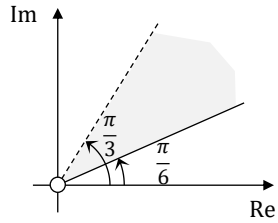
**Question 13a**

The region between the circles of radius $1$ and $3$ centred at $(-4,3)$.




**Question 13b**

The sector between the rays from the origin at arguments of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, including the ray at an argument of $\frac{\pi}{6}$ but not $\frac{\pi}{3}$.




**Question 14**

$$\begin{aligned}&\frac{12-6i}{4+3i} \times \frac{4-3i}{4-3i} \\ &= \frac{48-36i-24i-18}{4^2+3^2} \\ &= \frac{30}{25}-\frac{60}{25}i \\ &= 1.2-2.4i\end{aligned}$$

**Answer D**

**Question 15a**

$$\left(x^2-5\right)\left(x^2+2\right)$$

**Question 15b**

$$\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)\left(x^2+2\right)$$

**Question 15c**

$$\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)\left(x+\sqrt{2}i\right)\left(x-\sqrt{2}i\right)$$

**Question 16**

$(a+ib)^2 = 1+i\sqrt{3}$ by inspection

$a^2-b^2=1$

and $2ab=\sqrt{3}$

too hard - use another method:

$$1+i\sqrt{3} = 2\text{cis}\frac{\pi}{3}$$

The square roots of $1+i\sqrt{3}$ are

$$\begin{aligned}&\pm\left(2\text{cis}\frac{\pi}{3}\right)^{\frac{1}{2}} \\ &= \pm\sqrt{2}\text{cis}\frac{\pi}{6} \\ &= \pm\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) \\ &= \pm\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i\right)\end{aligned}$$

**Question 16 (continued)**

Alternatively:

$$a = \pm\sqrt{\frac{|z| + \text{Re}(z)}{2}}$$

$$= \pm\sqrt{\frac{\sqrt{(1)^2 + \left(\sqrt{3}\right)^2} + 1}{2}}$$

$$= \pm\sqrt{\frac{3}{2}} \times \sqrt{\frac{2}{2}}$$

$$= \pm\frac{\sqrt{6}}{2}$$

$$b = \text{sgn}\left(\sqrt{3}\right) \pm \sqrt{\frac{|z| - \text{Re}(z)}{2}}$$

$$= (+) \pm \sqrt{\frac{\sqrt{(1)^2 + \left(\sqrt{3}\right)^2} - 1}{2}}$$

$$= \pm\frac{1}{\sqrt{2}}$$

$$= \pm\frac{\sqrt{2}}{2}$$

The square roots are $\pm\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i\right)$

Alternative II:

Let $(c + id)^2 = 2 + 2\sqrt{3}i$

$$c^2 - d^2 = 2$$

$$2cd = 2\sqrt{3}$$

$c = \sqrt{3}, d = 1$ or $c = -\sqrt{3}, d = -1$

The square roots of $2 + 2\sqrt{3}i$ are $\pm\left(\sqrt{3} + i\right)$ so the square roots of $1 + \sqrt{3}i$ are

$$\pm\left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= \pm\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i\right)$$

**Question 17a**

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= 1$$

$$\theta = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{\pi}{6}$$

$$\frac{\sqrt{3}}{2} + \frac{i}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$

**Question 17b**

$$z^{15} = \left(\text{cis}\frac{\pi}{6}\right)^{15}$$

$$= \text{cis}\frac{15\pi}{6}$$

$$= \text{cis}\frac{\pi}{2}$$

$$= i$$

**Question 18a**

$$\overline{zw} = \bar{z} \times \overline{w}$$

$$= (3 + i)(2 - i)$$

$$= 6 - 3i + 2i + 1$$

$$= 7 - i$$

**Question 18b**

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

$$= \frac{\sqrt{(3)^2 + (-1)^2}}{\sqrt{(2)^2 + (1)^2}}$$

$$= \frac{\sqrt{10}}{\sqrt{5}}$$

$$= \sqrt{2}$$

**Question 19**

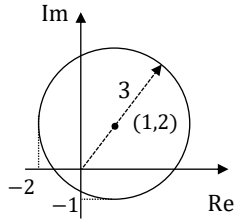
$$-2i \times \text{cis}\left(-\frac{\pi}{2}\right)$$

$$= -2i(-i)$$

$$= 2i^2$$

$$= -2$$

**Question 20a**




**Question 20b**

The maximum modulus occurs on the opposite side of the circle, on a line through the centre, and its length is the distance from the origin to the centre of the circle plus the radius

$$|z|_{\text{max}} = \sqrt{(1)^2 + (2)^2} + 3$$

$$= 3 + \sqrt{5}$$

**Question 21**

$$1 + \sqrt{3}i = 2\ \text{cis}\frac{\pi}{3}$$

$$\left(1 + \sqrt{3}i\right)^4 = 2^4\,\text{cis}\frac{4\pi}{3}$$

$$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 16\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -8 - 8\sqrt{3}i$$

**Answer D**

**Question 23a**

$$\text{Re}(z) - \text{Im}(w) = 3 - (-2) = 5$$

**Question 23b**

$$\frac{z}{iw} = \frac{3 + 4i}{i(1 - 2i)}$$

$$= \frac{3 + 4i}{2 + i} \times \frac{2 - i}{2 - i}$$

$$= \frac{6 - 3i + 8i + 4}{2^2 + 1^1}$$

$$= 2 + i$$

**Question 23c**

$(a+ib)^2 = 3+4i$

$a^2 - b^2 = 3$

$2ab = 4$

$a = 2, b = 1$ or $a = -2, b = -1$ by inspection

The square roots of $3+4i$ are $\pm(2+i)$

Alternative:

$$a = \pm\sqrt{\frac{|z| + \text{Re}(z)}{2}}$$

$$= \pm\sqrt{\frac{\sqrt{3^2+4^2}+3}{2}}$$

$$= \pm 2$$

$$b = \text{sgn}(4) \pm \sqrt{\frac{|z| - \text{Re}(z)}{2}}$$

$$= \pm\sqrt{\frac{\sqrt{3^2+4^2}-3}{2}}$$

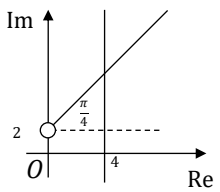
$$= \pm 1$$

The square roots of $3+4i$ are $\pm(2+i)$

**Question 24**

$|z-9| = |z+1|$ is the perpendicular bisector of the points $(9,0)$ and $(-1,0)$, so $\text{Re} = 4$.

$\arg(z-2i) = \frac{\pi}{4}$ is the ray form $(0,2)$ with an argument of $\frac{\pi}{4}$.




Substituting $x = 4$ into $y = x + 2$, the point $(4,6)$ is the point of intersection, so $4+6i$ is the only value of $z$ that satisfies both equations.

**Question 25**

$$\text{Arg}\,\bar{z} = \text{Arg}\left(1 + i\sqrt{3}\right)$$

$$= \tan^{-1}\sqrt{3}$$

$$= \frac{\pi}{3}$$

**Answer B**

**Question 26**

The shaded are is on and inside a circle centre the origin, radius 1, so $|z| \leq 1$ and also on or inside a circle centred at $1+i$ with radius 1, so $|z-(1+i)| \leq 1$

**Answer D**

**Question 27a**

$$AB = (3+4i)(5-13i)$$

$$= 15 - 39i + 20i + 52$$

$$= 67 - 19i$$

**Question 27b**

$$\frac{A}{B} = \frac{3+4i}{5-13i} \times \frac{5+13i}{5+13i}$$

$$= \frac{15+39i+20i-52}{5^2+13^2}$$

$$= -\frac{37}{194} + \frac{59}{194}i$$

**Question 27c**

$(a+ib)^2 = -3-4i$

$a^2 - b^2 = -3$

$2ab = -4$

by inspection $a = 1, b = -2$ or $a = -2, b = 1$

The square roots of $A$ are $\pm(1-2i)$

Alternatively:

$$a = \pm\sqrt{\frac{|z| + \text{Re}(z)}{2}}$$

$$= \pm\sqrt{\frac{\sqrt{(-3)^2+(-4)^2}-3}{2}}$$

$$= \pm 1$$

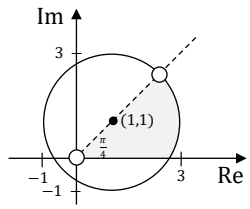
$$b = \text{sgn}(-4) \pm \sqrt{\frac{|z| - \text{Re}(z)}{2}}$$

$$= (-) \pm \sqrt{\frac{\sqrt{(-3)^2+(-4)^2}+3}{2}}$$

$$= \pm 2$$

**Question 28**

The region on or inside the circle centred at $(1,1)$ with radius 2 and also between rays with arguments of 0 and $\frac{\pi}{4}$, including the ray of argument 0 but not $\frac{\pi}{4}$. Note the points of discontinuity are not in the region.




**Question 29**

The distance from the origin is between 1 and 2, so $1 \leq |z| \leq 2$

**Answer B**

**Question 30a**

$$\left(x^2+4\right)\left(x^2-3\right)$$

**Question 30b**

$$\left(x^2+4\right)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$$

**Question 30c**

$$(x-2i)(x+2i)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$$

**Question 31**

Let $M$ be the midpoint of $AC$, $\therefore \overrightarrow{OM} = \frac{z_1+z_3}{2}$

Let $N$ be the midpoint of $BD$, $\therefore \overrightarrow{ON} = \frac{z_2+z_4}{2}$

If $z_1 + z_3 = z_2 + z_4$

$2\overrightarrow{OM} = 2\overrightarrow{ON}$

$\therefore \overrightarrow{ON} = \overrightarrow{ON}$

$\therefore$ the diagonals bisect so $ABCD$ is a parallelogram.

**Question 32**

$|z+2i| \geq |z+2|$ means that the region includes points that are closer to $(-2,0)$ than $(0,-2)$, so above the line $y=x$ (the perpendicular bisector), so C or D possible.

$\text{Im}\,(z) + \text{Re}\,(z) \geq 2$ is equivalent to $x+y \geq 2$, so on or above the line $x+y=2$, so C

**Answer C**

**Question 33**

$M$ is the midpoint of $AB$, so $\dfrac{z_1+z_2}{2}$. The vector $MQ$ is half the vector $AB$ rotated anti−clockwise 90 degrees, so $\dfrac{z_1-z_2}{2}i$.

So $\omega = \dfrac{z_1+z_2}{2} + \dfrac{z_1-z_2}{2}i$.

**Question 34**

If $\omega$ is a non-real cube root of unity, then

$$1+\omega+\omega^2 = 0$$

$$\begin{aligned}(1+\omega-\omega^2)^7 &= (1+\omega+\omega^2-2\omega^2)^7 \\ &= (-2\omega^2)^7 \\ &= -128\omega^{14} \\ &= -128\omega^{12}\omega^2 \\ &= -128\omega^2\end{aligned}$$

**Answer D**

**Question 35**

$$\begin{aligned}(z+\bar{z})^2 - (z-\bar{z})^2 &= 16 \\ (2x)^2 - (2iy)^2 &= 16 \\ 4x^2 + 4y^2 &= 16 \\ x^2+y^2 &= 4\end{aligned}$$

**Answer B**

**Question 36a**

$$\left|1+i\sqrt{3}\right| = \sqrt{1^2+\left(\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$$\arg(1+i\sqrt{3}) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$1+i\sqrt{3} = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

**Question 36b**

$$\begin{aligned}&z^5+16z \\ &= 2^5\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)+32\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) \\ &= 32\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)+32\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) \\ &= 64\cos\frac{\pi}{3} \\ &= 32\end{aligned}$$

**Question 37**

$$\begin{aligned}z+3i\bar{z} &= -1+13i \\ x+iy+3i(x-iy) &= -1+13i \\ x+iy+3ix+3y &= -1+13i \\ x+3y &= -1 \quad (1) \\ 3x+y &= 13 \quad (2) \\ 3\times(1)-(2) \quad 8y &= -16 \\ y &= -2 \\ \text{sub in } (1) \quad x+3(-2) &= -1 \\ x &= 5 \\ \therefore z &= 5-2i\end{aligned}$$

**Question 38**

$\angle POQ = \dfrac{\pi}{3}$ (angle in an equilateral triangle)

$$\therefore w = z\,\text{cis}\frac{\pi}{3}$$

$$\begin{aligned}\therefore w^3+z^3 &= \left(z\,\text{cis}\frac{\pi}{3}\right)^3+z^3 \\ &= z^3\text{cis}\frac{3\pi}{3}+z^3 \\ &= z^3(-1)+z^3 \\ &= 0\end{aligned}$$

**Question 39**

$$\begin{aligned}\left(\text{cis}\frac{\pi}{3}\right)^9 &= \text{cis}\,3\pi \\ &= \text{cis}\,\pi \\ &= -1\end{aligned}$$

**Question 40a**

$$\begin{aligned}(1-3i)^2 &= 1-6i-9 \\ &= -8-6i\end{aligned}$$

**Question 40b**

$$\begin{aligned}z &= \frac{8\pm\sqrt{64-4(2)(12+3i)}}{2(2)} \\ &= \frac{8\pm\sqrt{-32-24i}}{4} \\ &= \frac{4\pm\sqrt{-8-6i}}{2} \\ &= \frac{4\pm(1-3i)}{2} \\ &= \frac{5-3i}{2}, \frac{3+3i}{2}\end{aligned}$$

**Question 41**

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA}\times\text{cis}\left(2\left(\frac{\pi}{6}\right)\right) \\ &= (1+3i)\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) \\ &= (1+3i)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{2}+i\frac{\sqrt{3}}{2}+i\frac{3}{2}-\frac{3\sqrt{3}}{2} \\ &= \frac{1-3\sqrt{3}}{2}+i\frac{\sqrt{3}+3}{2} \\ \overrightarrow{OB} &= \overrightarrow{OA}+\overrightarrow{OC} \\ &= 1+3i+\frac{1-3\sqrt{3}}{2}+i\frac{\sqrt{3}+3}{2} \\ &= \frac{3-3\sqrt{3}}{2}+i\frac{\sqrt{3}+9}{2}\end{aligned}$$

**Question 42a**

$$\begin{aligned}&\Omega^3-1 = 0 \\ \therefore\ &(\Omega-1)(\Omega^2+\Omega+1) = 0\end{aligned}$$

Now $\Omega \neq 1$ since $\Omega$ is non−real

$$\therefore 1+\Omega+\Omega^2 = 0$$

**Question 42b**

$$\begin{aligned}1+\Omega+\bar{\Omega} &= \sum\alpha \\ &= -\frac{b}{a} \\ &= 0\end{aligned}$$

$$\begin{aligned}\therefore 1+\Omega+\bar{\Omega} &= 1+\Omega+\Omega^2 \\ \therefore \bar{\Omega} &= \Omega^2\end{aligned}$$

**Question 43**

$$1+i=\sqrt{2}\operatorname{cis}\frac{\pi}{4}$$
$$=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$
$$=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)-i\sin\left(-\frac{\pi}{4}\right)\right)$$

since cosine is even and sine is odd

$\therefore a=\sqrt{2}, b=-\frac{\pi}{4}$.

**Question 44a**

$$\left|\frac{\sqrt{3}+i}{1+i}\right|=\frac{|\sqrt{3}+i|}{|1+i|}$$
$$=\frac{\sqrt{\left(\sqrt{3}\right)^2+(1)^2}}{\sqrt{(1)^2+(1)^2}}$$
$$=\frac{2}{\sqrt{2}}$$
$$=\sqrt{2}$$
$$\arg\left(\frac{\sqrt{3}+i}{1+i}\right)=\arg\left(\sqrt{3}+i\right)-\arg(1+i)$$
$$=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}\left(\frac{1}{1}\right)$$
$$=\frac{\pi}{6}-\frac{\pi}{4}$$
$$=-\frac{\pi}{12}$$
$$\therefore\frac{\sqrt{3}+i}{1+i}=\sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)$$

**Question 44b**

$$\left(\frac{\sqrt{3}+i}{1+i}\right)^n=\left(\sqrt{2}\left(\operatorname{cis}\left(-\frac{\pi}{12}\right)\right)\right)^n$$
$$=2^{\frac{n}{2}}\operatorname{cis}\left(-\frac{n\pi}{12}\right)$$
$$\therefore-\frac{n\pi}{12}=k\pi\quad k\in\mathbb{Z}\text{ since }z^n\text{ is real}$$
$$n=-12k$$

$\therefore n=12$ is the smallest positive value of $n$ such that $z^n$ is purely real.

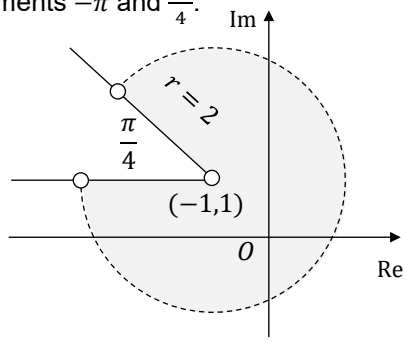
**Question 45**

$$\frac{1}{\bar{z}-i}=\frac{1}{x-iy-i}$$
$$=\frac{1}{x-i(y+1)}\times\frac{x+i(y+1)}{x+i(y+1)}$$
$$=\frac{x+i(y+1)}{x^2+(y+1)^2}$$
$$\therefore\operatorname{Im}\left(\frac{1}{\bar{z}-i}\right)=\frac{y+1}{x^2+(y+1)^2}=2$$
$$\therefore y+1=2x^2+2y^2+4y+2$$
$$x^2+y^2+\frac{3}{2}y=-\frac{1}{2}$$
$$x^2+\left(y+\frac{3}{4}\right)^2=-\frac{1}{2}+\frac{9}{16}$$
$$x^2+\left(y+\frac{3}{4}\right)^2=\frac{1}{16}$$

which is a circle.

**Question 46**

The region inside the circle of radius 2 centred at $(-1,1)$ and the major sector between the rays with arguments $-\pi$ and $\frac{3\pi}{4}$.




**Question 47**

$$\frac{1+i}{\sqrt{2}}=\frac{\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)}{\sqrt{2}}$$
$$=\operatorname{cis}\left(\frac{\pi}{4}\right)$$
$$\therefore(r\operatorname{cis}\theta)^3=\operatorname{cis}\frac{\pi}{4}$$
$$\therefore r^3=1\qquad 3\theta=2k\pi+\frac{\pi}{4}\quad k=-1,0,1$$
$$\theta=-\frac{7\pi}{12},\frac{\pi}{12},\frac{9\pi}{12}$$
$$z=\operatorname{cis}\left(-\frac{7\pi}{12}\right),\operatorname{cis}\left(\frac{\pi}{12}\right),\operatorname{cis}\left(\frac{3\pi}{4}\right)$$

**Question 48**

$$\frac{z}{\bar{z}}=\frac{z}{\bar{z}}\times\frac{z}{z}$$
$$=\frac{z^2}{37}\qquad\text{since }z\bar{z}=37$$

Let $z=a+ib$

$$\frac{z}{\bar{z}}=\frac{a^2-b^2+2abi}{37}\qquad(1)$$
$$\frac{z}{\bar{z}}=\frac{35}{37}+\frac{12i}{37}\qquad(2)\text{ given}$$

From (1) and (2):

$$\therefore\quad x^2-y^2=35\qquad(3)$$
$$xy=6\qquad(4)$$

by inspection $x=6, y=1$ or $x=-6, y=-1$

$\therefore z=\pm(6+i)$

**Question 49**

$$\arg\left(-1+i\sqrt{3}\right)=\pi-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$
$$=\frac{2\pi}{3}$$
$$\arg w=2012\arg\left(-1+i\sqrt{3}\right)$$
$$=\frac{4024\pi}{3}$$
$$=-\frac{2\pi}{3}$$

**Question 50**

$$-3\left(\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)\right)$$
$$=3\operatorname{cis}\pi\times\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$$
$$=3\operatorname{cis}\left(\frac{2\pi}{3}\right)$$

**Question 51a**

$\angle AOB = \dfrac{\pi}{4}$ (right−angled isosceles triangle)

$\angle COB = \pi - 2\left(\dfrac{7\pi}{24}\right)$ (angle sum isosceles Δ)

$= \dfrac{5\pi}{12}$

$\angle AOC = \dfrac{\pi}{4} + \dfrac{5\pi}{12}$

$= \dfrac{2\pi}{3}$

**Question 51b**

$\gamma = \left|\dfrac{OC}{OB} \times \dfrac{OB}{OA} \times OA\right| \text{cis}(\arg\alpha + \angle AOC)$

$= \left|1 \times \dfrac{\sqrt{2}|\alpha|}{|\alpha|} \times |\alpha|\right| \times \text{cis}\left(\arg(\alpha) + \dfrac{2\pi}{3}\right)$

$= \sqrt{2}|\alpha| \times \text{cis}(\arg(\alpha)) \times \text{cis}\left(\dfrac{2\pi}{3}\right)$

$= \sqrt{2}\text{cis}\left(\dfrac{2\pi}{3}\right)|\alpha|\text{cis}(\arg\alpha)$

$= \sqrt{2}\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)\alpha$

**Question 51c**

$2\alpha^2 + \gamma^2 + \alpha\gamma\sqrt{2}$

$= 2\alpha^2 + \left(\sqrt{2}\text{cis}\left(\dfrac{2\pi}{3}\right)\alpha\right)^2 + \alpha\left(\sqrt{2}\text{cis}\left(\dfrac{2\pi}{3}\right)\alpha\right)\sqrt{2}$

$= 2\alpha^2\left(1 + \text{cis}\dfrac{4\pi}{3} + \text{cis}\dfrac{2\pi}{3}\right)$

$= 2\alpha^2\left(1 + \text{cis}\left(-\dfrac{2\pi}{3}\right) + \text{cis}\dfrac{2\pi}{3}\right)$

$= 2\alpha^2\left(1 + 2\text{Re}\left(\text{cis}\dfrac{2\pi}{3}\right)\right)$

$= 2\alpha^2\left(1 + 2\cos\dfrac{2\pi}{3}\right)$

$= 2\alpha^2\left(1 + 2\left(-\dfrac{1}{2}\right)\right)$

$= 0$

**Question 52**

$z\bar{z} - 2(z + \bar{z}) = 5$

$|z|^2 - 2\text{Re}(z) = 5$

$Let\ z = x + iy$

$x^2 + y^2 - 4x = 5$

$x^2 - 4x + 4 + y^2 = 5 + 4$

$(x-2)^2 + y^2 = 3^2$

A circle centred at $(2,0)$ with radius $3$.

**Question 53**

$\sqrt{2} - \sqrt{2}i = 2\text{ cis}\left(-\dfrac{\pi}{4}\right)$

$\left(\sqrt{2} - \sqrt{2}i\right)^{10} = \left(2\text{ cis}\left(-\dfrac{\pi}{4}\right)\right)^{10}$

$= 2^{10}\text{cis}\left(-\dfrac{5\pi}{2}\right)$

$= 1024\text{ cis}\left(-\dfrac{\pi}{2}\right)$

$= -1024i$

**Question 54**

$\overrightarrow{OB} = \dfrac{3}{2} \times -i \times \overrightarrow{OP}$

$= -\dfrac{3}{2}i(6 + ai)$

$= -9i + \dfrac{3a}{2}$

$\therefore B$ represents $\dfrac{3a}{2} - 9i$.

**Question 55**

The vertical line is of the form $\text{Re}(z) = 2$

A is correct

B is $|z - (0 + 0i)| = |z - (4 + 0i)|$, the perpendicular bisector of $(0,0)$ and $(4,0)$, which is correct.

C would be an arc with ends $(0,0)$ and $(4,0)$. This is incorrect.

D $z + \bar{z} = 2\text{Re}(z) = 4 \rightarrow \text{Re}(z) = 2$ thus correct

**Answer C**

**Question 56a**

$\left|2 + 2\sqrt{3}i\right| = \sqrt{(2)^2 + \left(2\sqrt{3}\right)^2} = 4$

$\arg\left(2 + 2\sqrt{3}i\right) = \tan^{-1}\left(\dfrac{2\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$

$\therefore\ 2 + 2\sqrt{3}i = 4\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

**Question 56b**

$\left(2 + 2\sqrt{3}i\right)^3 = \left[4\ \text{cis}\dfrac{\pi}{3}\right]^3$

$= 4^3\,\text{cis}\,\pi$

$= -64$

**Question 56c**

$z^4 = 2 + 2\sqrt{3}i$

$\therefore r^4\text{cis}\,4\theta = 4\text{ cis}\left(\dfrac{\pi}{3} + 2k\pi\right) \quad k = -2, -1, 0, 1$

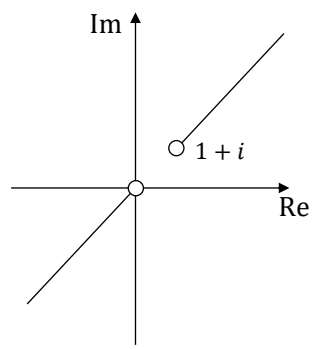
$\therefore r^4 = 4 \quad 4\theta = -\dfrac{11\pi}{3}, -\dfrac{5\pi}{3}, \dfrac{\pi}{3}, \dfrac{7\pi}{3}$

$r = \sqrt{2} \quad \theta = -\dfrac{11\pi}{12}, -\dfrac{5\pi}{12}, \dfrac{\pi}{12}, \dfrac{7\pi}{12}$

$z = \sqrt{2}\text{cis}\left(-\dfrac{11\pi}{12}\right), \sqrt{2}\text{cis}\left(-\dfrac{5\pi}{12}\right), \sqrt{2}\text{cis}\left(\dfrac{\pi}{12}\right), \sqrt{2}\text{cis}\left(\dfrac{7\pi}{12}\right)$

**Question 57**

The vectors $z - 0$ and $z - (1 + i)$ have the same principal argument, which occurs on the line through $(0,0)$ and $(1,1)$ outside the two points.



**Question 58**

$\overrightarrow{OR} = \frac{1}{2}i\overrightarrow{OS}$

$\therefore w = \frac{i}{2}z$

Since this is not one of the options, multiply the RHS top and bottom by $i$

$w = \frac{i^2 z}{2i}$

$= -\frac{z}{2i}$

**Answer D**

**Question 59**

Let $c = \cos\theta, s = \sin\theta$.

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta \quad (1)$

Also $(\cos\theta + i\sin\theta)^5$

$= c^5 + 5c^4 si + 10c^3 s^2 i^2 + 10c^2 s^3 i^3 + 5cs^4 i^4 + s^5 i^5$

$= c^5 + 5c^4 si - 10c^3 s^2 - 10c^2 s^3 i + 5cs^4 + s^5 i \quad (2)$

Equating real parts of $(1)$ and $(2)$:

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta\left(1 - \cos^2\theta\right) + 5\cos\theta\left(1 - \cos^2\theta\right)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

**Question 60**

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$= \overrightarrow{OA} + \overrightarrow{BC}$

$= z_1 + (z_3 - z_2)$

$\overrightarrow{DB} = z_2 - \left(z_1 + (z_3 - z_2)\right)$

$= 2z_2 - z_1 - z_3$

**Answer C**

**Question 61**

The cube roots of $-1$ are cis $\pm\frac{\pi}{3}$ and $-1$

A is true, as non-real complex roots of unity occur in conjugate pairs

B is true, as

$\omega^2 + 1 - \omega = 1 + \omega + \omega^2 - 2\omega$

$= 0 - 2\omega$ as $1 + \omega + \omega^2 = 0$

$\neq 0$ as $\omega$ is non−real

C is true, as when $|z| = 1\ \bar{z} = \frac{1}{z}$, so $\frac{1}{\omega} = \overline{w}$, and $\omega$ and $\bar{\omega}$ are roots

D is false, as

$(\omega - 1)^3 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - 1\right)^3$

$= \left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3$

$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$

$= \text{cis}\ 2\pi$

$= 1$

**Answer D**

**Question 62a**

Let $(a + ib)^2 = 16 - 30i$

$\therefore a^2 - b^2 = 16$

$2ab = -30$

by inspection $a = 5, b = -3$ or $a = -5, b = 3$

The square roots are $\pm(5 - 3i)$

Alternatively:

$a = \pm\sqrt{\frac{|z| + \text{Re}(z)}{2}}$

$= \pm\sqrt{\frac{\sqrt{(16)^2 + (-30)^2} + 16}{2}}$

$= \pm 5$

$b = \frac{\text{Im}(z)}{2a}$

$= \frac{-30}{2(\pm 5)}$

$= \mp 3$

The square roots are $\pm(5 - 3i)$

**Question 62b**

$z = \frac{2 \pm \sqrt{(-2)^2 + 4(1)(15 - 30i)}}{2(1)}$

$= \frac{2 \pm \sqrt{64 - 120i}}{2}$

$= 1 \pm \sqrt{16 - 30i}$

$= 1 \pm (5 - 3i)$

$= 6 - 3i, -4 + 3i$

**Question 63**

$z + \bar{z} = 2\text{Re}(z)$

$= 2\cos\theta$

$\therefore \cos\theta = \frac{z + \bar{z}}{2}$

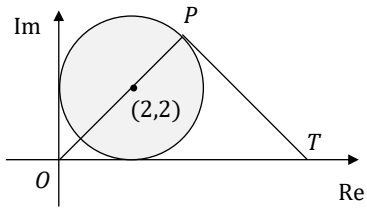
$z - \bar{z} = 2i\text{Im}(z)$

$= 2i\sin\theta$

$\therefore \sin\theta = \frac{z - \bar{z}}{2i}$

**Question 64a**

On and inside the circle centred at $(2,2)$ with radius 2.




**Question 64b**

The maximum modulus occurs on the far side of the circle on a line through the centre.

$\left|\overrightarrow{OP}\right| = \sqrt{(2)^2 + (2)^2} + 2 = 2\sqrt{2} + 2$

**Question 64c**

$\Delta OPT$ is right angled isosceles, so

$$\begin{aligned}\text{Area} &= \frac{1}{2}\times OP\times PT \\ &= \frac{1}{2}(OP)^2 \\ &= \frac{1}{2}\left(2\sqrt{2}+2\right)^2 \\ &= \frac{1}{2}\left(8+8\sqrt{2}+4\right) \\ &= 6+4\sqrt{2}\ \text{units}^2\end{aligned}$$

**Question 65**

$$\begin{aligned}\text{A: } \arg(z_1+z_2) &= \arg\left(1-\left(2+\sqrt{3}\right)i\right) \\ &= \tan^{-1}\left(-\left(2+\sqrt{3}\right)\right) \\ &= -1.309 \\ &= -\frac{5\pi}{12}\end{aligned}$$

**Answer A**

**Question 66a**

$$\begin{aligned}z+1 &= (\cos\theta+1)+i\sin\theta \\ &= \left(2\cos^2\frac{\theta}{2}-1+1\right)+i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \\ &= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right) \\ z-1 &= (\cos\theta-1)+i\sin\theta \\ &= \left(1-2\sin^2\frac{\theta}{2}-1\right)+i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \\ &= -2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)\end{aligned}$$

Alternatively:

$$\begin{aligned}z+1 &= \text{cis}\,\theta+\text{cis}\,0 \\ &= \text{cis}\frac{\theta}{2}\left(\text{cis}\frac{\theta}{2}+\text{cis}\left(-\frac{\theta}{2}\right)\right) \\ &= \text{cis}\frac{\theta}{2}\left(2\text{Re}\left(\text{cis}\frac{\theta}{2}\right)\right) \\ &= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right) \\ z-1\& &= \text{cis}\,\theta-\text{cis}\,0 \\ &= \text{cis}\frac{\theta}{2}\left(\text{cis}\frac{\theta}{2}-\text{cis}\left(-\frac{\theta}{2}\right)\right) \\ &= \text{cis}\frac{\theta}{2}\left(2i\text{Im}\left(\text{cis}\frac{\theta}{2}\right)\right) \\ &= 2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right) \\ &= 2i^2\sin\frac{\theta}{2}\left(\frac{1}{i}\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right) \\ &= -2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)\end{aligned}$$

**Question 66b**

$$\begin{aligned}&\frac{z-1}{z+1} \\ &= \frac{-2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}\times\frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}} \\ &= -\frac{\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}-\sin^2\frac{\theta}{2}i-\cos^2\theta\, i-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)} \\ &= -\frac{\sin\frac{\theta}{2}(-i)}{\cos\frac{\theta}{2}} \\ &= i\tan\frac{\theta}{2}\ \text{which is purely imaginary} \\ &\therefore \text{Re}\left(\frac{z-1}{z+1}\right)=0\end{aligned}$$

**Question 67**

$$\begin{aligned}\sqrt{3}+i &= 2\,\text{cis}\frac{\pi}{6} \\ \therefore\left(\sqrt{3}+i\right)^4 &= \left(2\,\text{cis}\frac{\pi}{6}\right)^4 \\ &= 16\,\text{cis}\frac{2\pi}{3} \\ &= 16\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\ &= -8+8\sqrt{3}i\end{aligned}$$

**Answer D**

**Question 68a**

**Question 68a**

$$\begin{aligned}\frac{1}{\sqrt{2}}+i\left(\frac{1}{\sqrt{2}}\right) &= \text{cis}\frac{\pi}{4} \\ \beta &= \alpha\,\text{cis}\frac{\pi}{3} \\ &= \text{cis}\frac{\pi}{4}\times\text{cis}\frac{\pi}{3} \\ &= \text{cis}\frac{7\pi}{12}\end{aligned}$$

**Question 68b**

$$\begin{aligned}\text{Area} &= \frac{1}{2}\times|\alpha|\times|\beta|\times\sin\angle AOB \\ &= \frac{1}{2}\times1\times1\times\sin\frac{\pi}{3} \\ &= \frac{\sqrt{3}}{4}\ \text{unit}^2\end{aligned}$$

**Question 69**

$$\begin{aligned}\text{Im}\left(\frac{1}{\bar{z}-i}\right) &= \text{Im}\left(\frac{1}{x-iy-i}\times\frac{x+i(y+1)}{x+i(y+1)}\right) \\ &= \text{Im}\left(\frac{x+i(y+1)}{x^2+(y+1)^2}\right) \\ &= \frac{y+1}{x^2+(y+1)^2} \\ \frac{y+1}{x^2+(y+1)^2} &= 1 \\ x^2+y^2+2y+1 &= y+1 \\ x^2+y^2+y+\frac{1}{4} &= \frac{1}{4} \\ x^2+\left(y+\frac{1}{2}\right)^2 &= \frac{1}{4}\end{aligned}$$

This is a circle centred at $\left(0,-\frac{1}{2}\right)$ with radius $\frac{1}{2}$.

**Question 70**

$$x+iy=\frac{1-t^2+2it}{1+t^2}$$

$$x=\frac{1-t^2}{1+t^2}$$

$$y=\frac{2t}{1+t^2}$$

$$\therefore\ x^2+y^2=\frac{(1-t^2)^2+(2t)^2}{(1+t^2)^2}$$

$$=\frac{1-2t^2+t^4+4t^2}{(1+t^2)^2}$$

$$=\frac{(1+t^2)^2}{(1+t^2)^2}$$

$$=1$$

$\therefore z$ lies on the unit circle.

**Question 71**

$$2\bar{z}-iz=2(a-bi)-i(a+bi)$$

$$=2a-2bi-ai+b$$

$$=(2a+b)-i(2b+a)$$

$$\therefore 2a+b=1 \quad (1)$$

$$2b+a=-4 \quad (2)$$

$2\times(1)-(2)$: $3a=6$

$$a=2$$

sub in (1): $2(2)+b=1$

$$b=-3$$

$$\therefore z=2-3i$$

**Question 72**

Let $z=\text{cis}\frac{\pi}{9}$

$$z^6-z^3+1$$

$$=\left(\text{cis}\frac{\pi}{9}\right)^6-\left(\text{cis}\frac{\pi}{9}\right)^3+1$$

$$=\text{cis}\frac{2\pi}{3}-\text{cis}\frac{\pi}{3}+1$$

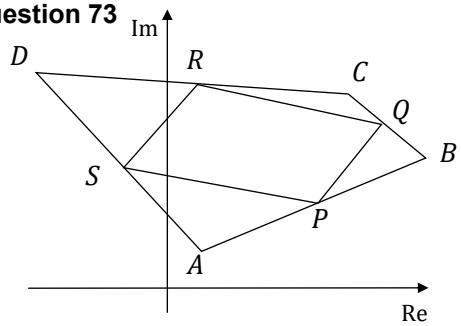
$$=-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}+1$$

$$=-\frac{1}{2}-\frac{1}{2}+1$$

$$=0$$

$\therefore z=\text{cis}\frac{\pi}{9}$ satisfies $z^6-z^3+1=0$.

**Question 73**




$P$ represents $\frac{a+b}{2}$, $Q$ represents $\frac{b+c}{2}$,

$R$ represents $\frac{c+d}{2}$, $S$ represents $\frac{a+d}{2}$.

**Question 73 continued**

$$\therefore \overrightarrow{OM}=\frac{1}{2}\left(\frac{a+b}{2}+\frac{c+d}{2}\right)$$

$$=\frac{1}{4}(a+b+c+d)$$

$$\overrightarrow{ON}=\frac{1}{2}\left(\frac{b+c}{2}+\frac{a+d}{2}\right)$$

$$=\frac{1}{4}(a+b+c+d)$$

Since $\overrightarrow{OM}=\overrightarrow{ON}$ the diagonals bisect each other, so $PQRS$ is a parallelogram.

**Question 74**

$$\left|2+2\sqrt{3}i\right|=\sqrt{(2)^2+\left(2\sqrt{3}\right)^2}$$

$$=4$$

$$\arg\left(2+2\sqrt{3}i\right)=\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$

$$=\frac{\pi}{3}$$

$$\therefore 2+2\sqrt{3}i=4\,\text{cis}\frac{\pi}{3}$$

$$(r\,\text{cis}\,\theta)^4=4\,\text{cis}\frac{\pi}{3}$$

$r^4=4 \quad 4\theta=\left(2k\pi+\frac{\pi}{3}\right) \quad k=-2,-1,0,1$

$r=\sqrt{2} \quad \theta=\frac{(6k+1)\pi}{12}$

$$=-\frac{11\pi}{12},-\frac{5\pi}{12},\frac{\pi}{12},\frac{7\pi}{12}$$

The roots are $\sqrt{2}\,\text{cis}\left(-\frac{11\pi}{12}\right)$, $\sqrt{2}\text{cis}\left(-\frac{5\pi}{12}\right)$, $\sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)$ and $\sqrt{2}\text{cis}\left(\frac{7\pi}{12}\right)$.

**Question 75**

$$(z-\bar{z})^2+18(z+\bar{z})=36$$

$$(2iy)^2+18(2x)=36$$

$$-4y^2+36x=36$$

$$4y^2=36x-36$$

$$y^2=9x-9$$

which is a concave right parabola.

**Question 76**

$$\frac{z}{z^2+a^2}$$

$$=\frac{a(\cos\theta+i\sin\theta)}{[a(\cos\theta+i\sin\theta)]^2+a^2}$$

$$=\frac{a(\cos\theta+i\sin\theta)}{a^2(\cos^2\theta-\sin^2\theta+2i\sin\theta\cos\theta)+a^2}$$

$$=\frac{a(\cos\theta+i\sin\theta)}{a^2(\cos^2\theta-\sin^2\theta+1\ +2i\sin\theta\cos\theta)}$$

$$=\frac{a(\cos\theta+i\sin\theta)}{a^2(2\cos^2\theta+2i\sin\theta\cos\theta)}$$

$$=\frac{a(\cos\theta+i\sin\theta)}{2a^2\cos\theta\,(\cos\theta+i\sin\theta)}$$

$$=\frac{1}{2a\cos\theta}$$

**Question 77**

Let $S = 1 + \omega^k + \omega^{2k}$

For $k = 3n \quad n \in \mathbb{Z}^+$

$S = 1 + \omega^{3n} + \omega^{6n}$

$= 1 + (\omega^3)^n + (\omega^3)^{2n}$

$= 1 + 1^n + 1^{2n}$

$= 1 + 1 + 1$

$= 3$

For $k = 3n + 1$

$S = 1 + \omega^{3n+1} + \omega^{6n+2}$

$= 1 + (\omega^3)^n \omega + (\omega^3)^{2n} \omega^2$

$= 1 + \omega + \omega^2$

$= 0 \quad$ since $\omega \neq 1$

For $k = 3n + 2$

$S = 1 + \omega^{3n+2} + \omega^{6n+4}$

$= 1 + (\omega^3)^n \omega^2 + (\omega^3)^{2n+1} \omega$

$= 1 + \omega^2 + \omega$

$= 0$

$\therefore 1 + \omega^k + \omega^{2k} = 0$ or 3.

**Question 78**

$$\frac{a}{1+i} + \frac{b}{1+2i} = 1$$

$$(1+2i)a + (1+i)b = (1+i)(1+2i)$$

$$a + 2ai + b + bi = 1 + 2i + i - 2$$

$$(a+b) + i(2a+b) = -1 + 3i$$

$$\therefore a + b = -1$$

$$2a + b = 3$$

$a = 4, b = -5$ by inspection.

**Question 79a**

$$z^5 = 1$$

$$z^5 - 1 = 0$$

$$\therefore (z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$

$\therefore z^4 + z^3 + z^2 + z + 1 = 0 \quad$ since $z \neq 1$

dividing by $z^2$:

$$z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$$

**Question 79b**

$$\left(z^2 + 2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) - 1 = 0$$

$$x^2 + x - 1 = 0$$

**Question 79c**

Now $\text{cis}\frac{2\pi}{5}$ and $\text{cis}\frac{4\pi}{5}$ are solutions of $z^5 = 1$, so $\text{cis}\frac{2\pi}{5} + \frac{1}{\text{cis}\frac{2\pi}{5}}$ and $\text{cis}\frac{4\pi}{5} + \frac{1}{\text{cis}\frac{4\pi}{5}}$ are solutions to the quadratic in part (b).

$$\left(\text{cis}\frac{2\pi}{5} + \frac{1}{\text{cis}\frac{2\pi}{5}}\right)\left(\text{cis}\frac{4\pi}{5} + \frac{1}{\text{cis}\frac{4\pi}{5}}\right) = \frac{e}{a}$$

$$\left(\text{cis}\frac{2\pi}{5} + \overline{\text{cis}\frac{2\pi}{5}}\right)\left(\text{cis}\frac{4\pi}{5} + \overline{\text{cis}\frac{4\pi}{5}}\right) = \frac{-1}{1}$$

$$\left(2\text{Re}\left(\text{cis}\frac{2\pi}{5}\right)\right)\left(2\text{Re}\left(\text{cis}\frac{4\pi}{5}\right)\right) = \frac{-1}{1}$$

$$\left(2\cos\frac{2\pi}{5}\right)\left(2\cos\frac{4\pi}{5}\right) = -1$$

$$\therefore \cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -\frac{1}{4}$$

**Question 80a**

$\angle AOB = \angle BOC = \angle COD = \frac{\pi}{4}$ (angle in a right-angled isosceles triangle)

$$\frac{|OA|}{|OB|} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore |OB| = \sqrt{2}|OA|$$

$$\arg\beta = \arg\alpha + \frac{\pi}{4}$$

$$\beta = |OB|\text{cis}\left(\arg\alpha + \frac{\pi}{4}\right)$$

$$= \sqrt{2}\text{cis}\frac{\pi}{4}(|OA|\text{cis}\,\alpha)$$

$$= \sqrt{2}\text{cis}\frac{\pi}{4}\alpha$$

**Question 80b**

Similarly $\overrightarrow{OD} = \sqrt{2}\,\text{cis}\frac{\pi}{4}\overrightarrow{OC}; \overrightarrow{OC} = \sqrt{2}\,\text{cis}\frac{\pi}{4}\overrightarrow{OB}$

$$\therefore \overrightarrow{OD} = \left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)\left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)\left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)\alpha$$

$$= 2\sqrt{2}\,\text{cis}\frac{3\pi}{4}\,\alpha$$

$$= -2\sqrt{2}\,\frac{\text{cis}\pi}{4}\,i\alpha$$

**Question 80c**

$64\alpha^4 + \beta^4$

$= 64\alpha^4 + \left(2\sqrt{2}\,\text{cis}\frac{\pi}{4}\,i\alpha\right)^4$

$= 64\alpha^4 + 64\,\text{cis}\,\pi(i^4)\alpha^4$

$= 64\alpha^4 - 64\alpha^4$

$= 0$

**Question 81a**

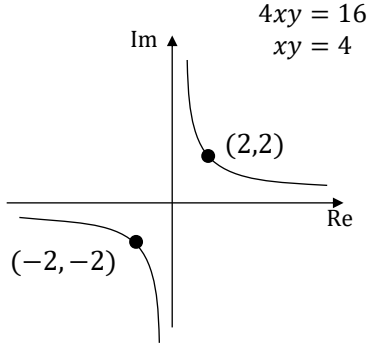
$$z^2 - (\bar{z})^2 = 16i$$

$$(x+iy)^2 - (x-iy)^2 = 16i$$

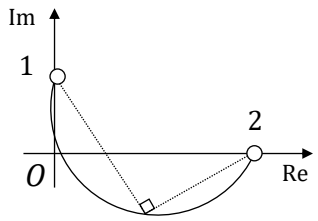
$$x^2 + 2xyi - y^2 - x^2 + 2xyi + y^2 = 16i$$

$$4xy = 16$$

$$xy = 4$$



**Question 81b**




**Question 82a**

$$\left|\frac{z}{\bar{z}}\right| = \frac{|z|}{|\bar{z}|}$$
$$= 1$$
$$\arg\left(\frac{z}{\bar{z}}\right) = \arg z - \arg \bar{z}$$
$$= \arg z - (-\arg z)$$
$$= 2\arg z$$

**Question 82b**

$$\frac{z}{\bar{z}} = i$$
$$\therefore 2\arg z = 2k\pi + \frac{\pi}{2} \text{ for } k = 0, 1$$
$$\arg z = k\pi + \frac{\pi}{4}$$
$$= \frac{\pi}{4}, -\frac{3\pi}{4}$$
$$x = y \text{ for } \arg z = \frac{\pi}{4}, -\frac{3\pi}{4}$$
$$\therefore z = (1+i)r$$
$$= r + ir \text{ for real } r, r \neq 0$$

**Question 83a**

$$\cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$$
since cosine is even and sine is odd
$$= 2\cos n\theta$$

**Question 83b**

$$\cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta)$$
$$- i\sin(n\theta) \text{ since cosine is even and sine is odd}$$
$$= 2\cos n\theta$$

**Question 83c**

$$(z + z^{-1})^3 = \left(z^3 + z^{-3}\right) + 3(z + z^{-1})$$
$$(2\cos\theta)^3 = (2\cos 3\theta) + 3(2\cos\theta) \text{ from (a)}$$
$$8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$$
$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

**Question 84a**

$$wz = \frac{3+4i}{5} \times \frac{5+12i}{13}$$
$$= \frac{15 + 36i + 20i - 48}{65}$$
$$= \frac{-33}{65} + \frac{56i}{65}$$

**Question 84b**

Since $|w| = |z| = 1$, $|wz|^2 = |w\bar{z}|^2 = |\overline{w}z|^2 = |\overline{w}\bar{z}|^2 = 1$.

Choosing say the first two:

$$|wz| = \left(-\frac{33}{65}\right)^2 + \left(\frac{56}{65}\right)^2$$
$$= 1$$
$$\therefore 33^2 + 56^2 = 65^2$$
$$w\bar{z} = \frac{3+4i}{5} \times \frac{5-12i}{13}$$
$$= \frac{15 - 36i + 20i + 48}{65}$$
$$= \frac{63}{65} - \frac{16i}{65}$$
$$\therefore 63^2 + 16^2 = 65^2$$

**Question 85**

$$\left|\left(1+\sqrt{3}i\right)z^2 + 2iz\right|$$
$$\leq \left|\left(1+\sqrt{3}i\right)z^2\right| + |2iz|$$
(by the triangle inequality)
$$= \left|1+\sqrt{3}i\right||z|^2 + 2|z|$$
$$< \sqrt{1^2 + \left(\sqrt{3}\right)^2} \times \left(\sqrt{3}\right)^2 + 2\left(\sqrt{3}\right) \text{ since } |z| < \sqrt{3}$$
$$= 6 + 2\sqrt{3}$$
$$< 6 + 2(2)$$
$$= 10$$

**Question 86a**

Produce $AZ$ to $N$, $BZ$ to $M$, and construct $ZP$ parallel ot the $x$-axis.

$\angle NZP = \text{Arg}(Z - 2i)$ (corresponding angles to parallel lines)

$\angle MZP = \text{Arg}(Z - 2\sqrt{3})$ (corresponding angles to parallel lines)

$\therefore \angle MZN = \text{Arg}(Z - 2\sqrt{3}) - \text{Arg}(Z - 2i)$

$\angle MZN = \angle AZB = \alpha$ (vertically opposite)

$\therefore \alpha = \frac{\pi}{3}$

**Question 86b**

The centre of the circle lies on the perpendicular bisector of $AB$. Let the centre of the circle be $C$. The midpoint of $AB$ is $(\sqrt{3}, 1)$, and interval $AB$ has gradient $-\frac{1}{\sqrt{3}}$, so the perpendicular bisector has gradient $\sqrt{3}$, so equation $y - 1 = \sqrt{3}(x - \sqrt{3})$.

$\angle ACB = 2\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}$ (angle at the centre is twice the angle at the circumference)

$\therefore \angle ACM = \frac{\pi}{3}$

$\therefore \angle CAM = \frac{\pi}{6}$ (angle sum $\Delta AOM$)

$\therefore \angle CAM = \angle ABO = \frac{\pi}{6}$

$\therefore AC || OB$ $\therefore$ the $y$-value of $C$ is 2. Sub in the equation:

$$2 - 1 = \sqrt{3}\left(x - \sqrt{3}\right) \quad \therefore x = \frac{4}{\sqrt{3}}$$

The centre is $\left(\frac{4}{\sqrt{3}}, 2\right)$ and the radius is $\frac{4}{\sqrt{3}}$ (since $AC$ is horizontal it equals the radius).

**Question 87a**

$$z^5 + 1 = 0$$
$$z^5 = -1$$
$$(r\,\text{cis}\theta)^5 = -1$$
$$\therefore\ r^5\text{cis}5\theta = \text{cis}\left((2k+1)\pi\right) \text{ for } k = -2, \ldots, 2$$
$$r^5 = 1 \qquad 5\theta = (2k+1)\pi$$
$$r = 1 \qquad 5\theta = -3\pi, -\pi, \pi, 3\pi, 5\pi$$
$$\theta = \pm\frac{3\pi}{5}, \pm\frac{\pi}{5}, \pi$$
$$z = \text{cis}\left(\pm\frac{3\pi}{5}\right), \text{cis}\left(\pm\frac{\pi}{5}\right), -1$$

**Question 87b**

$$z^5 + 1 = 0$$
$$(z+1)\left(z^4 - z^3 + z^2 - z + 1\right) = 0$$
$$\therefore z = -1 \quad \text{or} \quad z^4 - z^3 + z^2 - z + 1 = 0$$

Since $z = -1$ is real, the non-real solutions of $z^5 + 1 = 0$ must be the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$.

**Question 87c**

$$z^4 - z^3 + z^2 - z + 1 = 0$$
$$\therefore z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0$$
$$z^2 + z^{-2} - (z + z^{-1}) + 1 = 0$$
$$2\cos 2\theta - 2\cos\theta + 1 = 0$$
$$2\left(2\cos^2\theta - 1\right) - 2\cos\theta + 1 = 0$$
$$4\cos^2\theta - 2\cos\theta - 1 = 0$$

**Question 87d**

$z = \text{cis}\left(\frac{3\pi}{5}\right)$ is solution of $z^5 + 1 = 0$ from (i)

$$\therefore\ \cos\left(\frac{3\pi}{5}\right) = \frac{2 \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)}$$
$$= \frac{2 \pm 2\sqrt{5}}{8}$$
$$= \frac{1 - \sqrt{5}}{4} \qquad \text{since} \cos\left(\frac{3\pi}{5}\right) < 0$$
$$\therefore \sec\left(\frac{3\pi}{5}\right) = \frac{4}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}}$$
$$= \frac{4 + 4\sqrt{5}}{1 - 5}$$
$$= -1 - \sqrt{5}$$

**Question 88a**

Let $c = \cos\theta, s = \sin\theta$.

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta \qquad (1)$$

Also

$$(\cos\theta + i\sin\theta)^5$$
$$= c^5 + 5c^4si + 10c^3s^2i^2 + 10c^2s^3i^3 + 5cs^4i^4 + s^5i^5$$
$$= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i \quad (2)$$

Equating the real parts of (1) and (2) and the imaginary parts:

$$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\sin 5\theta = 5\cos^4\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$\therefore \tan 5\theta$$
$$= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} \div \frac{\cos^5\theta}{\cos^5\theta}$$
$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

**Question 88b**

$\tan 5\theta = 0$ occurs when $\sin 5\theta = 0$

$$\therefore 5\theta = k\pi$$
$$\theta = \frac{k\pi}{5}$$
$$= 0, \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}$$

also $5\tan\theta - 10\tan^3\theta + \tan^5\theta = 0$

$$\tan\theta\left(5 - 10\tan^2\theta + \tan^4\theta\right) = 0$$

Let $x = \tan\theta$

$$\therefore x\left(x^4 - 10x^2 + 5\right) = 0$$
$$\therefore x = 0 \text{ or } x^4 - 10x^2 + 5 = 0$$

$x = \tan 0$ is the solution to $x = 0$, so the other solutions to $\tan 5\theta = 0$ must be the solutions to $x^4 - 10x^2 + 5 = 0$, so its solutions are

$$x = \tan\left(\pm\frac{\pi}{5}\right), \tan\left(\pm\frac{2\pi}{5}\right)$$
$$= \pm\tan\frac{\pi}{5}, \pm\tan\frac{2\pi}{5} \text{ since tangent is odd}$$

**Question 88c**

$$\prod\alpha = \frac{e}{a}$$
$$\therefore\ \tan\frac{\pi}{5}\tan\frac{2\pi}{5}\left(-\tan\frac{2\pi}{5}\right)\left(-\tan\frac{\pi}{5}\right) = \frac{5}{1}$$
$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$$

**Question 89a**

$$(r\,\text{cis}\,\theta)^5 = 1$$
$$r^5\text{cis}\,5\theta = \text{cis}\,2k\pi$$
$$\therefore r = 1 \qquad 5\theta = 2k\pi \qquad k = -2, -1, 0, 1, 2$$
$$\theta = \frac{2k\pi}{5}$$
$$= 0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}$$

The roots of $z^5 - 1$ $= 0$ are $\text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right), 1$.

**Question 89b**

The sum of the roots is zero, so:

$$1 + \text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right) = 0$$
$$1 + 2\text{Re}\left(\text{cis}\left(\frac{2\pi}{5}\right)\right) + 2\text{Re}\left(\text{cis}\left(\frac{4\pi}{5}\right)\right) = 0$$
$$2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = -1$$
$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$$

**Question 90**

$$32\sin^4\theta\cos^2\theta$$
$$= 32\sin^4\theta - 32\sin^6\theta$$
$$= 32\left(\frac{z - z^{-1}}{2i}\right)^4 - 32\left(\frac{z - z^{-1}}{2i}\right)^6$$
$$= 2\left(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}\right)$$
$$+\frac{1}{2}\left(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}\right)$$
$$= \frac{1}{2}\left(z^6 + z^{-6}\right) - \left(z^4 + z^{-4}\right) - \frac{1}{2}\left(z^2 + z^{-2}\right) + 2$$
$$= \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$

**Question 91a**

Roots are separated by $\frac{2\pi}{5}$

$\omega_1 = \text{cis}\left(-\frac{3\pi}{5}\right), \omega_2 = \text{cis}\left(-\frac{\pi}{5}\right),$

$\omega_3 = \text{cis}\left(\frac{\pi}{5}\right), \omega_4 = \text{cis}\left(\frac{3\pi}{5}\right)$

$\overline{\omega_4} = \overline{\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}}$

$= \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

$= \cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)$

$= \text{cis}\left(-\frac{3\pi}{5}\right)$

$= \omega_1$

**Question 91b**

$z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$

Since $\omega \neq 1$, $\omega$ solves $z^4 - z^3 + z^2 - z + 1 = 0$

$\therefore \omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$

$\therefore \omega^4 + \omega^2 + 1 = \omega^3 + \omega$

**Question 91c**

$\omega_1^3 = \left(\text{cis}\left(-\frac{3\pi}{5}\right)\right)^3$

$= \text{cis}\left(-\frac{9\pi}{5}\right)$

$= \text{cis}\left(\frac{\pi}{5}\right)$

$= \omega_3$

**Question 91d**

$\omega_1^3 + \omega_2^3 + \omega_3^3 + \omega_4^3$

$= \omega_3 + \omega_1 + \omega_4 + \omega_2$

$= \sum\alpha - (-1)$

$= 0 - (-1)$

$= 1$

**Question 91e**

$\omega^4 = \omega^3 - \omega^2 + \omega - 1$ (from b)

$\therefore \omega_1^4 + \omega_2^4 + \omega_3^4 + \omega_4^4 = \omega_1^3 + \omega_2^3 + \omega_3^3 + \omega_4^3 - \omega_1^2 - \omega_2^2 - \omega_3^2 - \omega_4^2 + \omega_1 + \omega_2 + \omega_3 + \omega_4 - 4$

$\left(\text{cis}\left(-\frac{3\pi}{5}\right)\right)^4 + \left(\text{cis}\left(-\frac{\pi}{5}\right)\right)^4 + \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^4 + \left(\text{cis}\left(\frac{3\pi}{5}\right)\right)^4$

$= \omega_3 + \omega_1 + \omega_4 + w_2 - \text{cis}\left(\frac{4\pi}{5}\right) - \text{cis}\left(-\frac{2\pi}{5}\right) - \text{cis}\left(\frac{2\pi}{5}\right) - \text{cis}\left(-\frac{4\pi}{5}\right) + \omega_1 + \omega_2 + \omega_3 + \omega_4 - 4$

$2\left(\text{cis}\left(-\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(\frac{2\pi}{5}\right)\right) = 2(\omega_1 + \omega_2 + \omega_3 + \omega_4) - 4$

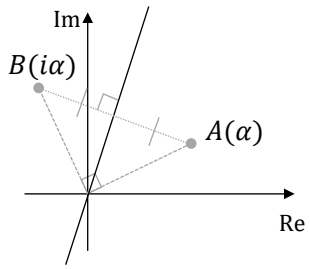
$\text{cis}\left(-\frac{4\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) = (1) - 2$

$2\cos\frac{4\pi}{5} + 2\cos\frac{2\pi}{5} = -1$

$\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = -\frac{1}{2}$

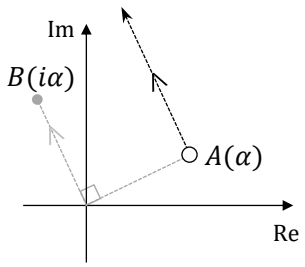
**Question 92a**

The perpendicular bisector of $AB$

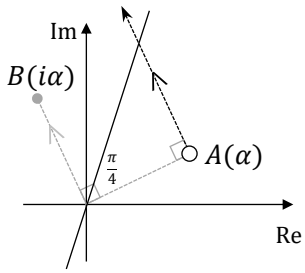



**Question 92b**

A ray from $A$ parallel to $\overrightarrow{OB}$




**Question 92c**




The point forms a right angled isosceles triangle with $O$ and $A$, so is $\sqrt{2}\text{cis}\frac{\pi}{4}\alpha$ or $(1+i)\alpha$.

**Question 93a**

$(1+ic)^5$

$= 1 + 5ic + 10i^2c^2 + 10i^3c^3 + 5i^4c^4 + i^5c^5$

$= 1 + 5ic - 10c^2 - 10c^3i + 5c^4 + c^5i$

**Question 93b**

$\text{Im}\left((1+ic)^5\right) = 0$ since $(1+ic)^5$ is real,

$\therefore 5c - 10c^3 + c^5 = 0$

$c(c^4 - 10c^2 + 5) = 0$

$\therefore c^4 - 10c^2 + 5 = 0$ since $c \neq 0$

**Question 93c**

$$c^2 = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{10 \pm 4\sqrt{5}}{2}$$

$$= 5 \pm 2\sqrt{5}$$

$$\therefore c = \sqrt{5-2\sqrt{5}}, -\sqrt{5-2\sqrt{5}}, \sqrt{5+2\sqrt{5}}, -\sqrt{5+2\sqrt{5}}$$

**Question 93d**

$1 + ic = r\,\text{cis}\,\theta$

$\therefore (1+ic)^5 = r^5\text{cis}\,5\theta$

since $(1+ic)^5$ is real, $\sin 5\theta = 0$

$\therefore 5\theta = k\pi \quad k \in \mathbb{Z}$

$\therefore \theta = \frac{k\pi}{5}$

The smallest positive value of $\theta$ is $\frac{\pi}{5}$

**Question 93e**

For the smallest possible value of $c$

$$1 + ic = r\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$$

$$\therefore \cos\frac{\pi}{5} = \frac{1}{r} \qquad \sin\frac{\pi}{5} = \frac{c}{r}$$

$$\therefore \tan\frac{\pi}{5} = \frac{\frac{c}{r}}{\frac{1}{r}}$$

$$= c$$

$$= \sqrt{5-2\sqrt{5}}$$

since the smallest value of $c$ corresponds to the smallest possible value of $\theta$.

**Question 94a**

$z^n + z^{-n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

**Question 94b**

The roots of $x^5 - 1 = 0$ are $1, \text{cis}\left(\pm\frac{2\pi}{5}\right)$, $\text{cis}\left(\pm\frac{4\pi}{5}\right)$.

$\therefore x^5 - 1$

$= (x-1)\left(x - \text{cis}\frac{2\pi}{5}\right)\left(x - \text{cis}\left(-\frac{2\pi}{5}\right)\right)\left(x - \text{cis}\frac{4\pi}{5}\right)\left(x - \text{cis}\left(-\frac{4\pi}{5}\right)\right)$

$= (x-1)\left(x^2 - 2\cos\frac{2\pi}{5}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5}x + 1\right)$

**Question 94c**

now $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$

$\left(x^2 - 2\cos\frac{2\pi}{5}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5}x + 1\right) = x^4 + x^3 + x^2 + x + 1$

Let $x = 1$

$$\left(1 - 2\cos\frac{2\pi}{5} + 1\right)\left(1 - 2\cos\frac{4\pi}{5} + 1\right) = 5$$

$$4\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = 5$$

$$\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = \frac{5}{4}$$

**Question 95a**

$$v = \text{cis}\frac{\pi}{6}$$

$$w = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$$

$$wv = 2\,\text{cis}\left(\frac{\pi}{6} - \frac{2\pi}{3}\right)$$

$$= 2\,\text{cis}\left(-\frac{\pi}{2}\right)$$

$$= -2i$$

$$iv = \text{cis}\frac{\pi}{2} \times \text{cis}\frac{\pi}{6}$$

$$= \text{cis}\frac{2\pi}{3}$$

**Question 95b**

sub $v = \text{cis}\,\frac{\pi}{6}$ into $Z^4 = iZ$

$\left(\text{cis}\,\frac{\pi}{6}\right)^4 = i\,\text{cis}\,\frac{\pi}{6}$

$\text{cis}\,\frac{2\pi}{3} = \text{cis}\,\frac{\pi}{2} \times \text{cis}\,\frac{\pi}{6}$

$\text{cis}\,\frac{2\pi}{3} = \text{cis}\,\frac{2\pi}{3}$

$\therefore v$ satisfies $Z^4 = iZ$

$\therefore Z(Z^3 - i) = 0$

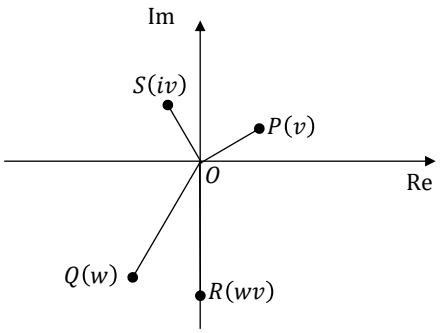
$\therefore Z = 0 \quad \text{or } Z = \sqrt[3]{i} = \text{cis}\left(\frac{2k\pi + \frac{\pi}{2}}{3}\right)$

$= \text{cis}\,\frac{(4k+1)\pi}{6}$

$= \text{cis}\left(-\frac{\pi}{2}\right), \text{cis}\,\frac{\pi}{6}, \text{cis}\left(\frac{5\pi}{6}\right)$

The other two non-zero solutions are $\text{cis}\left(-\frac{\pi}{2}\right) = -i$ and $\text{cis}\left(\frac{5\pi}{6}\right)$.

**Question 95c**




**Question 95d**

$OQR$ is isosceles, $\angle QOR = \frac{\pi}{6}$

$\angle QRO = \frac{1}{2}\left(\pi - \frac{\pi}{6}\right) = \frac{5\pi}{12}$,

$\therefore \arg RQ = \frac{\pi}{2} + \frac{5\pi}{12} = \frac{11\pi}{12}$.

$\overrightarrow{PS} = iv - v$

$= (-1+i)v$

$= \sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)\text{cis}\,\frac{\pi}{6}$

$= \sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right)$

$\therefore PS || RQ$ since $\arg\overrightarrow{RQ} = \arg\overrightarrow{PS} = \frac{11\pi}{12}$

**Question 95e**

Let $iv - v = u(w - wv)$

$u = \frac{|iv - v|}{|w - wv|}$

$= \frac{SP}{QR}$

$SP = \sqrt{OP^2 + OS^2}$

$= \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

$RQ = \sqrt{OR^2 + OQ^2 - 2\times OR \times OQ \times \cos\frac{\pi}{6}}$

$= \sqrt{2^2 + 2^2 - 2\times 2\times 2\cos\frac{\pi}{6}}$

$= \sqrt{8 - \frac{8\sqrt{3}}{2}}$

$= \sqrt{8 - 4\sqrt{3}}$

$u = \frac{\sqrt{2}}{\sqrt{8-4\sqrt{3}}} \times \frac{\sqrt{8+4\sqrt{3}}}{\sqrt{8+4\sqrt{3}}}$

$= \frac{\sqrt{2}\sqrt{8+4\sqrt{3}}}{\sqrt{64-48}}$

$= \sqrt{\frac{16+8\sqrt{3}}{16}}$

$= \frac{\sqrt{4+2\sqrt{3}}}{2}$

$= \frac{\sqrt{\left(1+\sqrt{3}\right)^2}}{2}$

$= \frac{1+\sqrt{3}}{2}$

**Question 96a**

$(1 + i\tan\theta)^n + (1 - i\tan\theta)^n$

$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$

$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos(-\theta) + i\sin(-\theta)}{\cos\theta}\right)^n$

$= \frac{\cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)}{\cos^n\theta}$

$= \frac{\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta}{\cos^n\theta}$

$= \frac{2\cos n\theta}{\cos^n\theta}$

**Question 96b**

**ii** $(1+z)^4+(1-z)^4=0$

Let $z=i\tan\theta$ since $z$ is purely imaginary

$(1+i\tan\theta)^4+(1-i\tan\theta)^4=0$

$$\frac{2\cos 4\theta}{\cos^4\theta}=0 \qquad \text{from (i)}$$

$$\therefore \cos 4\theta=0$$

$$4\theta=k\pi+\frac{\pi}{2} \quad \text{for } k=-2,-1,0,1$$

$$\theta=\frac{(2k+1)\pi}{8}$$

$$=\pm\frac{\pi}{8},\pm\frac{3\pi}{8}$$

$$\therefore z=i\tan\left(\pm\frac{\pi}{8}\right),i\tan\left(\pm\frac{3\pi}{8}\right)$$

**Question 97a**

$(w^n)^9-1$

$=\left(w^9\right)^n-1$

$=\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)^n-1$

$=\cos 2\pi+i\sin 2\pi-1$

$=0$

**Question 97b**

$w+w^8$

$=\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}+\cos\frac{16\pi}{9}+i\sin\frac{16\pi}{9}$

$=\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}+\cos\frac{2\pi}{9}-i\sin\frac{2\pi}{9}$

$=2\cos\frac{2\pi}{9}$

**Question 97c**

$(w^3+w^6)(w^2+w^7)$

$=w^5+w^{10}+w^8+w^{13}$

$=w^5+w+w^8+w^4$

**Question 97d**

the roots of $z^9-1=0$ are

$w^k=\text{cis}\left(\frac{2k\pi}{9}\right) \quad k=-4,-3,\ldots 4$

$$w^2+w^7=2\cos\frac{4\pi}{9}$$

$$w^3+w^6=2\cos\frac{6\pi}{9}$$

, and $w^4+w^5=2\cos\frac{8\pi}{9}$ (similarly to (ii))

$$\therefore 2\cos\frac{6\pi}{9}\times 2\cos\frac{4\pi}{9}=2\cos\frac{8\pi}{9}+2\cos\frac{2\pi}{9}$$

$$\therefore 2\cos\left(-\frac{\pi}{3}\right)\times 2\cos\frac{4\pi}{9}=-2\cos\frac{\pi}{9}+2\cos\frac{2\pi}{9}$$

$$-2\cos\frac{4\pi}{9}=-2\cos\frac{\pi}{9}+2\cos\frac{2\pi}{9}$$

$$\therefore \cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}=\cos\frac{\pi}{9}$$

**Question 98**

Let $w=x+iy$

$$\therefore (1-i)w=(1-i)(x+iy)$$

$$=x+y+i(y-x)$$

$$y=x \text{ since } (1-i)w \text{ is real}$$

$$\therefore w=x+ix$$

$$=(1+i)x \quad \left\{=\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)x\right\}$$

$$w^2=2ix^2$$

$$w^3=(2i-2)x^3$$

$$\therefore (2i-2)x^3+i\left(2ix^2\right)+ik\left((1+i)x\right)+2i=0$$

$$\therefore 2x^3+2x^2+kx=0 \qquad (1)$$

$$2x^3+kx+2=0 \qquad (2)$$

$$(1)-(2): \quad 2x^2-2=0$$

$$\therefore x=1 \quad \text{or} \quad x=-1$$

sub in (1): $k=-4$ $\quad k=0$

$\therefore k=-4$ or $0$

**Question 99a**

$z^n+z^{-n}=z^n+\overline{z^n}$ since $|z|=1$

$$=2\text{Re}(z^n)$$

$$=2\text{Re}(\cos(n\theta)+i\sin(n\theta))$$

$$=2\cos(n\theta)$$

Similarly $z^n-z^{-n}=2\text{Im}(z^n)=2i\sin(n\theta)$

**Question 99b**

$2^5\sin^4\theta\cos^2\theta$

$$=32\left(\frac{z-z^{-1}}{2i}\right)^4\left(\frac{z+z^{-1}}{2}\right)^2$$

$$=32\left(\frac{z^4-4z^2+6-4z^{-2}+z^{-4}}{16i^4}\right)\left(\frac{z^2+2+z^{-2}}{4}\right)$$

$$=\frac{1}{2}\left(z^6+2z^4+z^2-4z^4-8z^2-4+6z^2+12\right.$$

$$\left.+6z^{-2}-4-8z^{-2}-4z^{-4}+z^{-2}+2z^{-4}+z^{-6}\right)$$

$$=\frac{1}{2}\left(\left(z^6+z^{-6}\right)-2\left(z^4+z^{-4}\right)-\left(z^2+z^{-2}\right)+4\right)$$

$$=\frac{1}{2}(2\cos 6\theta-2(2\cos 4\theta)-(2\cos 2\theta)+4)$$

$$=\cos 6\theta-2\cos 4\theta-\cos 2\theta+2$$

**Question 100a**

$$\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}$$
$$=\frac{1+\dfrac{1-t^2}{1+t^2}+i\dfrac{2t}{1+t^2}}{1-\dfrac{1-t^2}{1+t^2}-i\dfrac{2t}{1+t^2}}$$
$$=\frac{1+t^2+1-t^2+2ti}{1+t^2-1+t^2-2ti}$$
$$=\frac{2+2ti}{2t^2-2ti}$$
$$=\frac{1+ti}{t(t-i)}\times\frac{t+i}{t+i}$$
$$=\frac{t+i+t^2i-t}{t(t^2+1)}$$
$$=\frac{i\left(1+t^2\right)}{t(1+t^2)}$$
$$=\frac{i}{t}$$
$$=i\cot\frac{\theta}{2}$$

**Question 100b**

$$\left(\frac{z-1}{z+1}\right)^8=-1$$
$$\therefore\frac{z-1}{z+1}=\sqrt[8]{-1}$$
$$=\operatorname{cis}\frac{(2k+1)\pi}{8}\quad k=0,1,\ldots,7$$
$$\therefore z-1=z\operatorname{cis}\frac{(2k+1)\pi}{8}+\operatorname{cis}\frac{(2k+1)\pi}{8}$$
$$z\left(1-\operatorname{cis}\frac{(2k+1)\pi}{8}\right)=\left(1+\operatorname{cis}\frac{(2k+1)\pi}{8}\right)$$
$$z=\frac{1+\operatorname{cis}\dfrac{(2k+1)\pi}{8}}{1-\operatorname{cis}\dfrac{(2k+1)\pi}{8}}$$
$$=i\cot\frac{(2k+1)\pi}{16}\ \text{from (i)}$$
$$=i\cot\left(\frac{\pi}{16}\right),i\cot\left(\frac{3\pi}{16}\right),\ldots i\cot\left(\frac{15\pi}{16}\right)$$

**Question 1a**
'$n$ is not a multiple of $3$'
**Question 1b**
'$n$ is greater than $20$'
**Question 1c**
'$n$ is not a multiple of $3$ or is greater than $20$'
**Question 1d**
'$n$ is not a multiple of $3$ and is greater than $20$'
**Question 2**
$P$ means that the triangle is isosceles. The complement is that the triangle is scalene or equilateral.
**Question 3**
'if $x^2$ is less than $4$ then $x$ is less than $2$'
**Question 4**
We can list the values of $n$ that satisfy $P$ and $Q$ to help us understand the implication.
$P$: $\{3, 6, 9, 12, \ldots\}$ $Q$: $\{9, 18, 27, 36, \ldots\}$
Since all elements of $Q$ are also elements of $P$ we can say that $Q$ implies $P$, $Q \Rightarrow P$, $P$ is implied by $Q$, so, **b** $Q \Rightarrow P$ and **d** $P$ is implied by $Q$ are true, and **a** $P \Rightarrow Q$ and **c** $P$ implies $Q$ are false.
**Question 5**
A rectangle is a parallelogram, but a parallelogram is not necessarily a rectangle, so $Q \Rightarrow P$.
**Question 6**
The implication on line $3$ is incorrect, as $x = -\sqrt{1}$ for $x = -1$.
**Question 7a**
If $2x + 1$ is not composite then $x$ is not prime
**Question 7b**
If $\frac{1}{x} \neq 1$ then $x^2 \neq 1$.
**Question 8a**
If $x$ is irrational then $x^3$ is irrational.
**Question 8b**
If $x^2 \geq 1$ then $x \leq -1$ or $x \geq 1$.
**Question 9**
$\neg Q \Rightarrow \neg P$: If it is not a duck then it is not a bird. The contrapositive is false, since some things that are not ducks are birds, for example, a magpie is not a duck but it is a bird. Note that since the original statement was false, the contrapositive is also false.
**Question 10**
Since the statement applies to all $n$ then the contrapositive must also apply to all $n$. Reverse the two parts of the statement and negate each part, so '$\forall n \in \mathbb{Z}$, if $n$ is not odd then $n^2$is not odd'.
**Question 11a**
$\neg P$: '$x$ is a positive integer and $-3x$ is not a negative integer'.
**Question 11b**
$\neg Q$: '$x$ is an odd integer and $4x$ is an odd integer'.
**Question 11c**
$\neg R$: 'an animal lives in the ocean and is not a fish'.
**Question 12a**
$\neg P$: '$x$ is a multiple of $6$ and $x$ is a multiple of $8$ and $x$ is not a multiple of $24$'.
**Question 12b**
$\neg Q$: '$ABCD$ is a parallelogram and $ABCD$ is a kite and $ABCD$ is not a rhombus'.
**Question 13a**
$\forall$ since doubling any integer results in an even integer.
**Question 13b**
$\exists$ since there is one real number where $e^m = 4$
**Question 13c**
$\exists$ since there is at least one natural number that is less than $2$ when we divide it by $3$ and square it. For example, $m = 3$ gives $\left(\frac{m}{3}\right)^2 = 1 \leq= 2$.
**Question 13d**
$\forall$ since the cosine of all real numbers lies in the interval $[-1, 1]$, $1 + \cos m$ is greater than or equal to $0$.
**Question 14**
We start by picking any integer value for $d$ from $0$ to $10$ inclusive, ($\forall\{d \in \mathbb{Z} | 0 \leq d \leq 10\}$), then find at least one positive value of $p$ ($\exists p > 0$) such that $p > d$. Answer C.
**Question 15a**
$\forall x \in \mathbb{Z}^+, 2x \neq x^2$
**Question 15b**
$\exists x \in \mathbb{R}, \forall m \in \mathbb{Z}^+, x \geq 2m + 1$
**Question 16a**
The quadrilateral does not have four equal sides.
**Question 16b**
The quadrilateral does not have four right angles.
**Question 16c**
The quadrilateral does not have four equal sides or does not have four right angles.
**Question 16d**
The quadrilateral does not have four equal sides and does not have four right angles.
**Question 17a**
Converse: If an animal is warm−blooded, then it is a mammal.
Original: True. Mammals are warm−blooded.
Converse: False. Birds are warm−blooded but are not mammals.
**Question 17b**
Converse: If the diagonals of a quadrilateral are perpendicular, then it is a rhombus.
Original: True. The diagonals of a rhombus are perpendicular.
Converse: False. A kite can have perpendicular diagonals without being a rhombus.
**Question 17c**
Converse: If $n$ is a multiple of $4$, then $n$ is a multiple of $12$.
Original: True, since every multiple of $12$ is a multiple of $4$.
Converse: False. For example, $8$ is a multiple of $4$ but not a multiple of $12$.

**Question 18a**
The triangle is right−angled and is isosceles and does not have two angles of $45°$.
**Question 18b**
$n$ is a positive integer and is divisible by $6$ and $n^2$ is not divisible by $12$.
**Question 18c**
The animal is a bird and cannot fly and is not a penguin.
**Question 19a**
$\forall$, since the product $n(n+1)$ of two consecutive integers is always even.
**Question 19b**
Neither quantifier makes the statement true, since $x^2+4x+5=(x+2)^2+1>0$ for all real $x$.
**Question 19c**
$\exists$, since the equation is true for $x \geq 3$, but not for $x<3$.
**Question 19d**
$\exists$, since the inequality is true for some positive integers, such as $n=1$, but not for all positive integers.
**Question 20a**
If $n$ is even, then $n^2$ is even.
The original implication is true.
**Question 20b**
If the diagonals of a quadrilateral are not equal in length, then the quadrilateral is not a square.
The original implication is true.
**Question 20c**
If $x \leq 3$, then $x^2 \leq 9$.
The original implication is false. For example, $x=-4$ gives $x^2=16>9$, but $x>3$ is false.
**Question 21a**
$\exists x \in \mathbb{Z}^+, 2x$ is not prime and $2x+1$ is even
**Question 21b**
$\forall x \in \mathbb{R}, x^2-7 \neq 9$ or $x \leq 2$
**Question 22**
The proposition is in the form $P \Rightarrow Q$, where:
$P$: each term in a geometric sequence is smaller than the previous term
$Q$: $0<r<1$.
To disprove the proposition we need to find $P$ and not $Q$ occurring simultaneously, so each term in a geometric sequence is smaller than the previous term AND $r$ is not between $0$ and $1$.
Writing out the first three terms in each sequence:
A: $2,4,8,$ which is not in descending order. Eliminate A.
B: $8,4,2,$ which is in descending order but has $0<r<1$. Eliminate B.
C: $-2,-4,-8,$ which is in descending order and where $r$ is not between $0$ and $1$. Correct.
D: $8,-4,2,$ which is not in descending order. Eliminate D.
**Answer C**

**Question 23a**
$\exists x \in \mathbb{R}, x^2<4 \wedge x \geq 1$
**Question 23b**
$\forall n \in \mathbb{Z}^+, n^2$ is odd $\vee$ $n^3$ is even
**Question 23c**
$\exists n \in \mathbb{Z}^+, (n$ is prime$) \wedge (n$ is even$)$
**Question 23d**
$\forall x \in \mathbb{R}, \left(x^2 \neq 16\right) \vee (x \leq -2) \vee (x \geq 5)$
**Question 24a**
$P: \forall n \in \mathbb{Z}^+, \exists m \in \mathbb{Z}^+, m>n$
$Q: \exists m \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+, m>n$
**Question 24b**
$P$ is true, since for every positive integer $n$, we can choose $m=n+1$.
$Q$ is false, since no positive integer is greater than every positive integer.
**Question 24c**
$\neg P: \exists n \in \mathbb{Z}^+, \forall m \in \mathbb{Z}^+, m \leq n$
$\neg Q: \forall m \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+, m \leq n$
**Question 25a**
$\forall x \in A, \left(M(x) \vee B(x)\right) \wedge \left(B(x) \Rightarrow F(x)\right)$
**Question 25b**
There is at least one animal in the sanctuary that is neither a mammal nor a bird, or there is at least one bird in the sanctuary that cannot fly.
**Question 25c**
The statement requires every bird to be able to fly, but places no restriction on the ability of mammals to fly. Therefore, a flightless mammal would not disprove the statement.
**Question 26a**
The implication on line $4$ is incorrect.
**Question 26b**
The correct identity is $\sqrt{x^2}=|x|$ not $\sqrt{x^2}=x$ for all real $x$.
**Question 26c**
For example, $x=-3$ is a real number, but $x \geq 0$ is false.
**Question 26d**
The correct conclusion is $|x| \geq 0$
**Question 27a**
$\forall x, y \in \mathbb{R} \setminus \{0\}, \quad xy>0 \Rightarrow (x>0 \wedge y>0)$
**Question 27b**
Converse: $(x>0 \wedge y>0) \Rightarrow xy>0$
**Question 27c**
Contrapositive: $(x \leq 0 \vee y \leq 0) \Rightarrow xy \leq 0$ where $x, y \neq 0$.
**Question 27d**
The original proposition is false. For example, $x=-2$ and $y=-3$ give $xy=6>0$, but neither number is positive.
The converse is true, since the product of two positive real numbers is positive.
The contrapositive is false, since it is logically equivalent to the original proposition. The same counterexample, $x=-2$ and $y=-3$, disproves it.

**Question 1**

Let $m = 2k$ for $k \in \mathbb{Z}$

$m^2 = (2k)^2$

$= 2(2k)$

$= 2p$ for $p \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore$ if $m$ is even, then $m^2$ is even.

**Question 2**

Let $m = 2k + 1, n = 2j + 1$ for j,$k \in \mathbb{Z}$

$mn = (2k + 1)(2j + 1)$

$= 4jk + 2k + 2j + 1$

$= 2(2jk + k + j) + 1$

$= 2p + 1$ for $p \in \mathbb{Z}$ since $j, k \in \mathbb{Z}$

$\therefore$ the product of any odd number $m$ and any odd number $n$ is odd

**Question 3**

$m^2 = (4k)^2$

$= 16k^2$

$= 16p$ for $p \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore \forall m = 4k$ for $k \in \mathbb{Z}, m^2 = 16p$ for $p \in \mathbb{Z}$.

**Question 4**

$(a + b)^2 + (a - b)^2$

$= a^2 + 2ab + b^2 + a^2 - 2ab + b^2$

$= 2(a^2 + b^2)$

$= 2p$ for $p \in \mathbb{Z}$ since $a, b \in \mathbb{Z}$

$\therefore \forall a, b \in \mathbb{Z}$, $(a + b)^2 + (a - b)^2$ is divisible by 2.

**Question 5**

Suppose by contrapositive that $n$ is even.

Let $n = 2k$ for $k \in \mathbb{Z}$.

$\therefore n^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$

$= 2j$ for $j \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore$ if $n$ is even then $n^2$ is even.

$\therefore$ if $n^2$ is odd then $n$ is odd by contrapositive.

**Question 6**

Suppose by contrapositive that neither $m$ nor $n$ are divisible by 5.

Let $m = 5j + p$ and $n = 5k + q$ for j,$k \in \mathbb{Z}$ and $p, q \in \{1,2,3,4\}$.

$mn = (5j + p)(5k + q)$

$= 25jk + 5jq + 5kp + pq$

$= 5(5jk + jq + kp) + pq$

$= 5c + pq$ for $c \in \mathbb{Z}$ since $j, k, p, q \in \mathbb{Z}$

Now $pq = 1, 2, 3, 4, 6, 8, 9, 12,$ or 16, so $pq$ is not a multiple of 5, so $mn$ is not a multiple of 5.

$\therefore$ if $m$ and $n$ are not divisible by 5 then $mn$ is not divisible by 5.

$\therefore$ if $mn$ is divisible by 5 then $m$ or $n$ must be divisible by 5 by contrapositive.

**Question 7**

If $x$ is even, let $x = 2k$ for $k \in \mathbb{Z}$.

$x^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$

$= 2j$ for $j \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore$ if $x$ is even then $x^2$ is even

Conversely, we will show that if $x^2$ is even then $x$ is even using proof by contrapositive.

Suppose by contrapositive that $x$ is odd

Let $x = 2k + 1$ for $k \in \mathbb{Z}$

$x^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2p + 1$ for $p \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\therefore$ if $x$ is odd $x^2$ is odd, hence if $x^2$ is even then $x$ is even by contrapositive.

$\therefore$ $x$ is even if and only if $x^2$ is even.

**Question 8**

Suppose by contrapositive that $a$ is odd

Let $a = 2n + 1$ for $n \in \mathbb{Z}$

$\therefore a^3 + 5 = (2n + 1)^3 + 5$

$= 8n^3 + 12n^2 + 6n + 1 + 5$

$= 2(4n^3 + 6n^2 + 3)$

$= 2p$ for $p \in \mathbb{Z}$ since $n \in \mathbb{Z}$

$\therefore$ if $a$ is odd then $a^3 + 5$ is even

$\therefore$ if $a^3 + 5$ is odd then $a$ is even by contrapositive.

**Question 9**

Let the consecutive numbers be $k$ and $k + 1$ for $k \in \mathbb{Z}$

$(k + 1)^2 - k^2$

$= k^2 + 2k + 1 - k^2$

$= 2k + 1$

$= k + (k + 1)$

$\therefore$ the difference between the squares of any two consecutive numbers equals their sum.

**Question 10**

Let the consecutive numbers be $k - 1, k$ and $k + 1$ for $k \in \mathbb{Z}$

$P = k(k - 1)(k + 1) = k^3 - k$

Case 1 – $k$ is even

Let $k = 2m$ for $m \in \mathbb{Z}$

$\therefore P = (2m)^3 - 2m$

$= 2(4m^3 - m)$

$= 2p$ for $p \in \mathbb{Z}$

$\therefore$ true if $k$ is even

Case 2 – $k$ is odd

Let $k = 2m + 1$ for $m \in \mathbb{Z}$

$\therefore P = (2m + 1)^3 - (2m + 1)$

$= 8m^3 + 12m^2 + 6m + 1 - 2m - 1$

$= 2(4m^3 + 6m^2 + 2m)$

$= 2p$ for $p \in \mathbb{Z}$

$\therefore$ true if $k$ is odd

$\therefore$ The product of any three consecutive numbers is even

**Question 11**

Let the consecutive numbers be $k, k+1$, $k+2$ and $k+3$ for $k \in \mathbb{Z}$

$S = k + k + 1 + k + 2 + k + 3$
$= 4k + 6$
$= 2(2k + 3)$
$= 2p$ for $p \in \mathbb{Z}$

$\therefore$ The sum of any four consecutive numbers is always even.

**Question 12**

Suppose by contrapositive that $m$ is even.
Let $m = 2k$ for $k \in \mathbb{Z}$
$m^2 = (2k)^2$
$= 4k^2$
$= 4p$ for $p \in \mathbb{Z}$
$\therefore$ if $m$ is even then $m^2$ is divisible by 4.
$\therefore$ if $m^2$ is not divisible by 4 then $m$ must be odd by contrapositive.

**Question 13**

Prove that if $x^2$ is divisible by 9 then $x$ is a multiple of 3 by contrapositive
Suppose by contrapositive that $x$ is not a multiple of 3
Let $x = 3k + j$ for $k \in \mathbb{Z}$ and $j \in \{1,2\}$
$\therefore x^2 = (3k + j)^2$
$= 9k^2 + 6jk + j^2$
$= 3(3k^2 + 2jk) + j^2$
$= 3p + 1$ or $3p + 4$ for $p \in \mathbb{Z}$ since $k \in \mathbb{Z}$
which are not multiples of 9
$\therefore$ if $x$ is not a multiple of 3 then $x^2$ is not a multiple of 9
$\therefore$ if $x^2$ is a multiple of 9 then $x$ is a multiple of 3 by contrapositive.
Conversely, if $x$ is a multiple of 3 let $x = 3j$ for $j \in \mathbb{Z}$
$x^2 = (3j)^2$
$= 9j^2$
$= 9p$ for $p \in \mathbb{Z}$ since $j \in \mathbb{Z}$
$\therefore$ if $x$ is a multiple of 3 then $x^2$ is divisible by 9.
$\therefore x^2$ is divisible by 9 if and only if $x$ is a multiple of 3

**Question 14**

Suppose by contrapositive that $m$ is odd and $n$ is odd.
Let $m = 2k + 1, n = 2j + 1$ for $k, j \in \mathbb{Z}$

$$\frac{mn}{2} = \frac{(2k+1)(2j+1)}{2}$$
$$= \frac{4kj + 2k + 2j + 1}{2}$$
$$= 2kj + k + j + \frac{1}{2}$$

Which is not integral since $k, j \in \mathbb{Z}$
$\therefore$ if $m$ and $n$ are both odd then $\frac{mn}{2}$ is not integral.
$\therefore$ if $\frac{mn}{2}$ is integral then $m$ or $n$ must be even.

**Question 15**

If $m^2 - n^2$ is even then $(m+n)(m-n)$ is even, using the difference of two squares.
$\therefore$ At least one of $m+n$ and $m-n$ is even, since two odd numbers have an odd product,
$\therefore$ If $m^2 - n^2$ is even at least one of the sum and difference of $m$ and $n$ are even.
Conversely, if at least one of $m+n$ and $m-n$ are even then $(m+n)(m-n)$ is even, since the product of two even numbers or an even and an odd number is even.
$\therefore m^2 - n^2$ is even, using the difference of two squares.
$\therefore$ If at least one of the sum and difference of $m$ and $n$ are even then $m^2 - n^2$ is even.
$\therefore m^2 - n^2$ is even iff at least one of the sum and difference of $m$ and $n$ are even.

**Question 16**

If both integers are odd, let the integers be $2m+1, 2n+1$ for $m, n \in \mathbb{Z}$.
$P = (2m + 1)(2n + 1)$
$= 4mn + 2m + 2n + 1$
$= 2(2mn + m + n) + 1$
$= 2p + 1$ for $p \in \mathbb{Z}$ since $m, n \in \mathbb{Z}$ #
$\therefore$ if both integers are odd then the product is odd.
By contrapositive, if the product is even then at least one of the integers must be even
Conversely, one or both of the integers is even, let the integers be $2m + j$ and $2n$, where $m, n \in \mathbb{Z}$ and $j = \{0,1\}$.
$p = (2m + j)(2n)$
$= 4mn + 2nj$
$= 2(2mn + nj)$
$= 2p$ for $p \in \mathbb{Z}$ since $m, n \in \mathbb{Z}$
$\therefore$ if at least one of the integers is even then the product is even
$\therefore$ the product of two integers is even if and only if one or both are even

**Question 17**

Let the rational numbers be $\frac{p}{q}$ and $\frac{m}{n}$, where $p, q, m, n \in \mathbb{Z}, q, m, n \neq 0$

$$\frac{p}{q} \div \frac{m}{n} = \frac{p}{q} \times \frac{n}{m}$$
$$= \frac{pn}{qm}$$
$$= \frac{a}{b} \quad \text{for } a, b \in \mathbb{Z} \text{ since } p, q, m, n \in \mathbb{Z}$$

$\therefore$ the quotient of two rational numbers is rational.

**Question 18**

$((k+1)^2 - 1)((k+2)^2 - 1)$
$= (k + 1 + 1)(k + 1 - 1)(k^2 + 4k + 4 - 1)$
$= (k + 2)(k)(k^2 + 4k + 3)$
$= (k + 2)(k)(k + 3)(k + 1)$
$= k(k + 1)(k + 2)(k + 3)$

**Question 19**

$3^k$
$= 3^k - 1^k + 1$
$= (3-1)\left(3^{k-1} + 3^{k-2}\times 1 + 3^{k-3}\times 1^2 \ldots + 1^{k-1}\right)$
$+1$
$= 2\left(3^{k-1} + 3^{k-2}\times 1 + 3^{k-3}\times 1^2 \ldots + 1^{k-1}\right) + 1$
$= 2p + 1$ for $p \in \mathbb{Z}$
$\therefore$ when $3^k$ is divided by 2, the remainder is 1.

**Question 20**

$a^2 + 2b$
$= (2-b)^2 + 2b$ since $a + b = 2$
$= 4 - 4b + b^2 + 2b$
$= b^2 + 4 - 2b$
$= b^2 + 2(2-b)$
$= b^2 + 2a$

Alternatively:

LHS – RHS
$= a^2 - b^2 + 2b - 2a$
$= (a+b)(a-b) - 2(a-b)$
$= (a-b)(a+b-2)$
$= (a-b)(0)$ since $a + b = 2$
$= 0$
$\therefore a^2 - b^2 + 2b - 2a = 0$
$\therefore a^2 + 2b = b^2 + 2a$

**Question 21**

$a^3 - a + 1$
$= a\left(a^2 - 1\right) + 1$
$= a(a+1)(a-1) + 1$
$= 2k + 1$ for $k \in \mathbb{Z}$,
since the product of three consecutive numbers is even.
$\therefore a^3 - a + 1$ is odd $\forall a \in \mathbb{Z}^+$

**Question 22**

Let $n = 3k \pm 1$ for $k \in \mathbb{Z}$
$n^2 - 1 = (3k \pm 1)^2 - 1$
$= 9k^2 \pm 6k + 1 - 1$
$= 9k^2 \pm 6k$
$= 3k(3k \pm 2)$
$= 3p$ for $p \in \mathbb{Z}$
$\therefore n^2 - 1$ is divisible by 3 if $n$ is not a multiple of 3.

**Question 23**

$(2^m - 1)^2 - 1 = (2^m - 1 + 1)(2^m - 1 - 1)$
$= 2^m(2^m - 2)$
$= 2^m \cdot 2(2^{m-1} - 1)$
$= 2^{m+1}(2^{m-1} - 1)$
$\therefore \forall m \in \mathbb{Z}^+$, $(2^m - 1)^2 - 1$ is divisible by $2^{m+1}$.

**Question 24**

Suppose by contrapositive that $n$ is composite.
Let $n = pq$ where $p, q \in \mathbb{Z}, p, q > 1$
$2^n - 1$
$= 2^{pq} - 1$
$= (2^p)^q - 1^q$
$= \left((2^p) - 1\right)\left((2^p)^{q-1} + (2^p)^{q-2}\right.$
$\left. + (2^p)^{q-3} + \ldots + 1\right)$
$= jk$ where $j, k \in \mathbb{Z}, j, k > 1$ since $p, q \in \mathbb{Z}, p, q > 1$
$\therefore$ if $n$ is composite then $2^n - 1$ is composite.
$\therefore$ if $2^n - 1$ is prime then $n$ is prime, for $n > 1$ by contrapositive.

**Question 25**

Let the number be $x = 100a + 10b + c$
where $a, b, c \in \mathbb{Z}$, a $\geq 0$ and $0 \leq b, c \leq 9$
If the last two digits are a multiple of 4 then $10b + c = 4m$ for $m \in \mathbb{Z}^+$
$\therefore x = 4(25a) + 4m$
$= 4(25a + m)$
$= 4p$ for $p \in \mathbb{Z}^+$ since $a, m \in \mathbb{Z}^+$
$\therefore$ if the last two digits are a multiple of 4 then the number is divisible by 4.

Conversely, we will show by contrapositive that if a number is divisible by 4 then the last two digits are a multiple of 4.

If the last two digits are not a multiple of 4 then $10b + c = 4m + k$ for $m, k \in \mathbb{Z}^+$ with $k$ not a multiple of 4.
$\therefore x = 4(25a) + 4m + k$
$= 4(25a + m) + k$
$\neq 4p$ for $p \in \mathbb{Z}^+$ since $a, m \in \mathbb{Z}^+$ and $k$ is not a multiple of 4
$\therefore$ if the last two digits are not a multiple of 4 then the number is not divisible by 4.
$\therefore$ if the number is divisible by 4 then the last two digits are a multiple of 4 by contrapositive.
$\therefore$ a number is divisible by 4 if and only if the last two digits are a multiple of 4.

**Question 26a**

Let $a = 5k + m$, where $m = \{1, 2, 3, 4\}$ and $k \in \mathbb{Z}$
$a^2 = (5k + m)^2$
$= 25k^2 + 10km + m^2$
$= 5\left(5k^2 + 2km\right) + m^2$
Now $m^2 = 1, 4, 9$ or $16$
$\therefore a^2 = 5\left(5k^2 + 2km\right) + 1, 5\left(5k^2 + 2km\right) + 4,$
$5\left(5k^2 + 2km + 1\right)$
$+ 4$ or $5\left(5k^2 + 2km + 3\right) + 1$
$\therefore a^2 = 5j + 1, 5j + 4$ where $j \in \mathbb{Z}$

**Question 26b**

$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$

**Case 1** – $a^2 = 5j + r, b^2 = 5i + r \;\; r = \{1, 4\}$ and $j, i \in \mathbb{Z}$, have the same remainder when divided by 5

$$a^4 - b^4 = \big(5j + r - (5i + r)\big)(5j + r + 5i + r)$$
$$= 5(j - i)(5j + 5i + 2r)$$

So $a^4 - b^4$ is divisible by 5 if $a$ and $b$ have the same remainder when divided by 5.

**Case 2** – $a^2 = 5j + 1, b^2$

$= 5i + 4$, wlog, where $j, i \in \mathbb{Z}$

$$a^4 - b^4$$
$$= \big((5j + 1) - (5i + 4)\big)\big((5j + 1) + (5i + 4)\big)$$
$$= (5j - 5i - 3)(5j + 5i + 5)$$
$$= 5(5j - 5i - 3)(j + i + 1)$$

So $a^4 - b^4$ is divisible by 5 if $a$ and $b$ have different remainders when divided by 5.

$\therefore a^4 - b^4$ is divisible by 5.

**Question 27**

$$(2k + 1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 8\left(\frac{k^2 + k}{2}\right) + 1$$
$$= 8\left(\frac{k(k + 1)}{2}\right) + 1$$
$$= 8T + 1$$

**Question 28**

Case 1 – if $n$ is odd

Let $n = 2k + 1$ for $k \in \mathbb{Z}$

$$(n + 1)^n - n^{n+1}$$
$$= (2k + 1 + 1)^{2k+1} - (2k + 1)^{2k+1+1}$$
$$= (2k + 2)^{2k+1} - (2k + 1)^{2k+2}$$

$2k + 2 = 2(k + 1)$ is even, and an even number to any integer power is even.

$2k + 1$ is odd, and an odd number to any integer power is odd.

The difference between an even number and an odd number is odd.

$\therefore (n + 1)^n - n^{n+1}$ is odd if $n$ is odd.

Case 2 – if $n$ is even

Let $n = 2k$ for $k \in \mathbb{Z}$

$(n + 1)^n - n^{n+1} = (2k + 1)^{2k} - (2k)^{2k+1}$

$2k + 1$ is odd, and an odd number to any integer power is odd.

$2k$ is even, and an even number to any integer power is even.

The difference between an odd number and an even number is odd.

$\therefore (n + 1)^n - n^{n+1}$ is odd if $n$ is even.

$\therefore (n + 1)^n - n^{n+1}$ is odd for all positive integers $n$.

**Question 29**

Case 1 – $\sqrt{2}^{\sqrt{2}}$ is rational

$\sqrt{2}^{\sqrt{2}}$ is an irrational number to the power is an irrational number, so if $\sqrt{2}^{\sqrt{2}}$ is rational then we have proved the problem.

Case 2 – $\sqrt{2}^{\sqrt{2}}$ is irrational

Consider $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$, which in this case is an irrational number to the power is an irrational number.

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}}$$
$$= \sqrt{2}^{2}$$
$$= 2$$

which is rational, so we have proved the problem.

$\therefore$ whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational we have proved that an irrational number raised to the power of an irrational number can be rational.

**Question 30**

Let $n = a_0 + a_1 \times 10^1 + a_2 \times 10^2 + \ldots + a_k \times 10^k$ for $a_0, a_1, a_2 \ldots a_k \in \mathbb{Z}, 0 <= a_0, a_1, a_2 \ldots a_k \leq 9$.

$\therefore n = a_0 + a_1 \times (3q_1 + 1) + a_2 \times (3q_2 + 1) + \ldots + a_k \times (3q_k + 1)$ for $q_1, q_2 \ldots q_k \in \mathbb{Z}^+$

$\therefore n = (a_0 + a_1 + a_2 + \ldots + a_k) + 3(a_1 q_1 + a_2 q_2 + \ldots + a_k q_k)$

$= (a_0 + a_1 + a_2 + \ldots + a_k) + 3m$ for $m \in \mathbb{Z}^+$ since $a_i, q_i \in \mathbb{Z}^+$

If $n$ is divisible by $3$, let $n = 3j$ for $j \in \mathbb{Z}^+$.

$\therefore (a_0 + a_1 + a_2 + \ldots + a_k) + 3m = 3j$

$a_0 + a_1 + a_2 + \ldots + a_k = 3(j - m)$

$= 3t$ for $t \in \mathbb{Z}^+$ since $j, m \in \mathbb{Z}^+$

$\therefore$ if $n$ is divisible by $3$ then the sum of its digits is divisible by $3$

Conversely, if the sum of its digits is divisible by $3$

$a_0 + a_1 + a_2 + \ldots + a_k = 3s$ for $s$

$\therefore n = 3s + 3m$

$= 3r$ for $r \in \mathbb{Z}^+$ since $s, m \in \mathbb{Z}^+$

$\therefore$ if the sum of its digits is divisible by $3$ then $n$ is divisible by $3$

$\therefore$ a positive integer $n$ is divisible by 3 if and only if the sum of its digits is divisible by 3.

**Question 1**

Suppose by contradiction that $2\sqrt{3}-\sqrt{5}\geq\sqrt{2}$

$\therefore \left(2\sqrt{3}-\sqrt{5}\right)^2 \geq \left(\sqrt{2}\right)^2$

$12-4\sqrt{15}+5\geq 2$

$15\geq 4\sqrt{15}$

$225\geq 240$

Which is a contradiction, so $2\sqrt{3}-\sqrt{5}<\sqrt{2}$.

**Question 2**

Suppose by contradiction that $2m+4n=51$ for $m,n\in\mathbb{Z}$

$\therefore 2(n+2m)=51$

Which is a contradiction since the LHS is even but the RHS is odd, so there are no integers $m$ and $n$ such that $2m+4n=51$.

**Question 3**

Suppose by contradiction that $n$ is an even integer and $n^2$ is odd. $*$

Let $n=2k$ for $k\in\mathbb{Z}$

$\therefore n^2=(2k)^2$

$=4k^2$

$=2\left(2k^2\right)$

$=2p$ for $p\in\mathbb{Z}$

$\therefore n^2$ is even #

Which contradicts $(*)$ since $n^2$ cannot be both odd and even, hence if $n$ is an even integer then $n^2$ is even.

**Question 4**

Suppose by contradiction that $n^2-1$ is even and $n$ is even. $*$

Let $n=2k$ for $k\in\mathbb{Z}$

$\therefore n^2-1=(2k)^2-1$

$=4k^2-1$

$=2\left(2k^2-1\right)+1$

$=2p+1$ for $p\in\mathbb{Z}$

$\therefore n^2-1$ is odd #

Which contradicts $(*)$ since $n^2-1$ cannot be both odd and even, hence if $n^2-1$ is even then $n$ is odd.

**Question 5**

Suppose by contradiction that at least one of the integers is even and the product is odd $(*)$

Let the integers be $2m+j$ and $2n$, where $m,n\in\mathbb{Z}$ and $j\in\{0,1\}$.

$p=(2m+j)(2n)$

$=4mn+2nj$

$=2(2mn+nj)$

$=2p$ for $p\in\mathbb{Z}$ since $m,n\in\mathbb{Z}$ #

Which contradicts $(*)$ since the product cannot be odd and even.

$\therefore$ if the product of any two integers is odd, then both of them must be odd

**Question 6**

Suppose by contradiction that $\sqrt{5}+\sqrt{7}\geq 5$

$\therefore \left(\sqrt{5}+\sqrt{7}\right)^2\geq 25$ since $\sqrt{5},\sqrt{7},5>0$

$5+2\sqrt{35}+7\geq 25$

$2\sqrt{35}\geq 13$

$\sqrt{140}\geq 13$

$140\geq 169$

Which is a contradiction, so $\sqrt{5}+\sqrt{7}<5$

**Question 7**

Suppose by contradiction that $\sqrt{3}$ is rational.

Let $\sqrt{3}=\frac{p}{q}$ where $p,q\in\mathbb{Z}^+$, $\text{HCF}(p,q)=1$ $*$

$\therefore 3=\frac{p^2}{q^2}$

$3q^2=p^2$

3 divides the LHS, so 3 divides $p^2$, so 3 also divides $p$ since 3 is prime.

Let $p=3k$ for some $k\in\mathbb{Z}$

$\therefore 3q^2=\left(3k^2\right)$

$q^2=3k^2$

So similarly 3 also divides $q$.

$\therefore p$ and $q$ have a common factor of 3 which contradicts $\text{HCF}(p,q)=1$ $(*)$.

$\therefore \sqrt{3}$ must be irrational.

**Question 8**

Suppose by contradiction $\log_2 5$ is rational.

$\therefore \log_2 5=\frac{p}{q}$ for $p,q\in\mathbb{Z}^+$, $\text{HCF}(p,q)=1$

$\therefore q\log_2 5=p$

$\log_2 5^q=p$

$5^q=2^p$

LHS is odd, since an odd number to any power is odd.

RHS is even, since an even number to any power is even.

This is a contradiction, so $\log_2 5$ must be irrational.

**Question 9**

Suppose by contradiction that $a+b\leq 5$ and $a>2$ and $b>2$ $*$

$\therefore a+b\geq 3+3$ since $a,b\in\mathbb{Z}$

$\geq 6$ #

Which contradicts $(*)$ since $a+b$ cannot be $\leq 5$ and $\geq 6$, hence $a\leq 2$ or $b\leq 2$.

**Question 10**

Suppose by contradiction that $4n+8m=102$ for $n,m\in\mathbb{Z}$

$\therefore 4(n+2m)=102$

Which is a contradiction since the LHS is a multiple of 4 but the RHS isn't, so there are no integers $a$ and $b$ such that $4n+8m=102$.

**Question 11**

Suppose by contradiction that the negation is true

$$\frac{1}{a-2b}=\frac{1}{a}-\frac{1}{2b}$$
$$\frac{1}{a-2b}=\frac{2b-a}{2ab}$$
$$\frac{1}{a-2b}=-\frac{a-2b}{2ab}$$
$$-2ab=(a-2b)^2$$

This is a contradiction as the LHS is negative as $a,b>0$ and the RHS is positive as $\mathbb{R}^2\geq 0$

$\therefore \frac{1}{a-2b}\neq\frac{1}{a}-\frac{1}{2b}$ for $a,b>0$ and $a\neq 2b$

**Alternative solution I:**

Suppose by contradiction that the negation is true

$$\frac{1}{a-2b}=\frac{1}{a}-\frac{1}{2b}$$
$$\frac{1}{a-2b}=\frac{2b-a}{2ab}$$
$$\frac{1}{a-2b}=-\frac{a-2b}{2ab}$$
$$-2ab=(a-2b)^2$$
$$-2ab=a^2-4ab+4b^2$$
$$\therefore\ a^2-2ab+4b^2=0$$
$$a^2-2ab+b^2+3b^2=0$$
$$(a-b)^2+3b^2=0$$

Now $(a-b)^2\geq 0$ since $\mathbb{R}^2\geq 0$, and $3b^2>0$ since $b>0$, so the LHS $>0$.

This is a contradiction as the RHS $=0$

$\therefore \frac{1}{a-2b}\neq\frac{1}{a}-\frac{1}{2b}$ for $a,b>0$ and $a\neq 2b$

**Alternative solution II:**

Suppose by contradiction that the negation is true

$$\frac{1}{a-2b}=\frac{1}{a}-\frac{1}{2b}$$
$$\frac{1}{a-2b}=\frac{2b-a}{2ab}$$
$$-2ab=(a-2b)^2$$
$$-2ab=a^2-4ab+4b^2$$
$$\therefore\ a^2-2ab+4b^2=0$$

Using the quadratic formula as a shortcut for completing the square − taking $a$ as the variable and $b$ as a constant

$$\Delta=(-2b)^2-4(1)(4b^2)$$
$$=-12b^2$$
$$<0 \quad \text{since } b^2>0$$

$\therefore$ no solutions, which is a contradiction, so the negation is false

$\therefore \frac{1}{a-2b}\neq\frac{1}{a}-\frac{1}{2b}$ for $a,b>0$ and $a\neq 2b$

**Question 12**

Suppose by contradiction that $n^3-1$ is odd and $n$ is odd. $\quad *$

Let $n=2k+1$ for $k\in\mathbb{Z}$

$$\therefore n^3-1=(2k+1)^3-1$$
$$=8k^3+12k^2+6k+1-1$$
$$=2\left(4k^3+6k^2+3k\right)$$
$$=2p \qquad \text{for } p\in\mathbb{Z} \text{ since } k\in\mathbb{Z}$$

$\therefore n^3-1$ is even $\quad \#$

Which contradicts $(*)$ since $n^3-1$ cannot be both odd and even, hence if $n^3-1$ is odd then $n$ is even.

**Question 13**

Suppose by contradiction that the product of a rational and irrational number is rational

Let $a=\frac{p}{r}$ and $ab=\frac{m}{n}$, where $p,q,m,n\in\mathbb{Z}$, $p,n,r\neq 0$ and $b$ is irrational $(*)$

$$\therefore \frac{p}{r}\times b=\frac{m}{n}$$
$$b=\frac{mr}{np}$$
$$=\frac{c}{d} \text{ for } c,d\in\mathbb{Z} \text{ since } p,q,m,n\in\mathbb{Z}\ \#$$

This contradicts $(*)$ since $b$ cannot be rational and irrational.

$\therefore$ the product of a rational and irrational number is irrational

**Question 14**

Suppose by contradiction that $m^2-4n=3$ where $m,n\in\mathbb{Z}$

Since 3 is odd $m^2-4n$ must be odd.

Let $m^2-4n=2k+1$ for $k\in\mathbb{Z}$

$$\therefore m^2=4n+2k+1$$
$$=2(2n+k)+1$$

$\therefore m^2$ must be odd.

$\therefore\ m$ must be odd.

Let $m=2p+1$ for $p\in\mathbb{Z}$.

$$\therefore (2p+1)^2-4n=3$$
$$4p^2+4p+1-4n=3$$
$$4(p^2+p-n)=2$$

This is contradiction since the LHS is a multiple of 4 but the RHS is not, so if $m^2-4n=3$ then $m$ and $n$ cannot both be integers

**Question 15**

Suppose by contradiction that there is a finite number of odd integers $(*)$

Let the largest odd integer be $k$

$k+2$ is also odd, so we have another odd integer which contradicts $(*)$

$\therefore$ there are infinitely many odd integers.

**Question 16**

Suppose by contradiction that $\exists n \in \mathbb{Z}$, $\log_2(2n+1)$ is rational.

Let $\log_2(2n+1) = \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$, $\text{HCF}(p,q) = 1$.

$q\log_2(2n+1) = p$

$\log_2[(2n+1)^q] = p$

$(2n+1)^q = 2^p$

$2n+1$ is odd so the LHS is odd since an odd number to any integer power is odd.

The RHS is even since an even number to any integer power is even.

This is a contradiction since an odd number cannot be equal to an even number.

Hence $\log_2(2n+1)$ must be irrational $\forall n \in Z^+$.

***Alternative solution I:***

Suppose by contradiction that $\exists n \in \mathbb{Z}$, $\log_2(2n+1)$ is rational.

Let $\log_2(2n+1) = \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$, $\text{HCF}(p,q) = 1$.

$2n+1 = 2^{\frac{p}{q}}$

$2n+1$ is odd $\forall n \in \mathbb{Z}^+$ so the LHS is odd, while the RHS is either even (if $\frac{p}{q}$ is an integer) or not an integer (if $\frac{p}{q}$ is not an integer).

This is a contradiction since an odd number cannot be equal to a number that is even or not an integer.

Hence $\log_2(2n+1)$ must be irrational $\forall n \in Z^+$.

**Question 17**

Suppose by contradiction that $\sqrt{\pi}$ is rational.

Let $\sqrt{\pi} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$, $\text{HCF}(p,q) = 1$

$q\sqrt{\pi} = p$

$\pi q^2 = p^2$

Now the RHS is an integer but the LHS is not since $\pi$ is irrational which is a contradiction, hence the square root of $\pi$ is also irrational.

**Question 18**

Suppose $\sin x + \cos x < 1$

$\therefore (\sin x + \cos x)^2 < 1$

since $\sin x, \cos x > 0$ in the given domain

$\sin^2 x + 2\sin x \cos x + \cos^2 x < 1$

$1 + 2\sin x \cos x < 1$

$2\sin x \cos x < 0$

But $\sin x, \cos x > 0$ for $0 < x < \frac{\pi}{2}$ so this is a contradiction, hence $\sin x + \cos x \geq 1$.

**Question 19**

Suppose by contradiction that $\sqrt{60}$ is rational.

Let $\sqrt{60} = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$, $\text{HCF}(p,q) = 1$ $*$

$\therefore 60 = \frac{p^2}{q^2}$

$60q^2 = p^2$

$30$ divides the LHS, so $30$ divides $p^2$, so $30$ divides $p$ since $30$ is square−free.

Let $p = 30k$ for some $k \in \mathbb{Z}$

$\therefore 60q^2 = (30k^2)$

$q^2 = 15k^2$

$15$ divides the RHS, so $15$ divides $q^2$, so $15$ divides $q$ since $15$ is square−free.

$\therefore p$ and $q$ have a common factor of $15$ which contradicts $\text{HCF}(p,q) = 1$ $(*)$

$\therefore \sqrt{60}$ must be irrational

**Question 20**

Suppose by contradiction that $a$ is rational, $b$ is irrational and $a+b$ is rational $(*)$.

Let $a = \frac{p}{q}, a+b = \frac{j}{k}$ for $p, q, j, k \in \mathbb{Z}$, $q, k \neq 0$, $\text{HCF}(p,q) = 1, \text{HCF}(j,k) = 1$

$\therefore \frac{p}{q} + b = \frac{j}{k}$

$b = \frac{j}{k} - \frac{p}{q}$

$= \frac{jq - kp}{kq}$

$= \frac{m}{n}$ for $m, n \in \mathbb{Z}$ since $p, q, j, k \in \mathbb{Z}$

$\therefore b$ is rational #

This contradicts $(*)$ since $b$ cannot be rational and irrational, so if $a$ is rational and $b$ is irrational, then $a+b$ is irrational.

**Question 21**

Suppose by contradiction that $b$ and $b+1$ are both divisible by $a$.

Let $b = ma$ $(1)$ and $b+1 = na$ $(2)$ for $m, n \in \mathbb{Z}$.

$\therefore ma + 1 = na$

$na - ma = 1$

$n - m = \frac{1}{a}$ #

This is a contradiction since the LHS is an integer but the RHS is not since $a > 1$,

$\therefore$ for positive integers $a, b$ and $a > 1$, $b$ is not divisible by $a$ or $b+1$ is not divisible by $a$.

**Question 22**

Suppose by contradiction that $x^2 - y^2 = 1$ has solutions $x, y \in \mathbb{Z}^+$ $(*)$

$\therefore (x+y)(x-y) = 1$

$x + y = 1$ and $x - y = 1$ since $x, y \in \mathbb{Z}^+$ #

This contradicts $(*)$ since $x + y = 1$ has no positive integral solutions, so there are no positive integers $x, y$ such that $x^2 - y^2 = 1$.

**Question 23**

Let $a = 2m + j, b = 2n + j$ for $m, n \in \mathbb{Z}$ and $j \in \{0,1\}$

$a + b = 2m + j + 2n + j$

$= 2(m + n + j)$

$= 2p$ for $p \in \mathbb{Z}$

$\therefore$ if $a, b$ have the same parity then $a + b$ is even

Conversely, we will show by contradiction that if the sum of two integers is even they must have the same parity.

Suppose by contradiction that $a, b$ have opposite parity and their sum is even $(*)$

Let $a = 2m + j, b = 2n + k$ for $m, n \in \mathbb{Z}$ and $j, k \in \{0,1\}$ and $j \neq k$

$a + b = 2m + 2n + j + k$

$= 2(m + n) + 1$

$= 2p + 1$ for $p \in \mathbb{Z}$

$\therefore a + b$ is odd #

This contradicts $(*)$ as $a + b$ cannot be odd and even.

$\therefore$ if two numbers have the same parity then their sum must be even.

$\therefore$ the sum of two integers is even if and only if they have the same parity (both odd or both even)

**Question 24**

Suppose by contradiction that $x = \frac{p}{q}$ for $p, q \in \mathbb{Z}$, $q \neq 0$, HCF$(p, q) = 1$ is a solution $*$

$\therefore \left(\frac{p}{q}\right)^3 + 3\left(\frac{p}{q}\right) + 5 = 0$

$p^3 + 3pq^2 + 5q^3 = 0$

Now the RHS is even

If $p$ and $q$ are odd then LHS = odd + odd + odd = odd which is a contradiction, so no solution.

If $p$ is odd and $q$ is even then LHS = odd + even + even = odd which is a contradiction, so no solution.

If $p$ is even and $q$ is odd then LHS = even + even + odd = odd which is a contradiction, so no solution.

If $p$ and $q$ are both even then they have a common factor of 2 which contradicts $(*)$

$\therefore$ there are no rational solutions to $x^3 + 3x + 5 = 0$

**Question 25**

Since $n$ is composite, let $n = ab$ for integers $1 < a, b < n$.

Suppose, by contradiction, that $a > \sqrt{n}$ and $b > \sqrt{n}$.

$\therefore ab > n$ which is a contradiction.

$\therefore$ all composite integers $n$ have a prime divisor $k$ where $k \leq \sqrt{n}$.

**Question 1**

7 is prime and it is divisible by 7, so the statement is false.

**Question 2**

$a < 0$

$\Delta = 2^2 - 4(-1)(-2)$

$= -4$

$< 0$

The quadratic is negative definite

$\therefore \forall x \in \mathbb{R},\ -x^2 + 2x - 2 < 0$, so the statement is false.

**Question 3**

Let $a = -2\ b = -3$

$\therefore a - b = -2 - (-3)$

$= 1$

$> 0$

$a^2 - b^2 = (-2)^2 - (-3)^2$

$= -5$

$< 0$

$a - b > 0$ and $a^2 - b^2 < 0$, so the statement is false.

**Question 4**

Let $a = -2, b = 1$

$a^2 - b^2 = (-2)^2 - (1)^2$

$= 3$

$> 0$

$a - b = -2 - 1$

$= -3$

$< 0$

$a^2 - b^2 > 0$ and $a - b < 0$, so the statement is false.

**Question 5**

The correct answer must be an increasing function in the whole domain, which only occurs for B.

**ANSWER B**

**Question 6**

Let $n = 3$

$3^3 + 4^3$

$= 27 + 64$

$= 91$

$< 125$

$= 5^3$

$\therefore 3^n + 4^n < 5^n$ for $n = 3$

$\therefore$ the statement is true.

**Question 7**

Let $x = -3$

$|2x + 1| = |2(-3) + 1|$

$= 5$

$\leq 5$

$|x| = |-3|$

$= 3$

$> 2$

$|2x + 1| \leq 5$ and $|x| > 2$, so the statement is false.

**Question 8**

Let the three consecutive even numbers be $2k, 2k + 2$ and $2k + 4$ for $k \in \mathbb{Z}$.

$(2k)^2 + (2k + 2)^2 + (2k + 4)^2$

$= 4k^2 + 4k^2 + 8k + 4 + 4k^2 + 16k + 16$

$= 12k^2 + 24k + 20$

$= 4(3k^2 + 6k + 5)$

$= 4p$ for $p \in \mathbb{Z}$ since $k \in \mathbb{Z}$

The statement is true.

**Question 9**

Here we will use a proof by contradiction to prove it is false for all positive integers.

Let the Pythagorean triad consist of two even integers $a, b$ and an odd integer $c$, such that $a = 2i, b = 2j$ and $c = 2k + 1$ for $i, j, k \in \mathbb{Z}$.

$\therefore (2i)^2 + (2j)^2 = (2k + 1)^2$

$4i^2 + 4j^2 = 4k^2 + 4k + 1$

$4(i^2 + j^2) = 4(k^2 + k) + 1$

$4m = 4n + 1$ for $m, n \in \mathbb{Z}$ since $i, j, k$

Now the LHS is a multiple of 4 yet the RHS isn't so we have a contradiction, so there is no Pythagorean triad where the two smallest numbers are even and the largest number is odd.

**Question 10**

True. Any number divisible by 6 has at least the two prime factors 2 and 3, so cannot be prime.

**Question 11**

False. $3^5 + 2 = 245$ which is divisible by 5.

**Question 12**

$\neg P: \forall m > 0, m^2 - 9 \neq 0$

The negation is false, since for $m = 3$, $m^2 - 9 = 0$.

**Question 13**

Let $x = \sqrt{2}, y = 2\sqrt{2}$

$\therefore xy = \sqrt{2} \times 2\sqrt{2}$

$= 4$

Which is rational, so the statement is false.

**Question 14**

The counterexample needs to be the negation of $P \Rightarrow Q$, which is $P$ and $\neg Q$, so in this case an integer that is 'not a perfect square and is prime'.

**ANSWER A**

**Question 15**

We are looking for the option where the contrapositive is false. The contrapositive is: $\forall n \in Z$, if $z$ is purely imaginary then $z^n$ is purely imaginary.

This will be disproved by an example which negates the contrapositive, so where $z$ is purely imaginary and $z^n$ is not purely imaginary.

This occurs with B since $z = 0 + i = i$, which is purely imaginary, while $z^4 = 1$, which is not purely imaginary.

**ANSWER B**

**Question 16**

Let $n = 1$.

$n^2 + 3n + 1 = 1 + 3 + 1 = 5$

Since $5$ is prime, but $n = 1$ is odd, the statement is false.

**Question 17**

Suppose there exist three consecutive positive integers whose squares sum to a perfect square.

Let the integers be $n - 1$, $n$, $n + 1$, where $n \geq 2$.

Then, for some positive integer $m$, $(n-1)^2 + n^2 + (n+1)^2 = m^2$.

Expanding and simplifying, $3n^2 + 2 = m^2$.

Every integer can be expressed in one of the forms $3k$, $3k + 1$ or $3k + 2$, where $k \in \mathbb{Z}$.

Case 1: $m = 3k$

$m^2 = 9k^2 = 3(3k^2)$.

Case 2: $m = 3k + 1$

$m^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$.

Case 3: $m = 3k + 2$

$m^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$.

Therefore, the square of any integer is either a multiple of $3$ or one more than a multiple of $3$.

However, $m^2 = 3n^2 + 2$ is two more than a multiple of $3$, which is a contradiction.

Therefore, no three consecutive positive integers have a sum of squares that is a perfect square, so the statement is false.

**Question 18**

Let $n = 1$.

Then $n^2 + 2 = 1 + 2 = 3$, which is divisible by $3$.

However, $n^2 + 5 = 1 + 5 = 6$, which is not divisible by $9$.

Therefore, the statement is false.

**Question 19**

Let $a = \sqrt{2}$, $b = -\sqrt{2}$.

Both $a$ and $b$ are irrational.

However, $a + b = \sqrt{2} - \sqrt{2} = 0$ and $ab = \sqrt{2}(-\sqrt{2}) = -2$.

Since $0$ and $-2$ are rational, the statement is true.

**Question 20**

Suppose there exist positive integers $a$ and $b$ such that $a^2 + b^2 = 4ab + 3$.

$\therefore a^2 + b^2 - 4ab = 3$.

Case 1: Both $a$ and $b$ are even.

Let $a = 2m$ and $b = 2n$, where $m, n \in \mathbb{Z}$.

$4m^2 + 4n^2 - 16mn = 3$

$4(m^2 + n^2 - 4mn) = 3$

The LHS is divisible by $4$, but the RHS is not, which is a contradiction.

Case 2: Both $a$ and $b$ are odd.

Let $a = 2m + 1$ and $b = 2n + 1$.

$(2m+1)^2 + (2n+1)^2 - 4(2m+1)(2n+1) = 3$

$4m^2 + 4m + 1 + 4n^2 + 4n + 1 - 4(4mn + 2m + 2n + 1) = 3$

$4(m^2 + n^2 - 4mn - m - n) - 2 = 3$

$4(m^2 + n^2 - 4mn - m - n) = 5$

The LHS is divisible by $4$, but the RHS is not, which is a contradiction.

Case 3: One of $a$ and $b$ is odd and the other is even.

Without loss of generality, let $a = 2m + 1$ and $b = 2n$.

$(2m+1)^2 + (2n)^2 - 4(2m+1)(2n) = 3$

$4m^2 + 4m + 1 + 4n^2 - 16mn - 8n = 3$

$4(m^2 + m + n^2 - 4mn - 2n) = 2$

The LHS is divisible by $4$, but the RHS is not, which is a contradiction.

Since all possible cases lead to a contradiction, there are no positive integers $a$ and $b$ satisfying the equation.

Therefore, the statement is false.

**Question 1a**
stays the same, as adding any number is equivalent to an increasing function
**Question 1b**
stays the same, as dividing by a positive number is equivalent to an increasing function
**Question 1c**
reverses, as multiplying by a negative number is equivalent to a decreasing function
**Question 1d**
we need more information, such as restricting the domain, since sine is not a monotonic function the whole domain
**Question 1e**
stays the same, as taking a natural logarithm is equivalent to an increasing function
**Question 2a**
$m > n > 0$
$\sqrt{m} > \sqrt{n}$
**Question 2b**
$p < 0 < n$
$-p > 0 > n$
$-n < -p$
**Question 2c**
$r < p$
$-n > -m$
There is no way to determine the relative size of $r - n$ and $p - m$
**Question 2d**
$r < p$
$n < m$
There is no way to determine the relative size of $\frac{r}{n}$ and $\frac{p}{m}$
**Question 2e**
$m > n > 0$
$\frac{1}{m} < \frac{1}{n}$
$-\frac{2}{m} > -\frac{2}{n}$
**Question 2f**
$m > n$
$-r > -p$
$m - r > n - p$
$(m - r)^3 > (n - p)^3$
since $x^3$ is an increasing function
**Question 2g**
$m > n$
$r^2 > p^2$
$\therefore mr^2 > np^2$
**Question 2h**
$m > n$
$p > r$
$mp > nr$
$\therefore \frac{1}{mp} < \frac{1}{nr}$
**Question 2i**
$m > 0 > p$
$m^3 > p^3$
**Question 2j**
$n^2 < m^2$
$p^2 < r^2$
$n^2 + p^2 < m^2 + r^2$
**Question 3**
LHS − RHS
$= \frac{1}{b} - \frac{1}{a}$
$= \frac{a - b}{ab}$
$> 0$ since $a > b > 0$
$\therefore -\frac{1}{a} > -\frac{1}{b}$
Alternative I:
Suppose by contradiction that
$-\frac{1}{a} \leq -\frac{1}{b}$
$-a \geq -b$
$a \leq b$
Which is a contradiction since $a > b$
$\therefore -\frac{1}{a} > -\frac{1}{b}$.
Alternative II:
$a > b$
$\frac{1}{a} < \frac{1}{b}$ since $a > b > 0$
$-\frac{1}{a} > -\frac{1}{b}$
**Question 4**
LHS − RHS
$= (2 + a)(2 - a) - (2 + b)(2 - b)$
$= \left(4 - a^2\right) - \left(4 - b^2\right)$
$= b^2 - a^2$
$< 0$ since $a > b > 0$
$\therefore (2 + a)(2 - a) < (2 + b)(2 - b)$
Alternative I:
Suppose by contradiction that
$(2 + a)(2 - a) \geq (2 + b)(2 - b)$
$4 - a^2 \geq 4 - b^2$
$-a^2 \geq -b^2$
$a^2 \leq b^2$
$a \leq b$ since $a, b > 0$
Which is a contradiction since $a > b$
$\therefore (2 + a)(2 - a) < (2 + b)(2 - b)$
Alternative II:
$a > b$
$a^2 > b^2$ since $a > b > 0$
$-a^2 < -b^2$
$4 - a^2 < 4 - b^2$
$(2 + a)(2 - a) < (2 + b)(2 - b)$

**Question 5a**

LHS

$= (a+b+c)^2 - (ab+bc+ac)$

$= a^2+b^2+c^2+2(ab+bc+ac)$
$\quad -(ab+bc+ac)$

$= a^2+b^2+c^2+ab+bc+ac$

$> 0+0+0+0+0+0$ since $a, b, c > 0$

$= 0$

$\therefore (a+b+c)^2 - (ab+bc+ac) > 0$

Alternative I:

Suppose by contradiction that

$$(a+b+c)^2 - (ab+bc+ac) \leq 0$$
$$a^2+b^2+c^2+2(ab+bc+ac) - (ab+bc+ac) \leq 0$$
$$a^2+b^2+c^2+ab+bc+ac \leq 0$$

Which is a contradiction since $a, b, c > 0$

$\therefore (a+b+c)^2 - (ab+bc+ac) > 0.$

Alternative II:

$$a^2+b^2+c^2+ab+bc+ac > 0 \text{ since } a, b, c > 0$$
$$a^2+b^2+c^2+2(ab+bc+ac) - (ab+bc+ac) > 0$$
$$(a+b+c)^2 - (ab+bc+ac) > 0$$

**Question 5b**

LHS $= a^2+b^2+c^2-(ab+bc+ca)$

$= \frac{1}{2}\left(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2\right)$

$= \frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$

$\geq \frac{1}{2}(0+0+0)$ *since* $\mathbb{R}^2 \geq 0$

$= 0$

$\therefore a^2+b^2+c^2-(ab+bc+ac) \geq 0$

Alternative I:

Suppose by contradiction that

$$a^2+b^2+c^2-(ab+bc+ac) < 0$$
$$\frac{1}{2}\left(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2\right) < 0$$
$$\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right) < 0$$

Which is a contradiction since $\mathbb{R}^2 \geq 0$

$\therefore a^2+b^2+c^2-(ab+bc+ac) \geq 0.$

Alternative II:

$\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right) \geq 0$ since $\mathbb{R}^2 \geq 0$

$$a^2+b^2+c^2-(ab+bc+ca) \geq 0$$

**Question 6**

$|a-b|+|b-a|$

$> |(a-b)+(b-a)|$ triangle inequality and since $a > b$

$= |0|$

$= 0$

$\therefore |a-b|+|b-a| > 0$ for $a > b > 0$.

**Question 7**

$(a+2b)(a-2b)$

$= a^2-4b^2$

$= \left|a^2-4b^2\right|$ since $a > 2b > 0$

$= \left|\left(a^2-2ab\right)+\left(2ab-4b^2\right)\right|$ triangle inequality

$\leq \left|a^2-2ab\right|+\left|2ab-4b^2\right|$

$= \left|a^2-2ab\right|+\left|4b^2-2ab\right|$

**Question 8**

If $x$ is the smallest side then $x + 5 > 7$, so $x > 2$.
If $x$ is the longest side then $5 + 7 > x$, so $x < 12$
$\therefore 2 < x < 12$.

**Question 9**

LHS – RHS
$= x^5 - x^4y + y^5 - xy^4$
$= x^4(x - y) + y^4(y - x)$
$= x^4(x - y) - y^4(x - y)$
$= (x^4 - y^4)(x - y)$
$= (x^2 + y^2)(x^2 - y^2)(x - y)$
$= (x^2 + y^2)(x + y)(x - y)(x - y)$
$= (x^2 + y^2)(x + y)(x - y)^2$
$\geq 0$ since all three expressions are non−negative
$\therefore x^5 + y^5 \geq x^4y + xy^4$

Alternative I:
Suppose by contradiction that

$$x^5 + y^5 < x^4y + xy^4$$
$$x^5 - x^4y + y^5 - xy^4 < 0$$
$$x^4(x - y) - y^4(x - y) < 0$$
$$(x^4 - y^4)(x - y) < 0$$
$$(x^2 + y^2)(x^2 - y^2)(x - y) < 0$$
$$(x^2 + y^2)(x + y)(x - y)^2 < 0$$

Which is a contradiction since $\mathbb{R}^2 \geq 0$ and $x, y > 0$
$\therefore x^5 + y^5 \geq x^4y + xy^4$

Alternative II:

$$(x^2 + y^2)(x + y)(x - y)^2 \geq 0 \quad \text{since } \mathbb{R}^2 \geq 0 \text{ and } x, y > 0$$
$$(x^2 + y^2)(x^2 - y^2)(x - y) \geq 0$$
$$(x^4 - y^4)(x - y) \geq 0$$
$$x^4(x - y) - y^4(x - y) \geq 0$$
$$x^5 - x^4y + y^5 - xy^4 \geq 0$$
$$x^5 + y^5 \geq x^4y + xy^4$$

**Question 10**

LHS $= 1 - \sin 2x$
$= \sin^2 x + \cos^2 x - 2 \sin x \cos x$
$= (\sin x - \cos x)^2$
$\geq 0$ since $\mathbb{R}^2 \geq 0$
$\therefore 1 - \sin 2x \geq 0.$

Alternative I:
Suppose by contradiction that
$1 - \sin 2x < 0$
$\sin 2x > 1$
Which is a contradiction since sine $\leq 1$
$\therefore 1 - \sin 2x \geq 0.$

Alternative II:
$\sin 2x \leq 1$ since sine $\leq 1$
$0 \leq 1 - \sin 2x$
$1 - \sin 2x \geq 0$

**Question 11**

LHS $= a^2 + b^2 - 2a - 2b + 2$
$= a^2 - 2a + 1 + b^2 - 2b + 1$
$= (a - 1)^2 + (b - 1)^2$
$\geq 0 + 0$
$= 0$
$\therefore a^2 + b^2 - 2(a + b) + 2 \geq 0$

Alternative I:
Suppose by contradiction that
$a^2 + b^2 - 2(a + b) + 2 < 0$
$a^2 - 2a + 1 + b^2 - 2b + 1 < 0$
$(a - 1)^2 + (b - 1)^2 < 0$
Which is a contradiction since $\mathbb{R}^2 \geq 0$
$\therefore a^2 + b^2 - 2(a + b) + 2 \geq 0.$

Alternative II:
$(a - 1)^2 + (b - 1)^2 \geq 0$ since $\mathbb{R}^2 \geq 0$
$a^2 - 2a + 1 + b^2 - 2b + 1 \geq 0$
$a^2 + b^2 - 2(a + b) + 2 \geq 0$

**Question 12**

$\text{LHS} - \text{RHS} = a^2 - bc - b^2 - c^2$

$= (b+c)^2 - bc - b^2 - c^2$

$= b^2 + 2bc + c^2 - bc - b^2 - c^2$

$= bc$

$> 0$ since $b, c > 0$

$\therefore a^2 - bc > b^2 + c^2$

Alternative I:

Suppose by contradiction that

$a^2 - bc \leq b^2 + c^2$

$(b+c)^2 - bc \leq b^2 + c^2$

$b^2 + 2bc + c^2 - bc \leq b^2 + c^2$

$bc \leq 0$

Which is a contradiction since $b, c > 0$

$\therefore a^2 - bc > b^2 + c^2$

Alternative II:

$bc > 0$ since $b, c > 0$

$b^2 + bc + c^2 > b^2 + c^2$

$(b+c)^2 - bc > b^2 + c^2$

$a^2 - bc > b^2 + c^2$ since $a = b + c$

**Question 13**

$\text{LHS} = a\left(a^2 - b^2\right) - b\left(a^2 - b^2\right)$

$= (a-b)\left(a^2 - b^2\right)$

$= (a-b)(a+b)(a-b)$

$= (a-b)^2(a+b)$

$\geq 0$ since $\mathbb{R}^2 \geq 0$ and $a, b > 0$

$\therefore a^3 - ab^2 - a^2b + b^3 \geq 0.$

Alternative I:

Suppose by contradiction that

$a^3 - ab^2 - a^2b + b^3 < 0$

$a\left(a^2 - b^2\right) - b\left(a^2 - b^2\right) < 0$

$\left(a^2 - b^2\right)(a-b) < 0$

$(a+b)(a-b)^2 < 0$

Which is a contradiction since $\mathbb{R}^2 \geq 0$ and $a, b > 0$

$\therefore a^3 - ab^2 - a^2b + b^3 \geq 0.$

Alternative II:

$(a+b)(a-b)^2 \geq 0$ since $\mathbb{R}^2 \geq 0$ and $a, b > 0$

$\left(a^2 - b^2\right)(a-b) \geq 0$

$a^3 - ab^2 - a^2b + b^3 \geq 0$

**Question 14**

$x > \sin x \quad (1)$

$\pi - x > \sin(\pi - x)$ also from $(1)$ since $\pi - x > 0$ in the domain

$\therefore \pi - x > \sin x$ since $\sin(\pi - x) = \sin x \quad (2)$

$(1) \times (2)$:

$\pi x - x^2 > \sin^2 x$

**Question 15**

If the sides are $4, 4$ and $6$ then the sides satisfy the triangle inequality in any order: $4 + 4 > 6$, $4 + 6 > 4$, $6 + 4 > 4$

If the sides are $4, 6$ and $6$ then the sides also satisfy the triangle inequality in any order: $4 + 6 > 6$, $6 + 4 > 6$, $6 + 6 > 4$

The third side of the isosceles triangle can be $4$ cm or $6$ cm.

**Question 16**

$$\begin{aligned}\text{LHS} - \text{RHS} &= a^2 + b^2 - 4a - 4b + 8 \\ &= a^2 - 4a + 4 + b^2 - 4b + 4 \\ &= (a-2)^2 + (b-2)^2 \\ &\geq 0 + 0 \\ &= 0\end{aligned}$$

$\therefore a^2 + b^2 + 8 \geq 4(a+b)$.

Alternative I:

Suppose by contradiction that

$$\begin{aligned}a^2 + b^2 + 8 &< 4(a+b) \\ a^2 + b^2 - 4a - 4b + 8 &< 0 \\ a^2 - 4a + 4 + b^2 - 4b + 4 &< 0 \\ (a-2)^2 + (b-2)^2 &< 0\end{aligned}$$

Which is a contradiction since $\mathbb{R}^2 \geq 0$

$\therefore a^2 + b^2 + 8 \geq 4(a+b)$.

Alternative II:

$(a-2)^2 + (b-2)^2 \geq 0$ since $\mathbb{R}^2 \geq 0$

$$\begin{aligned}a^2 - 4a + 4 + b^2 - 4b + 4 &\geq 0 \\ a^2 + b^2 + 8 &\geq 4(a+b)\end{aligned}$$

**Question 17**

$$\begin{aligned}\text{LHS} - \text{RHS} &= \frac{1}{x} + \frac{1}{y} - \frac{4}{x+y} \\ &= \frac{y(x+y) + x(x+y) - 4xy}{xy(x+y)} \\ &= \frac{xy + y^2 + x^2 + xy - 4xy}{xy(x+y)} \\ &= \frac{x^2 - 2xy + y^2}{xy(x+y)} \\ &= \frac{(x-y)^2}{xy(x+y)} \\ &\geq 0\end{aligned}$$

$\therefore \frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$

Alternative I:

Suppose by contradiction that

$$\begin{aligned}\frac{1}{x} + \frac{1}{y} &< \frac{4}{x+y} \\ \frac{x+y}{xy} &< \frac{4}{x+y} \\ (x+y)^2 &< 4xy \\ x^2 + 2xy + y^2 &< 4xy \\ x^2 - 2xy + y^2 &< 0 \\ (x-y)^2 &< 0\end{aligned}$$

Which is a contradiction since $\mathbb{R}^2 \geq 0$

$\therefore \frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$.

Alternative II:

$$\begin{aligned}(x-y)^2 &\geq 0 \text{ since } \mathbb{R}^2 \geq 0 \\ x^2 - 2xy + y^2 &\geq 0 \\ x^2 + 2xy + y^2 &\geq 4xy \\ (x+y)^2 &\geq 4xy \\ \frac{x+y}{xy} &\geq \frac{4}{x+y} \\ \frac{1}{x} + \frac{1}{y} &\geq \frac{4}{x+y}\end{aligned}$$

**Question 18**

$$\begin{aligned}\text{LHS} - \text{RHS} &= \frac{2k+2}{2k+3} - \frac{2k}{2k+1} \\ &= \frac{(2k+2)(2k+1) - 2k(2k+3)}{(2k+1)(2k+3)} \\ &= \frac{4k^2 + 6k + 2 - 4k^2 - 6k}{(2k+1)(2k+3)} \\ &= \frac{2}{(2k+1)(2k+3)} \\ &> 0 \quad \text{since } k \in \mathbb{Z}^+\end{aligned}$$

$\therefore \frac{2k+2}{2k+3} > \frac{2k}{2k+1}$

Alternative I:

Suppose by contradiction that

$$\begin{aligned}\frac{2k+2}{2k+3} &\leq \frac{2k}{2k+1} \\ (2k+2)(2k+1) &\leq 2k(2k+3) \\ 4k^2 + 6k + 2 &\leq 4k^2 + 6k \\ 2 &\leq 0\end{aligned}$$

Which is a contradiction

$\therefore \frac{2k+2}{2k+3} > \frac{2k}{2k+1}$.

**Question 19**

$$\begin{aligned}&\text{LHS} - \text{RHS} \\ &= \frac{x^2 + y^2}{2} - \frac{x^2 + 2xy + y^2}{4} \\ &= \frac{2x^2 + 2y^2 - x^2 - 2xy - y^2}{4} \\ &= \frac{x^2 - 2xy + y^2}{4} \\ &= \left(\frac{x}{2} - \frac{y}{2}\right)^2 \\ &\geq 0\end{aligned}$$

$\therefore \frac{x^2 + y^2}{2} \geq \left(\frac{x+y}{2}\right)^2$

Alternative I:

Suppose by contradiction that

$$\begin{aligned}\frac{x^2 + y^2}{2} &< \left(\frac{x+y}{2}\right)^2 \\ \frac{x^2 + y^2}{2} &< \frac{x^2 + 2xy + y^2}{4} \\ 2x^2 + 2y^2 &< x^2 + 2xy + y^2 \\ x^2 - 2xy + y^2 &< 0 \\ (x-y)^2 &< 0\end{aligned}$$

Which is a contradiction since $\mathbb{R}^2 \geq 0$.

$\therefore \frac{x^2 + y^2}{2} \geq \left(\frac{x+y}{2}\right)^2$.

Alternative II:

$$\begin{aligned}\left(\frac{x}{2} - \frac{y}{2}\right)^2 &\geq 0 \text{ since } \mathbb{R}^2 \geq 0 \\ \frac{x^2 - 2xy + y^2}{4} &\geq 0 \\ \frac{2x^2 + 2y^2 - x^2 - 2xy - y^2}{4} &\geq 0 \\ \frac{x^2 + y^2}{2} &\geq \frac{x^2 + 2xy + y^2}{4} \\ \frac{x^2 + y^2}{2} &\geq \left(\frac{x+y}{2}\right)^2\end{aligned}$$

**Question 20**

$$\begin{aligned}|x^2-x|+|x-1| &\geq |(x^2-x)+(x-1)| \\ &\geq |x^2-1| \\ &\geq |(x+1)(x-1)|\end{aligned}$$

**Question 21**

LHS – RHS

$$\begin{aligned}&= a^2+b^2+c^2-ab-bc-ca \\ &= \frac{1}{2}\left[(a^2-2ab+b^2)+(b^2-2bc+c^2)\right. \\ &\left.+(c^2-2ac+c^2)\right] \\ &\geq \frac{1}{2}[0+0+0] \\ &= 0\end{aligned}$$

$\therefore a^2+b^2+c^2 \geq ab+bc+ac$

Equality occurs when $a=b=c$, so if $a^2+b^2+c^2=ab+bc+ac$ then $a=b=c$ so $\Delta ABC$ is equilateral.

**Question 22**

In $\Delta AXB$, $AX+BX>AB$ $(1)$ (triangle inequality)

Similarly in $\Delta BXC$ and $\Delta AXC$:

$BX+CX>BC$ $(2)$ and $AX+CX>AC$ $(3)$

$(1)+(2)+(3)$:

$$2(AX+BX+CX)>AB+BC+AC$$

$$AX+BX+CX>\frac{AB+AC+BC}{2}$$

**Question 23**

$$\begin{aligned}&\frac{(x+y)^2}{x^2+y^2}-2 \\ &= \frac{x^2+2xy+y^2-2x^2-2y^2}{x^2+y^2} \\ &= \frac{-x^2+2xy-y^2}{x^2+y^2} \\ &= -\frac{(x-y)^2}{x^2+y^2} \leq 0\end{aligned}$$

$$\therefore \frac{(x+y)^2}{x^2+y^2} \leq 2.$$

Alternative I:

Suppose by contradiction that

$$\begin{aligned}\frac{(x+y)^2}{x^2+y^2} &> 2 \\ x^2+2xy+y^2 &> 2x^2+2y^2 \\ x^2-2xy+y^2 &< 0 \\ (x-y)^2 &< 0\end{aligned}$$

Which is a contradiction since $\mathbb{R}^2 \geq 0$

$$\therefore \frac{(x+y)^2}{x^2+y^2} \leq 2.$$

Alternative II:

$$\begin{aligned}(x-y)^2 &\geq 0 \text{ since } \mathbb{R}^2 \geq 0 \\ x^2-2xy+y^2 &\geq 0 \\ x^2+y^2 &\geq 2xy \\ 2x^2+2y^2 &\geq x^{\cdot 2}+2xy+y^2 \\ 2(x^2+y^2) &\geq (x+y)^2 \\ \frac{(x+y)^2}{x^2+y^2} &\leq 2\end{aligned}$$

**Question 23 (continued)**

Alternative III (uses AM−GM from next lesson):

$$\begin{aligned}\frac{(x+y)^2}{x^2+y^2} &= \frac{x^2+2xy+y^2}{x^2+y^2} \\ &= \frac{x^2+y^2}{x^2+y^2}+\frac{2xy}{x^2+y^2} \\ &= 1+\frac{2xy}{x^2+y^2}\end{aligned}$$

$x^2+y^2 \geq 2\sqrt{x^2y^2}$ (AM−GM)

$x^2+y^2 \geq 2xy$

$$\frac{2xy}{x^2+y^2} \leq 1$$

$$\therefore \frac{(x+y)^2}{x^2+y^2} \leq 1+1$$

$$= 2$$

**Question 24a**

$$\begin{aligned}x-1+3x+1 &> 5-x \\ 6x &> 5 \\ x &> \frac{5}{6} \quad (1) \\ 2x-1+5-x &> 3x+1 \\ 3 &> 2x \\ x &< \frac{3}{2} \quad (2) \\ 3x+1+5-x &> 2x-1 \\ 6 &> -1 \quad \text{true}\end{aligned}$$

From $(1)$ and $(2)$:

$$\frac{5}{6}<x<\frac{3}{2}$$

**Question 24b**

If $2x-1=3x+1 \rightarrow x=-2$ which is outside the domain from (a) so $2x-1 \neq 3x+1$

If $2x-1=5-x \rightarrow x=2$ which is outside the domain from (a) so $2x-1 \neq 5-x$

If $5-x=3x+1 \rightarrow 4x=4 \rightarrow x=1, 5-1=3(1)+1=4$ and $2(1)-1=1$

$$\begin{aligned}\cos\theta &= \frac{1^2+4^2-4^2}{2\times 1\times 4} \\ &= \frac{1}{8}\end{aligned}$$

**Question 25**

$$\begin{aligned}\text{LHS} - \text{RHS} &= (x-3)(x+1)(x-5)(x+3)+40\\ &= \left(x^2-2x-3\right)\left(x^2-2x-15\right)+40\\ &= \left(x^2-2x-9+6\right)\left(x^2-2x-9-6\right)+40\\ &= \left(x^2-2x-9\right)^2-6^2+40\\ &= \left(x^2-2x-9\right)^2+4\end{aligned}$$

$> 0$ since the expressions are non−negative and positive respectively

$\therefore (x-3)(x+1)(x-5)(x+3)+40>0$

**Question 26**

$$\begin{aligned}\text{LHS} - \text{RHS} &= \left(a^2-b^2\right)\left(x^2-y^2\right)-(ax-by)^2\\ &= a^2x^2-a^2y^2-b^2x^2+b^2y^2-a^2x^2+2abxy-b^2y^2\\ &= -\left(a^2y^2-2abxy+b^2x^2\right)\\ &= -(ay-bx)^2\\ &\leq 0\end{aligned}$$

$\therefore \left(a^2-b^2\right)\left(x^2-y^2\right) \leq (ax-by)^2$

**Question 27a**

$(a-b)^2>0$ since $a \neq b$

$a^2-2ab+b^2>0$

$a^2-ab+b^2>ab \qquad (1)$

$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$

$\therefore a^3+b^3>(a+b)ab$ using $(1)$

Alternative solution I:

$$\begin{aligned}a^3+b^3 &= (a+b)\left(a^2-ab+b^2\right)\\ &= (a+b)\left(a^2-2ab+b^2+ab\right)\\ &= (a+b)[(a-b)^2+ab]\\ &> (a+b)[0+ab] \text{ since } \mathbb{R}^2 \geq 0, a \neq b\\ &= ab(a+b)\end{aligned}$$

Alternative solution II:

$$\begin{aligned}\text{LHS} - \text{RHS} &= a^3+b^3-a^2b-ab^2\\ &= a\left(a^2-b^2\right)-b\left(a^2-b^2\right)\\ &= (a-b)\left(a^2-b^2\right)\\ &= (a-b)(a+b)(a-b)\\ &= (a-b)^2(a+b)\\ &> 0 \text{ since } \mathbb{R}^2 \geq 0 \text{ and } a,b>0\end{aligned}$$

$\therefore$ LHS $>$ RHS

$\therefore\ a^3+b^3>ab(a+b)$

Alternative solution III:

$$\begin{aligned}\text{LHS} - \text{RHS} &= a^3+b^3-ab(a+b)\\ &= (a+b)\left(a^2-ab+b^2\right)-ab(a+b)\\ &= (a+b)(a^2-ab+b^2-ab)\\ &= (a+b)(a^2-2ab+b^2)\\ &= (a+b)(a-b)^2\\ &> 0 \text{ since } \mathbb{R}^2 \geq 0 \text{ and } a,b>0\end{aligned}$$

$\therefore$ LHS $>$ RHS

$\therefore\ a^3+b^3>ab(a+b)$

**Question 27b**

Similarly from (a)

$a^3+c^3>ac(a+c)$

$b^3+c^3>bc(b+c)$

Adding the inequalities

$2\left(a^3+b^3+c^3\right)>ab(a+b)+ac(a+c)+bc(b+c)$

$2\left(a^3+b^3+c^3\right)>a^2b+ab^2+a^2c+ac^2+b^2c+bc^2$

$2\left(a^3+b^3+c^3\right)>a^2(b+c)+b^2(a+c)+c^2(a+b)$

**Question 28**

Suppose by contradiction that $\frac{x+s}{n} \leq \frac{y+s}{m}$.

$$\therefore \frac{x+s}{\sqrt{x^2+s^2}} \leq \frac{y+s}{\sqrt{y^2+s^2}} \quad \text{(Pythagoras' theorem)}$$

$$\frac{(x+s)^2}{x^2+s^2} \leq \frac{(y+s)^2}{y^2+s^2} \quad \text{(since all pronumerals are positive)}$$

$$\begin{aligned}
\left(x^2+2sx+s^2\right)\left(y^2+s^2\right) &\leq \left(y^2+2sy+s^2\right)\left(x^2+s^2\right) \\
x^2y^2+s^2x^2+2sxy^2+2s^3x+s^2y^2+s^4 &\leq x^2y^2+s^2y^2+2sx^2y+2s^3y+s^2x^2+s^4 \\
2sxy^2+2s^3x &\leq 2sx^2y+2s^3y \\
xy^2+s^2x &\leq x^2y+s^2yx \\
y^2-x^2y+s^2x-s^2y &\leq 0 \\
xy(y-x)+s^2(x-y) &\leq 0 \\
xy(y-x)-s^2(y-x) &\leq 0 \\
(y-x)\left(xy-s^2\right) &\leq 0 \quad \#
\end{aligned}$$

This is a contradiction since $y-x<0$ [$x>y$ given] and $xy-s^2<0$ [the area of the square is greater than the area of the rectangle].

The supposition is incorrect, so $\frac{x+s}{n} > \frac{y+s}{m}$.

**Question 29**

$$\begin{aligned}
a^n-b^n &= (a-b)\left(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}\right) \\
&< (a-b)\underbrace{\left(a^{n-1}+a^{n-1}+a^{n-1}+\ldots+a^{n-1}\right)}_{n \text{ times}} \\
&< na^{n-1}(a-b)
\end{aligned}$$

**Question 30**

$$\begin{aligned}
\sqrt{b} \leq \frac{\sqrt{a}+\sqrt{b}}{2} &\leq \sqrt{a} \\
\frac{1}{\sqrt{a}} \leq \frac{2}{\sqrt{a}+\sqrt{b}} &\leq \frac{1}{\sqrt{b}} \quad \text{since all positive} \\
\frac{1}{a} \leq \frac{4}{\left(\sqrt{a}+\sqrt{b}\right)^2} &\leq \frac{1}{b} \quad \text{since all positive} \\
b\left(\sqrt{a}+\sqrt{b}\right)^2 \leq 4ab &\leq a\left(\sqrt{a}+\sqrt{b}\right)^2 \\
b\left((\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})\right)^2 \leq 4ab\left(\sqrt{a}-\sqrt{b}\right)^2 &\leq a\left(\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)\right)^2 \\
b(a-b)^2 \leq 4ab\left(a-2\sqrt{ab}+b\right) &\leq a(a-b)^2 \\
b(a-b)^2 \leq 4ab(a+b)-8ab\sqrt{ab} &\leq a(a-b)^2 \\
\frac{(a-b)^2}{8a} \leq \frac{a+b}{2}-\sqrt{ab} &\leq \frac{(a-b)^2}{8b}
\end{aligned}$$

**Question 31**

$$\begin{aligned}
\text{LHS}-\text{RHS} &= \frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}}-\sqrt{a}-\sqrt{b} \\
&= \frac{a\sqrt{a}+b\sqrt{b}-a\sqrt{b}-b\sqrt{a}}{\sqrt{ab}} \\
&= \frac{\sqrt{a}(a-b)-\sqrt{b}(a-b)}{\sqrt{ab}} \\
&= \frac{(a-b)\left(\sqrt{a}-\sqrt{b}\right)}{\sqrt{ab}} \\
&= \frac{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)}{\sqrt{ab}} \\
&= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2\left(\sqrt{a}+\sqrt{b}\right)}{\sqrt{ab}} \\
&\geq 0
\end{aligned}$$

$$\therefore \frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}} \geq \sqrt{a}+\sqrt{b}$$

**Question 32**

$$\begin{aligned}\text{LHS} - \text{RHS} &= a^2 + b^2 + c^2 + d^2 + e^2 - a(b + c + d + e) \\ &= \left(\frac{a^2}{4} - ab + b^2\right) + \left(\frac{a^2}{4} - ac + c^2\right) + \left(\frac{a^2}{4} - ad + d^2\right) + \left(\frac{a^2}{4} - ae + e^2\right) \\ &= \left(\frac{a}{2} - b\right)^2 + \left(\frac{a}{2} - c\right)^2 + \left(\frac{a}{2} - d\right)^2 + \left(\frac{a}{2} - e\right)^2 \\ &\geq 0\end{aligned}$$

$$\therefore a^2 + b^2 + c^2 + d^2 + e^2 \geq a(b + c + d + e)$$

**Question 33a**

$$\left(\frac{a}{A} - \frac{b}{B}\right)^2 \geq 0$$

$$\frac{a^2}{A^2} - \frac{2ab}{AB} + \frac{b^2}{B^2} \geq 0$$

$$\frac{a^2}{A^2} + \frac{b^2}{B^2} \geq \frac{2ab}{AB}$$

$$\therefore \frac{ab}{AB} \leq \frac{1}{2}\left(\frac{a^2}{A^2} + \frac{b^2}{B^2}\right)$$

**Question 33b**

$$\frac{a_k b_k}{AB} \leq \frac{1}{2}\left(\frac{a_k^2}{A^2} + \frac{b_k^2}{B^2}\right) \text{ from (i)}$$

$$\therefore \sum_{k=1}^{n} \frac{a_k b_k}{AB} \leq \frac{1}{2}\sum_{k=1}^{n}\left(\frac{a_k^2}{A^2} + \frac{b_k^2}{B^2}\right)$$

$$\frac{1}{AB}\sum_{k=1}^{n} a_k b_k \leq \frac{1}{2}\left(\frac{\sum_{k=1}^{n} a_k^2}{A^2} + \frac{\sum_{k=1}^{n} b_k^2}{B^2}\right)$$

$$\frac{1}{AB}\sum_{k=1}^{n} a_k b_k \leq \frac{1}{2}\left(\frac{A^2}{A^2} + \frac{B^2}{B^2}\right)$$

$$\sum_{k=1}^{n} a_k b_k \leq AB$$

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sqrt{\sum_{k=1}^{n} a_k^2}\right)\left(\sqrt{\sum_{k=1}^{n} b_k^2}\right)$$

$$\therefore \left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

**Question 33c**

Let $a_k = \sqrt{\dfrac{S}{S - x_k}}, \qquad b_k = \sqrt{\dfrac{S - x_k}{S}}$

$$\therefore \left(\sum_{k=1}^{n} \sqrt{\frac{S}{S - x_k}} \times \sqrt{\frac{S - x_k}{S}}\right)^2 \leq \left(\sum_{k=1}^{n} \frac{S}{S - x_k}\right)\left(\sum_{k=1}^{n} \frac{S - x_k}{S}\right)$$

$$\left(\sum_{k=1}^{n} 1\right)^2 \leq \left(\sum_{k=1}^{n} \frac{S}{S - x_k}\right) \times \frac{1}{S}\left(nS - (x_1 + x_2 + x_3 + \ldots + x_k)\right)$$

$$n^2 \leq \left(\sum_{k=1}^{n} \frac{S}{S - x_k}\right) \times \frac{1}{S}(nS - S)$$

$$n^2 \leq \left(\sum_{k=1}^{n} \frac{S}{S - x_k}\right)(n - 1)$$

$$\therefore \frac{S}{S - x_1} + \frac{S}{S - x_2} + \frac{S}{S - x_3} + \ldots + \frac{S}{S - x_n} \geq \frac{n^2}{n - 1}$$

**Question 34a**

$$(2x-y)^2 \geq 0$$
$$4x^2-4xy+y^2 \geq 0$$
$$4x^2+5xy+y^2 \geq 9xy$$
$$(x+y)(4x+y) \geq 9xy$$
$$\frac{(x+y)(4x+y)}{xy} \geq 9$$
$$(x+y)\left(\frac{1}{x}+\frac{4}{y}\right) \geq 9 \qquad (1)$$

**Question 34b**

$$\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0$$
$$x-2\sqrt{xy}+y \geq 0$$
$$x+y \geq 2\sqrt{xy} \qquad (2)$$
$$\left(\frac{1}{\sqrt{x}}-\frac{2}{\sqrt{y}}\right)^2 \geq 0$$
$$\frac{1}{x}-\frac{4}{\sqrt{xy}}+\frac{4}{y} \geq 0$$
$$\frac{1}{x}+\frac{4}{y} \geq \frac{4}{\sqrt{xy}} \qquad (3)$$

$(2)\times(3)$:

$$\therefore (x+y)\left(\frac{1}{x}+\frac{4}{y}\right) \geq 8$$

**Question 34c**

The minimum value of each inequality occurs at equality, so $y=x$ in (2) and $y=4x$ in (3). Neither of these matches equality in (1), which is $y=2x$, which causes the disparity.

**Question 35**

Suppose $\dfrac{a+c}{\sqrt{a^2+c^2}}-\dfrac{b+c}{\sqrt{b^2+c^2}} \leq 0$

$$\therefore \frac{a+c}{\sqrt{a^2+c^2}} \leq \frac{b+c}{\sqrt{b^2+c^2}}$$
$$(a+c)\sqrt{b^2+c^2} \leq (b+c)\sqrt{a^2+c^2}$$
$$(a+c)^2\left(b^2+c^2\right) \leq (b+c)^2\left(a^2+c^2\right) \quad \text{since } a,b,c>0$$
$$\left(a^2+2ac+c^2\right)\left(b^2+c^2\right) \leq \left(b^2+2bc+c^2\right)\left(a^2+c^2\right)$$
$$2ab^2c+2ac^3 \leq 2a^2bc+2bc^3$$
$$ab^2+ac^2 \leq a^2b+bc^2$$
$$ab^2-a^2b+ac^2-bc^2 \leq 0$$
$$ab(b-a)+c^2(a-b) \leq 0$$
$$c^2(a-b)-ab(a-b) \leq 0$$
$$\left(c^2-ab\right)(a-b) \leq 0 \quad \#$$

This is a contradiction since $c^2>ab$ and $a>b$, so $\left(c^2-ab\right)(a-b)>0$

Hence $\dfrac{a+c}{\sqrt{a^2+c^2}}-\dfrac{b+c}{\sqrt{b^2+c^2}}>0$

**Question 1**

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$
$$a-2\sqrt{ab}+b \geq 0$$
$$a+b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$

**Question 2**

$$(a-b)^2 \geq 0$$
$$a^2-2ab+b^2 \geq 0$$
$$a^2+2ab+b^2 \geq 4ab$$
$$(a+b)^2 \geq 4ab$$
$$a+b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$

**Question 3**

$$\left(x-\frac{1}{x}\right)^2 \geq 0$$
$$x^2-2+\frac{1}{x^2} \geq 0$$
$$x^2+\frac{1}{x^2} \geq 2$$

Alternative I:

$$\frac{x^2+\frac{1}{x^2}}{2} \geq \sqrt{x^2\times\frac{1}{x^2}}$$
$$x^2+\frac{1}{x^2} \geq 2$$

Alternative II:

$$\text{LHS}-\text{RHS} = x^2-2+\frac{1}{x^2}$$
$$= \left(x-\frac{1}{x}\right)^2$$
$$\geq 0$$
$$\therefore x^2-2+\frac{1}{x^2} \geq 0$$
$$\therefore x^2+\frac{1}{x^2} \geq 2$$

Alternative III:

Suppose by contradiction that

$$x^2+\frac{1}{x^2} < 2$$
$$x^2-2+\frac{1}{x^2} < 0$$
$$\left(x-\frac{1}{x}\right)^2 < 0$$

which is a contradiction since $\mathbb{R}^2 \geq 0$

$$\therefore x^2+\frac{1}{x^2} \geq 2.$$

**Question 5**

LHS − RHS
$$= (a+2b)^2-8ab$$
$$= a^2+4ab+4b^2-8ab$$
$$= a^2-4ab+4b^2$$
$$= (a-2b)^2$$
$$\geq 0 \quad \text{since } \mathbb{R}^2 \geq 0$$
$$\therefore (a+2b)^2-8ab \geq 0$$
$$\therefore (a+2b)^2 \geq 8ab$$

**Question 5 (continued)**

Alternative I:

$$(a-2b)^2 \geq 0 \text{ since } \mathbb{R}^2 \geq 0$$
$$a^2-4ab+4b^2 \geq 0$$
$$a^2+4ab+4b^2 \geq 8ab$$
$$(a+2b)^2 \geq 8ab$$

Alternative II:

$$\frac{a+2b}{2} \geq \sqrt{a\times 2b}$$
$$a+2b \geq 2\sqrt{2ab}$$
$$(a+2b)^2 \geq 8ab$$

Alternative III:

Suppose by contradiction that

$$(a+2b)^2 < 8ab$$
$$a^2+4ab+4b^2 < 8ab$$
$$a^2-4ab+4b^2 < 0$$
$$(a-2b)^2 < 0$$

which is a contradiction since $\mathbb{R}^2 \geq 0$

$$\therefore (a+2b)^2 \geq 8ab.$$

**Question 6**

$$\left(1-\sqrt{x}\right)^2 \geq 0$$
$$1-2\sqrt{x}+x \geq 0$$
$$1+x \geq 2\sqrt{x}$$

Alternative I:

$$\frac{1+x}{2} \geq \sqrt{x\times 1}$$
$$1+x \geq 2\sqrt{x}$$

Alternative II:

$$\text{LHS}-\text{RHS} = 1-2\sqrt{x}+x$$
$$= \left(1-\sqrt{x}\right)^2$$
$$\geq 0 \text{ since } \mathbb{R}^2 \geq 0$$
$$\therefore 1+x \geq 2\sqrt{x}$$

Alternative III:

Suppose by contradiction that

$$1+x < 2\sqrt{x}$$
$$1-2\sqrt{x}+x < 0$$
$$\left(1-\sqrt{x}\right)^2 < 0$$

which is a contradiction since $\mathbb{R}^2 \geq 0$

$$\therefore 1+x \geq 2\sqrt{x}.$$

**Question 7**

$$\left(\sqrt{ac}-\sqrt{\frac{b}{c}}\right)^2 \geq 0$$
$$ac-2\sqrt{ab}+\frac{b}{c} \geq 0$$
$$\therefore ac+\frac{b}{c} \geq 2\sqrt{ab}$$

Alternative I:

$$\frac{ac+\frac{b}{c}}{2} \geq \sqrt{ac\cdot\frac{b}{c}}$$
$$\therefore ac+\frac{b}{c} \geq 2\sqrt{ab}$$

**Question 7 (continued)**

Alternative II:

$$\text{LHS} - \text{RHS} = ac - 2\sqrt{ab} + \frac{b}{c}$$
$$= \left(\sqrt{ac} - \sqrt{\frac{b}{c}}\right)^2$$
$$\geq 0 \text{ since } \mathbb{R}^2 \geq 0$$
$$\therefore ac + \frac{b}{c} \geq 2\sqrt{ab}$$

Alternative III:

Suppose by contradiction that

$$ac + \frac{b}{c} < 2\sqrt{ab}$$
$$ac - 2\sqrt{ab} + \frac{b}{c} < 0$$
$$\left(\sqrt{ac} - \sqrt{\frac{b}{c}}\right)^2 < 0$$

which is a contradiction since $\mathbb{R}^2 \geq 0$

$$\therefore ac + \frac{b}{c} \geq 2\sqrt{ab}$$

**Question 8**

$$P = 2(a+b), A = ab$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$
$$\frac{2(a+b)}{4} \geq \sqrt{ab}$$
$$\frac{P}{4} \geq \sqrt{A}$$
$$\frac{P^2}{16} \geq A$$
$$P^2 \geq 16A$$

**Question 9**

$$a^2 + 1 \geq 2\sqrt{a^2 \times 1}$$
$$a^2 + 1 \geq 2|a| \qquad \text{since } \sqrt{a^2} = |a|$$
$$\therefore |a| \leq \frac{a^2+1}{2}$$

Alternative I:

$$\text{LHS} - \text{RHS} = |a| - \frac{a^2+1}{2}$$
$$= -\frac{a^2 - 2|a| + 1}{2}$$
$$= -\frac{(|a|-1)^2}{2}$$
$$\leq 0 \quad \text{since } \mathbb{R}^2 \geq 0$$
$$\therefore \text{LHS} \leq \text{RHS}$$
$$\therefore |a| \leq \frac{a^2+1}{2}$$

**Question 10**

$$(x^2 - 2y)^2 \geq 0$$
$$x^4 - 4x^2y + 4y^2 \geq 0$$
$$x^4 + x^2y + 4y^2 \geq 5x^2y$$

Alternative I:

$$\text{LHS} - \text{RHS} = x^4 + x^2y + 4y^2 - 5x^2y$$
$$= x^4 - 4x^2y + 4y^2$$
$$= (x^2 + 2y)^2$$
$$\geq 0$$
$$\therefore x^4 + x^2y + 4y^2 \geq 5x^2y$$

Alternative II:

Suppose by contradiction that

$$x^4 + x^2y + 4y^2 < 5x^2y$$
$$x^4 + x^2y + 4y^2 - 5x^2y < 0$$
$$(x^2 + 2y)^2 < 0$$

which is a contradiction since $\mathbb{R}^2 \geq 0$

$$\therefore x^4 + x^2y + 4y^2 \geq 5x^2y$$

**Question 11**

$$(ad - bc)^2 \geq 0$$
$$a^2d^2 - 2abcd + b^2c^2 \geq 0$$
$$a^2d^2 + b^2c^2 \geq 2abcd$$
$$a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \geq a^2c^2 + 2abcd + b^2d^2$$
$$\therefore (a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$$

Alternative I:

$$\text{LHS} - \text{RHS}$$
$$= (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2$$
$$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 - a^2c^2 - 2abcd - b^2d^2$$
$$= a^2d^2 - 2abcd + b^2c^2$$
$$= (ad - bc)^2$$
$$\geq 0 \text{ since } \mathbb{R}^2 \geq 0$$
$$\therefore (a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$$

Alternative III:

Suppose by contradiction that

$$(a^2 + b^2)(c^2 + d^2) < (ac + bd)^2$$
$$a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 < a^2c^2 + 2abcd + b^2d^2$$
$$a^2d^2 - 2abcd + b^2c^2 < 0$$
$$(ad - bc)^2 < 0$$

which is a contradiction since $\mathbb{R}^2 \geq 0$

$$\therefore (a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$$

**Question 12**

$$\left(ac-\frac{b}{c}\right)^2 \geq 0$$

$$a^2c^2-2ab+\frac{b^2}{c^2} \geq 0$$

$$\therefore a^2c^2+\frac{b^2}{c^2} \geq 2ab$$

Alternative I:

$$\frac{a^2c^2+\frac{b^2}{c^2}}{2} \geq 2\sqrt{a^2c^2\cdot\frac{b^2}{c^2}}$$

$$= 2\sqrt{a^2b^2}$$

$$= 2ab$$

$$\therefore a^2c^2+\frac{b^2}{c^2} \geq 2ab$$

Alternative II:

$$\text{LHS}-\text{RHS} = a^2c^2-2ab+\frac{b^2}{c^2}$$

$$= \left(ac-\frac{b}{c}\right)^2$$

$$\geq 0$$

$$\therefore a^2c^2+\frac{b^2}{c^2} \geq 2ab$$

Alternative III:

Suppose by contradiction that

$$a^2c^2+\frac{b^2}{c^2} < 2ab$$

$$a^2c^2-2ab+\frac{b^2}{c^2} < 0$$

$$\left(ac-\frac{b}{c}\right)^2 < 0$$

which is a contradiction since $\mathbb{R}^2 \geq 0$

$$\therefore a^2c^2+\frac{b^2}{c^2} \geq 2ab$$

**Question 13**

$$\frac{\frac{x}{yz}+\frac{y}{xz}}{2} \geq \sqrt{\frac{xy}{xyz^2}} \qquad \text{(AM−GM)}$$

$$\frac{1}{2}\left(\frac{x}{yz}+\frac{y}{xz}\right) \geq \frac{1}{z} \qquad (1)$$

Similarly

$$\frac{1}{2}\left(\frac{x}{yz}+\frac{z}{xy}\right) \geq \frac{1}{y} \qquad (2)$$

$$\frac{1}{2}\left(\frac{y}{xz}+\frac{z}{xy}\right) \geq \frac{1}{x} \qquad (3)$$

$(1)+(2)+(3)$:

$$\frac{1}{2}\left(\frac{x}{yz}+\frac{y}{xz}\right)+\frac{1}{2}\left(\frac{x}{yz}+\frac{z}{xy}\right)+\frac{1}{2}\left(\frac{y}{xz}+\frac{z}{xy}\right)$$

$$\geq \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

**Question 14**

$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2}$$

$$\geq \sqrt{\frac{x^2}{y^2}\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)} \qquad \text{(AM − GM)}$$

$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2} \geq \sqrt{\frac{x^2}{y^2}\left(\frac{x^2y^2+z^4}{x^2z^2}\right)}$$

$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2} \geq \frac{1}{yz}\sqrt{x^2y^2+z^4} \qquad (1)$$

Similarly

$$\frac{\frac{y^2}{z^2}+\left(\frac{x^2}{y^2}+\frac{z^2}{x^2}\right)}{2} \geq \frac{1}{xz}\sqrt{y^2z^2+x^4} \qquad (2)$$

$(1)+(2)$:

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq \frac{1}{yz}\sqrt{x^2y^2+z^4}+\frac{1}{xz}\sqrt{y^2z^2+x^4}$$

**Question 15**

$$\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2} \geq \sqrt{\frac{a+b}{2}\times\frac{c+d}{2}}$$

$$\therefore \frac{a+b+c+d}{4} \geq \sqrt{\sqrt{ab}\times\sqrt{cd}}$$

since $\frac{(a+b)}{2} \geq \sqrt{ab}$, $\frac{c+d}{2} \geq \sqrt{cd}$

$$\therefore \frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

**Question 16**

$$\frac{\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{d}+\frac{d^3}{a}}{4}$$

$$\geq \sqrt[4]{\frac{a^3}{b}\times\frac{b^3}{c}\times\frac{c^3}{d}\times\frac{d^3}{a}} \qquad \text{(AM − GM)}$$

$$\frac{\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{d}+\frac{d^3}{a}}{4} \geq \sqrt[4]{a^2b^2c^2d^2}$$

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{d}+\frac{d^3}{a} \geq 4\sqrt{abcd}$$

**Question 17**

$$\frac{a+b+c+\left(\frac{a+b+c}{3}\right)}{4} \geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)}$$

$$\frac{\frac{4}{3}(a+b+c)}{4} \geq (abc)^{\frac{1}{4}}\left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{1} \geq (abc)^{\frac{1}{4}}\left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}} \geq (abc)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}\times\frac{4}{3}} \geq (abc)^{\frac{1}{4}\times\frac{4}{3}}$$

$$\therefore \frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

**Question 18**

$$\frac{a^3+a^3+b^3}{3} \geq \sqrt[3]{a^6b^3}$$

$$\therefore \frac{2a^3+b^3}{3} \geq a^2b \quad (1)$$

Similarly

$$\frac{2b^3+c^3}{3} \geq b^2c \quad (2) \qquad \frac{2c^3+a^3}{3} \geq c^2a \quad (3)$$

$(1)+(2)+(3)$:

$$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$$

**Question 19**

$$(a_1-\sqrt{a_1})^2 \geq 0$$

$$a_1^2-2a_1\sqrt{a_1}+a_1 \geq 0$$

$$a_1^2+a_1 \geq 2a_1\sqrt{a_1} \quad (1)$$

Similarly for $a_2$ to $a_n$ $\quad (2)\ldots(n)$

$(1)\times(2)\times(3)\times\ldots\times(n)$:

$$\therefore (a_1^2+a_1)(a_2^2+a_2)(a_3^2+a_3)\ldots(a_n^2+a_n)$$

$$\geq 2a_1\sqrt{a_1}\times 2a_2\sqrt{a_2}\times 2a_3\sqrt{a_3}\times\ldots\times 2a_n\sqrt{a_n}$$

$$\geq 2^n(a_1a_2a_3\ldots a_n)\sqrt{a_1a_2a_3\ldots a_n}$$

$$\geq 2^n$$

**Question 20a**

$$(a-b)^2 \geq 0$$

$$a^2-2ab+b^2 \geq 0$$

$$a^2+b^2 \geq 2ab \quad (1)$$

Similarly:

$$a^2+c^2 \geq 2ac \quad (2)$$

$$b^2+c^2 \geq 2bc \quad (3)$$

$(1)+(2)+(3)$:

$$2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$$

$$a^2+b^2+c^2 \geq ab+bc+ca$$

**Question 20b**

$$(a+b+c)^2$$

$$= a^2+b^2+c^2+2(ab+bc+ca)$$

$$\geq ab+bc+ca+2(ab+bc+ca) \quad \text{from (a).}$$

$$\geq 3(ab+bc+ca)$$

Alternatively

$$a^2+b^2+c^2-ab-bc-ca$$

$$=\frac{1}{2}(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2)$$

$$=\frac{1}{2}((a-b)^2+(b-c)^2+(c-a)^2)$$

$$\geq 0$$

$$\therefore a^2+b^2+c^2 \geq ab+bc+ca$$

**Question 21**

$\mathbb{R}^2 \geq \mathbf{0}$

$$(p-2q)^2 \geq 0$$

$$p^2-4pq+4q^2 \geq 0$$

$$p^2+4q^2 \geq 4pq$$

**Contradiction**

Suppose $p^2+4q^2<4pq$

$$\therefore p^2-4pq+4q^2<0$$

$$(p-2q)^2<0$$

Which is a contradiction since any real number squared must be greater than or equal to 0

$$\therefore p^2+4q^2 \geq 4pq$$

**Question 21 (continued)**

**LHS−RHS**

$$\text{LHS}-\text{RHS} = p^2-4pq+4q^2$$

$$= (p-2q)^2$$

$$\geq 0$$

$$\therefore \text{LHS} \geq \text{RHS}$$

$$\therefore p^2+4q^2 \geq 4pq$$

**AM−GM**

Let $a=p^2, b=4q^2$ in (i):

$$\frac{p^2+4q^2}{2} \geq \sqrt{p^2\times 4q^2}$$

$$p^2+4q^2 \geq 4|pq| \geq 4pq$$

**Question 22a**

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$

$$a-2\sqrt{ab}+b \geq 0$$

$$a+b \geq 2\sqrt{ab}$$

**Question 22b**

Let $a=\sec^2 x-1, b=1$ in (a)

$$\sec^2 x-1+1 \geq 2\sqrt{(\sec^2 x-1)(1)}$$

$$\sec^2 x \geq 2\sqrt{\sec^2 x-1}$$

$$\sec^2 x \geq 2\sqrt{\tan^2 x}$$

$$\sec^2 x \geq 2\left|\tan x\right| \text{ since } \sqrt{a^2}=|a|$$

$\therefore$ the graph of $y = \sec^2 x$ lies on or above the graph of $y = 2\left|\tan x\right|$.

Alternative I

$$\left(\left|\tan x\right|-1\right)^2 \geq 0$$

$$\tan^2 x-2\left|\tan x\right|+1 \geq 0 \text{ since } \left|\tan x\right|^2=\tan^2 x$$

$$\tan^2 x+1 \geq 2\left|\tan x\right|$$

$$\therefore \sec^2 x \geq 2\left|\tan x\right|$$

$\therefore$ the graph of $y = \sec^2 x$ lies on or above the graph of $y = 2\left|\tan x\right|$.

Alternative II

$$\text{LHS}-\text{RHS} = \sec^2 x-2\left|\tan x\right|$$

$$= \tan^2 x+1-2\left|\tan x\right|$$

$$= \left(\left|\tan x\right|-1\right)^2 \text{ since } \left|\tan x\right|^2$$

$$\geq 0$$

$$\therefore \sec^2 x \geq 2\left|\tan x\right|$$

$\therefore$ the graph of $y = \sec^2 x$ lies on or above the graph of $y = 2\left|\tan x\right|$.

**Question 23**

$\frac{a^4+a^2}{2} \geq \sqrt{a^4 \times a^2}$ AM – GM

$a^4+a^2 \geq 2|a^3|$

$\geq 2a^3$ since $a \geq 0$

$a^4 \geq 2a^3 - a^2$

Alternative I:

$(a^2-a)^2 \geq 0$

$a^4 - 2a^3 + a^2 \geq 0$

$a^4 \geq 2a^3 - a^2$

Alternative II:

LHS – RHS $= a^4 - 2a^3 + a^2$

$= a^2(a^2 - 2a + 1)$

$= a^2(a-1)^2 \geq 0$ since $\mathbb{R}^2 \geq 0$

$\therefore$ LHS $\geq$ RHS

$a^4 \geq 2a^3 - a^2$

Alternative III:

Suppose $a^4 < 2a^3 - a^2$

$\therefore a^4 - 2a^3 + a^2 < 0$

$a^2(a^2 - 2a + 1) < 0$

$a^2(a-1)^2 < 0$

$[a(a-1)]^2 < 0$

Which is a contradiction since $\mathbb{R}^2 \geq 0$

$\therefore a^4 \geq 2a^3 - a^2$

**Question 24**

$a^2b + b^2c + c^2a$

$\geq 3\sqrt[3]{(a^2b)(b^2c)(c^2a)}$ AM–GM

$\geq 3\sqrt[3]{a^3b^3c^3}$

$= 3abc$

**Question 25**

$\frac{a^7+a^7+a^7+a^7+b^7+b^7+b^7}{7} \geq \sqrt[7]{a^{28}b^{21}}$

$\therefore \frac{4a^7+3b^7}{7} \geq a^4b^3$

Similarly

$\frac{4b^7+3c^7}{7} \geq b^4c^3$

$\frac{4c^7+3a^7}{7} \geq c^4a^3$

Summing the above gives

$a^7 + b^7 + c^7 \geq a^4b^3 + b^4c^3 + c^4a^3$

**Question 26a**

$\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^2 \geq 0$

$\frac{x}{y} - 2 + \frac{y}{x} \geq 0$

$\frac{x}{y} + \frac{y}{x} \geq 2$

**Question 26b**

Similarly:

$$\frac{x}{w}+\frac{w}{x}\geq 2,\frac{x}{z}+\frac{z}{x}\geq 2,\frac{w}{y}+\frac{y}{w}\geq 2,\frac{w}{z}+\frac{z}{w}\geq 2,\frac{y}{z}+\frac{z}{y}\geq 2$$

Summing the 6 inequalities:

$$\frac{x}{y}+\frac{y}{x}+\frac{x}{w}+\frac{w}{x}+\frac{x}{z}+\frac{z}{x}+\frac{w}{y}+\frac{y}{w}+\frac{w}{z}+\frac{z}{w}+\frac{y}{z}+\frac{z}{y}\geq 12$$

$$\frac{x+y+z}{w}+\frac{w+y+z}{x}+\frac{w+x+z}{y}+\frac{w+x+y}{z}\geq 12$$

**Question 26c**

Let $w+x+y+z=1$

$$\therefore \frac{1-w}{w}+\frac{1-x}{x}+\frac{1-y}{y}+\frac{1-z}{z}\geq 12$$

$$\frac{1}{w}-1+\frac{1}{x}-1+\frac{1}{y}-1+\frac{1}{z}-1\geq 12$$

$$\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 16$$

**Question 27**

$$\left(a^2-b^2\right)^2\geq 0$$

$$a^4-2a^2b^2+b^4\geq 0$$

$$2(a^4+b^4)-\left(a^2+b^2\right)^2\geq 0$$

$$\therefore a^4+b^4\geq \frac{\left(a^2+b^2\right)^2}{2} \quad (1)$$

Similarly

$$a^2+b^2\geq \frac{(a+b)^2}{2}$$

$$\therefore a^2+b^2\geq \frac{1^2}{2}$$

$$a^2+b^2\geq \frac{1}{2} \quad (2)$$

From (1) and (2):

$$a^4+b^4\geq \frac{\left(\frac{1}{2}\right)^2}{2}$$

$$a^4+b^4\geq \frac{1}{8}$$

**Question 28**

$$a+b+c\geq 3\sqrt[3]{abc}$$

$$a+b+c\geq 3 \quad (1)$$

$$\frac{a^2+1}{2}\geq \sqrt{a^2\cdot 1}$$

$$\geq a$$

Similarly

$$\frac{b^2+1}{2}\geq b \qquad \frac{c^2+1}{2}\geq c$$

Summing the inequalities:

$$\frac{a^2+1}{2}+\frac{b^2+1}{2}+\frac{c^2+1}{2}\geq a+b+c$$

$$a^2+b^2+c^2+3\geq 2(a+b+c)$$

$$a^2+b^2+c^2\geq 2(a+b+c)-3$$

$$\geq 2(a+b+c)-(a+b+c) \quad \text{from (1)}$$

$$\geq a+b+c$$

**Question 29**

$$\frac{a^4+a^4+b^4+c^4}{4} \geq \sqrt[4]{a^8b^4c^4}$$

$$\therefore \frac{2a^4+b^4+c^4}{4} \geq a^2bc$$

Similarly

$$\frac{2b^4+a^4+c^4}{4} \geq b^2ac$$

$$\frac{2c^4+a^4+b^4}{4} \geq c^2ab$$

Summing the above gives

$$a^4+b^4+c^4 \geq a^2bc+b^2ca+c^2ab$$

**Question 30**

$$\frac{1+ab}{1+a} = \frac{abc+ab}{abc+a} = \frac{ab(c+1)}{a(bc+1)} = \frac{b(c+1)}{bc+1} \quad (3)$$

Similarly

$$\frac{1+bc}{1+b} = \frac{c(a+1)}{ac+1} \quad (2)$$

$$\frac{1+ac}{1+c} = \frac{a(b+1)}{ab+1} \quad (3)$$

$$\begin{aligned}
\text{LHS} &= \frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \\
&= \frac{1}{2}\left(\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c}+\frac{b(c+1)}{bc+1}+\frac{c(a+1)}{ac+1}+\frac{a(b+1)}{ab+1}\right) \\
&\geq \frac{1}{2}\left(6\times\sqrt[6]{\frac{1+ab}{1+a}\cdot\frac{1+bc}{1+b}\cdot\frac{1+ac}{1+c}\cdot\frac{b(c+1)}{bc+1}\cdot\frac{c(a+1)}{ac+1}\cdot\frac{a(b+1)}{ab+1}}\right) \\
&\geq \frac{1}{2}\times 6\times\sqrt{1} \\
&\geq 3
\end{aligned}$$

**Question 31**

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$
$$a-2\sqrt{ab}+b \geq 0$$
$$a+b \geq 2\sqrt{ab}$$
$$\sqrt{ab} \leq \frac{a+b}{2} \quad (1)$$

$(1)\times\sqrt{ab}$: $$ab \leq \frac{(a+b)\sqrt{ab}}{2}$$
$$\frac{2ab}{a+b} \leq \sqrt{ab} \quad (2)$$

$$\left(\frac{a}{2}-\frac{b}{2}\right)^2 \geq 0$$
$$\frac{a^2-2ab+b^2}{4} \geq 0$$
$$\frac{a^2-2ab+b^2}{4}+\frac{(a+b)^2}{4} \geq \frac{(a+b)^2}{4}$$
$$\frac{2a^2+2b^2}{4} \geq \frac{(a+b)^2}{4}$$
$$\frac{(a+b)^2}{4} \leq \frac{a^2+b^2}{2}$$
$$\sqrt{\frac{(a+b)^2}{4}} \leq \sqrt{\frac{a^2+b^2}{2}}$$
$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \quad (3)$$

From (2), (1)and (3):

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

*Alternative for GM and QM*

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

$$\left(\frac{a+b}{2}\right)^2-\left(\sqrt{\frac{a^2+b^2}{2}}\right)^2$$
$$=\frac{a^2+2ab+b^2}{4}-\frac{a^2+b^2}{2}$$
$$=-\frac{a^2-2ab+b^2}{4}$$
$$=-\left(\frac{a-b}{2}\right)^2$$
$$\leq 0$$
$$\therefore \left(\frac{a+b}{2}\right)^2 \leq \left(\sqrt{\frac{a^2+b^2}{2}}\right)^2$$
$$\therefore \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

**Question 32a**

$$3\left(a^2+b^2+c^2\right)-(a+b+c)^2$$
$$=3a^2+3b^2+3c^2-\left(\left(a^2+b^2+c^2\right)+2(ab+bc+ca)\right)$$
$$=2\left(a^2+b^2+c^2\right)-2(ab+bc+ca)$$
$$=\left(a^2-2ab+b^2\right)+\left(b^2-2bc+c^2\right)+\left(c^2-2ca+a^2\right)$$
$$=(a-b)^2+(b-c)^2+(c-a)^2$$
$$\geq 0$$
$$\therefore 3\left(a^2+b^2+c^2\right) \geq (a+b+c)^2 \quad (1)$$

$$(a+b+c)^2-3(ab+bc+ac)$$
$$=a^2+b^2+c^2+2(ab+ca+ca)-3(ab+bc+ca)$$
$$=a^2+b^2+c^2-(ab+bc+ca)$$
$$=\frac{a^2-2ab+b^2}{2}+\frac{b^2-2bc+c^2}{2}+\frac{c^2-2ac+a^2}{2}$$
$$=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}$$
$$\geq 0$$
$$\therefore (a+b+c)^2 \geq 3(ab+bc+ac) \quad (2)$$

From (1) and (2):

$$3\left(a^2+b^2+c^2\right) \geq (a+b+c)^2 \geq 3(ab+bc+ac)$$

**Question 32b**

$$\begin{aligned}\text{LHS} - \text{RHS} &= a^2+b^2+c^2+ab+bc+ca-6\\ &= \frac{a^2+2ab+b^2}{2}+\frac{b^2+2bc+c^2}{2}+\frac{c^2+2ac+a^2}{2}-2(a+b+c)\\ &= \frac{(a+b)^2+(b+c)^2+(c+a)^2}{2}-\frac{12}{2} \quad \text{since } a+b+c=3\\ &= \frac{(3-c)^2+(3-a)^2+(3-b)^2}{2}-\frac{12}{2}\\ &= \frac{\left((3-c)^2-4\right)+\left((3-a)^2-4\right)+\left((3-b)^2-4\right)}{2}\\ &= \frac{(5-c)(1-c)+(5-a)(1-a)+(5-b)(1-b)}{2}\\ &= \frac{\left(5-6c+c^2\right)+\left(5-6a+a^2\right)+\left(5-6b+b^2\right)}{2}\\ &= \frac{\left(a^2+b^2+c^2\right)-6(a+b+c)+15}{2}\\ &= \frac{a^2+b^2+c^2-18+15}{2}\\ &= \frac{a^2+b^2+c^2-3}{2}\end{aligned}$$

Now $3\left(a^2+b^2+c^2\right) \geq (a+b+c)^2$ from (a)

$$\begin{aligned}\therefore a^2+b^2+c^2 &\geq \frac{(a+b+c)^2}{3}\\ &\geq \frac{3^3}{3}\\ &\geq 3\\ &\geq 0\end{aligned}$$

$\therefore a^2+b^2+c^2+ab+bc+ca \geq 6$

**Question 33a**

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$

$$\begin{aligned}a-2\sqrt{ab}+b &\geq 0\\ a+b &\geq 2\sqrt{ab}\\ \frac{a+b}{2} &\geq \sqrt{ab}\end{aligned}$$

**Question 33b**

$$\frac{a_1+a_2+a_3+\ldots+a_n}{n} \geq \sqrt[n]{a_1a_2a_3\ldots a_n}$$

$$\therefore a_1a_2a_3\ldots a_n \leq \left(\frac{a_1+a_2+a_3+\ldots+a_n}{n}\right)^n \qquad (1)$$

Let $a_1=1, a_2=2, a_3=3, \ldots, a_n=n$

$a_1a_2a_3\ldots a_n = 1\times2\times3\times\ldots\times n = n!$ (2)

$a_1+a_2+a_3+\ldots+a_n = 1+2+3+\ldots+n = \frac{n}{2}(1+n)$ (3)

sub (2), (3) in (1)

$$n! \leq \left(\frac{\frac{n}{2}(1+n)}{n}\right)^n$$

$$n! \leq \left(\frac{n+1}{2}\right)^n$$

**Question 34a**

$$\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0$$

$$x-2\sqrt{xy}+y \geq 0$$

$$x+y \geq 2\sqrt{xy} \qquad (1)$$

**Question 34b**

Similarly

$x + z \geq 2\sqrt{xz}$ (2)

$y + z \geq 2\sqrt{yz}$ (3)

(1)×(2)×(3):

$(x + y)(x + z)(y + z) \geq 8\sqrt{x^2y^2z^2}$

$(x + y)(x + z)(y + z) \geq 8xyz$

**Question 34c**

Let $x = S - a, y = S - b, z = S - c$ in (ii)

$\therefore (2S - a - b)(2S - a - c)(2S - b - c)$

$\geq 8(S - a)(S - b)(S - c)$

$(c)(b)(a) \geq 8(S - a)(S - b)(S - c)$

$(S - a)(S - b)(S - c) \leq \dfrac{abc}{8}$

$\therefore S(S - a)(S - b)(S - c) \leq \dfrac{abc \times S}{8}$

$\left(\sqrt{S(S - a)(S - b)(S - c)}\right)^2 \leq \dfrac{abc \times \frac{1}{2}(a + b + c)}{8}$

$\therefore A^2 \leq \dfrac{(a + b + c)abc}{16}$

**Question 35**

**Rearranging LHS**

$ab \times bc \times ca \leq \left(\dfrac{ab + ba + ca}{3}\right)^3$ (given result)

$a^2b^2c^2 \leq \left(\dfrac{2(ab + ba + ca)}{6}\right)^3$

$abc \leq \left(\dfrac{S}{6}\right)^{\frac{3}{2}}$ (square root)

**Rearranging RHS**

$$\left(\frac{S}{6}\right)^{\frac{3}{2}} = \left(\frac{2(ab + bc + ca)}{6}\right)^{\frac{3}{2}}$$
$$= \sqrt{\left(\frac{ab + bc + ca}{3}\right)^3}$$
$$\geq \sqrt{ab \times bc \times ca} \quad \text{(given result)}$$
$$= abc$$

$abc \leq \left(\dfrac{S}{6}\right)^{\frac{3}{2}}$

**LHS-RHS**

$$abc - \left(\frac{S}{6}\right)^{\frac{3}{2}} = abc - \sqrt{\left(\frac{2(ab + bc + ca)}{6}\right)^3}$$
$$= abc - \sqrt{\left(\frac{ab + bc + ca}{3}\right)^3}$$
$$\leq abc - \sqrt{ab \times bc \times ca} \quad \text{(given result)}$$
$$= 0$$

$abc \leq \left(\dfrac{S}{6}\right)^{\frac{3}{2}}$

continued over page

**Question 35 (continued)**
**LHS Using AM−GM with 3 terms**

$$abc = \left(\sqrt[3]{(ab \times bc \times ca)}\right)^{\frac{3}{2}}$$

$$\leq \left(\frac{ab + bc + ca}{3}\right)^{\frac{3}{2}} \qquad \text{(AM−GM with 3 terms)}$$

$$= \left(\frac{2(ab + bc + ca)}{6}\right)^{\frac{3}{2}}$$

$$= \left(\frac{S}{6}\right)^{\frac{3}{2}}$$

$$abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$$

**RHS Using AM−GM with 3 terms**

$$\left(\frac{S}{6}\right)^{\frac{3}{2}} = \left(\frac{2(ab + bc + ca)}{6}\right)^{\frac{3}{2}}$$

$$= \left(\frac{ab + bc + ca}{3}\right)^{\frac{3}{2}}$$

$$\geq \left(\sqrt[3]{ab \times bc \times ca}\right)^{\frac{3}{2}} \qquad \text{(AM−GM with 3 terms)}$$

$$= \left(a^2 b^2 c^2\right)^{\frac{1}{2}}$$

$$= abc$$

$$\therefore abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$$

**Question 1a**

$$(a-b)^2 \geq 0$$
$$a^2 - 2ab + b^2 \geq 0$$
$$2a^2 + 2b^2 \geq a^2 + 2ab + b^2$$
$$2(a^2 + b^2) \geq (a+b)^2$$

**Question 1b**

$d^2 = a^2 + b^2$ Pythagoras' theorem

$\therefore 2d^2 \geq \left(\frac{P}{2}\right)^2$ from part (a)

$$d^2 \geq \frac{P^2}{8}$$
$$d \geq \frac{P}{2\sqrt{2}} \text{ since } d > 0$$
$$\therefore P \leq 2\sqrt{2}d$$

**Question 2**

$$-1 \leq \cos x \leq 1$$
$$0 \leq \cos^2 x \leq 1$$
$$0 \leq e^{-x}\cos^2 x \leq e^{-x}$$

as $x \to \infty, e^{-x} \to 0$

$\therefore \lim_{x\to\infty}\left[e^{-x}\cos^2 x\right] = 0$ by the squeeze theorem.

**Question 3**

The area under the curve $y = \ln x$ is greater than the area of the triangle, so:

$$\int_1^t \ln x \; dx > \frac{1}{2}(t-1)\times \ln t$$
$$\left[x\ln x - x\right]_1^t > \frac{1}{2}(t-1)\times \ln t$$
$$t\ln t - t + 1 > \frac{1}{2}(t-1)\times \ln t$$
$$\frac{1}{2}(t-1)\times \ln t - t\ln t < -(t-1)$$
$$-\frac{1}{2}(t+1)\ln t < -(t-1)$$
$$\ln t \; > 2\left(\frac{t-1}{t+1}\right)$$

**Question 4**

$3 - x^2 \leq f(x) \leq 3 + 2x^2$ for $x \neq 0$

as $x \to 0, 3 - x^2 \to 3, 3 + 2x^2 \to 3$

$\therefore \lim_{x\to 0} f(x) = 3$ by the squeeze theorem.

**Question 5**

Let $f(x) = e^{\frac{x}{2}} - \frac{x}{2} - 1$

$$f'(x) = \frac{1}{2}e^{\frac{x}{2}} - \frac{1}{2}$$
$$f(0) = e^0 - 0 - 1 = 0$$
$$f'(x) \geq 0 \text{ for } x \geq 0 \text{ since } e^{\frac{x}{2}} \geq 1$$
$$\therefore e^{\frac{x}{2}} - \frac{x}{2} - 1 \geq 0 \text{ for } x \geq 0$$

since $f(0) = 0$ and $f'(x) \geq 0$

$$\therefore e^{\frac{x}{2}} \geq \frac{x}{2} + 1 \text{ for } x \geq 0$$

**Question 6**

$$-1 \leq \cos\left(\frac{2}{x^2}\right) \leq 1 \text{ for } x \neq 0$$
$$-|x| \leq x\cos\left(\frac{2}{x^2}\right) \leq |x|$$

as $x \to 0, -|x| \to 0, |x| \to 0$

$\therefore \lim_{x\to 0}\left[x\cos\left(\frac{2}{x^2}\right)\right] = 0$ by the squeeze theorem.

**Question 7**

Lower Rectangles < Exact Area < Upper Rectangles

$$(p-1)\times 1^2 < \int_1^p x^2 dx < (p-1)\times p^2$$
$$p - 1 < \left[\frac{x^3}{3}\right]_1^p < p^3 - p^2$$
$$p - 1 < \frac{p^3}{3} - \frac{1}{3} < p^3 - p^2$$

**Question 8**

Let $f(x) = x - \ln(x+1), x > -1$

$$f'(x) = 1 - \frac{1}{x+1}$$
$$f''(x) = \frac{1}{(x+1)^2} > 0$$

Let $f'(x) = 0$

$$1 - \frac{1}{x+1} = 0$$
$$x = 0$$
$$f(0) = 0 - \ln 1 = 0$$

$\therefore$ minimum turning point at $(0,0)$

$\therefore f(x) \geq 0$ for $x > -1$

$\therefore x \geq \ln(x+1)$ for $x > -1$

**Question 9**

$$A = \frac{1}{2}ab\sin\theta$$
$$\leq \frac{1}{2}ab(1) \text{ since } 0 < \sin\theta \leq 1 \text{ for } 0 < \theta < \pi$$
$$\leq \frac{ab}{2}$$

**Question 10**

$$-1 \leq \sin\left(\frac{1}{x-2}\right) \leq 1 \text{ for } x \neq 2$$
$$-(x-2)^2 \leq (x-2)^2\sin\left(\frac{1}{x-2}\right) \leq (x-2)^2$$

as $x \to 2, -(x-2)^2 \to 0, (x-2)^2 \to 0$

$\therefore \lim_{x\to 2}\left[(x-2)^2\sin\left(\frac{1}{x-2}\right)\right] = 0$ by the squeeze theorem.

**Question 11**

$\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$

$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ since $\sin x > 0$

$\therefore \cos x < \frac{\sin x}{x} < 1$ taking reciprocals

as $x \to 0^+, \cos x \to 1$

$\therefore \lim_{x\to 0}\left[\frac{\sin x}{x}\right] = 1$ by the squeeze theorem,

since $\frac{\sin x}{x}$ is an even function.

**Question 12**

Let the length and width of the rectangle be $a$ and $b$.

$$P = 2(a+b) \quad (1)$$
$$A = ab \quad (2)$$
$$(a-b)^2 \geq 0$$
$$a^2 - 2ab + b^2 \geq 0$$
$$a^2 + 2ab + b^2 \geq 4ab$$
$$(a+b)^2 \geq 4ab$$
$$\left(\frac{P}{2}\right)^2 \geq 4A \quad \text{from } (1) \text{ and } (2)$$
$$\frac{P^2}{4} \geq 4A$$
$$\therefore A \leq \frac{P^2}{16}$$

**Question 13a**

$f(0) = 0 - \ln(1+0+0) = 0$

**Question 13b**

$$f'(x) = 1 - \frac{1+x}{1+x+\frac{x^2}{2}}$$
$$= \frac{1+x+\frac{x^2}{2} - 1 - x}{1+x+\frac{x^2}{2}}$$
$$= \frac{x^2}{2+2x+x^2}$$
$$= \frac{x^2}{(x+1)^2+1}$$

$\geq 0$ for all $x$ as $x_2, (x+1)^2$ and 1 are all $\geq 0$ (with equality only when $x = 0$).

$\therefore f(x)$ is an increasing function for $x < 0$

**Question 13c**

Since $f(0) = 0$ and $f(x)$ is an increasing function for $x < 0$, thus $f(x) < 0$ for $x < 0$.

$$x - \ln\left(1 + x + \frac{x^2}{2}\right) < 0$$
$$x < \ln\left(1 + x + \frac{x^2}{2}\right)$$
$$e^x < 1 + x + \frac{x^2}{2} \text{ for } x < 0$$

**Question 14**

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \quad (1)$$
$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad (2)$$
$$\left|\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right| \leq 2 \text{ from } (1) \text{ and } (2)$$

$$\therefore 0 \leq x^2\left|\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right| \leq 2x^2$$

as $x \to 0, 2x^2 \to 0$

$$\therefore \lim_{x\to 0}\left[x^2\left|\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right|\right] = 0$$

**Question 15**

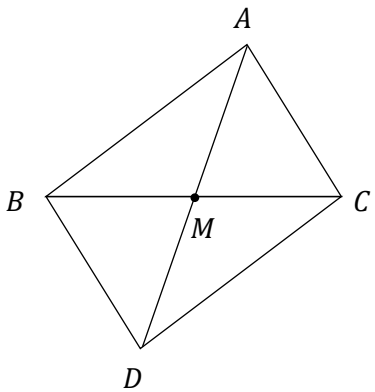
$$\cos\left(\frac{1}{x^2}\right) \leq f(x) \leq 1 - \sin\left(\frac{1}{x^2}\right)$$

as $x \to \infty, \cos\left(\frac{1}{x^2}\right) \to \cos 0 = 1, 1 - \sin\left(\frac{1}{x^2}\right)$

$\to 1 - \sin 0 = 1$

$\therefore \lim_{x\to\infty} f(x) = 1$ by the squeeze theorem.

**Question 16**




$M$ is the midpoint of $AD$, since the diagonals of a parallelogram bisect each other. In $\Delta ABD$:

$AD < AB + BD$ triangle inequality

$\therefore 2AM < AB + BD$

since $M$ is the midpoint of $AD$

$2AM < AB + AC$

opposite sides of a parallelogram are

$\therefore AM < \frac{AB + AC}{2}$

**Question 17**

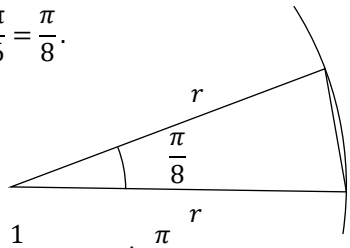
$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$
$$-\frac{x^2}{1+x^2} \leq \frac{x^2 \sin\left(\frac{1}{x}\right)}{1+x^2} \leq \frac{x^2}{1+x^2}$$

as $x \to 0, -\frac{x^2}{1+x^2} \to 0, \frac{x^2}{1+x^2} \to 0$

$$\therefore \lim_{x\to 0}\left[\frac{x^2 \sin\left(\frac{1}{x}\right)}{1+x^2}\right] = 0 \text{ by the squeeze theorem.}$$

**Question 18**

The $16-$sided can be divided into $16$ isosceles triangles with equal sides of $r$ where the included angle is $\frac{2\pi}{16}=\frac{\pi}{8}$.




$$\text{Area}_{\text{polygon}} = 16\times\frac{1}{2}\times r\times r\times\sin\frac{\pi}{8}$$
$$= 8r^2\times\frac{\sqrt{2-\sqrt{2}}}{2}$$
$$= 4\sqrt{2-\sqrt{2}}\,r^2 \text{ units}^2$$
$$\text{Area}_{\text{circle}} = \pi r^2$$

$\text{Area}_{\text{polygon}} < \text{Area}_{\text{circle}}$ since the polygon lies on or inside the circle

$$4\sqrt{2-\sqrt{2}}\,r^2 < \pi r^2$$
$$4\sqrt{2-\sqrt{2}} < \pi$$
$$\sqrt{2-\sqrt{2}} < \frac{\pi}{4}$$
$$2-\sqrt{2} < \frac{\pi^2}{16}$$
$$\frac{\pi^2}{16}+\sqrt{2} > 2$$

**Question 19**

$$\big(n-(n-1)\big)\times\frac{1}{n} < \int_{n-1}^{n}\frac{1}{x}dx$$
$$< \big(n-(n-1)\big)\times\frac{1}{n-1}$$
$$\frac{1}{n} < \Big[\ln x\Big]_{n-1}^{n} < \frac{1}{n-1}$$
$$\frac{1}{n} < \ln n-\ln(n-1) < \frac{1}{n-1}$$
$$\frac{1}{n} < \ln\frac{n}{n-1} < \frac{1}{n-1}$$
$$e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$$
$$e^{-\frac{1}{n-1}} < \frac{n-1}{n} < e^{-\frac{1}{n}}$$
$$e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$$

**Question 20**

as $x\to\infty, 2(e^{-x}+2)\to 2(0+2)=4$

as $x\to\infty, \frac{3x+4x^2}{x^2}=\frac{\frac{3}{x}+4}{1}, x\neq 0\to\frac{0+4}{1}=4$

$\therefore \lim\limits_{x\to\infty} f(x)=4.$

**Question 21a**

$$\int_1^n \ln x\,dx$$
$$= \Big[x\ln x-x\Big]_1^n$$
$$= n\ln n-n+1$$

**Question 21b**

$$S_{LR} = 1\times(\ln 1+\ln 2+\ln 3+\ldots+\ln(n-1))$$
$$= \ln\big(1\times2\times3\times\ldots\times(n-1)\big)$$
$$= \ln(n-1)!$$

**Question 21c**

$$S_{UR} = 1\times(\ln 2+\ln 3+\ln 4\ldots+\ln n)$$
$$= \ln(2\times3\times\ldots\times n)$$
$$= \ln n!$$

**Question 21d**

$$S_{LR} < \int_1^n \ln x\,dx < S_{UR}$$
$$\ln(n-1)! < n\ln n-n+1 < \ln n!$$
$$e^{\ln(n-1)!} < e^{n\ln n-n+1} < e^{\ln n!}$$
$$(n-1)! < e^{\ln(n^n)}e^{1-n} < n!$$
$$(n-1)! < n^n e^{1-n} < n!$$

**Question 22a**

$$f'(x) = \frac{(n+x)^n(n+1)(n+1+x)^n - (n+1+x)^{n+1}n(n+x)^{n-1}}{(n+x)^{2n}}$$
$$= \frac{(n+x)^{n-1}(n+1+x)^n\big((n+x)(n+1) - n(n+1+x)\big)}{(n+x)^{2n}}$$
$$= \frac{x(n+x)^{n-1}(n+1+x)^n}{(n+x)^{2n}}$$
$$\geq 0 \text{ since } n, x \geq 0$$

$\therefore f(x)$ is an increasing function of $x$ for $x > 0$.

**Question 22b**

Let $x = 1$

$$\therefore \left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$$
$$\left(\frac{n+2}{n+1}\right)^{n+1} > \left(\frac{n+1}{n}\right)^n$$
$$(n+2)^{n+1}n^n > (n+1)^{2n+1}$$

**Question 23**

$$-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$$
$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$
$$\therefore -1 \leq \sin\left(\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right) \leq 1$$
$$\therefore -x^2 \leq x^2\sin\left(\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right) \leq x^2 \text{ since } x^2 \geq 0$$

as $x \to 0, -x^2 \to 0, x^2 \to 0$

$$\therefore \lim_{x\to 0}\left[x^2\sin\left(\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right)\right] = 0 \text{ by the squeeze theorem.}$$

**Question 24a**

$$1 - x^2 \leq 1 \leq 1 + x \text{ for } x \geq 0$$
$$(1+x)(1-x) \leq 1 \leq 1+x$$
$$1 - x \leq \frac{1}{1+x} \leq 1 \quad \text{since } x \geq 0$$

**Question 24b**

$$1 - x \leq \frac{1}{1+x} \leq 1$$
$$\int_0^{\frac{1}{n}}(1-x)dx \leq \int_0^{\frac{1}{n}}\frac{1}{1+x}dx \leq \int_0^{\frac{1}{n}}dx$$
$$\left[x - \frac{x^2}{2}\right]_0^{\frac{1}{n}} \leq \Big[\ln(1+x)\Big]_0^{\frac{1}{n}} \leq \Big[x\Big]_0^{\frac{1}{n}}$$
$$\frac{1}{n} - \frac{1}{2n^2} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$$
$$1 - \frac{1}{2n} \leq n\ln\left(1+\frac{1}{n}\right) \leq 1$$

**Question 24c**

$$e^{1-\frac{1}{2n}} \leq e^{n\ln\left(1+\frac{1}{n}\right)} \leq e^1$$
$$e^{1-\frac{1}{2n}} \leq e^{\ln\left[\left(1+\frac{1}{n}\right)^n\right]} \leq e$$
$$e^{1-\frac{1}{2n}} \leq \left(1+\frac{1}{n}\right)^n \leq e$$

as $n \to \infty$ $e^{1-\frac{1}{2n}} \to e^1 = e$

$$\therefore \lim_{n\to\infty}\left(n+\frac{1}{n}\right)^n = e$$

**Question 25a**

$f(x) = \ln x - x + 1 \quad (x > 0)$

$f'(x) = \dfrac{1}{x} - 1$

$f''(x) = -\dfrac{1}{x^2} < 0$ for all $x$

$\therefore$ maximum turning point when $f'(x) = 0$

$\therefore \dfrac{1}{x} - 1 = 0 \Rightarrow x = 1 \Rightarrow f(1) = \ln 1 - 1 + 1 = 0$

Since the maximum turning point occurs at $(1,0)$, $f(x) \leq 0$ for $x > 0$

**Question 25b**

$\ln x \leq x - 1 \qquad (*) \qquad$ from (i)

$$\sum_{r=1}^{n} \ln(np_r)$$

$$\leq \sum_{r=1}^{n} (np_r - 1) \qquad \text{from } (*)$$

$\leq np_1 + np_2 + np_3 + \ldots np_n - n$

**Question 25c**

$$\sum_{r=1}^{n} \ln(np_r) \leq n(p_1 + p_2 + p_3 + \ldots + p_n) - n$$

$$\leq n(1) - n$$

$$\leq 0$$

**Question 25d**

$\therefore \ln n + \ln p_1 + \ln p_2 + \ln p_3 + \ldots + \ln p_n \leq 0$

$\ln(n^n p_1 p_2 p_3 \ldots p_n) \leq 0$

$e^{\ln(n^n p_1 p_2 p_3 \ldots p_n)} < e^0$

$n^n p_1 p_2 p_3 \ldots p_n \leq 1$

but $p_1, p_2, p_3, \ldots p_n, n$ are all positive

$\therefore 0 < n^n p_1 p_2 p_3 \ldots p_n \leq 1$

**Question 26**

Splitting each triangle into two right−angled triangles, we can show that

$$a = 2r \sin\frac{\theta}{2} \qquad (1)$$

$$b = 2r \sin\theta \qquad (2)$$

Now $2r \sin\dfrac{\theta}{2} < 2r \sin\theta$ since $\sin\theta$ is increasing for $\left[0, \dfrac{\pi}{2}\right]$

$\therefore a < b \qquad (3)$

From (2): $b = 2r\left[2 \sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\right]$ double angle results

$$= 2\left[2r \sin\frac{\theta}{2}\right]\cos\frac{\theta}{2}$$

$$= 2a \cos\frac{\theta}{2}$$

$< 2a$ since $0 < \cos\dfrac{\theta}{2} < 1$ for $\left[0, \dfrac{\pi}{2}\right]$

$\therefore b < 2a \qquad (4)$

$\therefore a < b < 2a$ from (3) and (4)

**Question 27a**

$y' = 81x^2 - 54x$

Let $y' = 0$

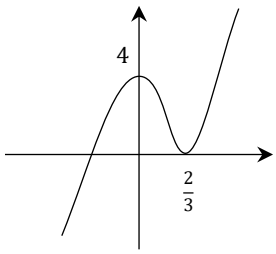
$27x(3x - 2) = 0$

$x = 0, \frac{2}{3}$

$y = 4, 0$

The turning points are $(0,4)$ and $\left(\frac{2}{3}, 0\right)$

**Question 27b**




$27x^3 - 27x^2 + 4 \geq 0$ for $x \geq 0$

$27x^2(x - 1) \geq -4$

$x^2(x - 1) \geq -\frac{4}{27}$

$x^2(1 - x) \leq \frac{4}{27}$

**Question 27c**

(Proof by contradiction)

Assume all three numbers are greater than $\frac{4}{27}$:

$bc(1 - a) > \frac{4}{27}$ (1)

$ac(1 - b) > \frac{4}{27}$ (2)

$ab(1 - c) > \frac{4}{27}$ (3)

$(1)\times(2)\times(3)$:

$$a^2b^2c^2(1 - a)(1 - b)(1 - c) > \left(\frac{4}{27}\right)^3$$

$$a^2(1 - a)\times b^2(1 - b)\times c^2(1 - c) > \left(\frac{4}{27}\right)^3$$

But $a^2(1 - a) \leq \frac{4}{27}$ from (i), and similarly

$b^2(1 - b) \leq \frac{4}{27}$, $\quad c^2(1 - c) \leq \frac{4}{27}$

$\therefore a^2(1 - a)\times b^2(1 - b)\times c^2(1 - c) \leq \left(\frac{4}{27}\right)^3$

$\therefore$ at least one of the assumptions (1), (2) or (3) is false.

$\therefore$ at least one of the numbers $ab(1 - c)$, $ac(1 - b)$ and $bc(1 - a)$ is less than or equal to $\frac{4}{27}$

**Question 28**

$$\left|f(x)-2\right| \leq (x-1)^2 \left|\sin\left(\frac{1}{x-1}\right)\right|$$

$\therefore \left|f(x)-2\right| \leq (x-1)^2$ since $\left|\sin\left(\frac{1}{x-1}\right)\right| \leq 1$

$$-(x-1)^2 \leq f(x)-2 \leq (x-1)^2$$

$$2-(x-1)^2 \leq f(x) \leq 2+(x-1)^2$$

as $x \to 1, 2-(x-1)^2 \to 2, 2+(x-1)^2 \to 2$

$\therefore \lim\limits_{x\to 1} f(x) = 2$ by the squeeze theorem.

**Question 29a**

$$1-t^2+t^4-t^6+\ldots+t^{4n}$$

$$= \frac{1\left(1-\left(-t^2\right)^{2n+1}\right)}{1-(-t^2)}$$

$$= \frac{1+t^{4n+2}}{1+t^2}$$

now:

$$1 < 1+t^{4n+2} < 1+t^{4n+2}+t^{4n+4}$$

$$\therefore \frac{1}{1+t^2} < \frac{1+t^{4n+2}}{1+t^2} < \frac{1+t^{4n+2}+t^{4n+4}}{1+t^2}$$

$$\therefore \frac{1}{1+t^2} < 1-t^2+t^4-t^6+\ldots+t^{4n} < \frac{1}{1+t^2}+t^{4n+2}$$

**Question 29b**

Integrating all parts from 0 to $x$ with respect to $t$:

$$\int_0^x \frac{1}{1+t^2}\,dt$$

$$= \left[\tan^{-1} t\right]_0^x$$

$$= \tan^{-1} x - \tan^{-1} 0$$

$$= \tan^{-1} x$$

$$\int_0^x (1-t^2+t^4-t^6+\ldots+t^{4n})\,dt$$

$$= \left[t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\cdot+\frac{t^{4n+1}}{4n+1}\right]_0^x$$

$$= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+\frac{x^{4n+1}}{4n+1}$$

$$\int_0^x \left(\frac{1}{1+t^2}+t^{4n+2}\right)dt$$

$$= \left[\tan^{-1} t+\frac{t^{4n+3}}{4n+3}\right]_0^x$$

$$= \tan^{-1} x+\frac{x^{4n+3}}{4n+3}$$

Since $0 < t < x$ all functions are positive

**Question 29c**

$$\therefore \tan^{-1} x < x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+\frac{x^{4n+1}}{4n+1} < \tan^{-1} x+\frac{x^{4n+3}}{4n+3}$$

Let $n \to \infty \Rightarrow \frac{x^{4n+3}}{4n+3} \to 0$ since $0 < x \leq 1$

$\therefore \tan^{-1} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$ by the squeeze theorem.

**Question 29d**

Let $x = 1$

$$\therefore \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

$$\pi = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$$

**Question 30a**

Let $u = \ln x$ for $x > 0$

$$\therefore e^{\ln x} \geq 1 + \ln x$$

$$x \geq 1 + \ln x$$

$$\ln x \leq x - 1$$

**Question 30b**

Let $x = \frac{c_1}{\mu}$

$$\ln\left(\frac{c_1}{\mu}\right) \leq \frac{c_1}{\mu} - 1 \quad (1)$$

Similarly

$$\ln\left(\frac{c_2}{\mu}\right) \leq \frac{c_2}{\mu} - 1 \quad (2), \qquad \ln\left(\frac{c_3}{\mu}\right) \leq \frac{c_3}{\mu} - 1 \quad (3), \ldots, \ln\left(\frac{c_n}{\mu}\right) \leq \frac{c_n}{\mu} - 1 \quad (n)$$

Summing inequalities $(1)$ to $(n)$:

$$\ln\left(\frac{c_1}{\mu}\right) + \ln\left(\frac{c_2}{\mu}\right) + \ln\left(\frac{c_3}{\mu}\right) + \ldots + \ln\left(\frac{c_n}{\mu}\right) \leq \frac{c_1}{\mu} - 1 + \frac{c_2}{\mu} - 1 + \frac{c_3}{\mu} - 1 + \ldots + \frac{c_n}{\mu} - 1$$

$$\ln\left(\frac{c_1}{\mu} \times \frac{c_2}{\mu} \times \frac{c_3}{\mu} \times \ldots \times \frac{c_n}{\mu}\right) \leq \left(\frac{c_1}{\mu} + \frac{c_2}{\mu} + \frac{c_3}{\mu} + \ldots + \frac{c_n}{\mu}\right) - n$$

$$\therefore \log_e\left(\frac{c_1 c_2 c_3 \ldots c_n}{\mu^n}\right) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\mu} - n$$

**Question 30c**

$$\log_e\left(\frac{c_1 c_2 c_3 \ldots c_n}{\mu^n}\right) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\mu} - n$$

$$\log_e(c_1 c_2 c_3 \ldots c_n) - \log_e(\mu^n) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\mu} - n$$

$$\log_e(c_1 c_2 c_3 \ldots c_n) - n \log_e(\mu) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\mu} - n$$

$$\log_e(c_1 c_2 c_3 \ldots c_n) - n \log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}} - n$$

$$\log_e(c_1 c_2 c_3 \ldots c_n) - n \log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right) \leq n - n$$

$$\log_e(c_1 c_2 c_3 \ldots c_n) \leq n \log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right)$$

$$\frac{1}{n}\log_e(c_1 c_2 c_3 \ldots c_n) \leq \log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right)$$

$$\log_e \sqrt[n]{c_1 c_2 c_3 \ldots c_n} \leq \log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right)$$

$$\therefore \sqrt[n]{c_1 c_2 c_3 \ldots c_n} \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}$$

**Question 30d**

$$\therefore c_1 + c_2 + c_3 + \ldots + c_n \geq n\sqrt[n]{c_1 c_2 c_3 \ldots c_n}$$

$$\frac{101}{103} + \frac{103}{105} + \frac{105}{107} + \ldots + \frac{197}{199} + \frac{199}{101} \geq 50 \sqrt[50]{\frac{101}{103} \times \frac{103}{105} \times \frac{105}{107} \times \ldots \times \frac{197}{199} \times \frac{199}{101}}$$

$$\geq 50\sqrt[50]{1}$$

$$\geq 50$$

**Question 1**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $1 < 2^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$k < 2^k$$

RTP $P(k+1)$: $k+1 < 2^{k+1}$

LHS $= k+1$

$< 2^k + 1$ from $P(k)$

$< 2^k + 2^k$ for $k \geq 1$

$= 2 \cdot 2^k$

$= 2^{k+1}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 2**

Let $P(n)$ represent the proposition $a_n$

$= 5(3^{n-1})$.

$P(1)$ is true since LHS $= a_1 = 5(3^{1-1}) = 5$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$a_k = 5\left(3^{k-1}\right)$$

RTP $P(k+1)$: $a_{k+1} = 5\left(3^k\right)$

LHS $= a_{k+1}$

$= 3a_k$ from the recursive formula

$= 3\left(5\left(3^{k-1}\right)\right)$ from $P(k)$

$= 5\left(3^k\right)$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 3**

Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= 2^{1+4} = 32$; and RHS $= 2(1) + 9 = 11$ so $2^{1+4} > 2(1) + 9$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$2^{k+4} > 2k + 9$$

RTP $P(k+1)$ $2^{k+5} > 2k + 11$

LHS $= 2^{k+5}$

$= 2\left(2^{k+4}\right)$

$> 2(2k+9)$ from $P(k)$

$= 4k + 18$

$> 2k + 11 + 2k + 7$

$> 2k + 11$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 4**

Let $P(n)$ represent the proposition

$u_n = 3 + 2^{n-1}$.

$P(1)$ is true since LHS $= u_1 = 3 + 2^{1-1} = 4$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$u_k = 3 + 2^{k-1}$$

RTP $P(k+1)$: $u_{k+1} = 3 + 2^k$

LHS $= u_{k+1}$

$= 2u_k - 3$ from the recursive formula

$= 2\left(3 + 2^{k-1}\right) - 3$ from $P(k)$

$= 6 + 2 \times 2^{k-1} - 3$

$= 3 + 2^k$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 5**

Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= 2(1) = 2$; RHS $= (1+2)! = 6$, so $2(1) < (1+2)!$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$2k < (k+2)!$$

RTP $P(k+1)$: $2(k+1) < (k+3)!$

LHS $= 2k + 2$

$< (k+2)! + 2$ from $P(k)$

$= \left(1 + \dfrac{2}{(k+2)!}\right)(k+2)!$

$< (k+3)(k+2)!$ since $k \geq 1$

$< (k+3)!$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 6**

If $P(1)$ is true, and for all $k \geq 1$ if $P(k)$ is true then $P(k+1)$ is true, then for all $n \geq 1$ $P(n)$ is true. We know this statement as mathematical induction, specifically for a statement that is true for all positive integers.

**Question 7**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $a_1 = 2$ and $a_1 = 2\times5^{1-1} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$a_k = 2\times5^{k-1}$$

RTP $P(k+1)$: $a_{k+1} = 2\times5^k$

LHS $= a_{k+1}$

$= 5a_k$ from the recursive formula

$= 5\left(2\times5^{k-1}\right)$ from $P(k)$

$= 2\times5^k$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

**Question 8**

Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= 2^1 + 1 = 3$;

RHS $= 3^1 = 3$, so $2^1 + 1 \leq 3^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$2^k + 1 \leq 3^k$$

RTP $P(k+1)$: $2^{k+1} + 1 \leq 3^{k+1}$

LHS $= 2^{k+1} + 1$

$= 2\left(2^k + 1\right) - 1$

$\leq 2\left(3^k\right) - 1$ from $P(k)$

$\leq 3\left(3^k\right) - 3^k - 1$

$\leq 3^{k+1} - \left(3^k + 1\right)$

$\leq 3^{k+1}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 9**

Let $P(n)$ represent the proposition.

$P(0)$ is true since LHS $= a_0 = A$ and

RHS=$a_0 = A(1+r)^0 = A$.

If $P(k)$ is true for some arbitrary $k \geq 0$ then

$$a_k = A(1+r)^k$$

RTP $P(k+1)$: $a_{k+1} = A(1+r)^{k+1}$

LHS $= a_{k+1}$

$= (1+r)a_k$ from the recursive formula

$= (1+r)\times A(1+r)^k$ from $P(k)$

$= A(1+r)^{k+1}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 0$ by induction

**Question 10**

Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= 2^{1+4} = 32$ and

RHS $= (1+4)^2 = 25$ so $2^{1+4} > (1+4)^2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$2^{k+4} > (k+4)^2$$

RTP $P(k+1)$: $2^{k+5} > (k+5)^2$

LHS $= 2^{k+5}$

$= 2\left(2^{k+4}\right)$

$> 2(k+4)^2$ from $P(k)$

$= 2k^2 + 16k + 32$

$= \left(k^2 + 10k + 25\right) + \left(k^2 + 6k + 7\right)$

$> (k+5)^2$ since $k \geq 0$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 11**

Let $P(n)$ represent the proposition.

$P(2)$ is true since LHS $= 2^2 = 4$;

RHS $= 2 + 1 = 3$ so $2^2 > 2 + 1$

If $P(k)$ is true for some arbitrary $k \geq 2$ then

$$k^2 > k + 1$$

RTP $P(k+1)$: $(k+1)^2 > k + 2$

LHS $= k^2 + 2k + 1$

$> k + 1 + 2k + 1$ from $P(k)$

$> 3k + 2$

$> k + 2$ since $k \geq 2$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction.

**Question 12**

Let $P(n)$ represent the proposition.

$P(4)$ is true since LHS $= 4! = 24$; RHS $= 2(4) = 8$ so $4! > 2(4)$

If $P(k)$ is true for some arbitrary $k \geq 4$ then

$k! > 2k$

RTP $P(k+1)$ $(k+1)! > 2k + 2$

LHS $= (k+1)\times k!$

$> (k+1)\times 2k$ from $P(k)$

$= k(2k+2)$

$> 2k + 2$ for $k \geq 4$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 4$ by induction.

**Question 13**

Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= a_1 = \dfrac{2}{1!} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$a_k = \frac{2}{k!}$$

RTP $P(k+1)$: $a_{k+1} = \dfrac{2}{(k+1)!}$

LHS $= a_{k+1}$

$= \dfrac{a_k}{k+1}$ from the recursive formula

$= \dfrac{2}{k!} \div (k+1)$ from $P(k)$

$= \dfrac{2}{(k+1)!}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

**Question 14**

Let $P(n)$ represent the proposition.

$P(6)$ is true since LHS $= 6(6)+6 = 42$; RHS $= 2^6 = 64 \therefore 6(6)+6 < 2^6$.

If $P(k)$ is true for some arbitrary $k \geq 6$ then

$$6k + 6 < 2^k$$

RTP $P(k+1)$: $6k + 12 < 2^{k+1}$

LHS $= 6k + 6 + 6$

$< 2^k + 6$ from $P(k)$

$< 2^k + 2^k$ for $k \geq 6$

$= 2^{k+1}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 6$ by induction

**Question 15**

Let $P(n)$ represent the proposition.

$P(8)$ is true since LHS $= 3(8)^2 + 3(8) + 1$

$= 217$; RHS $= 3^1 = 256$.

If $P(k)$ is true for some arbitrary $k \geq 8$ then

$$3k^2 + 3k + 1 < 2^k$$

RTP $P(k+1)$: $3(k+1)^2+3(k+1)+1 < 2^{k+1}$

LHS $= 3(k+1)^2+3(k+1)+1$

$= 3k^2 + 6k + 3 + 3k + 3 + 1$

$= 3k^2 + 3k + 1 + 6k + 6$

$< 2^k + 6k + 6$ from $P(k)$

$< 2^k + 2^k$ for $k \geq 8$

$= 2^{k+1}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 8$ by induction.

**Question 16a**

$u_2 = \sqrt{3u_1} = \sqrt{3(1)} = 3^{\frac{1}{2}}$

$u_3 = \sqrt{3u_2} = \sqrt{3\left(3^{\frac{1}{2}}\right)} = \left(3^{\frac{3}{2}}\right)^{\frac{1}{2}} = 3^{\frac{3}{4}} = 3^{\frac{1}{2}+\frac{1}{4}}$

$u_4 = \sqrt{3u_3} = \sqrt{3\left(3^{\frac{3}{4}}\right)} = \left(3^{\frac{7}{4}}\right)^{\frac{1}{2}} = 3^{\frac{7}{8}} = 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$

**Question 16b**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $u_1 = 3^{1-2^{1-1}} = 3^{1-1} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$u_k = 3^{1-2^{1-k}}$$

RTP $P(k+1)$: $u_{k+1} = 3^{1-2^{-k}}$

LHS $= u_{k+1}$

$= \sqrt{3u_k}$

$= \sqrt{3\times 3^{1-2^{1-k}}}$ from $P(k)$

$= \sqrt{3^{1+1-2^{1-k}}}$

$= \sqrt{3^{2-2^{1-k}}}$

$= \sqrt{3^{2\left(1-2^{-k}\right)}}$

$= 3^{1-2^{-k}}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

**Question 16c**

$u_n = 3^{1-2^{1-n}}$

as $n \to \infty, 2^{1-n} \to 0, u_n \to 3^{1-0} \to 3$

**Question 17**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $1^2 < 4^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$k^2 < 4^k$$

RTP $P(k+1)$: $(k+1)^2 < 4^{k+1}$

LHS $= (k+1)^2$

$= k^2 + 2k + 1$

$< 4^k + 2k + 1$ from $P(k)$

$< 4^k + 3\times 4^k$

since $2k+1 \leq 3k < 3\left(4^k\right)$ for $k \geq 1$

$= 4\left(4^k\right)$

$= 4^{k+1}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 18**

Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= \frac{1}{1} = 1$;

RHS $= \ln(1+1) = \ln 2 \approx 0.693$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k} > \ln(1+k)$$

RTP $P(k+1)$:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\frac{1}{k+1} > \ln(2+k)$$

$$\begin{aligned}\text{LHS} &= \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\frac{1}{k+1} \\ &> \ln(1+k)+\frac{1}{k+1} \qquad \text{from } P(k) \\ &> \ln(1+k)+\ln\left(1+\frac{1}{k+1}\right) \\ &= \ln(1+k)+\ln\left(\frac{k+2}{k+1}\right) \\ &= \ln(1+k)+\ln(k+2)-\ln(k+1) \\ &= \ln(k+2) \\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 19**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $F_0 = 2^{2^0}+1 = 3$; $F_1 = 2^{2^1}+1 = 5$ and $F_0 = F_1 - 2 \to 3 = 5-2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$F_0 \times F_1 \times F_2 \times \ldots \times F_{k-1} = F_k - 2$$

RTP $P(k+1)$:

$$F_0 \times F_1 \times F_2 \times \ldots \times F_{k-1} \times F_k = F_{k+1} - 2$$

$$\begin{aligned}\text{LHS} &= F_0 \times F_1 \times F_2 \times \ldots \times F_{k-1} \times F_k \\ &= (F_k - 2) \times F_k \quad \text{from } P(k) \\ &= \left(2^{2^k}+1-2\right)\left(2^{2^k}+1\right) \\ &= \left(2^{2^k}-1\right)\left(2^{2^k}+1\right) \\ &= 2^{2^{k+1}}-1 \\ &= F_{k+1}-2 \\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 0$ by induction.

**Question 20**

Let $P(n)$ represent the proposition.

$P(2)$ is true since LHS $= 12^2 = 144$;

RHS $= 7^2 + 5^2 = 74$

If $P(k)$ is true for some arbitrary $k \geq 2$ then

$$12^k > 7^k + 5^k$$

RTP $P(k+1)$: $12^{k+1} > 7^{k+1} + 5^{k+1}$

$$\begin{aligned}\text{LHS} &= 12(12^k) \\ &> 12\left(7^k+5^k\right) \qquad \text{from } P(k) \\ &= 12(7^k)+12(5^k) \\ &> 7\left(7^k\right)+5\left(5^k\right) \\ &= 7^{k+1}+5^{k+1} \\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction.

**Question 21a**

Let $P(1)$ represent the proposition.

$P(1)$ is true since $\frac{a_1-\sqrt{2}}{a_1+\sqrt{2}} = \frac{1-\sqrt{2}}{1+\sqrt{2}}$

$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^1 = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{1-1}}$$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$\frac{a_k-\sqrt{2}}{a_k+\sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}$$

RTP $P(k+1)$: $\frac{a_{k+1}-\sqrt{2}}{a_{k+1}+\sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^k}$

$$\begin{aligned}\text{LHS} &= \frac{a_{k+1}-\sqrt{2}}{a_{k+1}+\sqrt{2}} \\ &= \frac{\frac{1}{2}\left(a_k+\frac{2}{a_k}\right)-\sqrt{2}}{\frac{1}{2}\left(a_k+\frac{2}{a_k}\right)+\sqrt{2}} \\ &= \frac{a_k+\frac{2}{a_k}-2\sqrt{2}}{a_k+\frac{2}{a_k}+2\sqrt{2}} \\ &= \frac{a_k^2+2-2\sqrt{2}a_k}{a_k^2+2+2\sqrt{2}a_k} \\ &= \frac{a_k^2-2\sqrt{2}a_k+2}{a_k^2+2\sqrt{2}a_k+2} \\ &= \frac{\left(a_k-\sqrt{2}\right)^2}{\left(a_k+\sqrt{2}\right)^2} \\ &= \left(\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}\right)^2 \\ &= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^k} \\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

**Question 21b**

as $n \to \infty$,

$$\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}} \to 0$$

$$\frac{a_n-\sqrt{2}}{a_n+\sqrt{2}} \to 0$$

$$a_n - \sqrt{2} \to 0$$

$$a_n \to \sqrt{2}$$

**Question 22**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $1! \geq \left((1)!\right)^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $1!\times3!\times5!\times\ldots\times(2k-1)! \geq (k!)^k$

RTP $P(k+1)$ $\quad 1!\times3!\times5!\times\ldots\times(2k-1)!\times(2k+1)! \geq ((k+1)!)^{k+1}$

LHS $= 1!\times3!\times5!\times\ldots\times(2k-1)!\times(2k+1)! \geq (k!)^k\times(2k+1)!$ $\quad$ from $P(k)$

$$= (k!)^k\left(\underbrace{(2k+1)(2k)(2k-1)\ldots(k+2)}_{k\text{ terms}}(k+1)k!\right)$$

$$\geq (k!)^k\left((k+1)^k(k+1)!\right)$$

$$= \left((k+1)k!\right)^k(k+1)!$$

$$= \left((k+1)!\right)^k(k+1)!$$

$$= \left((k+1)!\right)^{k+1}$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 23**

Let $P(n)$ represent the proposition.

$P(2)$ is true since $1^3 < \frac{2^4}{4} < 1^3 + 2^3 \;\rightarrow\; 1 < 4 < 9$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $\quad 1^3 + 2^3 + \ldots + (k-1)^3 < \frac{k^4}{4} < 1^3 + 2^3 + \ldots + k^3$

RTP $P(k+1)$ $\quad 1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{(k+1)^4}{4} < 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$

$1^3 + 2^3 + \ldots + (k-1)^3 < \frac{k^4}{4} < 1^3 + 2^3 + \ldots + k^3$ $\quad$ from $P(k)$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 < 1^3 + 2^3 + \ldots + k^3 + k^3$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 < 1^3 + 2^3 + \ldots + k^3 + k^3$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 + \frac{3}{2}k^2 + k + \frac{1}{4} < 1^3 + 2^3 + \ldots + k^3 + k^3 + \frac{3}{2}k^2 + k + \frac{1}{4}$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{4} < 1^3 + 2^3 + \ldots + k^3 + k^3 + 3k^2 + 3k + 1$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{(k+1)^4}{4} < 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction

**Question 24**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $1 > \frac{1}{2} \geq \frac{1}{2}$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then $k > \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^k} \geq \frac{k}{2}$ (1)

RTP $P(k+1)$: $k+1 > \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^{k+1}} \geq \frac{k+1}{2}$

Now since $0 < 1, 2, 3, \ldots, 2^{k-1} < 2^k$:

$$\underbrace{\frac{1}{2^k} + \frac{1}{2^k} + \ldots + \frac{1}{2^k}}_{2^k \text{ times}} > \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^k+2^k} > \underbrace{\frac{1}{2^k+2^k} + \frac{1}{2^k+2^k} + \ldots + \frac{1}{2^k+2^k}}_{2^k \text{ times}}$$

$$\therefore \frac{2^k}{2^k} > \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^k+2^k} > \frac{2^k}{2^{k+1}}$$

$$1 > \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^k+2^k} > \frac{1}{2} \quad (2)$$

$(1) + (2)$: $k+1 > \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^{k+1}} > \frac{k}{2} + \frac{1}{2}$

$\therefore k+1 > \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^{k+1}} \geq \frac{k+1}{2}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 25**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $x + \frac{1}{x} \geq 2\sqrt{x \cdot \frac{1}{x}} = 2$

since the arithmetic mean is greater than or equal to the geometric mean.

Similarly $P(2)$ is true since $x^2 + \frac{1}{x^2} \geq 2\sqrt{x^2 \cdot \frac{1}{x^2}} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $x^k + x^{k-2} + x^{k-4} + \ldots + \frac{1}{x^{k-4}} + \frac{1}{x^{k-2}} + \frac{1}{x^k} \geq k+1$

RTP $P(k+2)$: $x^{k+2} + x^k + x^{k-2} + x^{k-4} + \ldots + \frac{1}{x^{k-4}} + \frac{1}{x^{k-2}} + \frac{1}{x^k} + \frac{1}{x^{k+2}} \geq k+3$

$$\text{LHS} = x^{k+2} + x^k + x^{k-2} + x^{k-4} + \ldots + \frac{1}{x^{k-4}} + \frac{1}{x^{k-2}} + \frac{1}{x^k} + \frac{1}{x^{k+2}}$$

$$= x^k + x^{k-2} + x^{k-4} + \ldots + \frac{1}{x^{k-4}} + \frac{1}{x^{k-2}} + \frac{1}{x^k} + x^{k+2} + \frac{1}{x^{k+2}}$$

$$\geq k + 1 + x^{k+2} + \frac{1}{x^{k+2}} \quad \text{from } P(k)$$

$$\geq k + 1 + 2\sqrt{x^{k+2} \cdot \frac{1}{x^{k+2}}} \quad \text{AM–GM}$$

$$\geq k + 1 + 2$$

$$\geq k + 3$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd and even $n \geq 1$ by induction.

**Question 26**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\sqrt{1} \leq \frac{1}{\sqrt{1}} \leq 2\sqrt{1} - 1 \rightarrow 1 \leq 1 \leq 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\sqrt{k} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} \leq 2\sqrt{k} - 1$

$RTP$ $P(k+1)$: $\sqrt{k+1} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$

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**Question 26 (continued)**

$$\sqrt{k} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}} \le 2\sqrt{k}-1 \qquad \text{from } P(k)$$

$$\sqrt{k}+\frac{1}{\sqrt{k+1}} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \le 2\sqrt{k}-1+\frac{1}{\sqrt{k+1}}$$

$$\frac{\sqrt{k}\times\sqrt{k+1}+1}{\sqrt{k+1}} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \le \frac{2\sqrt{k}\times\sqrt{k+1}+1}{\sqrt{k+1}}-1$$

$$\frac{\left(\sqrt{k}\times\sqrt{k}+1\right)}{\sqrt{k+1}} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \le \frac{2\sqrt{k^2+k}+1}{\sqrt{k+1}}-1$$

$$\frac{k+1}{\sqrt{k+1}} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \le \frac{2\sqrt{k^2+k+\frac{1}{2}}+1}{\sqrt{k+1}}-1$$

$$\frac{k+1}{\sqrt{k+1}} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \le \frac{2\sqrt{k^2+k+\frac{1}{2}}+1}{\sqrt{k+1}}-1$$

$$\sqrt{k+1} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \le \frac{2\sqrt{\left(k+\frac{1}{2}\right)^2}+1}{\sqrt{k+1}}-1$$

$$\sqrt{k+1} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \le \frac{2\left(k+\frac{1}{2}\right)+1}{\sqrt{k+1}}-1$$

$$\sqrt{k+1} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \le \frac{2(k+1)}{\sqrt{k+1}}-1$$

$$\sqrt{k+1} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \le 2\sqrt{k+1}-1$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \ge 1$ by induction.

**Question 27**

Let $P(n)$ represent the proposition

$P(1)$ is true since LHS $= 1$; RHS $= \frac{4(1)+3}{6}\sqrt{1} = \frac{7}{6}$

If $P(k)$ is true for some arbitrary $k \ge 1$ then

$$1+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{k} \le \frac{4k+3}{6}\sqrt{k}$$

RTP: $P(k+1)$

$$1+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{k}+\sqrt{k+1} \le \frac{4k+7}{6}\sqrt{k+1}$$

$$\text{LHS} \le \frac{4k+3}{6}\sqrt{k}+\sqrt{k+1} \qquad \text{from } P(k)$$

$$= \frac{(4k+3)\sqrt{k}+6\sqrt{k+1}}{6}$$

$$= \frac{\sqrt{(4k+3)^2\times k}+6\sqrt{k+1}}{6}$$

$$= \frac{\sqrt{16k^3+24k^2+9k}+6\sqrt{k+1}}{6}$$

$$\le \frac{\sqrt{16k^3+24k^2+9k+1}+6\sqrt{k+1}}{6}$$

$$= \frac{\sqrt{(k+1)(16k^2+8k+1)}+6\sqrt{k+1}}{6}$$

$$= \frac{\sqrt{(k+1)(4k+1)^2}+6\sqrt{k+1}}{6}$$

$$= \frac{(4k+1)\sqrt{(k+1)}+6\sqrt{k+1}}{6}$$

$$= \frac{4k+7}{6}\sqrt{k+1}$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \ge 1$ by induction.

**Question 28a**

Let $P(1)$ represent the proposition.

$P(1)$ is true since $a_{1+1} = a_1(a_1 + 1) = 1(1 + 1) =$

$1 + \sum_{r=1}^{1} a_r^2 = 1 + (a_1)^2 = 1 + 1^2 = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$a_{k+1} = 1 + \sum_{r=1}^{k} a_r^2$

RTP $P(k + 1)$

$a_{k+2} = 1 + \sum_{r=1}^{k+1} a_r^2$

LHS $= a_{k+2}$

$= a_{k+1}(a_{k+1} + 1)$

$= (a_{k+1})^2 + a_{k+1}$

$= (a_{k+1})^2 + 1 + \sum_{r=1}^{k} a_r^2$ from $P(k)$

$= 1 + \sum_{r=1}^{k+1} a_r^2$

$=$ RHS

$\therefore P(k) \Rightarrow P(k + 1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

**Question 28b**

$(2a_{n+1} + 1)^2$

$= 4a_{n+1}^2 + 4a_{n+1} + 1$

$= (2a_{n+1})^2 + 4a_n(a_n + 1) + 1$

$= 4a_n^2 + 4a_n + 1 + (2a_{n+1})^2$

$= (2a_n + 1)^2 + (2a_{n+1})^2$

**Question 28c**

$(2a_{n+1} + 1)^2 - (2a_n + 1)^2 = (2a_{n+1})^2$

$\sum_{r=1}^{n} \left[(2a_{r+1} + 1)^2 - (2a_r + 1)^2\right] = \sum_{r=1}^{n} (2a_{r+1})^2$

$(2a_{n+1} + 1)^2 - (2a_1 + 1)^2 = \sum_{r=2}^{n+1} (2a_r)^2$

$\therefore (2a_{n+1} + 1)^2 = (2a_1 + 1)^2 + \sum_{r=2}^{n+1} (2a_r)^2$

**Question 28d**

$a_2 = a_1(a_1 + 1) = 1(1 + 1) = 2$

$a_3 = a_2(a_2 + 1) = 2(2 + 1) = 6$

$a_4 = a_3(a_3 + 1) = 6(6 + 1) = 42$

$a_5 = a_4(a_4 + 1) = 42(42 + 1) = 1806$

**Question 28e**

$a_5 = 1 + \sum_{r=1}^{4} a_r^2$

$= 1 + a_1^2 + a_2^2 + a_3^2 + a_4^2$

$= 1 + 1^2 + 2^2 + 6^2 + 42^2$

**Question 28f**

$(2a_1 + 1)^2 + \sum_{r=2}^{5} (2a_r)^2 = (2a_5 + 1)^2$ from (c)

$(2a_1 + 1)^2 + (2a_2)^2 + (2a_3)^2 + (2a_4)^2 + (2a_5)^2 = (2a_5 + 1)^2$

$(2(1) + 1)^2 + \left(2(2)\right)^2 + \left(2(6)\right)^2 + \left(2(42)\right)^2 + \left(2(1806)\right)^2 = (2(1806) + 1)^2$

$3^2 + 4^2 + 12^2 + 84^2 + 3612^2 = 3613^2$

**Question 29**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $a_1 = 2^{1-2^{1-1}} = 2^{1-1} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = 2^{(1-2^{(1-k)})}$

RTP $P(k+1)$ $\quad a_{k+1} = 2^{(1-2^{-k})}$

LHS $= a_{k+1}$

$= \sqrt{2a_k}$

$= \sqrt{2 \times 2^{(1-2^{(1-k)})}}$ from $P(k)$

$= \sqrt{2^{1+1-2^{1-k}}}$

$= \sqrt{2^{2-2^{1-k}}}$

$= \sqrt{2^{2(1-2^{-k})}}$

$= 2^{1-2^{-k}}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for $n \geq 1$ by induction

**Question 1**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{d}{dx}(e^x) = e^x$ and $1e^{1x} = e^x$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}.$$

RTP $P(k+1)$: $\frac{d}{dx}\left(e^{(k+1)x}\right) = (k+1)e^{(k+1)x}$

LHS $= \frac{d}{dx}\left(e^x \times e^{kx}\right)$

$= e^x \times \frac{d}{dx}\left(e^{kx}\right) + e^{kx} \times \frac{d}{dx}(e^x)$

by the product rule

$= e^x \times ke^{kx} + e^{kx} \times e^x$ from $P(k)$

$= ke^{kx+x} + e^{kx+x}$

$= (k+1)e^{(k+1)x}$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 2**

Let $P(n)$ represent the proposition

$P(2)$ is true since LHS $= {}^2C_2 = 1$; RHS $= 2^2 = 4$.

If $P(k)$ is true for some arbitrary $k \geq 2$ then

$${}^kC_2 \leq k^2$$

RTP $P(k+1)$: ${}^{k+1}C_2 \leq (k+1)^2$

LHS $= \frac{(k+1)!}{(k-1)!\,2!}$

$= \frac{k+1}{k-1} \times \frac{k!}{(k-2)!\,2!}$

$= \frac{k+1}{k-1} \times {}^kC_2$

$< \frac{k+1}{k-1} \times k^2$ from $P(k)$

$< k^2$ since $k+1 > k-1$

$< k^2 + 2k + 1$ since $k > 0$

$= (k+1)^2$

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for $n \geq 2$ by induction.

**Question 3**

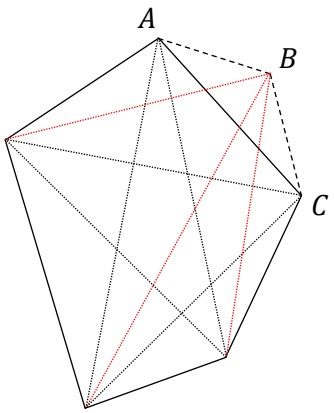
Let $P(n)$ represent the proposition.

$P(3)$ is true since a 3 sided plane convex polygon is a triangle which has no diagonals, and $\frac{3(3-3)}{2} = 0$.

If $P(k)$ is true for some arbitrary $k \geq 3$, then a $k$ sided plane convex polygon has $\frac{k(k-3)}{2}$ diagonals.

RTP $P(k+1)$: a $k+1$ sided plane convex polygon has $\frac{(k+1)(k-2)}{2}$ diagonals.

Consider the $k+1$ sided polygon, formed by adding $\Delta ABC$ to one side of the $k$ sided polygon.




The diagonals of the $k+1$ sided polygon include all the diagonals of the $k$ sided polygon, plus the new diagonals from $B$ to each of the vertices except for $A, B$ and $C$, so $k+1-3 = k-2$ new diagonals, plus the side $AC$ which is now a diagonal.

The $k+1$ sided polygon has:

$\frac{k(k-3)}{2} + k - 2 + 1$

$= \frac{k^2 - 3k + 2k - 4 + 2}{2}$

$= \frac{k^2 - k - 2}{2}$

$= \frac{(k+1)(k-2)}{2}$ diagonals

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 3$ by induction.

**Question 4a**

If there are $n$ students then there will be $\binom{n}{2}$ handshakes.

**Question 4b**

Let $P(n)$ represent the proposition.

$P(2)$ is true since 2 people involves 1 handshake, and $\binom{2}{2} = 1$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $k$ students will involve $\binom{k}{2}$ handshakes

RTP $P(k+1)$ $k+1$ students will involve $\binom{k+1}{2}$ handshakes

Consider a group of $k$ students, who have shaken each other's hands, which involves $\binom{k}{2}$ handshakes from $P(k)$.

The $k+1^{th}$ person joins them and shakes hands with each of the $k$ students already there.

The number of handshakes is

$$\begin{aligned}\binom{k}{2} + k &= \frac{k!}{2!\,(k-2)!} + k \\ &= \frac{k! + 2k(k-2)!}{2!\,(k-2)!} \\ &= \frac{(k-1)k! + 2k(k-1)(k-2)!}{2!\,(k-1)(k-2)!} \\ &= \frac{(k-1)k! + 2k!}{2!\,(k-1)!} \\ &= \frac{k!\,(k-1+2)}{2!\,(k-1)!} \\ &= \frac{(k+1)!}{2!\,(k-1)!} \\ &= \binom{k+1}{2} \\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction

**Question 5**

Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= (a-b)^1 = a - b$ and

$$\text{RHS} = \sum_{r=0}^{1} (-1)^r \binom{1}{r} a^{1-r} b^r$$

$$= 1\times\binom{1}{0} a^1 b^0 - 1\times\binom{1}{1} a^0 b^1 = a - b.$$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$(a-b)^k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} a^{k-r} b^r.$$

RTP $P(k+1)$:

$$(a-b)^{k+1} = \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} a^{k+1-r} b^r$$

continued next column

**Question 5 (continued)**

LHS

$$\begin{aligned}&= (a-b)(a-b)^k \\ &= (a-b)\sum_{r=0}^{k} (-1)^r \binom{k}{r} a^{k-r} b^r \qquad \text{from } P(k) \\ &= \sum_{r=0}^{k} (-1)^r \binom{k}{r} a^{k-r+1} b^r \\ &\qquad - \sum_{r=0}^{k} (-1)^r \binom{k}{r} a^{k-r} b^{r+1} \\ &= \sum_{r=0}^{k} (-1)^r \binom{k}{r} a^{k-r+1} b^r \\ &\qquad - \sum_{r=1}^{k+1} (-1)^{r-1} \binom{k}{r-1} a^{k-r+1} b^r \\ &= \binom{k}{0} a^{k-0+1} b^0 + \sum_{r=1}^{k} (-1)^r \binom{k}{r} a^{k-r+1} b^r \\ &\qquad - \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r-1} a^{k-r+1} b^r \\ &\qquad + (-1)^{k+1} \binom{k}{k+1-1} a^{k-(k+1)+1} b^{k+1} \\ &= a^{k+1} + \sum_{r=1}^{k} (-1)^r \left[\binom{k}{r} + \binom{k}{r-1}\right] a^{k-r+1} b^r \\ &\qquad + (-1)^{k+1} b^{k+1} \\ &= a^{k+1} + \sum_{r=1}^{k} (-1)^r \binom{k+1}{r} a^{k-r+1} b^r \\ &\qquad + (-1)^{k+1} b^{k+1} \\ &\qquad\qquad \text{since} \binom{k+1}{r} = \binom{k}{r} + \binom{k}{r-1} \\ &= (-1)^0 \binom{k+1}{0} a^{k-0+1} b^0 + \sum_{r=1}^{k} {}^{k+1}C_r a^{k-r+1} b^r \\ &\qquad + (-1)^{k+1} \binom{k+1}{k+1} a^{k-(k+1)+1} b^{k+1} \\ &= \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} a^{k+1-r} b^r \\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 6**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{d}{dx}(\cos^1 x) = -\sin x$

and $-1\cos^{1-1} x \sin x = -\sin x$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$\frac{d}{dx}(\cos^k x) = -k\cos^{k-1} x \sin x$$

RTP $P(k+1)$:

$$\frac{d}{dx}(\cos^{k+1} x) = -(k+1)\cos^k x \sin x$$

$$\begin{aligned}\text{LHS} &= \frac{d}{dx}(\cos x \cos^k x)\\ &= \cos x \frac{d}{dx}(\cos^k x) + \cos^k x \times \frac{d}{dx}(\cos x)\\ &= \cos x\left(-k\cos^{k-1} x \sin x\right)\\ &\quad + \cos^k x\,(-\sin x) \quad \text{from } P(k)\\ &= -k\cos^k x \sin x - \cos^k x \sin x\\ &= -(k+1)\cos^k x \sin x\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 7**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\binom{1}{1} = \frac{1!}{0!\,1!} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$\binom{k}{1} = k$$

RTP $P(k+1)$: $\binom{k+1}{1} = k+1$

$$\begin{aligned}\text{LHS} &= \binom{k+1}{1}\\ &= \binom{k}{1} + \binom{k}{0} \quad \text{since } \binom{n}{r} = \binom{n-1}{r}\\ &= k+1 \qquad \text{from } P(k)\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 8**

Let $P(n)$ represent the proposition.

$P(1)$ is true since with one circle we can colour the inside of the circle with one colour and the outside the other.

If $P(k)$ is true for some arbitrary $k \geq 1$, then with $k$ circles we can colour the resulting regions with only two colours so no two regions sharing an edge are the same colour.

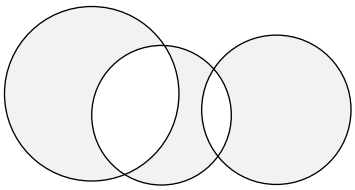
RTP $P(k+1)$ We can colour the regions that result from $k+1$ circles with only two colours so no two regions sharing an edge are the same colour.

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**Question 8 (continued)**

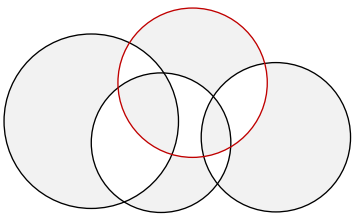
Consider the $k$ circles on the plane, with no two regions sharing a border being the same colour as shown.

$k$ circles



Add a circle, here shown in red, cutting some regions in two, and possibly leaving others unaffected.

$k+1$ circles



For all regions on the inside of the circle swap their colour, as shown. No two regions sharing an edge are the same colour.

$\therefore$ We can colour the regions that result from $k+1$ circles with only two colours so no two regions sharing an edge are the same colour.

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 9**

Let $P(n)$ represent the proposition.

$P(1)$ is true since

LHS $= \sin(\pi + x) = \sin\pi\cos x + \cos\pi \sin x$

$= 0 + (-1)\sin x$;

RHS $= (-1)^1 \sin x = (-1)\sin x$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$\sin(k\pi + x) = (-1)^k \sin x$$

RTP $P(k+1)$:

$$\sin((k+1)\pi + x) = (-1)^{k+1}\sin x$$

$$\begin{aligned}\text{LHS} &= \sin\left((k\pi + x) + \pi\right)\\ &= -\sin(k\pi + x)\\ &= -(-1)^k \sin x \ \text{ from } P(k)\\ &= (-1)^{k+1}\sin x\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 10**

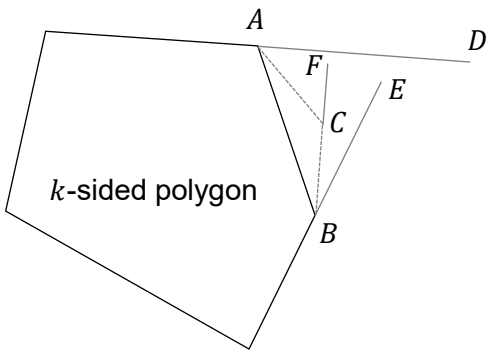
Let $P(n)$ represent the proposition.

$P(3)$ is true since the sum of the interior and exterior angle at each of the three vertices is $180°$, totalling $540°$. Subtracting the interior angle sum of a triangle, $180°$, we are left with an exterior angle sum of $360°$.

If $P(k)$ is true for some arbitrary $k \geq 3$, then the sum of the exterior angles of a $k-$sided plane convex polygon is $360°$

RTP $P(k+1)$: The sum of the exterior angles of a $(k+1)-$sided plane convex polygon is $360°$

Consider the $(k+1)$ − sided polygon below, formed by adding $\Delta ABC$ to one side of the $k-$sided polygon.




The exterior angle sum of the new polygon will be equal to that of the $k-$sided polygon:

- minus $\angle CAB$ from exterior angle $\angle DAB$
- minus $\angle CBA$ from exterior angle $\angle EBA$
- plus the new exterior angle $\angle ACF$

$\therefore$ the sum of the exterior angles of a $(k+1)-$sided polygon is $360° + \angle ACF - \angle CAB - \angle CBA$ from $P(k)$.

Now $\angle ACF$

$= \angle CAB + \angle CBA$ (exterior angle of $\Delta ABC$).

The sum of the exterior angles of a $k+1-$sided plane convex polygon is

$360° + \angle ACF - \angle ACF = 360°$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 3$ by induction.

**Question 11**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{d}{dx}(xe^{2x}) = e^{2x} + x \cdot 2e^{2x}$

$= (2x+1)e^{2x} = 2^{1-1}(2x+1)e^{2x}$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{d^k}{dx^k}(xe^{2x}) = 2^{k-1}(2x+k)e^{2x}$

RTP $P(k+1)$ $\frac{d^{k+1}}{dx^{k+1}}(xe^{2x})$

$= 2^k(2x+k+1)e^{2x}$

continued next column

**Question 11 (continued)**

$$\text{LHS} = \frac{d^{k+1}}{dx^{k+1}}(xe^{2x})$$
$$= \frac{d}{dx}\left(\frac{d^k}{dx^k}\right)$$
$$= \frac{d}{dx}\left(2^{k-1}(2x+k)e^{2x}\right) from P(k)$$
$$= 2^{k-1} \times [e^{2x}(2) + (2x+k) \times 2e^{2x}]$$
$$= 2^{k-1} \times 2e^{2x}[1 + 2x + k]$$
$$= 2^k(2x+k+1)e^{2x}$$
$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 12**

Let $P(n)$ represent the proposition.

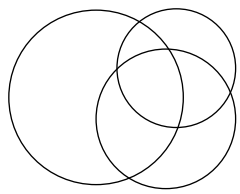
$P(1)$ is true since one circle divides the plane into two regions (one inside and one outside), and $1^2 - 1 + 2 = 2$.

If $P(k)$ is true for some arbitrary $k \geq 1$, then $k$ such circles divide the plane into $k^2 - k + 2$ regions.

RTP $P(k+1)$: $k+1$ such circles divide the plane into $(k+1)^2 - (k+1) + 2 = k^2 + k + 2$ regions.

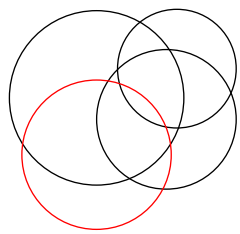
Consider the $k$ circles intersecting each other at two distinct points, with no three circles intersecting at the same point.

$k$ circles



Add a circle, here shown in red, cutting some regions in two, and possibly leaving others unaffected.

$k+1$ circles



The new circle cuts each of the $k$ existing circles in two places. The circumference between successive points of intersection adds one new region to the plane, one on the inside and one on the outside of the new circle.

The number of regions with $k+1$ circles is:

$k^2 - k + 2 + 2k$ from $P(k)$

$= k^2 + k + 2$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 13a**

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
$= \cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2$
$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$
$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

**Question 13b**

Let $P(n)$ represent the proposition.
$P(1)$ is true since $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1 + i\sin\theta_1$
If $P(k)$ is true for some arbitrary $k \geq 1$ then
$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_k + i\sin\theta_k) = \cos(\theta_1 + \theta_2 + \ldots + \theta_k) + i\sin(\theta_1 +$
RTP $P(k+1)$:
$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_k + i\sin\theta_k)(\cos\theta_{k+1} + i\sin\theta_{k+1})$
$= \cos(\theta_1 + \theta_2 + \ldots + \theta_k + \theta_{k+1}) + i\sin(\theta_1 + \theta_2 + \ldots$
LHS $= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_k + i\sin\theta_k)(\cos\theta_{k+1} + i\sin\theta_{k+1})$
$= (\cos(\theta_1 + \theta_2 + \ldots + \theta_k) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_k))(\cos\theta_{k+1} + i\sin\theta_{k+1})$
$= \cos(\theta_1 + \theta_2 + \ldots + \theta_k)\cos\theta_{k+1} + i\cos(\theta_1 + \theta_2 + \ldots + \theta_k)\sin\theta_{k+1} + i\sin(\theta_1 + \theta_2 + \ldots + \theta_k)$
$+ i^2\sin(\theta_1 + \theta_2 + \ldots + \theta_k)\sin\theta_{k+1}$
$= \cos(\theta_1 + \theta_2 + \ldots + \theta_k)\cos\theta_{k+1} - \sin(\theta_1 + \theta_2 + \ldots + \theta_k)\sin\theta_{k+1}$
$+ i(\cos(\theta_1 + \theta_2 + \ldots + \theta_k)\sin\theta_{k+1} + \sin(\theta_1 + \theta_2 + \ldots + \theta_k)\cos\theta_{k+1})$
$= \cos(\theta_1 + \theta_2 + \ldots + \theta_{k+1}) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_{k+1})$
$=$ RHS
$\therefore P(k) \Rightarrow P(k+1)$
$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 14**

Let $P(n)$ represent the proposition.
$P(0)$ is true since $\text{LHS} = \cos(2^0\alpha) = \cos\alpha$ ; $\text{RHS} = \dfrac{\sin(2\alpha)}{2\sin\alpha} = \dfrac{2\sin\alpha\cos\alpha}{2\sin\alpha} = \cos\alpha$
If $P(k)$ is true for some arbitrary $k \geq 0$ then
$\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^k\alpha = \dfrac{\sin(2^{k+1}\alpha)}{2^{k+1}\sin\alpha}$
RTP $P(k+1)$
$\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^k\alpha\cos 2^{k+1}\alpha = \dfrac{\sin(2^{k+2}\alpha)}{2^{k+2}\sin\alpha}$

$\text{LHS} = \cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^k\alpha\cos 2^{k+1}\alpha$
$= \dfrac{\sin(2^{k+1}\alpha)}{2^{k+1}\sin\alpha} \times \cos 2^{k+1}\alpha$ from $P(k)$
$= \dfrac{2\sin(2^{k+1}\alpha)\cos 2^{k+1}\alpha}{2^{k+2}\sin\alpha}$
$= \dfrac{\sin 2^{k+2}\alpha}{2^{k+2}\sin\alpha}$
$=$ RHS
$\therefore P(k) \Rightarrow P(k+1)$
$\therefore P(n)$ is true for $n \geq 0$ by induction

**Question 15a**

$T(k,x) - T(k,x+1)$
$= \dfrac{k!}{x(x+1)(x+2)\ldots(x+k)} - \dfrac{k!}{(x+1)(x+2)\ldots(x+k+1)}$
$= \dfrac{k!\,(x+k+1) - k!\,x}{x(x+1)(x+2)\ldots(x+k)(x+k+1)}$
$= \dfrac{(k+1)!}{x(x+1)(x+2)\ldots(x+k)(x+k+1)}$
$= T(k+1,x)$

**Page 13 – Question**

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**Page 13 – Question**

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**Question 15b**

Let $P(n,x)$ represent the proposition

$P(1,x)$ is true since $T(1,x)=\dfrac{{}^1C_0}{x}-\dfrac{{}^1C_1}{x+1}=\dfrac{1}{x}-\dfrac{1}{x+1}=\dfrac{x+1-x}{x(x+1)}=\dfrac{1!}{x(x+1)}$

If $P(k,x)$ is true for some arbitrary $k\geq 1$, with $x$ fixed, then

$$T(k,x)=\frac{{}^kC_0}{x}-\frac{{}^kC_1}{x+1}+\frac{{}^kC_2}{x+2}-\ldots+(-1)^k\frac{{}^kC_k}{x+k}=\frac{k!}{x(x+1)(x+2)\ldots(x+k)}$$

RTP $T(k+1,x)$:

$$T(k+1,x)=\frac{{}^{k+1}C_0}{x}-\frac{{}^{k+1}C_1}{x+1}+\frac{{}^{k+1}C_2}{x+2}-\ldots+(-1)^k\frac{{}^kC_k}{x+k}+(-1)^{k+1}\frac{{}^{k+1}C_{k+1}}{x+k+1}$$

$$=\frac{(k+1)!}{x(x+1)(x+2)\ldots(x+k+1)}$$

$$\text{LHS}=\frac{{}^{k+1}C_0}{x}-\frac{{}^{k+1}C_1}{x+1}+\frac{{}^{k+1}C_2}{x+2}-\ldots+(-1)^k\frac{{}^{k+1}C_k}{x+k}+(-1)^{k+1}\frac{{}^{k+1}C_{k+1}}{x+k+1}$$

$$=\frac{{}^kC_0}{x}-\frac{{}^kC_0+{}^kC_1}{x+1}+\frac{{}^kC_1+{}^kC_2}{x+2}-\ldots+(-1)^k\frac{{}^kC_{k-1}+{}^kC_k}{x+k}+(-1)^{k+1}\frac{{}^{k+1}C_{k+1}}{x+k+1}$$

$$=\left(\frac{{}^kC_0}{x}-\frac{{}^kC_1}{x+1}+\frac{{}^kC_2}{x+2}-\ldots+(-1)^k\frac{{}^kC_k}{x+k}\right)-\left(\frac{{}^kC_0}{x+1}-\frac{{}^kC_1}{x+2}+\frac{{}^kC_2}{x+3}\ldots+(-1)^k\frac{{}^kC_{k-1}}{x+k}+(-1)^{k+1}\frac{{}^kC_k}{x+k+1}\right)$$

$$=T(k,x)-T(k,x+1)$$

$$=T(k+1,x)\ \text{ from (a)}$$

$$=\frac{(k+1)!}{x(x+1)(x+2)\ldots(x+k+1)}$$

$\therefore P(k,x)\Rightarrow P(k+1,x)$

$\therefore P(n,x)$ is true for $n\geq 1$ by induction

**Question 15c**

$$\frac{{}^nC_0}{1}-\frac{{}^nC_1}{3}+\frac{{}^nC_2}{5}-\ldots+(-1)^n\frac{{}^nC_n}{2n+1}$$

$$=\frac{1}{2}\left(\frac{{}^nC_0}{1/2}-\frac{{}^nC_1}{1+1/2}+\frac{{}^nC_2}{2+1/2}-\ldots+(-1)^n\frac{{}^nC_n}{n+1/2}\right)$$

$$=\frac{1}{2}\times T\left(n,\frac{1}{2}\right)$$

$$=\frac{1}{2}\times\frac{n!}{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\ldots\left(\frac{1}{2}+n\right)}$$

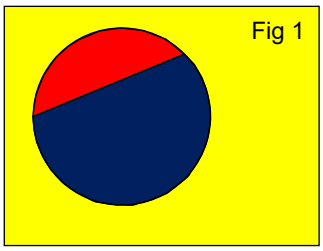
$$=\frac{1}{2}\times\frac{2^{n+1}n!}{1\times3\times5\times\ldots\times(2n+1)}$$

$$=\frac{2^nn!}{1\times3\times5\times\ldots\times(2n+1)}$$

**Question 16**

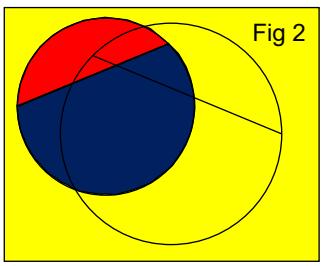
Let $P(n)$ represent the proposition.

$P(1)$ is true since when one circle and a chord of that circle are drawn in a plane the two segments can be coloured in the first two colours, and the outside of the circle in the third colour. Say red, blue and yellow respectively (Fig 1).



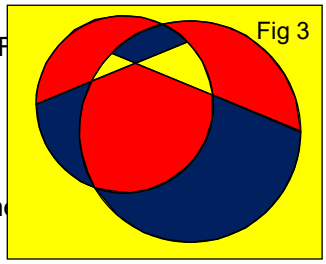

If $P(k)$ is true for some arbitrary $k \geq 1$ then when $k$ circles and chords are drawn on a plane then the regions created can be coloured with three colours in such a way that no two regions sharing the same length of border are the same colour.

RTP $P(k+1)$ when $k+1$ circles and chords are drawn on a plane then the regions created can be coloured with three colours in such a way that no two regions sharing the same length of border are the same colour.




To the $k$ circles and chords from $P(k)$ add a circle and a chord to that circle (Fig 2):

- For regions intersected by the minor segment change yellow to red, red to blue and blue to yellow.
- For regions intersected by the major segment change the colours in the reverse order, so red to yellow, yellow to blue and blue to red (F
- There are three cases to consider:
  - regions that are outside the new circle do not change colour, so no regions sharing the same border are the same colour, from $P(k)$.
  - any region that is cut by one segment has the outside stay the same colour and the inside change colour, so no regions sharing the sam border are the same colour.
  - any region that is cut by two segments has the outside stay the same colour and the two inside regions change colour in opposite directions, so no regions sharing the same border are the same colour.




$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 17**

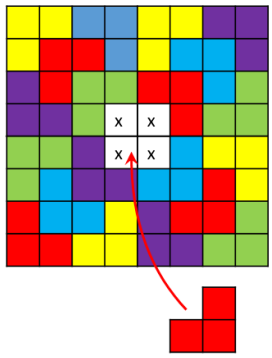
Let $P(n)$ represent the proposition.

$P(1)$ is true since a $2^1 \times 2^1$ grid with one square removed can be covered by one L tile.

If $P(k)$ is true for some arbitrary $k$ then it is possible to completely cover a $2^k \times 2^k$ grid with L tiles consisting of 3 squares if 1 square is removed.

RTP $P(k+1)$ It is possible to completely cover a $2^{k+1} \times 2^{k+1}$ grid with L tiles consisting of 3 squares if 1 square is removed.

Arrange 4 of the $2^k \times 2^k$ grids in a square, removing the square from each grid that is in the centre of the new square, which is a $2^{k+1} \times 2^{k+1}$ grid, as shown at right.




Place one extra tile in the centre, so only 1 square has been removed from the $2^{k+1} \times 2^{k+1}$ grid and the rest is covered.

$\therefore$ It is possible to completely cover a $2^{k+1} \times 2^{k+1}$ grid with L tiles consisting of 3 squares if 1 square is removed.

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 18**

Let $P(n)$ represent the proposition.

$P(1)$ is true since the one disk can be moved to the third peg in 1 move, and $2^1 - 1 = 1$

If $P(k)$ is true for some arbitrary $k$ then it takes $2^k - 1$ moves to move $k$ disks from the first peg to the third peg.

RTP $P(k+1)$: It takes $2^{k+1} - 1$ moves to move $k+1$ disks from the first peg to the third peg.

Move the first $k$ disks to the second peg instead of the third peg, in $2^k - 1$ moves, from $P(k)$.

Move the bottom disk to the third peg in one move.

Move the first $k$ disks again until they are on the third peg on top of the largest disk, in $2^k - 1$ moves, again from $P(k)$.

Total moves $= (2^k - 1) + 1 + (2^k - 1)$

$= 2 \cdot 2^k - 1$

$= 2^{k+1} - 1$ as required

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

**Question 19**

Let $P(n)$ represent the proposition.

$P(12)$ is true since we can use three 4 cent stamps.

If $P(k)$ is true for some arbitrary $k \geq 12$ then we can pay for it exactly using only 4 cent and 5 cent stamps.

RTP $P(k+1)$: We can pay for $k+1$ cents of postage exactly using only 4 cent and 5 cent stamps.

Case 1: We have used at least one 4 cent stamp to make $k$ cents postage

Remove one 4 cent stamp and replace it with a 5 cent stamp, increasing the postage by 1 cent from $k$ to $k+1$ cents.

Case 2: There are no 4 cent stamps used to make $k$ cents postage.

There must be at least three 5 cent stamps here, and if we remove three 5 cent stamps and replace them with four 4 cent stamps the postage increases by 1 cent from $k$ to $k+1$ cents.

$\therefore$ We can pay for $k+1$ cents postage.

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 12$ by induction.

**Question 20**

$T_1 = 3, d = 4$

$S_n = \frac{n}{2}\big(2(3) + 4(n-1)\big) = \frac{n}{2}(6 + 4n - 4) = n(2n+1)$

Let $P(n)$ represent the proposition, that $R_n = S_n + 1$.

$P(1)$ is true since $S_1 = 3$, and 2 such chords will divide the circle into 4 regions, $R_1 = 4 = 3 + 1 = S_1 + 1$.

If $P(k)$ is true for some arbitrary $k \geq 1$, then $2k$ such chords divide the circle into

$R_k = k(2k+1) + 1 = 2k^2 + k + 1$ regions.

RTP $P(k+1)$: $2(k+1)$ such chords divide the circle into

$R_{k+1} = 2(k+1)^2 + (k+1) + 1 = 2k^2 + 5k + 4$ regions.

The $(2k+1)^{\text{st}}$ chord intersects the $2k$ previous chords in $2k$ distinct places, dividing $2k+1$ of the existing regions in two, as shown below where the 5th chord passes through 5 regions, increasing the number of regions by 5.

So the $(2k+1)^{\text{st}}$ chord creates an extra $2k+1$ regions.

Similarly the $(2k+2)^{\text{nd}}$ such chord creates an extra $2k+2$ regions,

so on Erin's $(k+1)^{\text{th}}$ turn the number of regions increases by $2k + 1 + 2k + 2 = 4k + 3$.

$\therefore R_{k+1} = R_k + 4k + 3$

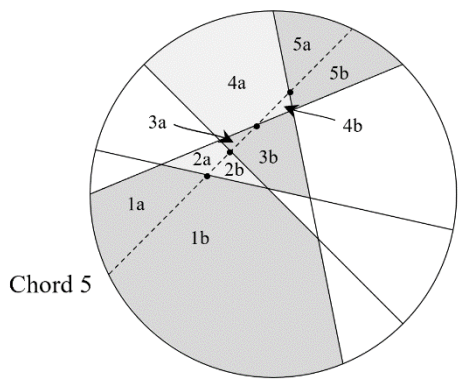
$= k(2k+1) + 1 + 4k + 3$ from $P(k)$

$= 2k^2 + k + 4k + 4$

$= 2k^2 + 5k + 4$

$\therefore P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for $n \geq 1$ by induction.



**Question 1**

$$\int_{-7}^{1} \frac{dx}{x^2+6x+25}$$
$$= \int_{-7}^{1} \frac{dx}{x^2+6x+9+16}$$
$$= \int_{-7}^{1} \frac{dx}{(x+3)^2+4^2}$$
$$= \frac{1}{4}\left[\tan^{-1}\left(\frac{x+3}{4}\right)\right]_{-7}^{1}$$
$$= \frac{1}{4}\left(\tan^{-1}1 - \tan^{-1}(-1)\right)$$
$$= \frac{1}{4}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$
$$= \frac{\pi}{8}$$

**Question 2**

$$\int_{0}^{\frac{3\sqrt{3}}{4}} \frac{1}{\sqrt{9-4x^2}}dx$$
$$= \frac{1}{2}\int_{0}^{\frac{3\sqrt{3}}{4}} \frac{2}{\sqrt{3^2-(2x)^2}}dx$$
$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_{0}^{\frac{3\sqrt{3}}{4}}$$
$$= \frac{1}{2}\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}0\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{3} - 0\right)$$
$$= \frac{\pi}{6}$$

**Question 3**

$$\int \frac{dx}{\sqrt{20-8x-x^2}}dx$$
$$= \int \frac{dx}{\sqrt{-(x^2+8x-20)}}$$
$$= \int \frac{dx}{\sqrt{-\left((x+4)^2-36\right)}}$$
$$= \int \frac{dx}{\sqrt{6^2-(x+4)^2}}$$
$$= \sin^{-1}\left(\frac{x+4}{6}\right) + c$$

**Question 4**

$$\int \frac{dx}{12x^2+12x+78}$$
$$= \frac{1}{3}\int \frac{dx}{4x^2+4x+26}$$
$$= \frac{1}{6}\int \frac{2dx}{(2x+1)^2+5^2}$$
$$= \frac{1}{6}\times\frac{1}{5}\tan^{-1}\left(\frac{2x+1}{5}\right) + c$$
$$= \frac{1}{30}\tan^{-1}\left(\frac{2x+1}{5}\right) + c$$

**Question 5**

$$\int \frac{x^2+2x}{x^2+2x+5}dx$$
$$= \int \frac{x^2+2x+5-5}{x^2+2x+5}dx$$
$$= \int \left(1 - \frac{5}{(x+1)^2+2^2}\right)dx$$
$$= x - \frac{5}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$$

**Question 6**

$$\int \frac{e^{-x}}{e^{-x}-1}dx$$
$$= -\int \frac{-e^{-x}}{e^{-x}-1}dx$$
$$= -\ln\left|e^{-x}-1\right| + c$$

**Question 7**

$$\int \frac{1}{x^2-3x+3}dx$$
$$= \int \frac{1}{x^2-3x+\frac{9}{4}+\frac{3}{4}}dx$$
$$= \int \frac{1}{\left(x-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{3}{2}}{\sqrt{3}/2}\right) + c$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-3}{\sqrt{3}}\right) + c$$

**Question 8**

$$\int_{-1}^{1} \frac{1}{5-2t+t^2}dt$$
$$= \int_{-1}^{1} \frac{dt}{(t-1)^2+2^2}$$
$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{t-1}{2}\right)\right]_{-1}^{1}$$
$$= \frac{1}{2}(0 - \left(-\frac{\pi}{4}\right)$$
$$= \frac{\pi}{8}$$

**Question 9**

$$\int \frac{2}{4x^2-4x+17}\,dx$$
$$=\int \frac{2}{(2x-1)^2+4^2}\,dx$$
$$=\frac{1}{4}\tan^{-1}\left(\frac{2x-1}{4}\right)+c$$

**Question 10**

$$\int \frac{2x}{x^4+2x^2+5}\,dx$$
$$=\int \frac{2x}{(x^2+1)^2+2^2}\,dx$$
$$=\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{2}\right)+c$$

**Question 11**

$$\int \frac{2}{e^x+1}\,dx$$
$$=\int \frac{2e^{-x}}{1+e^{-x}}\,dx$$
$$=-2\int \frac{-e^{-x}}{1+e^{-x}}\,dx$$
$$=-2\ln\left|1+e^{-x}\right|+c$$

**Question 12**

$$-\int \frac{2x}{\sqrt{3-2x^2-x^4}}\,dx$$
$$=-\int \frac{2x}{\sqrt{4-(1+x^2)^2}}\,dx$$
$$=\cos^{-1}\left(\frac{x^2+1}{2}\right)+c \quad \text{or} \quad -\sin^{-1}\left(\frac{x^2+1}{2}\right)+c$$

**Question 13**

$$\int \frac{2\cos x\,(\sin x+1)}{(\sin x+3)(\sin x-1)}\,dx$$
$$=\int \frac{2\sin x\cos x+2\cos x}{\sin^2 x+2\sin x-3}\,dx$$
$$=\ln\left|\sin^2 x+2\sin x-3\right|+c$$

**Question 14**

$$\int \frac{2}{x\ln 7}\,dx$$
$$=\frac{2}{\ln 7}\int \frac{dx}{x}$$
$$=\frac{2}{\ln 7}\ln|x|+c$$
$$=\frac{2\ln|x|}{\ln 7}+c$$
$$=\frac{\ln\left(x^2\right)}{\ln 7}+c$$
$$=\log_7\left(x^2\right)+c$$

**Question 15**

$$\int \frac{dx}{(x^2+4)\tan^{-1}\left(\frac{x}{2}\right)}$$
$$=\frac{1}{2}\int \frac{\frac{2}{x^2+4}}{\tan^{-1}\left(\frac{x}{2}\right)}\,dx$$
$$=\frac{1}{2}\int \frac{\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{2}\right)\right)}{\tan^{-1}\left(\frac{x}{2}\right)}\,dx$$
$$=\frac{1}{2}\ln\left|\tan^{-1}\left(\frac{x}{2}\right)\right|+c$$

**Question 16**

$$\frac{dy}{dx}=\frac{\cos x}{1+\sin^2 x}$$
$$\int_1^y dy=\int_{\frac{\pi}{2}}^x \frac{\cos x}{1+\sin^2 x}\,dx$$
$$\left[y\right]_1^y=\int_{\frac{\pi}{2}}^x \frac{\frac{d}{dx}(\sin x)}{1+(\sin x)^2}\,dx$$
$$y-1=\left[\tan^{-1}\left(\sin x\right)\right]_{\frac{\pi}{2}}^x$$
$$=\tan^{-1}\left(\sin x\right)-\tan^{-1}\left(\sin\frac{\pi}{2}\right)$$
$$y=\tan^{-1}\left(\sin x\right)-\frac{\pi}{4}+1$$

**Question 17**

$$\int_{-\frac{\pi}{2}}^0 \frac{\cos\theta}{\sin^2\theta+2\sin\theta+4}\,d\theta$$
$$=\int_{-\frac{\pi}{2}}^0 \frac{\cos\theta}{(\sin\theta+1)^2+\sqrt{3}^2}\,d\theta$$
$$=\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{\sin\theta+1}{\sqrt{3}}\right)\right]_{-\frac{\pi}{2}}^0$$
$$=\frac{1}{\sqrt{3}}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}0\right)$$
$$=\frac{1}{\sqrt{3}}\left(\frac{\pi}{6}-0\right)$$
$$=\frac{\pi}{6\sqrt{3}} \qquad \left(\text{or } =\frac{\pi\sqrt{3}}{18}\right)$$

**Question 18**

$$\int \frac{e^{3x}}{\sqrt{7+12e^{3x}-4e^{6x}}}\,dx$$
$$=\int \frac{e^{3x}}{\sqrt{-(4e^{6x}-12e^{3x}-7)}}\,dx$$
$$=\int \frac{e^{3x}}{\sqrt{-((2e^{3x}-3)^2-4^2)}}\,dx$$
$$=\frac{1}{6}\int \frac{6e^{3x}}{\sqrt{4^2-(2e^{3x}-3)^2}}\,dx$$
$$=\frac{1}{6}\sin^{-1}\left(\frac{2e^{3x}-3}{4}\right)+c$$

**Question 19**

$$\int \frac{\cos x - \sin x}{2 + \sin 2x}\,dx$$
$$= \int \frac{\cos x - \sin x}{2 + 2\sin x \cos x}\,dx$$
$$= \int \frac{\cos x - \sin x}{\sin^2 x + 2\sin x \cos x + \cos^2 x + 1}\,dx$$
$$= \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2 + 1}\,dx$$
$$= \tan^{-1}\left(\sin x + \cos x\right) + c$$

**Question 20**

$$\int \frac{\sin^3 x}{\cos x - 1}\,dx$$
$$= \int \frac{\sin x \sin^2 x}{\cos x - 1}\,dx$$
$$= \int \frac{\sin x\,(1 - \cos^2 x)}{\cos x - 1}\,dx$$
$$= \int \frac{\sin x\,(1 + \cos x)(1 - \cos x)}{\cos x - 1}\,dx$$
$$= -\int (\sin x + \sin x \cos x)\,dx$$
$$= -\int \left(\sin x + \frac{1}{2}\sin 2x\right)dx$$
$$= \cos x + \frac{1}{4}\cos 2x + c$$

**Question 21**

$$\int \frac{e^{\log_2 x}}{x}\,dx$$
$$= \int \frac{e^{\left(\frac{\ln x}{\ln 2}\right)}}{x}\,dx$$
$$= \int \frac{\left(e^{\ln x}\right)^{\frac{1}{\ln 2}}}{x}\,dx$$
$$= \int \frac{x^{\frac{1}{\ln 2}}}{x}\,dx$$
$$= \int x^{\frac{1}{\ln 2}-1}\,dx$$
$$= \ln 2\, x^{\frac{1}{\ln 2}} + c$$

**Question 22**

$$\int \frac{\sin 2x + \cos 2x}{\cos x}\,dx$$
$$= \int \frac{2\sin x \cos x + 2\cos^2 x - 1}{\cos x}\,dx$$
$$= \int (2\sin x + 2\cos x - \sec x)\,dx$$
$$= -2\cos x + 2\sin x - \ln\left|\tan x + \sec x\right| + c$$

**Question 23**

$$\int \frac{dx}{\sec x - 1}$$
$$= \int \frac{\cos x}{1 - \cos x}\,dx$$
$$= \int \frac{\cos x}{1 - \cos x} \times \frac{1 + \cos x}{1 + \cos x}\,dx$$
$$= \int \frac{\cos x + \cos^2 x}{1 - \cos^2 x}\,dx$$
$$= \int \frac{\cos x + \cos^2 x}{\sin^2 x}\,dx$$
$$= \int \cos x\,(\sin x)^{-2}\,dx + \int \cot^2 x\,dx$$
$$= -\frac{1}{\sin x} + \int \left(\operatorname{cosec}^2 x - 1\right)dx$$
$$= -\operatorname{cosec} x - \cot x - x + c$$

**Question 24**

$$\int (2x \operatorname{cosec} x - x^2 \cot x \operatorname{cosec} x)\,dx$$
$$= \int \left(\frac{2x}{\sin x} - \frac{x^2 \cos x}{\sin^2 x}\right)dx$$
$$= \int \frac{2x\sin x - x^2 \cos x}{\sin^2 x}\,dx$$
$$= \int \frac{d}{dx}\left(\frac{x^2}{\sin x}\right)dx$$
$$= \frac{x^2}{\sin x} + c$$

**Alt:**

$$\int (2x \operatorname{cosec} x - x^2 \cot x \operatorname{cosec} x)\,dx$$
$$= \int \left(\frac{d}{dx}(x^2)\operatorname{cosec} x + x^2 \frac{d}{dx}(\operatorname{cosec} x)\right)dx$$
$$= \int \frac{d}{dx}(x^2 \operatorname{cosec} x)\,dx$$
$$= x^2 \operatorname{cosec} x + c$$

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$$\int \frac{\cos x\,(2a\sin x + b)}{(\sin x - \alpha)(\sin x - \beta)}\,dx$$
$$= \int \frac{\cos x\,(2a\sin x + b)}{\sin^2 x - (\alpha + \beta)\sin x + \alpha\beta}\,dx$$
$$= \int \frac{\cos x\,(2a\sin x + b)}{\sin^2 x + \frac{b}{a}\sin x + \frac{c}{a}}\,dx$$
$$= \int \frac{a\cos x\,(2a\sin x + b)}{a\sin^2 x + b\sin x + c}\,dx$$
$$= a\int \frac{2a\sin x\cos x + b\cos x}{a\sin^2 x + b\sin x + c}\,dx$$
$$= a\ln\left|a\sin^2 x + b\sin x + c\right| + k$$

**Question 1**

$$\int x^2(x^3+4)^5\,dx$$
$$=\frac{1}{3}\int_{-7}^{1} 3x^2(x^3+4)^5\,dx$$
$$=\frac{1}{3}\times\frac{(x^3+4)^6}{6}+c$$
$$=\frac{(x^3+4)^6}{18}+c$$

**Alt:**

$$\int x^2(x^3+4)^5\,dx$$

$$u=x^3+4$$
$$du=3x^2dx$$

$$=\frac{1}{3}\int(x^3+4)^5\times 3x^2dx$$
$$=\frac{1}{3}\int u^5\,du$$
$$=\frac{1}{3}\times\frac{u^6}{6}+c$$
$$=\frac{(x^3+4)^6}{18}+c$$

**Question 2**

$$\int_1^2\frac{x^2}{\sqrt{2x^3-1}}dx$$
$$=\frac{1}{6}\int_1^2 6x^2(2x^3-1)^{-\frac{1}{2}}dx$$
$$=\frac{1}{6}\times 2\left[(2x^3-1)^{\frac{1}{2}}\right]_1^2$$
$$=\frac{\sqrt{15}-1}{3}$$

**Alt:**

$$\int_1^2\frac{x^2}{\sqrt{2x^3-1}}dx$$

$$u^2=2x^3-1$$
$$2u\,du=6x^2dx$$

$$=\frac{1}{6}\int_1^2\frac{1}{\sqrt{2x^3-1}}\times 6x^2dx$$
$$=\frac{1}{6}\int_1^{\sqrt{15}}\frac{1}{u}\times 2u\,du$$
$$=\frac{1}{3}\int_1^{\sqrt{15}}du$$
$$=\frac{1}{3}\left[u\right]_1^{\sqrt{15}}$$
$$=\frac{\sqrt{15}-1}{3}$$

**Question 3**

$$\int e^{2x}\sqrt{e^{2x}-1}\,dx$$
$$=\frac{1}{2}\int 2e^{2x}(e^{2x}-1)^{\frac{1}{2}}dx$$
$$=\frac{1}{2}\times\frac{2}{3}(e^{2x}-1)^{\frac{3}{2}}+c$$
$$=\frac{\sqrt{(e^{2x}-1)^3}}{3}+c$$

**Alt:**

$$\int e^{2x}\sqrt{e^{2x}-1}\,dx$$

$$u^2=e^{2x}-1$$
$$2u\,du=2e^{2x}dx$$

$$=\frac{1}{2}\int\sqrt{e^{2x}-1}\times 2e^{2x}dx$$
$$=\frac{1}{2}\int u\times 2u\,du$$
$$=\int u^2$$
$$=\frac{u^3}{3}+c$$
$$=\frac{\sqrt{(e^{2x}-1)^3}}{3}+c$$

**Question 4**

$$\int x\cos(x^2)\,dx$$
$$=\frac{1}{2}\int 2x\cos(x^2)\,dx$$
$$=\frac{1}{2}\sin(x^2)+c$$

**Alt:**

$$\int x\cos(x^2)\,dx$$

$$u=x^2$$
$$du=2xdx$$

$$=\frac{1}{2}\int\cos(x^2)\times 2x\,dx$$
$$=\frac{1}{2}\int\cos u\,du$$
$$=\frac{1}{2}\sin u+c$$
$$=\frac{1}{2}\sin x^2+c$$

**Question 5**

$$\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx$$
$$= \int_0^{\frac{\pi}{4}} \sec^2 x \, (\tan x)^3 \, dx$$
$$= \left[\frac{\tan^4 x}{4}\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{4} - 0$$
$$= \frac{1}{4}$$

**Alt:**

$$\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx \qquad \boxed{\begin{array}{c} u = \tan x \\ du = \sec^2 x \, dx \end{array}}$$
$$= \int_0^1 u^3 \, du$$
$$= \left[\frac{u^4}{4}\right]_0^1$$
$$= \frac{1}{4} - 0$$
$$= \frac{1}{4}$$

**Question 6**

$$\int \frac{\cos x}{\sin^5 x} dx$$
$$= \int \cos x \, (\sin x)^{-5} \, dx$$
$$= \frac{(\sin x)^{-4}}{-4} + c$$
$$= -\frac{1}{4 \sin^4 x} + c$$

**Alt:**

$$\int \frac{\cos x}{\sin^5 x} dx \qquad \boxed{\begin{array}{c} u = \sin x \\ du = \cos x \, dx \end{array}}$$
$$= \int \frac{1}{\sin^5 x} \times \cos x \, dx$$
$$= \int u^{-5} \, du$$
$$= \frac{u^{-4}}{-4} + c$$
$$= -\frac{1}{4 \sin^4 x} + c$$

**Question 7**

$$\int \sin^{\frac{3}{2}} 2x \cos 2x \, dx$$
$$= \frac{1}{2} \int 2 \cos 2x \, (\sin 2x)^{\frac{3}{2}} \, dx$$
$$= \frac{1}{2} \times \frac{2}{5} (\sin 2x)^{\frac{5}{2}} + c$$
$$= \frac{\sin^{\frac{5}{2}} 2x}{5} + c$$

**Alt:**

$$\int \sin^{\frac{3}{2}} 2x \cos 2x \, dx \qquad \boxed{\begin{array}{c} u = \sin 2x \\ du = 2 \cos 2x \, dx \end{array}}$$
$$= \frac{1}{2} \int \sin^{\frac{3}{2}} 2x \times 2 \cos 2x \, dx$$
$$= \frac{1}{2} \int u^{\frac{3}{2}} \, du$$
$$= \frac{1}{2} \times \frac{2}{5} u^{\frac{5}{2}} + c$$
$$= \frac{\sin^{\frac{5}{2}} 2x}{5} + c$$

**Question 8**

$$\int \sqrt{\sin 2x} \cos 2x \, dx$$
$$= \frac{1}{2} \int (2 \cos 2x)(\sin 2x)^{\frac{1}{2}} \, dx$$
$$= \frac{1}{2} \times \frac{2}{3} (\sin 2x)^{\frac{3}{2}} + c$$
$$= \frac{\sqrt{\sin^3 2x}}{3} + c$$

**Alt:**

$$\int \sqrt{\sin 2x} \cos 2x \, dx \qquad \boxed{\begin{array}{c} u^2 = \sin 2x \\ 2u \, du = 2 \cos 2x \end{array}}$$
$$= \frac{1}{2} \int \sqrt{\sin 2x} \times 2 \cos 2x$$
$$= \frac{1}{2} \int u \times 2u \, du$$
$$= \int u^2 \, du$$
$$= \frac{u^3}{3} + c$$
$$= \frac{\sqrt{\sin^3 2x}}{3} + c$$

**Question 9**

$$\int e^x \cos e^x \, dx$$
$$= \sin e^x + c$$

**Alt:**

$$\int e^x \cos e^x \, dx \qquad \boxed{\begin{array}{c} u = e^x \\ du = e^x dx \end{array}}$$
$$= \int \cos e^x \times e^x dx$$
$$= \int \cos u \, du$$
$$= \sin u + c$$
$$= \sin e^x + c$$

**Question 10**

$$\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}} dx$$
$$= -\int \frac{-1}{\sqrt{1-x^2}} \times e^{\cos^{-1} x} \, dx$$
$$= -e^{\cos^{-1} x} + c$$

**Alt:**

$$\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}} dx \qquad \boxed{\begin{array}{c} u = \cos^{-1} x \\ du = -\dfrac{dx}{\sqrt{1-x^2}} \end{array}}$$
$$= -\int e^{\cos^{-1} x} \times \left( -\frac{dx}{\sqrt{1-x^2}} \right)$$
$$= -\int e^u \, du$$
$$= -e^u + c$$
$$= -e^{\cos^{-1} x} + c$$

**Question 11**

$$\int_1^{e^{\frac{\pi}{2}}} \frac{\cos(\ln x)}{x} dx$$
$$= \int_1^{e^{\frac{\pi}{2}}} \frac{1}{x} \cos(\ln x) \, dx$$
$$= \Big[ \sin(\ln x) \Big]_1^{e^{\frac{\pi}{2}}}$$
$$= \sin\left(\frac{\pi}{2}\right) - \sin 0$$
$$= 1 - 0$$
$$= 1$$

**Alt:**

$$\int_1^{e^{\frac{\pi}{2}}} \frac{\cos(\ln x)}{x} dx \qquad \boxed{\begin{array}{c} u = \ln x \\ du = \dfrac{dx}{x} \end{array}}$$
$$= \int_1^{e^{\frac{\pi}{2}}} \cos(\ln x) \times \frac{dx}{x}$$
$$= \int_0^{\frac{\pi}{2}} \cos u \, du$$
$$= \Big[ \sin u \Big]_0^{\frac{\pi}{2}}$$
$$= \sin\left(\frac{\pi}{2}\right) - \sin 0$$
$$= 1 - 0$$
$$= 1$$

**Question 12**

$$\int (3x^2 + 2x)\sqrt{x^3 + x^2} \, dx$$
$$= \int (3x^2 + 2x)(x^3 + x^2)^{\frac{1}{2}} dx$$
$$= \frac{2(x^3 + x^2)^{\frac{3}{2}}}{3} + c$$
$$= \frac{2\sqrt{(x^3 + x^2)^3}}{3} + c$$

**Alt:**

$$\int (3x^2 + 2x) \, dx \qquad \boxed{\begin{array}{c} u^2 = x^3 + x^2 \\ 2u \, du = (3x^2 + 2x) dx \end{array}}$$
$$= \int \sqrt{x^3 + x^2} \times (3x^2 + 2x) dx$$
$$= \int u \times 2u \, du$$
$$= 2 \int u^2 \, du$$
$$= 2 \times \frac{u^3}{3} + c$$
$$= \frac{2(x^3 + x^2)^{\frac{3}{2}}}{3} + c$$
$$= \frac{2\sqrt{(x^3 + x^2)^2}}{3} + c$$

**Question 13**

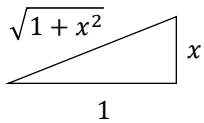
$$\int x^2 e^{x^3}\,dx$$
$$= \frac{1}{3}\int 3x^2 e^{x^3}\,dx$$
$$= \frac{1}{3}e^{x^3} + c$$

**Alt:**

$$\int x^2 e^{x^3}\,dx \qquad \boxed{\begin{array}{c} u = x^3 \\ du = 3x^2 dx \end{array}}$$
$$= \frac{1}{3}\int e^{x^3} \times 3x^2 dx$$
$$= \frac{1}{3}\int e^u\,du$$
$$= \frac{1}{3}e^u + c$$
$$= \frac{1}{3}e^{x^3} + c$$

**Question 14**

$$\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$$
$$= \int \frac{1}{1+x^2} \times \sin(\tan^{-1}x)\,dx$$
$$= -\cos(\tan^{-1}x) + c$$
$$= -\frac{1}{\sqrt{1+x^2}} + c$$




**Alt:**

$$\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx \qquad \boxed{\begin{array}{c} u = \tan^{-1}x \\ du = \dfrac{dx}{1+x^2} \end{array}}$$
$$= \int \sin(\tan^{-1}x) \times \frac{dx}{1+x^2}$$
$$= \int \sin u\,du$$
$$= -\cos u + c$$
$$= -\cos(\tan^{-1}x) + c$$
$$= -\frac{1}{\sqrt{1+x^2}} + c$$

**Alt II:**

$$\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$$
$$= \int \frac{\left({}^{x}/_{\sqrt{1+x^2}}\right)}{1+x^2}dx$$
$$= \int x\left(1+x^2\right)^{-\frac{3}{2}}dx$$
$$= \frac{1}{2}\int 2x\left(1+x^2\right)^{-\frac{3}{2}}dx$$
$$= \frac{1}{2}\times(-2)\times\left(1+x^2\right)^{-\frac{1}{2}} + c$$
$$= -\frac{1}{\sqrt{1+x^2}} + c$$

**Question 15**

$$\int (e^{t^2}+16)te^{t^2}\,dt$$
$$= \frac{1}{2}\int 2te^{t^2}\left(e^{t^2}+16\right)dt$$
$$= \frac{1}{2}\times\frac{\left(e^{t^2}+16\right)^2}{2} + c$$
$$= \frac{1}{4}\left(e^{t^2}+16\right)^2 + c$$

**Alt:**

$$\int (e^{t^2}+16)te^{t^2}\,dt \qquad \boxed{\begin{array}{c} u = e^{t^2}+16 \\ du = 2te^{t^2}dt \end{array}}$$
$$= \frac{1}{2}\int (e^{t^2}+16) \times 2te^{t^2}dt$$
$$= \frac{1}{2}\int u\,du$$
$$= \frac{1}{2}\times\frac{u^2}{2} + c$$
$$= \frac{1}{4}\left(e^{t^2}+16\right)^2 + c$$

**Question 16**

$$\int \frac{\operatorname{cosec}x\cot x}{1+\operatorname{cosec}^2 x}dx$$
$$= -\int \frac{-\operatorname{cosec}x\cot x}{1+(\operatorname{cosec}x)^2}dx$$
$$= -\tan^{-1}(\operatorname{cosec}x) + c$$

**Alt:**

$$\int \frac{\operatorname{cosec}x\cot x}{1+\operatorname{cosec}^2 x}dx \qquad \boxed{\begin{array}{c} u = \operatorname{cosec}x \\ du = -\cot x\operatorname{cosec}x\,dx \end{array}}$$
$$= -\int \frac{1}{1+\operatorname{cosec}^2 x} \times (-\cot x\operatorname{cosec}x\,dx)$$
$$= -\int \frac{1}{1+u^2}du$$
$$= -\tan^{-1}u + c$$
$$= -\tan^{-1}(\operatorname{cosec}x) + c$$

**Question 17**

$$\int e^x e^{e^x}\,dx$$
$$= e^{e^x} + c$$

**Alt:**

$$\int e^x e^{e^x}\,dx \qquad \boxed{\begin{array}{c} u = e^x \\ du = e^x dx \end{array}}$$
$$= \int e^{e^x} \times e^x dx$$
$$= \int e^u\,du$$
$$= e^u + c$$
$$= e^{e^x} + c$$

**Question 18**

$$\int \frac{\ln^3 x}{x}\,dx$$
$$= \int \frac{1}{x}(\ln x)^3\,dx$$
$$= \frac{\ln^4 x}{4} + c$$

**Alt:**

$$\int \frac{\ln^3 x}{x}\,dx$$

$$u = \ln x$$
$$du = \frac{1}{x}dx$$

$$= \int (\ln x)^3 \times \frac{1}{x}\,dx$$
$$= \int u^3\,du$$
$$= \frac{u^4}{4} + c$$
$$= \frac{\ln^4 x}{4} + c$$

**Question 19**

$$\int \sqrt{\frac{2+x}{2-x}}\,dx$$
$$= \int \sqrt{\frac{2+x}{2-x}} \times \sqrt{\frac{2+x}{2+x}}\,dx$$
$$= \int \frac{2+x}{\sqrt{4-x^2}}\,dx$$
$$= \int \left(\frac{2}{\sqrt{4-x^2}} + \frac{1}{2}\times(-2x)\left(4-x^2\right)^{-\frac{1}{2}}\right)dx$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\times 2\left(4-x^2\right)^{\frac{1}{2}} + c$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \sqrt{4-x^2} + c$$

**Question 20**

$$\int \frac{\sin(\tan\theta)}{\cos^2\theta}\,d\theta$$
$$= \int \sec^2\theta \sin(\tan\theta)\,d\theta$$
$$= -\cos(\tan\theta) + c$$

**Question 21**

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$$
$$= \int \frac{\cos^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \frac{1-\sin^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \cos x\left((\sin x)^{-\frac{1}{2}} - (\sin x)^{\frac{3}{2}}\right)dx$$
$$= 2(\sin x)^{\frac{1}{2}} - \frac{2}{5}(\sin x)^{\frac{5}{2}} + c$$
$$= 2\sqrt{\sin x} - \frac{2\sqrt{\sin^5 x}}{5} + c$$

**Alt:**

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$$

$$u^2 = \sin x$$
$$2u\,du = \cos x\,dx$$

$$= \int \frac{\cos^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \frac{1-\sin^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \frac{1-u^4}{u} \times 2u\,du$$
$$= 2\int (1-u^4)\,du$$
$$= 2u - \frac{2u^5}{5} + c$$
$$= 2\sqrt{\sin x} - \frac{2\sqrt{\sin^5 x}}{5} + c$$

**Question 22**

$$\int \frac{x\sin\left(\sqrt{2x^2-1}\right)}{\sqrt{2x^2-1}}\,dx$$
$$= \frac{1}{2}\int \left(\frac{1}{2}\left(2x^2-1\right)^{-\frac{1}{2}} \times 4x\right)\sin\left(\sqrt{2x^2-1}\right)dx$$
$$= -\frac{1}{2}\cos\sqrt{2x^2-1} + c$$

**Alt:**

$$\int \frac{x\sin\left(\sqrt{2x^2-1}\right)}{\sqrt{2x^2-1}}\,dx$$

$$u = \left(2x^2-1\right)^{\frac{1}{2}}$$
$$du = \frac{1}{2}\left(2x^2-1\right)^{-\frac{1}{2}}(4x)dx$$
$$= \frac{2x}{\sqrt{2x^2-1}}\,dx$$

$$= \frac{1}{2}\int \sin\left(\sqrt{2x^2-1}\right) \times \frac{2x}{\sqrt{2x^2-1}}\,dx$$
$$= \frac{1}{2}\int \sin u\,du$$
$$= -\frac{1}{2}\cos u + c$$
$$= -\frac{1}{2}\cos\sqrt{2x^2-1} + c$$

**Question 23**

$$\int \frac{e^{\sqrt{\sin x}}}{\sec x \sqrt{\sin x}}\,dx$$
$$= 2\int \frac{\cos x}{2\sqrt{\sin x}} e^{\sqrt{\sin x}}\,dx$$
$$= 2e^{\sqrt{\sin x}} + c$$

**Alt:**

$$\int \frac{e^{\sqrt{\sin x}}}{\sec x \sqrt{\sin x}}\,dx \qquad \boxed{\begin{aligned} u^2 &= \sin x \\ 2u\,du &= \cos x\,dx \end{aligned}}$$
$$= \int \frac{e^{\sqrt{\sin x}}}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \frac{e^u}{u} \times 2u\,du$$
$$= 2\int e^u\,du$$
$$= 2e^u + c$$
$$= 2e^{\sqrt{\sin x}} + c$$

**Question 24**

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$$
$$= \int \frac{1}{x^3\left(\frac{1}{x^2}+1\right)^{\frac{3}{2}}}\,dx$$
$$= \int \left(x^{-2}+1\right)^{-\frac{3}{2}}\left(x^{-3}\right)dx$$
$$= \left(x^{-2}+1\right)^{-\frac{1}{2}} + c$$
$$= \frac{1}{\sqrt{\frac{1}{x^2}+1}} + c$$
$$= \frac{x}{\sqrt{1+x^2}} + c$$

**Question 25**

$$\int_1^{e^{\frac{\pi}{2}}} \frac{\sin(\ln|x|)}{x}\,dx$$
$$= \int_1^{e^{\frac{\pi}{2}}} \frac{1}{x}\sin(\ln|x|)\,dx$$
$$= -\left[\cos(\ln|x|)\right]_1^{e^{\frac{\pi}{2}}}$$
$$= -\left(\cos\frac{\pi}{2} - \cos 0\right)$$
$$= -(0-1)$$
$$= 1$$

**Alt:**

$$\int_1^{e^{\frac{\pi}{2}}} \frac{\sin(\ln|x|)}{x}\,dx \qquad \boxed{\begin{aligned} u &= \ln|x| \\ du &= \frac{dx}{x} \end{aligned}}$$
$$= \int_1^{e^{\frac{\pi}{2}}} \sin(\ln|x|) \times \frac{dx}{x}$$
$$= \int_0^{\frac{\pi}{2}} \sin u\,du$$
$$= -\left[\cos u\right]_0^{\frac{\pi}{2}}$$
$$= -\left(\cos\frac{\pi}{2} - \cos 0\right)$$
$$= -(0-1)$$
$$= 1$$

**Question 26**

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{2e^{\frac{2}{x^2}}}{x^3}\,dx$$
$$= -\frac{1}{2}\int_{\sqrt{2}}^{\sqrt{3}} \left(-4x^{-3}\right)e^{2x^{-2}}\,dx$$
$$= -\frac{1}{2}\left[e^{\frac{2}{x^2}}\right]_{\sqrt{2}}^{\sqrt{3}}$$
$$= -\frac{1}{2}\left(e^{\frac{2}{3}} - e\right)$$
$$= \frac{e - e^{\frac{2}{3}}}{2}$$

**Alt:**

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{2e^{\frac{2}{x^2}}}{x^3}\,dx \qquad \boxed{\begin{aligned} u &= e^{\frac{2}{x^2}} \\ &= e^{2x^{-2}} \\ \frac{du}{dx} &= -4x^{-3}e^{\frac{2}{x^2}} \\ -\frac{du}{4} &= \frac{e^{\frac{2}{x^2}}}{x^3}\,dx \end{aligned}}$$
$$= 2\int_e^{e^{\frac{2}{3}}} -\frac{du}{4}$$
$$= -\frac{1}{2}\int_e^{e^{\frac{2}{3}}} du$$
$$= -\frac{1}{2}\left[u\right]_e^{e^{\frac{2}{3}}}$$
$$= -\frac{1}{2}\left(e^{\frac{2}{3}} - e\right)$$
$$= \frac{e - e^{\frac{2}{3}}}{2}$$

**Question 27**

$$\frac{d}{dx}\left(e^{\sqrt{2x^2-1}}\right)=\frac{1}{2}\left(2x^2-1\right)^{-\frac{1}{2}}(4x)e^{\sqrt{2x^2-1}}$$
$$=\frac{2xe^{\sqrt{2x^2-1}}}{\sqrt{2x^2-1}}$$
$$\int\frac{xe^{\sqrt{2x^2-1}}\sin\left(e^{\sqrt{2x^2-1}}\right)}{\sqrt{2x^2-1}}dx$$
$$=\frac{1}{2}\int\frac{2xe^{\sqrt{2x^2-1}}}{\sqrt{2x^2-1}}\sin\left(e^{\sqrt{2x^2-1}}\right)dx$$
$$=-\frac{1}{2}\cos\left(e^{\sqrt{2x^2-1}}\right)+c$$

**Question 28a**

$$\int\sin^3 2x\cos 2x\,dx$$
$$=\frac{1}{2}\int(2\cos 2x)(\sin 2x)^3\,dx$$
$$=\frac{1}{2}\times\frac{(\sin 2x)^4}{4}+c$$
$$=\frac{\sin^4 2x}{8}+c$$

**Alt:**

$$\int\sin^3 2x\cos 2x\,dx \qquad \boxed{\begin{array}{c}u=\sin 2x\\ du=2\cos 2x\,dx\end{array}}$$
$$=\frac{1}{2}\int\sin^3 2x\times 2\cos 2x\,dx$$
$$=\frac{1}{2}\int u^3\,du$$
$$=\frac{1}{2}\times\frac{u^4}{4}+c$$
$$=\frac{1}{8}\sin^4 2x+c$$

**Question 28b**

$$\int\sin^3 2x\cos 2x\,dx$$
$$=\int\sin 2x\left(1-\cos^2 2x\right)\cos 2x\,dx$$
$$=\int\sin 2x\cos 2x\,dx-\int\sin 2x\cos^3 2x\,dx$$
$$=\frac{1}{2}\int\sin 4x\,dx+\frac{1}{2}\int\left(-2\sin 2x\cos^3 2x\right)dx$$
$$=-\frac{1}{2}\times\frac{\cos 4x}{4}+\frac{1}{2}\times\frac{\cos^4 2x}{4}+c$$
$$=\frac{\cos^4 2x}{8}-\frac{\cos 4x}{8}+c$$

**Question 28c**

$$\int\sin^3 2x\cos 2x\,dx$$
$$=\int\sin 2x\times\sin^2 2x\times\cos 2x\,dx$$
$$=\int\sin 2x\times(1-\cos^2 2x)\times\cos 2x\,dx$$
$$=\int\left(\sin 2x\cos 2x-\sin 2x\cos^3 2x\right)dx$$
$$=\frac{1}{2}\int(2\cos 2x\sin 2x)\,dx$$
$$+\frac{1}{2}\int\left(-2\sin 2x\cos^3 2x\right)dx$$
$$=\frac{1}{2}\times\frac{\sin^2 2x}{2}+\frac{1}{2}\times\frac{\cos^4 2x}{4}+c$$
$$=\frac{\cos^4 2x}{8}+\frac{\sin^2 2x}{4}+c$$

**Question 28d**

$$\int\sin^3 2x\cos 2x\,dx$$
$$=\int\sin^2 2x\sin 2x\cos 2x\,dx$$
$$=\int\left(1-\cos^2 2x\right)\sin 2x\cos 2x\,dx$$
$$=\int\left(\sin 2x\cos 2x-\sin 2x\cos^3 2x\right)dx$$
$$=\frac{1}{2}\int\left(2\sin 2x\cos 2x-2\sin 2x\cos^3 2x\right)dx$$
$$=-\frac{1}{2}\times\frac{\cos^2 2x}{2}+\frac{1}{2}\times\frac{\cos^4 2x}{4}+c$$
$$=\frac{\cos^4 2x}{8}-\frac{\cos^2 2x}{4}+c$$

**Alt:**

$$\int\sin^3 2x\cos 2x\,dx \qquad \boxed{\begin{array}{c}u=\cos 2x\\ du=-2\sin 2x\,dx\end{array}}$$
$$=-\frac{1}{2}\int\sin^2 2x\cos 2x\times(-2\sin 2x\,dx)$$
$$=-\frac{1}{2}\int(1-\cos^2 2x)\cos 2x\times(-2\sin 2x\,dx)$$
$$=-\frac{1}{2}\int\left(1-u^2\right)u\,du$$
$$=-\frac{1}{2}\int\left(u-u^3\right)du$$
$$=-\frac{1}{2}\times\frac{u^2}{2}+\frac{1}{2}\times\frac{u^4}{4}+c$$
$$=\frac{u^4}{8}-\frac{u^2}{4}+c$$
$$=\frac{\cos^4 2x}{8}-\frac{\cos^2 2x}{4}+c$$

**Question 28e**

$$\int \sin^3 2x \cos 2x \, dx$$

$$= \int \sin^2 2x \times \sin 2x \cos 2x \, dx$$

$$= \frac{1}{2}\int \sin^2 2x \sin 4x \, dx$$

$$= \frac{1}{2}\int \frac{1}{2}(1 - \cos 4x) \sin 4x \, dx$$

$$= \frac{1}{4}\int (\sin 4x - \sin 4x \cos 4x) \, dx$$

$$= \frac{1}{4}\int \sin 4x \, dx - \frac{1}{8}\int \sin 8x \, dx$$

$$= -\frac{1}{4} \times \frac{\cos 4x}{4} + \frac{1}{8} \times \frac{\cos 8x}{8} + c$$

$$= -\frac{\cos 4x}{16} + \frac{\cos 8x}{64} + c$$

**Question 28f**

$$\int \sin^3 2x \cos 2x \, dx$$

$$= \int \sin^2 2x \sin 2x \cos 2x \, dx$$

$$= \frac{1}{2}\int \sin^2 2x \sin 4x \, dx$$

$$= \frac{1}{2}\int \frac{1}{2}(1 - \cos 4x) \sin 4x \, dx$$

$$= \frac{1}{4}\int (\sin 4x - \sin 4x \cos 4x) \, dx$$

$$= \frac{1}{4}\int \sin 4x \, dx - \frac{1}{4}\int 4 \cos 4x \sin 4x \, dx$$

$$= -\frac{1}{4} \times \frac{\cos 4x}{4} - \frac{1}{4} \times \frac{\sin^2 4x}{4} + c$$

$$= -\frac{\cos 4x}{16} - \frac{\sin^2 4x}{16} + c$$

**Question 1**

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx \qquad \boxed{\begin{aligned} u^2 &= 1+x^2 \\ 2u\,du &= 2x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}} \times 2x\,dx$$

$$= \frac{1}{2}\int \frac{u^2-1}{u} \times 2u\,du$$

$$= \int (u^2-1)\,du$$

$$= \frac{u^3}{3} - u + c$$

$$= \frac{\sqrt{(1+x^2)^3}}{3} - \sqrt{1+x^2} + c$$

**Question 2**

$$\int x^3\sqrt{x^2+1}\,dx \qquad \boxed{\begin{aligned} u^2 &= x^2+1 \\ 2u\,du &= 2x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int x^2\sqrt{x^2+1} \times 2x\,dx$$

$$= \frac{1}{2}\int (u^2-1) \times u \times 2u\,du$$

$$= \int (u^4-u^2)\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + c$$

$$= \frac{\sqrt{(x^2+1)^5}}{5} - \frac{\sqrt{(x^2+1)^3}}{3} + c$$

**Question 3**

$$\int x^5\sqrt{x^2+1}\,dx \qquad \boxed{\begin{aligned} u^2 &= x^2+1 \\ 2u\,du &= 2x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int x^4\sqrt{x^2+1} \times 2x\,dx$$

$$= \frac{1}{2}\int (u^2-1)^2 \times u \times 2u\,du$$

$$= \int (u^4-2u^2+1)u^2\,du$$

$$= \int (u^6-2u^4+u^2)\,du$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + c$$

$$= \frac{\sqrt{(x^2+1)^7}}{7} - \frac{2\sqrt{(x^2+1)^5}}{5} + \frac{\sqrt{(x^2+1)^3}}{3} + c$$

**Question 4**

$$\int_0^1 x^3\sqrt{1-x^2}\,dx \qquad \boxed{\begin{aligned} u^2 &= 1-x^2 \\ 2u\,du &= -2x\,dx \end{aligned}}$$

$$= -\frac{1}{2}\int_0^1 x^2\sqrt{1-x^2} \times (-2x)\,dx$$

$$= -\frac{1}{2}\int_1^0 (1-u^2)u \times 2u\,du$$

$$= \int_0^1 (u^2-u^4)\,du$$

$$= \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1$$

$$= \left(\left(\frac{1}{3} - \frac{1}{5}\right) - (0-0)\right)$$

$$= \frac{2}{15}$$

**Question 5**

$$\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\,dx \qquad \boxed{\begin{aligned} u^2 &= 1-x \\ 2u\,du &= -dx \end{aligned}}$$

$$= \int_1^{\frac{1}{2}} \frac{1-u^2}{u}(-2u\,du)$$

$$= 2\int_{\frac{1}{2}}^1 (1-u^2)\,du$$

$$= 2\left[u - \frac{u^3}{3}\right]_{\frac{1}{2}}^1$$

$$= 2\left(\left(1 - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{24}\right)\right)$$

$$= \frac{5}{12}$$

**Question 6**

$$\int_0^3 x\sqrt{x+1}\,dx \qquad \boxed{\begin{aligned} u^2 &= x+1 \\ 2u\,du &= dx \end{aligned}}$$

$$= \int_1^2 (u^2-1)u \times 2u\,du$$

$$= 2\int_1^2 (u^4-u^2)\,du$$

$$= 2\left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^2$$

$$= 2\left(\left(\frac{2^5}{5} - \frac{2^3}{3}\right) - \left(\frac{1^5}{5} - \frac{1^3}{3}\right)\right)$$

$$= \frac{116}{15}$$

**Question 7**

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx \qquad \boxed{\begin{array}{r} u^2 = e^x + 1 \\ 2u\,du = e^x dx \end{array}}$$

$$= \int \frac{e^x}{\sqrt{e^x+1}} \times e^x dx$$

$$= \int \frac{u^2-1}{u} \times 2u\,du$$

$$= 2\int (u^2-1)\,du$$

$$= \frac{2u^3}{3} - 2u + c$$

$$= \frac{2\sqrt{(e^x+1)^3}}{3} - 2\sqrt{e^x+1} + c$$

**Question 8**

$$\int \frac{1}{2+\sqrt{x}}\,dx \qquad \boxed{\begin{array}{r} u^2 = x \\ 2u\,du = dx \end{array}}$$

$$= \int \frac{1}{2+u} \times 2u\,du$$

$$= 2\int \frac{u}{u+2}\,du$$

$$= 2\int \frac{u+2-2}{u+2}\,du$$

$$= 2\int \left(1 - \frac{2}{u+2}\right) du$$

$$= 2u - 4\ln\left|u+2\right| + c$$

$$= 2\sqrt{x} - 4\ln\left|2+\sqrt{x}\right| + c$$

**Question 9**

$$\int x^5\sqrt{2-x^3}\,dx \qquad \boxed{\begin{array}{r} u^2 = 2 - x^3 \\ 2u\,du = -3x^2 dx \end{array}}$$

$$= -\frac{1}{3}\int x^3\sqrt{2-x^3} \times (-3x^2) dx$$

$$= -\frac{1}{3}\int (2-u^2) \times u \times 2u\,du$$

$$= -\frac{2}{3}\int (2-u^2)u^2\,du$$

$$= -\frac{2}{3}\int (2u^2 - u^4)\,du$$

$$= -\frac{2}{3}\left(\frac{2u^3}{3} - \frac{u^5}{5}\right) + c$$

$$= -\frac{4\sqrt{(2-x^3)^3}}{9} + \frac{2\sqrt{(2-x^3)^5}}{15} + c$$

**Question 10**

$$\int \frac{2x^3}{(x^2-1)^2}\,dx \qquad \boxed{\begin{array}{r} u = x^2 - 1 \\ du = 2x\,dx \end{array}}$$

$$= \int \frac{x^2}{(x^2-1)^2} \times 2x\,dx$$

$$= \int \frac{u+1}{u^2}\,du$$

$$= \int \left(\frac{1}{u} + u^{-2}\right) du$$

$$= \ln\left|u\right| - \frac{1}{u} + c$$

$$= \ln\left|x^2-1\right| - \frac{1}{x^2-1} + c$$

**Question 11**

$$\int \frac{x^3}{(x^2+1)^2}\,dx \qquad \boxed{\begin{array}{r} u = x^2 + 1 \\ du = 2x\,dx \end{array}}$$

$$= \frac{1}{2}\int \frac{x^2}{(x^2+1)^2} \times 2x\,dx$$

$$= \frac{1}{2}\int \frac{u-1}{u^2}\,du$$

$$= \frac{1}{2}\int \left(\frac{1}{u} - u^{-2}\right) du$$

$$= \frac{1}{2}\ln\left|u\right| + \frac{1}{2u} + c$$

$$= \frac{1}{2}\ln\left|x^2+1\right| + \frac{1}{2(x^2+1)} + c$$

**Question 12**

$$\int \frac{e^{2x}}{e^x+1}\,dx \qquad \boxed{\begin{array}{r} u = e^x \\ du = e^x dx \end{array}}$$

$$= \int \frac{e^x}{e^x+1} \times e^x dx$$

$$= \int \frac{u}{u+1} \times du$$

$$= \int \left(\frac{u+1-1}{u+1}\right) du$$

$$= \int \left(1 - \frac{1}{u+1}\right) du$$

$$= u - \ln\left|u+1\right| + c$$

$$= e^x - \ln\left|e^x+1\right| + c$$

**Question 13**

$$\int_1^2 \frac{dx}{x\sqrt{2x-1}} \qquad \boxed{\begin{array}{r} u^2 = 2x-1 \\ 2u\,du = 2dx \end{array}}$$

$$= \frac{1}{2}\int_1^2 \frac{1}{x\sqrt{2x-1}} \times 2dx$$

$$= \frac{1}{2}\int_1^{\sqrt{3}} \frac{1}{\frac{u^2+1}{2}} \times 2u\,du$$

$$= 2\int_1^{\sqrt{3}} \frac{1}{u^2+1}\,du$$

$$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{6}$$

**Question 14**

$$\int \frac{x}{\sqrt{x+5}}\,dx \qquad \boxed{\begin{array}{r} u^2 = x+5 \\ 2u\,du = dx \end{array}}$$

$$= \int \frac{u^2-5}{u} \times 2u\,du$$

$$= 2\int (u^2-5)\,du$$

$$= \frac{2u^3}{3} - 10u + c$$

$$= \frac{2\sqrt{(x+5)^3}}{3} - 10\sqrt{x+5} + c$$

$$= \frac{2(x+5)\sqrt{x+5}}{3} - 10\sqrt{x+5} + c$$

$$= \frac{2}{3}\sqrt{x+5}(x+5-15) + c$$

$$= \frac{2}{3}(x-10)\sqrt{x+5} + c$$

**Question 15**

$$d = \int_0^{0.5} t\sqrt{\frac{1+2t^2}{1-2t^2}}\,dx \qquad \boxed{\begin{array}{r} u = t^2 \\ du = 2t\,dt \end{array}}$$

$$= \frac{1}{2}\int_0^{0.5} \sqrt{\frac{1+2t^2}{1-2t^2}} \times 2t\,dt$$

$$= \frac{1}{2}\int_0^{0.25} \sqrt{\frac{1+2u}{1-2u}}\,du$$

$$= \frac{1}{2}\int_0^{0.25} \sqrt{\frac{1+2u}{1-2u}} \times \sqrt{\frac{1+2u}{1+2u}}\,du$$

$$= \frac{1}{2}\int_0^{0.25} \frac{1+2u}{\sqrt{1-4u^2}}\,du$$

$$= \frac{1}{2}\int_0^{0.25} \frac{du}{\sqrt{1-4u^2}} + \int_0^{0.25} \frac{u}{\sqrt{1-4u^2}}\,du$$

$$= \frac{1}{4}\int_0^{0.25} \frac{2}{\sqrt{1-(2u)^2}}\,du$$

$$-\frac{1}{8}\int_0^{0.25} (-8u)\left(1-4u^2\right)^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\left[\sin^{-1}(2u)\right]_0^{0.25} - \frac{1}{8}\left[2\left(1-4u^2\right)^{\frac{1}{2}}\right]_0^{0.25}$$

$$= \frac{1}{4}\left(\frac{\pi}{6} - 0\right) - \frac{1}{8}\left(2\sqrt{\frac{3}{4}} - 2\right)$$

$$= \frac{\pi}{24} - \frac{\sqrt{3}}{8} + \frac{1}{4} \text{ metres}$$

$$= 164 \text{ mm (nearest millimetre)}$$

**Question 16**

$$\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\,dx \qquad \boxed{\begin{array}{c} u = 1+x^2 \\ du = 2x\,dx \end{array}}$$

$$= \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{(1+x^2)^{\frac{3}{2}}} \times 2x\,dx$$

$$= \frac{1}{2}\int_1^4 \frac{u-1}{u^{\frac{3}{2}}}\,du$$

$$= \frac{1}{2}\int_1^4 \left(u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\right)du$$

$$= \frac{1}{2}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]_1^4$$

$$= \frac{1}{2}\left((4+1) - (2+2)\right)$$

$$= \frac{1}{2}$$

**Question 17a**

$$\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx \qquad \boxed{\begin{array}{c} u^2 = x \\ 2u\,du = dx \end{array}}$$

$$= \int \frac{u}{1-u} \times 2u\ du$$

$$= 2\int \frac{u^2}{1-u}\,du$$

$$= 2\int \frac{-u(1-u) - (1-u) + 1}{1-u}\,du$$

$$= 2\int \left(-u - 1 + \frac{1}{1-u}\right)du$$

$$= -u^2 - 2u - 2\ln|1-u| + c$$

$$= -x - 2\sqrt{x} - 2\ln\left|1-\sqrt{x}\right| + c$$

**Question 17b**

$$\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx \qquad \boxed{\begin{array}{c} u = 1 - x^{\frac{1}{2}} \\ du = -\frac{1}{2}x^{-\frac{1}{2}} \\ du = -\frac{du}{2\sqrt{x}} \end{array}}$$

$$= \int \frac{1-u}{u} \times \left(-2(1-u)\right)du$$

$$= -2\int \frac{(1-u)^2}{u}\,du$$

$$= -2\int \left(\frac{1}{u} - 2 + u\right)du$$

$$= -2\ln|u| + 4u - u^2 + c$$

$$= -2\ln\left|1-\sqrt{x}\right| + 4\left(1-\sqrt{x}\right) - \left(1-\sqrt{x}\right)^2 + c$$

$$= -2\ln\left|1-\sqrt{x}\right| + 4 - 4\sqrt{x} - 1 + 2\sqrt{x} - x + c$$

$$= -x - 2\sqrt{x} - 2\ln\left|1-\sqrt{x}\right| + c$$

**Question 18**

$$\int \frac{dx}{\sqrt{e^{2x}-1}} \qquad \boxed{\begin{array}{c} u = \frac{1}{e^x} \Leftrightarrow e^x = \frac{1}{u} \\ = e^{-x} \\ du = -e^{-x}dx \\ dx = -e^x du = -\frac{1}{u}du \end{array}}$$

$$= \int \frac{-\frac{1}{u}}{\sqrt{\left(\frac{1}{u}\right)^2 - 1}}\,du$$

$$= -\int \frac{du}{u\sqrt{\left(\frac{1}{u}\right)^2 - 1}}$$

$$= -\int \frac{du}{\sqrt{1-u^2}}$$

$$= \cos^{-1}u + c$$

$$= \cos^{-1}\left(\frac{1}{e^x}\right) + c$$

**Question 19**

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx \qquad \boxed{\begin{array}{c} u = -x \\ du = -dx \\ dx = -du \end{array}}$$

$$= \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\sin^2(-u)}{1+2^{-u}}(-du)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(u)}{1+2^{-u}} \times \frac{2^u}{2^u}\,du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^u \sin^2(u)}{2^u+1}\,du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^x \sin^2 x}{1+2^x}\,dx \qquad \text{change of variable}$$

$$\therefore 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^x \sin^2 x}{1+2^x}\,dx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+2^x)\sin^2 x}{1+2^x}\,dx$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\left(\frac{\pi}{2} - 0\right) - \left(-\frac{\pi}{2} - 0\right)\right)$$

$$= \frac{\pi}{4}$$

**Question 20**

$$\int \sqrt{\frac{x}{x+1}}\,dx$$

$$= \int \sqrt{\frac{x}{x+1}} \times \sqrt{\frac{x}{x}}\,dx$$

$$= \int \frac{x}{\sqrt{x^2+x}}\,dx$$

$$= \int \frac{x}{\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}}}\,dx$$

$$= \int \frac{x}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$

$$= \int \frac{x+\frac{1}{2}-\frac{1}{2}}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$

$$= \int \frac{x+\frac{1}{2}}{\sqrt{x^2+x}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$

$$= \frac{1}{2}\int (2x+1)\left(x^2+x\right)^{-\frac{1}{2}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$

$$= \frac{1}{2}\times 2\left(x^2+x\right)^{\frac{1}{2}} - \frac{1}{2}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$$

$$= \sqrt{x^2+x} - \frac{1}{2}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$$

$$= \sqrt{x^2+x} - \frac{1}{2}\ln\left|\left(\sqrt{\frac{x}{2}}+\sqrt{\frac{x+1}{2}}\right)^2\right| + c$$

$$= \sqrt{x^2+x} - \ln\left|\sqrt{\frac{x}{2}}+\sqrt{\frac{x+1}{2}}\right| + c$$

$$= \sqrt{x^2+x} - \ln\left(\sqrt{x}+\sqrt{x+1}\right) + \ln\sqrt{2} + c$$

$$= \sqrt{x^2+x} - \ln\left(\sqrt{x}+\sqrt{x+1}\right) + c$$

**Question 1**

$$\int \frac{x^2-2x-2}{x+1}\,dx$$
$$= \int \frac{x(x+1)-3(x+1)+1}{x+1}\,dx$$
$$= \int \left(x-3+\frac{1}{x+1}\right)dx$$
$$= \frac{x^2}{2}-3x+\ln\left|x+1\right|+c$$

**Question 2**

$$\int_1^3 \frac{2x-3}{x+1}\,dx$$
$$= \int_1^3 \frac{2(x+1)-5}{x+1}\,dx$$
$$= \int_1^3 \left(2-\frac{5}{x+1}\right)dx$$
$$= \Big[2x-5\ln\left|x+1\right|\Big]_1^3$$
$$= (6-5\ln 4)-(2-5\ln 2)$$
$$= 6-5\ln 4-2+5\ln 2$$
$$= 4-10\ln 2+5\ln 2$$
$$= 4-5\ln 2$$

**Question 3**

$$\int \frac{2x^2+19}{x^2+9}\,dx$$
$$= \int \frac{2(x^2+9)+1}{x^2+9}\,dx$$
$$= \int \left(2+\frac{1}{x^2+3^2}\right)dx$$
$$= 2x+\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$$

**Question 4**

$$\int \frac{3x+2}{x^2+6x+10}\,dx$$
$$= \int \frac{\frac{3}{2}(2x+6)-7}{x^2+6x+9+1}\,dx$$
$$= \frac{3}{2}\int \frac{2x+6}{x^2+6x+10}\,dx-7\int \frac{1}{(x+3)^2+1^2}\,dx$$
$$= \frac{3}{2}\ln\left|x^2+6x+10\right|-7\tan^{-1}\left(x+3\right)+c$$

**Question 5**

$$\int \frac{2}{x(x-1)}\,dx$$
$$= 2\int \left(\frac{1}{x-1}-\frac{1}{x}\right)dx$$
$$= 2\ln\left|x-1\right|-2\ln\left|x\right|+c$$

**Question 6**

$$\int \frac{2}{x(x^2+4)}\,dx$$
$$= \frac{1}{2}\int \left(\frac{1}{x}-\frac{x}{x^2+4}\right)dx$$
$$= \frac{1}{2}\int \left(\frac{1}{x}-\frac{1}{2}\times\frac{2x}{x^2+4}\right)dx$$
$$= \frac{1}{2}\ln\left|x\right|-\frac{1}{4}\ln\left|x^2+4\right|+c$$

**Question 7**

$$\int_{-2}^2 \frac{2x}{x^2+4x+20}\,dx$$
$$= \int_{-2}^2 \frac{2x+4-4}{x^2+4x+4+16}\,dx$$
$$= \int_{-2}^2 \frac{2x+4}{x^2+4x+20}\,dx-4\int_{-2}^2 \frac{dx}{(x+2)^2+4^2}$$
$$= \Big[\ln\left|x^2+4x+20\right|\Big]_{-2}^2 -4\times\frac{1}{4}\left[\tan^{-1}\left(\frac{x+2}{4}\right)\right]_{-2}^2$$
$$= \Big(\ln(4+8+20)-\ln(4-8+20)\Big) -\Big(\tan^{-1}(1)-\tan^{-1}0\Big)$$
$$= \ln 32-\ln 16-\frac{\pi}{4}+0$$
$$= \ln 2-\frac{\pi}{4}$$

**Question 8**

$$\int \frac{e^{2x}+e^x}{e^{2x}+1}\,dx$$
$$= \int \left(\frac{1}{2}\times\frac{2e^{2x}}{e^{2x}+1}+\frac{e^x}{(e^x)^2+1}\right)dx$$
$$= \frac{1}{2}\ln\left|e^{2x}+1\right|+\tan^{-1}(e^x)+c$$

**Question 9**

$$\int \frac{x^4+1}{x^2+2}\,dx$$
$$= \int \frac{x^2(x^2+2)-2(x^2+2)+5}{x^2+2}\,dx$$
$$= \int \left(x^2-2+\frac{5}{x^2+2}\right)dx$$
$$= x^3-2x+\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$$

**Question 10**

$$\int_0^2 \frac{x^3}{x^2+2}\,dx$$
$$= \int_0^2 \frac{x(x^2+2)-2x}{x^2+2}\,dx$$
$$= \int_0^2 \left(x - \frac{2x}{x^2+2}\right)dx$$
$$= \left[\frac{x^2}{2} - \ln\left|x^2+2\right|\right]_0^2$$
$$= (2 - \ln 6) - (0 - \ln 2)$$
$$= 2 - \ln 6 + \ln 2$$
$$= 2 - \ln 3$$

**Question 11**

$$\int \frac{x^6+3x^2-1}{x^3+1}\,dx$$
$$= \int \frac{x^3(x^3+1)-(x^3+1)+3x^2}{x^3+1}\,dx$$
$$= \int \left(x^3 - 1 + \frac{3x^2}{x^3+1}\right)dx$$
$$= \frac{x^4}{4} - x + \ln\left|x^3+1\right| + c$$

**Question 12**

$$\int \frac{x^2-4x+2}{(x-2)^3}\,dx$$
$$= \int \frac{(x-2)^2-2}{(x-2)^3}\,dx$$
$$= \int \left(\frac{1}{x-2} - 2(x-2)^{-3}\right)du$$
$$= \ln\left|x-2\right| - 2\times\left(-\frac{1}{2}\right)(x+1)^{-2} + c$$
$$= \ln\left|x-2\right| + \frac{1}{(x-2)^2} + c$$

**Question 13**

$$\int_0^1 \frac{e^{2x}-e^x}{e^x+1}\,dx$$
$$= \int_0^1 \frac{(e^x(e^x+1)-2e^x)}{e^x+1}\,dx$$
$$= \int_0^1 \left(e^x - \frac{2e^x}{e^x+1}\right)dx$$
$$= \left[e^x - 2\ln\left|e^x+1\right|\right]_0^1$$
$$= \Big(e - 2\ln(e+1)\Big) - \Big(1 - 2\ln 2\Big)$$
$$= e - 2\ln(e+1) - 2 + 2\ln 2$$
$$= \ln(e^e) - \ln\big((e+1)^2\big) - \ln\big(e^2\big) + \ln 4$$
$$= \ln\left(\frac{4e^e}{[e(e+1)]^2}\right)$$

**Question 14**

$$V = \pi\int_0^{\ln 2}\left(\frac{e^{2x}}{\sqrt{e^{2x}+1}}\right)^2 dx$$
$$= \pi\int_0^{\ln 2}\frac{e^{4x}}{e^{2x}+1}\,dx$$
$$= \pi\int_0^{\ln 2}\frac{e^{2x}\big(e^{2x}+1\big)-\big(e^{2x}+1\big)+1}{e^{2x}+1}\,dx$$
$$= \pi\int_0^{\ln 2}\left(e^{2x} - 1 + \frac{1}{e^{2x}+1}\right)dx$$
$$= \pi\int_0^{\ln 2}\left(e^{2x} - 1 + \frac{e^{-2x}}{1+e^{-2x}}\right)dx$$
$$= \pi\left[\frac{1}{2}e^{2x} - x - \frac{1}{2}\ln(1+e^{-2x})\right]_0^{\ln 2}$$
$$= \pi\left(\left(\frac{1}{2}e^{2\ln 2} - 1 - \frac{1}{2}\ln\big(1+e^{-2\ln 2}\big)\right)\right.$$
$$\left.-\left(\frac{1}{2} - 0 - \frac{1}{2}\ln 2\right)\right)$$
$$= 2.309076\ldots$$
$$= 2.31 \text{ units}^3 \text{ (2 dp)}$$

**Question 15**

$$\int \frac{\cos x}{\sin^2 x - 3\sin x + 2}\,dx$$
$$= \int \frac{\cos x}{(\sin x - 2)(\sin x - 1)}\,dx$$
$$= \int \left(\frac{\cos x}{\sin x - 2} - \frac{\cos x}{\sin x - 1}\right)dx$$
$$= \ln\left|\sin x - 2\right| - \ln\left|\sin x - 1\right| + c$$
$$= \ln\left|\frac{\sin x - 2}{\sin x - 1}\right| + c$$

**Question 16**

$$\int \frac{8x}{1+e^2x}\,dx$$
$$= \int \frac{\frac{8}{e^2}\big(1+e^2x\big)-\frac{8}{e^2}}{1+e^2x}\,dx$$
$$= \int \left(\frac{8}{e^2} - \frac{8}{e^4}\times\frac{e^2}{1+e^2x}\right)dx$$
$$= \frac{8x}{e^2} - \frac{8\ln\left|1+e^2x\right|}{e^4} + c$$

**Question 17**

$$\int \frac{\sec x \tan x}{\sec x + \sec^2 x}\,dx \qquad \boxed{\begin{array}{l} u = \sec x \\ du = \sec x \tan x\,dx \end{array}}$$

$$= \int \frac{1}{\sec x + \sec^2 x} \times \sec x \tan x\,dx$$

$$= \int \frac{1}{u + u^2}\,du$$

$$= \int \frac{1}{u(1+u)}\,du$$

$$= \int \left(\frac{1}{u} - \frac{1}{1+u}\right) du$$

$$= \ln|u| - \ln|1+u| + c$$

$$= \ln\left|\frac{u}{1+u}\right| + c$$

$$= \ln\left|\frac{\sec x}{1+\sec x}\right| + c$$

$$= \ln\left|\frac{1}{\cos x + 1}\right| + c$$

$$= -\ln\left|\cos x + 1\right| + c$$

**Question 18**

$$\int \frac{\tan^2 x + \tan x + \sec^2 x - 2}{\tan x - 1}\,dx$$

$$= \int \frac{\tan x\,(\tan x - 1) + 2(\tan x - 1) + \sec^2 x}{\tan x - 1}\,dx$$

$$= \int \left(\tan x + 2 + \frac{\sec^2 x}{\tan x - 1}\right) dx$$

$$= -\ln|\cos x| + 2x + \ln|\tan x - 1| + c$$

$$= \ln\left|\frac{\tan x - 1}{\cos x}\right| + 2x + c$$

**Question 19a**

$$I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx \qquad \boxed{\begin{array}{l} u = 4 - x \\ du = -dx \end{array}}$$

$$= -\int_3^1 \frac{\cos^2\left(\frac{\pi}{8}(4-u)\right)}{(4-u)u}\,du$$

$$= \int_1^3 \frac{\cos^2\left(\frac{\pi}{2} - \frac{\pi}{8}u\right)}{u(4-u)}\,du$$

$$= \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

**Question 19b**

$$2I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx + \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

$$= \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx + \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$$

$$= \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right) + \sin^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$$

$$= \int_1^3 \frac{1}{x(4-x)}\,dx$$

$$= \frac{1}{4}\int_1^3 \left(\frac{1}{x} + \frac{1}{4-x}\right) dx$$

$$\therefore I = \frac{1}{8}\Big[\ln x - \ln|4-x|\Big]_1^3$$

$$= \frac{1}{8}(\ln 3 - \ln 1 - \ln 1 + \ln 3)$$

$$= \frac{\ln 3}{4}$$

**Question 20**

$$\int \frac{\sin x}{\sin x + \cos x + e^x}\,dx$$

$$= -\int \frac{\cos x - \sin x + e^x - \cos x - e^x}{\sin x + \cos x + e^x}\,dx$$

$$= -\int \frac{\cos x - \sin x + e^x}{\sin x + \cos x + e^x}\,dx + \int \frac{\sin x + \cos x + e^x - \sin x}{\sin x + \cos x + e^x}\,dx$$

$$= -\ln|\sin x + \cos x + e^x| + \int \left(1 - \frac{\sin x}{\sin x + \cos x + e^x}\right) dx$$

$$= -\ln|\sin x + \cos x + e^x| + x - \int \frac{\sin x}{\sin x + \cos x + e^x}\,dx$$

$$\therefore 2\int \frac{\sin x}{\sin x + \cos x + e^x}\,dx = -\ln|\sin x + \cos x + e^x| + x + c$$

$$\int \frac{\sin x}{\sin x + \cos x + e^x}\,dx = \frac{x}{2} - \frac{1}{2}\ln|\sin x + \cos x + e^x| + c$$

**Question 1a**

$$\frac{2x+1}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2}$$
$$\frac{2x+1}{(x+1)(x+2)} = \frac{a(x+2)+b(x+1)}{(x+1)(x+2)}$$
$$2x+1 = (a+b)x + 2a + b$$

$$\therefore 2 = a + b \qquad (1)$$
$$1 = 2a + b \qquad (2)$$

$$(2)-(1) \quad -1 = a \quad \rightarrow \quad a = -1$$
$$\text{sub in (1)} \quad 2 = -1 + b \quad \rightarrow \quad b = 3$$
$$\therefore \frac{2x+1}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{3}{x+2}$$

**Question 1b**

$$\int \frac{2x+1}{(x+1)(x+2)}\,dx$$
$$= \int \left(-\frac{1}{x+1} + \frac{3}{x+2}\right)dx$$
$$= -\ln\left|x+1\right| + 3\ln\left|x+2\right| + c$$

**Question 2a**

$$\frac{2x+3}{x^2(x+1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1}$$
$$2x+3 = ax(x+1) + b(x+1) + cx^2$$

Let $x = 0 \;\therefore 2(0)+3 = a(0)+b(0+1)+c(0)$
$\rightarrow \; 3 = b \;\rightarrow\; b = 3$
Let $x = -1 \;\therefore 2(-1)+3 = a(0)+b(0)+c(1)$
$\rightarrow \; 1 = c \;\rightarrow c = 1$
equating coefficients of $x^2$:
$0 = a + c \;\rightarrow\; 0 = a + 1 \;\rightarrow\; a = -1$
$$\therefore \frac{2x+3}{x^2(x+1)} = -\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}$$

**Question 2b**

$$\int_2^3 \frac{2x+3}{x^2(x+1)}\,dx$$
$$= \int_2^3 \left(-\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}\right)dx$$
$$= \left[-\ln\left|x\right| - \frac{3}{x} + \ln\left|x+1\right|\right]_2^3$$
$$= \left(-\ln 3 - 1 + \ln 4\right) - \left(-\ln 2 - \frac{3}{2} - \ln 3\right)$$
$$= \ln 4 + \ln 2 - 2\ln 3 + \frac{1}{2}$$
$$= \ln\left(\frac{8}{9}\right) + \frac{1}{2}$$

**Question 3**

$$a = \frac{2(-1)+1}{-1+2} = -1$$
$$b = \frac{2(-2)+1}{(-2)+1} = 3$$
$$\therefore \frac{2x+1}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{3}{x+2}$$

**Question 4**

$$b = \frac{2(0)+3}{0+1} = 3$$
$$c = \frac{2(-1)+3}{(-1)^2} = 1$$
equating coefficients of $x^2$:
$0 = a + c \;\rightarrow\; 0 = a + 1 \;\rightarrow\; a = -1$
$$\therefore \frac{2x+3}{x^2(x+1)} = -\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}$$

**Question 5a**

$$a = \frac{1-(-1)}{(-1)+2} = 2$$
$$b = \frac{1-(-2)}{(-2)+1} = -3$$

**Question 5b**

b $$\int \frac{1-x}{(x+1)(x+2)}\,dx$$
$$= \int \left(\frac{2}{x+1} - \frac{3}{x+2}\right)dx$$
$$= 2\ln\left|x+1\right| - 3\ln\left|x+2\right| + c$$

**Question 6**

$$\frac{x+1}{x(x-1)^2} \equiv \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$$
$$a = \frac{(0)+1}{\left((0)-1\right)^2} = 1$$
$$c = \frac{(1)+1}{1} = 2$$
coefficients of $x^2$: $a + b = 0 \;\rightarrow b = -1$
$$\therefore \frac{x+1}{x(x-1)^2} \equiv \frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

**Question 7**

$$\frac{x+2}{x^2+7x+12} = \frac{x+2}{(x+3)(x+4)} = \frac{a}{x+3} + \frac{b}{x+4}$$
$$a = \frac{(-3)+2}{(-3)+4} = -1$$
$$b = \frac{(-4)+2}{(-4)+3} = 2$$
$$\int_0^1 \frac{x+2}{x^2+7x+12}\,dx$$
$$= \int_0^1 \left(-\frac{1}{x+3} + \frac{2}{x+4}\right)dx$$
$$= \left[-\ln\left|x+3\right| + 2\ln\left|x+4\right|\right]_0^1$$
$$= (-\ln 4 + 2\ln 5) - (-\ln 3 + 2\ln 4)$$
$$= -\ln 4 + \ln 25 + \ln 3 - \ln 16$$
$$= \ln\frac{75}{64}$$

**Question 8**

$$a = \frac{5(-2)-5}{(-2)^2+1} = -3$$

Let $x = i \quad bi + c = \frac{5i-5}{i+2} \times \frac{i-2}{i-2}$

$$= \frac{-5-10i-5i+10}{i^2-2^2}$$

$$bi + c = -1 + 3i$$

$$\therefore c = -1, b = 3$$

$$\therefore \frac{5x-5}{(x+2)(x^2+1)} = -\frac{3}{x+2} + \frac{3x-1}{x^2+1}$$

$$\therefore \int_3^4 \frac{5x-5}{(x+2)(x^2+1)}\,dx$$

$$= \int_3^4 \left(-\frac{3}{x+2} + \frac{3x-1}{x^2+1}\right)dx$$

$$= \int_3^4 \left(-\frac{3}{x+2} + \frac{3}{2} \times \frac{2x}{x^2+1} - \frac{1}{1+x^2}\right)dx$$

$$= \left[-3\ln|x+2| + \frac{3}{2}\ln|x^2+1| - \tan^{-1}x\right]_3^4$$

$$= \left(-3\ln 6 + \frac{3}{2}\ln 17 - \tan^{-1}4\right) - \left(-3\ln 5 + \frac{3}{2}\ln 10 - \tan^{-1}3\right)$$

$$= \frac{3}{2}\ln\frac{17}{10} + 3\ln\frac{5}{6} + \tan^{-1}3 - \tan^{-1}4$$

**Question 9**

$$\int_{\frac{1}{2}}^2 \frac{2}{x^3+x^2+x+1}\,dx$$

$$= \int_{\frac{1}{2}}^2 \left(\frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right)dx$$

$$= \left[\ln\left|x+1\right| - \frac{1}{2}\ln\left|x^2+1\right| + \tan^{-1}x\right]_{\frac{1}{2}}^2$$

$$= \left(\ln 3 - \frac{1}{2}\ln 5 + \tan^{-1}2\right) - \left(\ln\frac{3}{2} - \frac{1}{2}\ln\frac{5}{4} + \tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$= \ln 3 - \ln\frac{3}{2} - \frac{1}{2}\ln 5 + \frac{1}{2}\ln\frac{5}{4} + \tan^{-1}2 - \tan^{-1}\frac{1}{2}$$

$$= \ln 2 - \frac{1}{2}\ln 4 + \tan^{-1}2 - \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}2 - \tan^{-1}\frac{1}{2}$$

**Question 10**

$$\int_0^1 \frac{8(1-x)}{(2-x^2)(2-2x+x^2)}\,dx$$

$$= \int_0^1 \left(\frac{4-2x}{2-2x+x^2} - \frac{2x}{2-x^2}\right)dx$$

$$= \int_0^1 \left(-\frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1^2} - \frac{2x}{2-x^2}\right)dx$$

$$= \left[-\ln\left|x^2-2x+2\right| + 2\tan^{-1}(x-1) + \ln\left|2-x^2\right|\right]_0^1$$

$$= (0+0+0) - \left(-\ln 2 + 2\left(-\frac{\pi}{4}\right) + \ln 2\right)$$

$$= \frac{\pi}{2}$$

**Question 11**

$$\frac{x-6}{x^2+3x-4} = \frac{x-6}{(x+4)(x-1)} = \frac{a}{x+4} + \frac{b}{x-1}$$

$$a = \frac{(-4)-6}{(-4)-1} = 2$$

$$b = \frac{(1)-6}{(1)+4} = -1$$

$$\therefore \int_2^5 \frac{x-6}{x^2+3x-4}\,dx$$

$$= \int_2^5 \left(\frac{2}{x+4} - \frac{1}{x-1}\right)dx$$

$$= \left[2\ln\left|x+4\right| - \ln\left|x-1\right|\right]_2^5$$

$$= (2\ln 9 - \ln 4) - (2\ln 6 - 0)$$

$$= \ln 81 - \ln 4 - \ln 36$$

$$= \ln\frac{81}{144}$$

$$= \ln\frac{9}{16}$$

**Question 12a**

$$b = \frac{-5(0)+2}{(0)-1} = -2$$

$$c = \frac{-5(1)+2}{(1)^2} = -3$$

coefficients of $x^2$: $a + c = 0 \rightarrow a = 3$

**Question 12b**

$$\int_2^3 \frac{-5x+2}{x^2(x-1)}\,dx$$

$$= \int_2^3 \left(\frac{3}{x} - \frac{2}{x^2} - \frac{3}{x-1}\right)dx$$

$$= \left[3\ln|x| + \frac{2}{x} - 3\ln|x-1|\right]_2^3$$

$$= \left(3\ln 3 + \frac{2}{3} - 3\ln 2\right) - (3\ln 2 + 1 + 0)$$

$$= 3\ln\frac{3}{4} - \frac{1}{3}$$

**Question 13a**

$$\frac{16x-43}{(x-3)^2(x+2)} = \frac{a}{(x-3)^2} + \frac{b}{x-3} + \frac{c}{x+2}$$

$$a = \frac{16(3)-43}{(3)+2} = 1$$

$$c = \frac{16(-2)-43}{\big((-2)-3\big)^2} = -3$$

equating coefficients of $x^2$: $b + c = 0 \;\rightarrow\; b = 3$

**Question 13b**

$$\int \frac{16x-43}{(x-3)^2(x+2)}\,dx$$

$$= \int \left(\frac{1}{(x-3)^2} + \frac{3}{x-3} - \frac{3}{x+2}\right)dx$$

$$= -\frac{1}{x-3} + 3\ln\left|x-3\right| - 3\ln\left|x+2\right| + c$$

$$= 3\ln\left|\frac{x-3}{x+2}\right| - \frac{1}{x-3} + c$$

**Question 14**

$$a = \frac{2(-1)+3}{(-1)^2+4} = \frac{1}{5}$$

equating coefficients of $x^2$:

$$a + b = 0 \;\rightarrow\; b = -\frac{1}{5}$$

equating constants:

$$4a + c = 3 \;\rightarrow\; \frac{4}{5} + c = 3 \;\rightarrow c = \frac{11}{5}$$

$$\int_0^2 \frac{2x+3}{(x+1)(x^2+4)}\,dx$$

$$= \frac{1}{5}\int_0^2 \left(\frac{1}{x+1} - \frac{x-11}{x^2+4}\right)dx$$

$$= \frac{1}{5}\int_0^2 \left(\frac{1}{x+1} - \frac{1}{2}\times\frac{2x}{x^2+4} + \frac{11}{x^2+4}\right)dx$$

$$= \frac{1}{5}\left[\ln|x+1| - \frac{1}{2}\ln\left|x^2+4\right| + \frac{11}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \frac{1}{5}\left(\left(\ln 3 - \frac{1}{2}\ln 8 + \frac{11}{2}\times\left(\frac{\pi}{4}\right)\right) - \left(0 - \frac{1}{2}\ln 4 - 0\right)\right)$$

$$= \frac{1}{5}\left(\ln 3 + \frac{1}{2}\ln\frac{4}{8} + \frac{11\pi}{8}\right)$$

$$= \frac{1}{5}\left(\ln\frac{3}{\sqrt{2}} + \frac{11\pi}{8}\right)$$

$$= \frac{1}{5}\left(\ln\frac{3\sqrt{2}}{2} + \frac{11\pi}{8}\right)$$

**Question 15**

$$\frac{3x^2+8}{x(x^2+4)} = \frac{a}{x} + \frac{bx+c}{x^2+4}$$

$$a = \frac{3(0)+8}{(0)^2+4} = 2$$

$$b(2i) + c = \frac{3(2i)^2+8}{2i}$$

$$c + 2bi = -\frac{4}{2i}\times\frac{i}{i}$$

$$= 2i$$

$$\therefore b = 1, c = 0$$

$$\int \frac{3x^2+8}{x(x^2+4)}\,dx = \int \left(\frac{2}{x} + \frac{x}{x^2+4}\right)dx$$

$$= 2\ln\left|x\right| + \frac{1}{2}\ln\left|x^2+4\right| + c$$

**Question 16**

$$\int \frac{2x^3-x^2-8x-2}{x(x-2)}\,dx$$

$$= \int \frac{2x\big(x^2-2x\big) + 3\big(x^2-2x\big) - 2x - 2}{x(x-2)}\,dx$$

$$= \int \left(2x + 3 - \frac{2x+2}{x(x-2)}\right)dx \qquad (*)$$

Let $\frac{2x+2}{x(x-2)} = \frac{a}{x} + \frac{b}{x-2}$

$$a = \frac{2(0)+2}{(0)-2} = -1$$

$$b = \frac{2(2)+2}{(2)} = 3$$

sub in $(*)$

$$\int \frac{2x^3-x^2-8x-2}{x(x-2)}\,dx$$

$$= \int \left(2x + 3 + \frac{1}{x} - \frac{3}{x-2}\right)dx$$

$$= x^2 + 3x + \ln\left|x\right| - 3\ln\left|x-2\right| + c$$

**Question 17**

$$\frac{e^x+1}{e^{2x}-5e^x+6}=\frac{e^x+1}{(e^x-3)(e^x-2)}=\frac{a}{e^x-3}+\frac{b}{e^x-2}$$

$$a=\frac{e^{\ln 3}+1}{e^{\ln 3}-2}=\frac{3+1}{3-2}=4$$

$$b=\frac{e^{\ln 2}+1}{e^{\ln 2}-3}=\frac{2+1}{2-3}=-3$$

$$\int\frac{e^x+1}{e^{2x}-5e^x+6}dx$$

$$=\int\left(\frac{4}{e^x-3}-\frac{3}{e^x-2}\right)dx$$

$$=\int\left(\frac{4e^{-x}}{1-3e^{-x}}-\frac{3e^{-x}}{1-2e^{-x}}\right)dx$$

$$=\frac{4}{3}\ln\left|1-3e^{-x}\right|-\frac{3}{2}\ln\left|1-2e^{-x}\right|+c$$

**Question 18**

$$\int_0^1\frac{3x^3+x^2+12x-12}{(x^2+4)(x^2+2x+4)}dx$$

$$=\int_0^1\left(\frac{x-3}{x^2+2x+4}+\frac{2x}{x^2+4}\right)dx$$

$$=\int_0^1\left(\frac{\frac{1}{2}(2x+2)-4}{x^2+2x+4}+\frac{2x}{x^2+4}\right)dx$$

$$=\int_0^1\left(\frac{1}{2}\times\frac{2x+2}{x^2+2x+4}-\frac{4}{(x+1)^2+\left(\sqrt{3}\right)^2}+\frac{2x}{x^2+4}\right)dx$$

$$=\left[\frac{1}{2}\ln\left|x^2+2x+4\right|-\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+\ln\left|x^2+4\right|\right]_0^1$$

$$=\left(\frac{1}{2}\ln 7-\frac{4}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}+\ln 5\right)$$

$$-\left(\frac{1}{2}\ln 4-\frac{4}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}+\ln 4\right)$$

$$=\ln\sqrt{7}-\frac{4}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}+\ln 5-\ln 2-\ln 4+\frac{4}{\sqrt{3}}\left(\frac{\pi}{6}\right)$$

$$=\ln\frac{5\sqrt{7}}{8}+\frac{4}{\sqrt{3}}\left(\frac{\pi}{6}-\tan^{-1}\frac{2}{\sqrt{3}}\right)$$

**Question 19**

Let $\frac{3!}{x(x+1)(x+2)(x+3)}=\frac{a}{x}+\frac{b}{x+1}+\frac{c}{x+2}+\frac{d}{x+3}$

$$a=\frac{3!}{(0+1)(0+2)(0+3)}=1$$

$$b=\frac{3!}{(-1)(-1+2)(-1+3)}=-3$$

$$c=\frac{3!}{(-2)(-2+1)(-2+3)}=3$$

$$d=\frac{3!}{(-3)(-3+1)(-3+2)}=-1$$

$$\therefore\frac{3!}{x(x+1)(x+2)(x+3)}=\frac{1}{x}-\frac{3}{x+1}+\frac{3}{x+2}-\frac{1}{x+3}$$

**Question 20a**

RHS $= \dfrac{(A+2t)\left(t^2-2t+2\right)+(B-2t)\left(t^2+2t+2\right)}{(t^4+4)}$

equating coefficients of $t \quad -2A+4t+2B-4t=0 \quad \therefore A=B$

equating the constant terms $\quad 2A+2B=16 \quad \therefore A=B=4$

**Question 20b**

$$x=\int_0^t \frac{16}{t^4+4}\,dt$$

$$=\int_0^t \left(\frac{4+2t}{t^2+2t+2}+\frac{4-2t}{t^2-2t+2}\right)dt$$

$$=\int_0^t \left(\frac{2t+2}{t^2+2t+2}+\frac{2}{(t+1)^2+1}-\frac{2t-2}{t^2-2t+2}+\frac{2}{(t-1)^2+1}\right)dt$$

$$=\left[\ln\left(t^2+2t+2\right)+2\tan^{-1}(t+1)-\ln\left(t^2-2t+2\right)+2\tan^{-1}(t-1)\right]_0^t$$

$$=\left(\ln\left(t^2+2t+2\right)+2\tan^{-1}(t+1)-\ln\left(t^2-2t+2\right)+2\tan^{-1}(t-1)\right)$$
$$-(\ln 2+2\tan^{-1}1-\ln 2+2\tan^{-1}1)$$

$$=\ln\left(\frac{t^2+2t+2}{t^2-2t+2}\right)+2\tan^{-1}(t+1)+2\tan^{-1}(t-1)$$

**Question 20c**

$$x=\ln\left(\frac{1+\frac{2}{t}+\frac{2}{t^2}}{1-\frac{2}{t}+\frac{2}{t^2}}\right)+2\tan^{-1}(t+1)+2\tan^{-1}(t-1)$$

As $t\to\infty, x\to \ln(1)+2\tan^{-1}(\infty)+2\tan^{-1}(\infty)$

$\to \ln(1)+2\left(\dfrac{\pi}{2}\right)+2\left(\dfrac{\pi}{2}\right)$

$=2\pi$ metres

Since $\dfrac{16}{t^4+4}>0$ for all $t$ the particle is always moving to the right, so the particle approaches but never reaches a displacement of $x=2\pi$ metres.

**Question 1**

$\int_0^a f(a-x)dx$ $\quad \boxed{\begin{array}{l} u = a-x \\ du = -dx \\ dx = -du \end{array}}$

$= \int_a^0 f(u)\times(-du)$

$= \int_0^a f(u)du$

$= \int_0^a f(x)dx$

$\int_0^{2\pi} x\cos x\,dx$

$= \int_0^{2\pi} (2\pi - x)\cos(2\pi - x)\,dx$

$= 2\pi\int_0^{2\pi} \cos(2\pi - x)\,dx - \int_0^{2\pi} x\cos(2\pi - x)\,dx$

$= 2\pi\int_0^{2\pi} \cos x\,dx - \int_0^{2\pi} x\cos x\,dx$

$\therefore 2\int_0^{2\pi} x\cos x\,dx = 2\pi\int_0^{2\pi} \cos x\,dx$

$\int_0^{2\pi} x\cos x\,dx = \pi\Big[\sin x\Big]_0^{2\pi}$

$= \pi(0-0)$

$= 0$

**Question 2**

$\int_0^2 x\sqrt{2-x}\,dx$

$= \int_0^2 (2-x)\sqrt{x}\,dx$

$= \int_0^2 \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)dx$

$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$

$= \frac{4}{3}\times 2\sqrt{2} - \frac{2}{5}\times 4\sqrt{2}$

$= \frac{16\sqrt{2}}{15}$

**Question 3**

$\int_{-2}^2 \left(x + x^3 + x^5\right)\left(1 + x^2 + x^4\right)dx = 0$

[since an odd function × an even function is an odd function].

**Question 4**

$\int_{-\pi}^0 \sin\left(x + \frac{\pi}{2}\right)\cos\left(x + \frac{\pi}{2}\right)dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\cos x\,dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx$ where $f(x)$ is odd

$= 0$

**Question 5**

$x^3 > x^2$ for $x > 1$, so $1 \le a < b$.

**Question 6**

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\therefore 2\int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(a-x)\,dx$

$\int_0^a f(x)\,dx = \frac{1}{2}\int_0^a \big(f(x) + f(a-x)\big)\,dx$

$= \frac{1}{2}\int_0^a f(a)\,dx$

$= \frac{f(a)}{2}\Big[x\Big]_0^a$

$= \frac{f(a)}{2}(a-0)$

$= \frac{a}{2}f(a)$

**Question 7**

$A$: $e^{x^2} > 0$ and $\cos x \ge 0$ in the domain. True.

$B$: $x^3\cos x$ is odd since $x^3$ is odd and $\cos x$ is even. False

$C$: $\sin^2 x - \cos^2 x = -\cos 2x$ which is negative in the domain. False

$D$; $\sin^{-1}\left(x^3\right)$ is the odd function of an odd function so is odd. False.

**Answer A**

**Question 8**

$\int_{-a}^a f(x)\,dx$

$= \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx$

$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$

$= \int_0^a \big(f(x) + f(-x)\big)\,dx$

**Answer C**

**Question 10**

$\int_a^{3a} f(x-a)\,dx$ $\quad \boxed{\begin{array}{l} u = 2a - x \\ du = -dx \\ dx = -du \end{array}}$

$= \int_a^{-a} f(a-u)\,du$

$= \int_{-a}^{a} f(a-x)\,dx$

**Question 11**

for $0 \le x \le \frac{\pi}{6}$ $\sin^2 x < \sin x < 1 - \sin x < 1 - \sin^2 x$

$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x\,dx$ is the smallest integral

**Answer B**

**Question 12**

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx \qquad (1)$

$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$

$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx \qquad (2)$

From $(1)$ and $(2)$:

$2\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{1}{2}\left[x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{4}$

**Question 13**

$\int_0^2 \left(1 + \sin\left(\pi(1-x)^3\right)\right) dx$

$= \int_0^2 \left(1 - \sin\left(\pi(x-1)^3\right)\right) dx$ sine is odd

$= \int_{-1}^1 \left(1 - \sin \pi x^3\right) dx$ translating 1 unit left

$= \int_{-1}^1 1\,dx - \int_{-1}^1 \sin \pi x^3\,dx$

$= \left[x\right]_{-1}^1 - 0$ since sine and $x^3$ are odd

$= \left(1 - (-1)\right)$

$= 2$

**Question 14**

$x - a \ge 0$ for all $x \in [0,2]$

$\therefore a \le 0 \qquad (1)$

$1 - \sqrt{x-a} \ne 0$

$\sqrt{x-a} \ne 1$

$x - a \ne 1$ for all $x \in [0,2]$

$\therefore a < -1, a > 1 \qquad (2)$

The integral is defined only if $a < -1$, as both $(1)$ and $(2)$ are satisfied.

**Question 15**

$\int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2\left(x - \frac{\pi}{4}\right)}$

$= \int_0^{\frac{\pi}{2}} \sec^2\left(x - \frac{\pi}{4}\right) dx$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\,dx$

$= \left[\tan x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= 1 - (-1)$

$= 2$

**Question 16**

$xy - x = 1$

$x(y-1) = 1$

$x = \frac{1}{y-1}$

$\int_3^5 x\,dy = \int_3^5 \frac{dy}{y-1}$

$= \left[\ln(y-1)\right]_3^5$

$= \ln 4 - \ln 2$

$= \ln 2$

**Question 17**

Since $\tan^{-1} x$ is odd and $1 + \sin^2 x$ is even then $\frac{\tan^{-1} x}{1+\sin^2 x}$ is odd.

$\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd

$\therefore \int_{-1}^{1} \frac{\tan^{-1} x}{1 + \sin^2 x}\,dx = 0$

**Question 18**

$$\int_0^1 \left(x^2 - a\right)^2 dx$$
$$= \int_0^1 \left(x^4 - 2ax^2 + a^2\right) dx$$
$$= \left[\frac{x^5}{5} - \frac{2a}{3}x^3 + a^2 x\right]_0^1$$
$$= \left(\frac{1}{5} - \frac{2a}{3} + a^2\right) - (0)$$
$$= a^2 - \frac{2a}{3} + \frac{1}{5}$$

Minimum when $a = -\frac{-\frac{2}{3}}{2(1)} = \frac{1}{3}$

$$\min = \left(\frac{1}{3}\right)^2 - \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{5} = \frac{4}{45}$$

**Question 19**

If $f(a+x) = f(a-x)$ then $f(x)$ has line symmetry on $x = a$ (1).

**Question 19a**

$$\int_0^{2a} f(x)\,dx$$
$$= \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \text{ splitting the integral}$$
$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx \text{ from (1)}$$
$$= 2\int_0^a f(x)\,dx$$

**Question 19b**

$$\int_0^a f(2x)\,dx$$
$$= \frac{1}{2}\int_0^{2a} f(x)\,dx \text{ using a horizontal dilation}$$

of $f(x)$ by a factor of 2

$$= \int_0^a f(x)\,dx \text{ from part (a)}$$

**Question 19c**

$$\int_0^a f(x+a)\,dx$$
$$= \int_0^a f(a-x)\,dx \text{ since } f(x+a) = f(a-x)$$
$$= \int_0^a f(x)\,dx \text{ reflecting about } x = \frac{a}{2}$$

**Question 19d**

$$\int_0^a f(x-a)\,dx$$
$$= \int_{-a}^0 f(x)\,dx \text{ translating } a \text{ units to the left}$$
$$= \int_{2a}^{3a} f(x)\,dx \text{ since } f(x) \text{ is symmetrical on } x = a$$

**Question 20**

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x + \cos^2 x - 2\sin x \cos x}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\sin x - \cos x)^2}\,dx \qquad (1)$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right)}{1 + \left(\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)\right)^2}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + (\cos x - \sin x)^2}\,dx \qquad (2)$$

(1) + (2):

$$2\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1 + (\sin x - \cos x)^2}\,dx$$
$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\,dx = \frac{1}{2}\left[\tan^{-1}(\sin x - \cos x)\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$
$$= \frac{\pi}{4}$$

**Question 21**

$$\int_0^a [f(a-x) + f(a+x)]dx$$
$$= \int_0^a f(a-x)\,dx + \int_0^a f(x+a)\,dx$$
$$= \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$
$$= \int_0^{2a} f(x)\,dx$$

**Answer B**

**Question 22**

$$\int_0^{\pi} \frac{x\sin x}{1 + \cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$$
$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\,dx$$
$$= \pi\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\,dx$$
$$2\int_0^{\pi} \frac{x\sin x}{1 + \cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\,dx$$
$$\int_0^{\pi} \frac{x\sin x}{1 + \cos^2 x}\,dx = -\frac{\pi}{2}\left[\tan^{-1}(\cos x)\right]_0^{\pi}$$
$$= -\frac{\pi}{2}(\tan^{-1}(-1) - \tan^{-1}(1))$$
$$= -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$
$$= \frac{\pi^2}{4}$$

**Question 23a**

$$\int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}\,dx$$

$$u=\frac{1}{x}=x^{-1}$$
$$\frac{du}{dx}=-x^{-2}=-\frac{1}{x^2}$$
$$dx=-\frac{du}{u^2}$$

$$=\int_{a}^{\frac{1}{a}} \frac{f\left(\frac{1}{u}\right)}{\left(\frac{1}{u}\right)\left(f\left(\frac{1}{u}\right)+f(u)\right)}\left(-\frac{du}{u^2}\right)$$

$$=\int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{u}\right)}{u\left(f(u)+f\left(\frac{1}{u}\right)\right)}\,du$$

$$=\int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x}\right)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}\,dx$$

**Question 23b**

$$I=\int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x+\sin\frac{1}{x}\right)}\,dx$$

$$=\int_{\frac{1}{2}}^{2} \frac{\sin\frac{1}{x}}{x\left(\sin x+\sin\frac{1}{x}\right)}\,dx$$

$$\therefore 2I=\int_{\frac{1}{2}}^{2} \frac{\sin x+\sin\frac{1}{x}}{x\left(\sin x+\sin\frac{1}{x}\right)}\,dx$$

$$I=\frac{1}{2}\int_{\frac{1}{2}}^{2}\frac{dx}{x}$$
$$=\frac{1}{2}\Big[\ln x\Big]_{\frac{1}{2}}^{2}$$
$$=\frac{1}{2}\left(\ln 2-\ln\left(\frac{1}{2}\right)\right)$$
$$=\frac{1}{2}\big(\ln 2-(-\ln 2)\big)$$
$$=\ln 2$$

**Question 24**

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x\sin^2 x}{1+e^x}\,dx$$

$$=\int_{0}^{\frac{\pi}{2}}\left(\frac{e^x\sin^2 x}{1+e^x}+\frac{e^{-x}\sin^2(-x)}{1+e^{-x}}\right)dx \quad \text{given result}$$

$$=\int_{0}^{\frac{\pi}{2}}\left(\frac{e^x\sin^2 x}{1+e^x}+\frac{e^{-x}\sin^2 x}{1+e^{-x}}\times\frac{e^x}{e^x}\right)dx \ \sin^2 x \text{ even}$$

$$=\int_{0}^{\frac{\pi}{2}}\left(\frac{e^x\sin^2 x}{1+e^x}+\frac{\sin^2 x}{1+e^x}\right)dx$$

$$=\int_{0}^{\frac{\pi}{2}}\left(\frac{(1+e^x)\sin^2 x}{1+e^x}\right)dx$$

$$=\int_{0}^{\frac{\pi}{2}}\sin^2 x\,dx$$

$$=\frac{1}{2}\int_{0}^{\frac{\pi}{2}}(1-\cos 2x)\,dx$$

$$=\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_{0}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left(\left(\frac{\pi}{2}-0\right)-(0-0)\right)$$

$$=\frac{\pi}{4}$$

**Question 25a**

$$I=\int_{a}^{b}\frac{f(x)}{f(a+b-x)+f(x)}\,dx$$

$$=\int_{a}^{b}\frac{f(a+b-x)}{f(x)+f(a+b-x)}\,dx$$

$$2I=\int_{a}^{b}\frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}\,dx$$

$$=\Big[x\Big]_{a}^{b}$$

$$=b-a$$

$$\therefore I=\frac{b-a}{2}$$

**Question 25b**

$$\int_2^5 \frac{x^2}{\sqrt{29-x^2}+x}\,dx \qquad \boxed{\begin{array}{c} u = x^2 \\ du = 2x\,dx \\ dx = \dfrac{du}{2\sqrt{u}} \end{array}}$$

$$= \int_4^{25} \frac{u}{\sqrt{29-u}+\sqrt{u}} \times \frac{du}{2\sqrt{u}}$$

$$= \frac{1}{2}\int_4^{25} \frac{\sqrt{u}}{\sqrt{4+25-u}+\sqrt{u}}\,du$$

$$= \frac{1}{2}\int_4^{25} \frac{\sqrt{x}}{\sqrt{4+25-x}+\sqrt{x}}\,du \qquad \text{change of variable}$$

$$= \frac{1}{2}\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,dx \text{ where } f(x)=\sqrt{x}, a=4, b=25$$

$$= \frac{1}{2} \times \frac{25-4}{2} \text{ from (a)}$$

$$= \frac{21}{4}$$

**Question 26**

$$\int_3^7 \frac{\ln(x+2)}{\ln(24+10x-x^2)}\,dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln(x+2)}{\ln(24+10x-x^2)} + \frac{\ln\big((10-x)+2\big)}{\ln(24+10(10-x)-(10-x)^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln(x+2)}{\ln(24+10x-x^2)} + \frac{\ln(12-x)}{\ln(24+100-10x-100+20x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln(x+2)}{\ln(24+10x-x^2)} + \frac{\ln(12-x)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln(x+2)+\ln(12-x)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln\big((x+2)(12-x)\big)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln\big(12x-x^2+24-2x\big)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln\big(24+10x-x^2\big)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 dx$$

$$= \frac{1}{2}\Big[x\Big]_3^7$$

$$= \frac{1}{2}(7-3)$$

$$= 2$$

**Question 1**

$$\sin 4x \cos 3x$$
$$= \frac{1}{2}\Big[\sin(4x+3x) + \sin(4x-3x)\Big]$$
$$= \frac{1}{2}(\sin 7x + \sin x)$$

**Question 2**

$$\cos 75° \cos 15°$$
$$= \frac{1}{2}\Big(\cos(75°-15°) + \cos(75°+15°)\Big)$$
$$= \frac{1}{2}\Big(\cos 60° + \cos 90°\Big)$$
$$= \frac{1}{2}\Big(\frac{1}{2} + 0\Big)$$
$$= \frac{1}{4}$$

**Question 3**

$$\frac{\sin(A+B) + \sin(A-B)}{\cos(A-B) + \cos(A+B)}$$
$$= \frac{\sin A \cos B}{\cos A \cos B}$$
$$= \tan A$$

**Question 4**

$$\cos 130° + \cos 10°$$
$$= 2\cos\left(\frac{130+10}{2}\right)\cos\left(\frac{130-10}{2}\right)$$
$$= 2\cos 70° \cos 60°$$
$$= 2\cos 70° \times \frac{1}{2}$$
$$= \cos 70°$$

**Question 5**

$$\cos\frac{\pi}{12} - \cos\frac{5\pi}{12}$$
$$= \cos\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) - \cos\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right)$$
$$= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$
$$= 2\sin\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$= 2\times\frac{1}{\sqrt{2}}\times\frac{1}{2}$$
$$= \frac{1}{\sqrt{2}}$$
$$= \cos\frac{\pi}{4}$$

**Question 6**

$$\cos(A-B) - \cos(A+B)$$
$$= \cos A \cos B + \sin A \sin B$$
$$- (\cos A \cos B - \sin A \sin B)$$
$$= 2\sin A \sin B$$
$$\therefore \sin A \sin B = \frac{1}{2}\Big[\cos(A-B) - \cos(A+B)\Big]$$

**Question 7a**

$$\int \cos 3x \cos 2x \, dx$$
$$= \frac{1}{2}\int \Big(\cos(3x-2x) + \cos(3x+2x)\Big)\, dx$$
$$= \frac{1}{2}\int (\cos x + \cos 5x)\, dx$$
$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + c$$

**Question 7b**

$$\int \sin 8x \cos 5x \, dx$$
$$= \frac{1}{2}\int (\sin(8x+5x) + \sin(8x-5x))\, dx$$
$$= \frac{1}{2}\int (\sin 13x + \sin 3x)\, dx$$
$$= -\frac{1}{26}\cos 13x - \frac{1}{6}\cos 3x + c$$

**Question 8a**

$$\int_0^{\frac{\pi}{4}} \sin 4x \sin 3x \, dx \text{ (note the changed order)}$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos(4x-3x) - \cos(4x+3x))\, dx$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos x - \cos 7x)\, dx$$
$$= \frac{1}{2}\left[\sin x - \frac{1}{7}\sin 7x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left(\left(\frac{1}{\sqrt{2}} - \frac{1}{7}\left(-\frac{1}{\sqrt{2}}\right)\right) - (0-0)\right)$$
$$= \frac{1}{2}\times\frac{8}{7}\times\frac{1}{\sqrt{2}}$$
$$= \frac{2\sqrt{2}}{7}$$

**Question 8b**

$$\int_{\frac{\pi}{2}}^{\pi} \cos 5x \sin 3x \, dx$$
$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (\sin(5x+3x) - \sin(5x-3x))\, dx$$
$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (\sin 8x - \sin 2x)\, dx$$
$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\pi}$$
$$= \frac{1}{2}\left(\left(-\frac{1}{8}\cos 8\pi + \frac{1}{2}\cos 2\pi\right)\right.$$
$$\left. - \left(-\frac{1}{8}\cos 4\pi + \frac{1}{2}\cos \pi\right)\right)$$
$$= \frac{1}{2}\left(\left(-\frac{1}{8} + \frac{1}{2}\right) - \left(-\frac{1}{8} - \frac{1}{2}\right)\right)$$
$$= \frac{1}{2}$$

**Question 9a**

$$\sin 4x - \sin 2x = 0$$
$$2\sin 2x \cos 2x - \sin 2x = 0$$
$$\sin 2x\,(2\cos 2x - 1) = 0$$
$$\sin 2x = 0 \qquad \cos 2x = \frac{1}{2}$$
$$2x = 0, \pi, 2\pi \qquad 2x = \frac{\pi}{3}, \frac{5\pi}{3}$$
$$x = 0, \frac{\pi}{2}, \pi \qquad x = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$$

**Question 9b**

$$\sin 4x - \sin 2x = 0$$
$$\sin(3x + x) - \sin(3x - x) = 0$$
$$2\cos 3x \sin x = 0$$
$$\cos 3x = 0 \qquad \sin x = 0$$
$$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \qquad x = 0, \pi$$
$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$
$$\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$$

**Question 10a**

$$\int \sin x \sin 2x\, dx$$
$$= \int 2\sin^2 x \cos x\, dx$$
$$= 2\int (\sin x)^2 \times \frac{d}{dx}(\sin x)\, dx$$
$$= \frac{2\sin^3 x}{3} + c$$

**Question 10b**

$$\int \sin x \sin 2x\, dx$$
$$= \int \frac{1}{2}\Big[\cos(2x - x) - \cos(2x + x)\Big]\, dx$$
$$= \frac{1}{2}\int \Big[\cos x - \cos 3x\Big]\, dx$$
$$= \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + c$$
$$= \frac{1}{2}\sin x - \frac{1}{6}(\sin 2x \cos x + \cos 2x \sin x) + c$$
$$= \frac{1}{2}\sin x - \frac{1}{6}\left(2\sin x \cos^2 x + \sin x - 2\sin^3 x\right) + c$$
$$= \frac{1}{2}\sin x - \frac{1}{3}\sin x\left(1 - \sin^2 x\right) - \frac{1}{6}\sin x + \frac{1}{3}\sin^3 x + c$$
$$= \frac{1}{2}\sin x - \frac{1}{3}\sin x + \frac{1}{3}\sin^3 x - \frac{1}{6}\sin x + \frac{1}{3}\sin^3 x + c$$
$$= \frac{2\sin^3 x}{3} + c$$

**Question 10c**

The double angle results are much quicker and less complex, so are more efficient.

**Question 11**

$$\text{LHS} = \frac{1}{2}\Big[\cos(6x - 3x) + \cos(6x + 3x)\Big] - \frac{1}{2}\Big[\cos(7x - 4x) - \cos(7x + 4x)\Big]$$
$$= \frac{1}{2}\Big[\cos 3x + \cos 9x - \cos 3x + \cos 11x\Big]$$
$$= \frac{1}{2}\Big[\cos 9x + \cos 11x\Big]$$
$$= \frac{1}{2}\Big[\cos(10x - x) + \cos(10x + x)\Big]$$
$$= \frac{1}{2}\Big[2\cos 10x \cos x\Big]$$
$$= \cos 10x \cos x$$
$$\therefore A = 10 \text{ and } B = 1 \text{ (or vice versa)}$$

**Question 12**

$$\cos x \sin y = \frac{1}{2}\Big[\sin(x + y) - \sin(x - y)\Big]$$
$$= \frac{1}{2}(N - M)$$
$$= \frac{N - M}{2}$$

**Question 13**

$$\frac{\cos(A - B) + \cos(A + B)}{\cos(A - B) - \cos(A + B)}$$
$$= \frac{2\cos A \cos B}{2\sin A \sin B}$$
$$= \cot A \cot B$$

**Question 14**

$$\sin 3x + \sin x$$
$$= 2\sin\left(\frac{3x + x}{2}\right)\cos\left(\frac{3x - x}{2}\right)$$
$$= 2\sin 2x \cos x$$
$$= 2(2\sin x \cos x)\cos x$$
$$= 4\sin x \cos^2 x$$
$$= 4\sin x\left(1 - \sin^2 x\right)$$
$$= 4\sin x - 4\sin^3 x$$

**Question 15**

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)}{2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)}$$
$$= \frac{\sin\left(\frac{A + B}{2}\right)}{\cos\left(\frac{A + B}{2}\right)}$$
$$= \tan\left(\frac{A + B}{2}\right)$$

**Question 16**

$$\sin 3x + \cos 2x - \sin x = 0$$
$$\sin(2x + x) - \sin(2x - x) + \cos 2x = 0$$
$$2\cos 2x \sin x + \cos 2x = 0$$
$$\cos 2x\,(2\sin x + 1) = 0$$
$$\cos 2x = 0 \quad \sin x = -\frac{1}{2}$$
$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \qquad x = \frac{7\pi}{6}, \frac{11\pi}{6}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$$

**Question 17**

$$\int_0^{\pi} \sin(mx)\cos(nx)\,dx$$
$$= \frac{1}{2}\int_0^{\pi} (\sin(mx + nx) + \sin(mx - nx))\,dx$$
$$= \frac{1}{2}\int_0^{\pi} (\sin(m+n)x + \sin(m-n)x)\,dx$$
$$= \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right]_0^{\pi}$$
$$= \frac{1}{2}\left(\left(-\frac{\cos(m+n)\pi}{m+n} - \frac{\cos(m-n)\pi}{m-n}\right)\right.$$
$$\left.-\left(-\frac{\cos 0}{m+n} - \frac{\cos 0}{m-n}\right)\right)$$
$$= \frac{1}{2}\left(\left(-\frac{1}{m+n} - \frac{1}{m-n}\right)\right.$$
$$\left.-\left(-\frac{1}{m+n} - \frac{1}{m-n}\right)\right) \text{ since } \cos 2k\pi = 1$$
$$= 0$$

**Question 18a**

$$\sin 2\theta + \sin 4\theta - \sin 6\theta$$
$$= \sin(3\theta + \theta) + \sin(3\theta - \theta) - \sin 2(3\theta)$$
$$= 2\sin 3\theta \cos\theta - 2\sin 3\theta \cos 3\theta$$
$$= 2\sin 3\theta\,(\cos\theta - \cos 3\theta)$$
$$= 2\sin 3\theta\,(\cos(2\theta - \theta) - \cos(2\theta + \theta))$$
$$= 2\sin 3\theta\,(2\sin 2\theta \sin\theta)$$
$$= 4\sin\theta \sin 2\theta \sin 3\theta$$

**Question 18b**

$$\sin 6\theta = \sin 2\theta + \sin 4\theta$$
$$\sin 2\theta + \sin 4\theta - \sin 6\theta = 0$$
$$4\sin\theta \sin 2\theta \sin 3\theta = 0$$
$$\sin\theta = 0 \quad \sin 2\theta = 0 \quad \sin 3\theta = 0$$
$$\theta = 0, \pi \quad 2\theta = 0, \pi, 2\pi \quad 3\theta = 0, \pi, 2\pi, 3\pi$$
$$\theta = 0, \frac{\pi}{2}, \pi \qquad \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$$
$$\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$$

**Question 19**

$$\sin 2x + \sin 3x + \cos 2x + \cos 3x = 0$$
$$\sin\left(\frac{5x}{2} + \frac{x}{2}\right) + \sin\left(\frac{5x}{2} - \frac{x}{2}\right) + \cos\left(\frac{5x}{2} + \frac{x}{2}\right) + \cos\left(\frac{5x}{2} - \frac{x}{2}\right) = 0$$
$$2\sin\frac{5x}{2}\cos\frac{x}{2} + 2\cos\frac{5x}{2}\cos\frac{x}{2} = 0$$
$$2\cos\frac{x}{2}\left(\sin\frac{5x}{2} + \cos\frac{5x}{2}\right) = 0$$
$$\cos\frac{x}{2} = 0 \quad \sin\frac{5x}{2} + \cos\frac{5x}{2} = 0$$
$$\frac{x}{2} = \frac{\pi}{2} \quad \sin\frac{5x}{2} = -\cos\frac{5x}{2}$$
$$x = \pi \quad \tan\frac{5x}{2} = -1$$
$$\frac{5x}{2} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}$$
$$x = \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{11\pi}{10}, \frac{3\pi}{2}, \frac{19\pi}{10}$$
$$\therefore x = \frac{3\pi}{10}, \frac{7\pi}{10}, \pi, \frac{11\pi}{10}, \frac{3\pi}{2}, \frac{19\pi}{10}$$

**Question 20a**

$$\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x}$$
$$= \frac{\sin(3x + 2x) + \sin(3x - 2x) + \sin 3x}{\cos(3x + 2x) + \cos(3x - 2x) + \cos 3x}$$
$$= \frac{2\sin 3x \cos 2x + \sin 3x}{2\cos 3x \cos 2x + \cos 3x}$$
$$= \frac{\sin 3x\,(2\cos 2x + 1)}{\cos 3x\,(2\cos 2x + 1)}$$
$$= \frac{\sin 3x}{\cos 3x} \quad \text{since } \cos 2x \neq -\frac{1}{2}$$
$$= \tan 3x \quad \text{since } \cos 3x \neq 0$$

**Question 20b**

$$\sin x + \sin 3x + \sin 5x = \cos x + \cos 3x + \cos 5x$$
$$\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = 1$$
$$\tan 3x = 1$$
$$3x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$$
$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$$

**Question 21**

$$\cos\left(x + \frac{\pi}{6}\right)\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}\left[\sin\left(x + \frac{\pi}{6} + x - \frac{\pi}{6}\right) - \sin\left(x + \frac{\pi}{6} - x + \frac{\pi}{6}\right)\right]$$
$$= \frac{1}{2}\left[\sin 2x - \sin\frac{\pi}{3}\right]$$
$$= \frac{1}{2}\left[\sin 2x - \frac{\sqrt{3}}{2}\right]$$
$$\therefore \frac{1}{2}\left[\sin 2x - \frac{\sqrt{3}}{2}\right] = -\frac{\sqrt{3}}{4}$$
$$\sin 2x - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$
$$\sin 2x = 0$$
$$2x = 0, \pi, 2\pi, 3\pi, 4\pi$$
$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

**Question 22**

a $$\text{LHS} = \frac{1}{2}\left[\cos\left[(a + b) - (a - b)\right] - \cos\left[(a + b) + (a - b)\right]\right]$$
$$= \frac{1}{2}\left[\cos 2b - \cos 2a\right]$$
$$= \frac{1}{2}\left[\left(1 - 2\sin^2 b\right) - \left(1 - 2\sin^2 a\right)\right]$$
$$= \frac{1}{2}\left[2\sin^2 a - 2\sin^2 b\right]$$
$$= \sin^2 a - \sin^2 b$$

b $$\sin(4x + 2x)\sin(4x - 2x) = \sin 6x \quad \text{from (a)}$$
$$\sin 6x \sin 2x = \sin 6x$$
$$\sin 6x\,(\sin 2x - 1) = 0$$

$\sin 6x = 0$ or $\sin 2x - 1 = 0$

$6x = 0, \pi, 2\pi, 3\pi$ or $\sin 2x = 1$

$x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ or $2x = \frac{\pi}{2}$

$x = \frac{\pi}{4}$

$$\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$$

**Question 23**

$$4\cos 2x \sin x + \cos 3x - 2\sin x + 1 = 2(\sin 3x - \sin x) + \cos 3x - 2\sin x + 1$$
$$= 2\sin 3x + \cos 3x + 1$$
$$R = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 0.464$$
$$\therefore 4\cos 2x \sin x + \cos 3x - 2\sin x + 1 = \sqrt{5}\sin\left(3x + 0.464\right) + 1$$

**Question 24**

a $\text{LHS} = \frac{1}{2}(\sin A + \sin B) + \frac{1}{2}(\sin A - \sin B)$

$= \frac{1}{2}(\sin A + \sin B + \sin A - \sin B)$

$= \frac{1}{2} \times 2 \sin A$

$= \sin A$

b $\text{LHS} = \sin\left(\frac{2x + \frac{\pi}{3}}{2}\right)\cos\left(\frac{2x - \frac{\pi}{3}}{2}\right) + \cos\left(\frac{2x + \frac{\pi}{3}}{2}\right)\sin\left(\frac{2x - \frac{\pi}{3}}{2}\right)$

$= \sin 2x$

$\therefore \sin 2x = -\frac{\sqrt{3}}{2}$

$2x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

$x = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$

**Question 25**

$\int x \cos 4x^2 \cos 2x^2 \, dx$

$= \int x \times \frac{1}{2}(\cos 2x^2 + \cos 6x^2) \, dx$

$= \frac{1}{2}\int x \cos 2x^2 \, dx + \frac{1}{2}\int x \cos 6x^2 \, dx$

$= \frac{1}{8}\int 4x \cos 2x^2 \, dx + \frac{1}{24}\int 12x \cos 6x^2 \, dx$

$= \frac{\sin 2x^2}{8} + \frac{\sin 6x^2}{24} + c$

**Question 1a**

$$\int \sin x \cos^3 x \ dx$$

$$= -\int (-\sin x)(\cos x)^3 \, dx$$

$$= -\frac{\cos^4 x}{4} + c$$

**Question 1b**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 x \, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \cos^2 x \, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, (1 - \sin^2 x) \, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\cos x - \cos x \, (\sin x)^2\right) dx$$

$$= \left[\sin x - \frac{\sin^3 x}{3}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(1 - \frac{1}{3}\right) - \left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right)$$

$$= \frac{2}{3} - \frac{5}{6\sqrt{2}}$$

$$= \frac{8 - 5\sqrt{2}}{12}$$

**Question 2a**

$$\int \sec^4 x \tan^2 x \, dx$$

$$= \int \sec^2 x \sec^2 x \tan^2 x \, dx$$

$$= \int \sec^2 x \, (1 + \tan^2 x) \tan^2 x \, dx$$

$$= \int \sec^2 x \, (\tan^2 x + \tan^4 x) \, dx$$

$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + c$$

**Question 2b**

$$\int \sec^4 x \, dx$$

$$= \int \sec^2 x \sec^2 x \, dx$$

$$= \int \sec^2 x \, (\tan^2 x + 1) \, dx$$

$$= \int \sec^2 x \, (\tan x)^2 dx + \int \sec^2 x \, dx$$

$$= \frac{\tan^3 x}{3} + \tan x + c$$

**Question 3**

$$\int \operatorname{cosec} x \, dx$$

$$= \int \operatorname{cosec} x \times \frac{\cot x + \operatorname{cosec} x}{\operatorname{cosec} x + \cot x} dx$$

$$= \int \frac{\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x} dx$$

$$= -\int \frac{-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x} dx$$

$$= -\ln|\cot x + \operatorname{cosec} x| + c$$

**Question 4**

$$\int \tan x \sec^3 x \, dx$$

$$= \int \tan x \sec x \, (\sec x)^2 \, dx$$

$$= \frac{\sec^3 x}{3} + c$$

**Question 5**

$$\int \cos^4 x \, dx$$

$$= \int \left(\cos^2 x\right)^2$$

$$= \int \left(\frac{1}{2}(1 + \cos 2x)\right)^2 dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx$$

$$= \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}\left(\frac{1}{4}\sin 4x + x\right)\right) + c$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + \frac{x}{2} + c$$

$$= \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + \frac{3x}{8} + c$$

**Question 6**

$$\int \tan^3 x \, dx$$

$$= \int \tan^2 x \tan x \, dx$$

$$= \int \left(\sec^2 x - 1\right) \tan x \, dx$$

$$= \int (\sec^2 x \tan x - \tan x) \, dx$$

$$= \frac{\tan^2 x}{2} + \ln|\cos x| + c$$

**Question 7**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot^3 x \operatorname{cosec}^2 x \, dx$$

$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (-\operatorname{cosec}^2 x)(\cot x)^3 \, dx$$

$$= -\left[\frac{\cot^4 x}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= -\frac{1}{4}\left(\left(\frac{1}{\sqrt{3}}\right)^4 - \left(\sqrt{3}\right)^4\right)$$

$$= -\frac{1}{4}\left(\frac{1}{9} - 9\right)$$

$$= \frac{20}{9}$$

**Question 8**

$$\int \operatorname{cosec}^4 x \, dx$$

$$= \int \operatorname{cosec}^2 x \times \operatorname{cosec}^2 x \, dx$$

$$= \int \operatorname{cosec}^2 x \, (1 + \cot^2 x) \, dx$$

$$= \int (\operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cot^2 x) \, dx$$

$$= -\cot x - \frac{\cot^3 x}{3} + c$$

**Question 9**

$$\int \sqrt{\cos x} \sin^3 x \, dx$$

$$= \int \sin x \, (\cos x)^{\frac{1}{2}} \left(1 - \cos^2 x\right) dx$$

$$= -\int (-\sin x)\left((\cos x)^{\frac{1}{2}} - (\cos x)^{\frac{5}{2}}\right) dx$$

$$= -\frac{2}{3}(\cos x)^{\frac{3}{2}} + \frac{2}{7}(\cos x)^{\frac{7}{2}} + c$$

$$= \frac{2\sqrt{\cos^7 x}}{7} - \frac{2\sqrt{\cos^3 x}}{3} + c$$

**Question 10**

$$\int \sin^2 x \cos^4 x \; dx$$

$$= \int \frac{1}{2}(1 - \cos 2x)\left(\frac{1}{2}(1 + \cos 2x)\right)^2 \; dx$$

$$= \frac{1}{8}\int (1 - \cos 2x)\left((1 + \cos 2x)\right)^2 \, dx$$

$$= \frac{1}{8}\int (1 - \cos 2x)\left(1 + 2\cos 2x + \cos^2 2x\right) \, dx$$

$$= \frac{1}{8}\int (1 + 2\cos 2x + \cos^2 2x - \cos 2x - 2\cos^2 2x - \cos^3 2x) dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx$$

$$= \frac{1}{8}\int \left(1 + \cos 2x - \frac{1}{2}(1 + \cos 4x) - \cos 2x \left(1 - \sin^2 2x\right)\right) dx$$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x + \cos 2x \sin^2 2x\right) dx$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + c$$

**Question 11**

$$\int \frac{\cos 2x}{\cos x} dx$$

$$= \int \frac{2\cos^2 x - 1}{\cos x} dx$$

$$= \int (2\cos x - \sec x) \, dx$$

$$= 2\sin x - \ln|\tan x + \sec x| + c$$

**Question 12**

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan x}{\cos^2 x} dx$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$$

$$= \left[\frac{\tan^2 x}{2}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(1^2 - (-1)^2\right)$$

$$= 0$$

Alternative solution:

$\tan x$ is odd and $\cos^2 x$ is even, so $\frac{\tan x}{\cos^2 x}$ is odd

$$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan x}{\cos^2 x} dx = 0$$

**Question 13**

$$\int (\cos^4 x - \sin^4 x) \, dx$$

$$= \int (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) \, dx$$

$$= \int (\cos^2 x - \sin^2 x) \, dx$$

$$= \int \cos 2x \, dx$$

$$= \frac{1}{2}\sin 2x + c$$

**Question 14**

$$\int (\tan^4 x - \sec^4 x) \, dx$$

$$= \int (\tan^2 x - \sec^2 x)(\tan^2 x + \sec^2 x) \, dx$$

$$= \int (-1)\left(2\sec^2 x - 1\right) dx$$

$$= -2\tan x + x + c$$

**Question 15**

$$\int \sec^6 x\, dx$$

$$= \int \sec^2 x \sec^4 x\, dx$$

$$= \int \sec^2 x \left(\tan^2 x + 1\right)^2 dx$$

$$= \int \sec^2 x \left(\tan^4 x + 2\tan^2 x + 1\right) dx$$

$$= \frac{\tan^5 x}{5} + \frac{2\tan^3 x}{3} + \tan x + c$$

**Question 16**

$$V = \pi \int_0^{\frac{\pi}{4}} \left(\sec^2 x \tan^3 x\right)^2 dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^4 x \tan^6 x\, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \sec^2 x \tan^6 x\, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \left(\tan^2 x + 1\right) \tan^6 x\, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \left(\tan^8 x + \tan^6 x\right) dx$$

$$= \pi \left[\frac{\tan^9 x}{9} + \frac{\tan^7 x}{7}\right]_0^{\frac{\pi}{4}}$$

$$= \pi \left(\left(\frac{1}{9} + \frac{1}{7}\right) - (0 + 0)\right)$$

$$= \frac{16\pi}{63} \text{ units}^3$$

**Question 17**

$$\int \frac{(\cos x + \sin x)^3}{\sin 2x + 1} dx$$

$$= \int \frac{(\cos x + \sin x)^3}{2\sin x \cos x + \cos^2 x + \sin^2 x} dx$$

$$= \int \frac{(\cos x + \sin x)^3}{(\cos x + \sin x)^2} dx$$

$$= \int (\cos x + \sin x)\, dx$$

$$= \sin x - \cos x + c$$

**Question 18**

$$\int \frac{\sqrt{1 + \sin x}}{\sec x} dx$$

$$= \int \cos x\, (1 + \sin x)^{\frac{1}{2}} dx$$

$$= \frac{2}{3}(1 + \sin x)^{\frac{3}{2}} + c$$

$$= \frac{2}{3}\sqrt{(1 + \sin x)^3} + c$$

**Question 19**

$$\int \cot^2 x\, dx$$

$$= \int (\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\cot x - x + c$$

**Question 20**

$$\int \frac{\sin^5 x}{\cos^4 x} dx$$

$$= \int \sin x \frac{\sin^4 x}{\cos^4 x} dx$$

$$= \int \sin x \times \frac{\left(1 - \cos^2 x\right)^2}{\cos^4 x} dx$$

$$= \int \sin x \times \frac{1 - 2\cos x + \cos^2 x}{\cos^4 x} dx$$

$$= \int \sin x \times \big((\cos x)^{-4} - 2(\cos x)^{-3}$$

$$+ (\cos x)^{-2}\big)\, dx$$

$$= \frac{(\cos x)^{-3}}{-3} - \frac{2(\cos x)^{-2}}{-2} + \frac{(\cos x)^{-1}}{-1} + c$$

$$= -\frac{\sec^3 x}{3} + \sec^2 x + \sec x + c$$

**Question 21**

$$\int \cot^3 x\, dx$$

$$= \int \cot^2 x \cot x\, dx$$

$$= \int \left(\operatorname{cosec}^2 x - 1\right) \cot x\, dx$$

$$= \int \left(\cot x \operatorname{cosec}^2 x - \cot x\right) dx$$

$$= \int \left(-(-\cot x) \operatorname{cosec}^2 x - \frac{\cos x}{\sin x}\right) dx$$

$$= -\frac{\operatorname{cosec}^3 x}{3} - \ln|\sin x| + c$$

**Question 22**

$$\int \cot^3 x \operatorname{cosec}^3 x\, dx$$

$$= \int \cot x \operatorname{cosec} x \cot^2 x \operatorname{cosec}^2 x\, dx$$

$$= \int \cot x \operatorname{cosec} x \left(\operatorname{cosec}^2 x - 1\right) \operatorname{cosec}^2 x\, dx$$

$$= -\int -\cot x \operatorname{cosec} x \left(\operatorname{cosec}^4 x - \operatorname{cosec}^2 x\right) dx$$

$$= -\frac{\operatorname{cosec}^5 x}{5} + \frac{\operatorname{cosec}^3 x}{3} + c$$

**Question 23**

$$\int \sqrt{1 + \sin x}\, dx$$

$$= \int \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + \sin 2\left(\frac{x}{2}\right)} dx$$

$$= \int \sqrt{\cos^2 \frac{x}{2} + 2\cos \frac{x}{2} \sin \frac{x}{2} + \sin^2 \frac{x}{2}} dx$$

$$= \int \sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2} dx$$

$$= \int \left(\cos \frac{x}{2} + \sin \frac{x}{2}\right) dx$$

$$= 2\sin \frac{x}{2} - 2\cos \frac{x}{2} + c$$

**Question 1**

$$\begin{aligned}\cos A &= \cos\left[2\left(\frac{A}{2}\right)\right] \\ &= \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \\ &\qquad \text{using the double angle result} \\ &= \frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2}} \\ &\qquad \text{since } \cos^2\theta + \sin^2\theta = 1 \\ &= \frac{1 - \tan^2\frac{A}{2}}{1 + \tan^2\frac{A}{2}} \\ &\qquad \text{dividing the numerator and} \\ &\qquad \text{denominator by } \cos^2\frac{A}{2} \\ &= \frac{1 - t^2}{1 + t^2}\end{aligned}$$

**Question 2a**

$$\begin{aligned}&\frac{1 + \cos A}{\sin A} \\ &= \frac{1 + \frac{1 - t^2}{1 + t^2}}{\frac{2t}{1 + t^2}} \\ &= \frac{1 + t^2 + 1 - t^2}{2t} \\ &= \frac{2}{2t} \\ &= \frac{1}{t} \\ &= \cot\frac{A}{2}\end{aligned}$$

**Question 2b**

$$\begin{aligned}\cot 15° &= \cot\left(\frac{30°}{2}\right) \\ &= \frac{1 + \cos 30°}{\sin 30°} \qquad \text{from (a)} \\ &= \frac{1 + \frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= 2(1) + 2\left(\frac{\sqrt{3}}{2}\right) \\ &= 2 + \sqrt{3}\end{aligned}$$

**Question 3**

$$\begin{aligned}2\tan\frac{x}{2} &= \sqrt{3}\left(1 - \tan^2\frac{x}{2}\right) \\ \frac{2\tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}} &= \sqrt{3} \\ \tan x &= \sqrt{3} \\ x &= \frac{\pi}{3}, \frac{4\pi}{3}\end{aligned}$$

**Question 4a**

$$\begin{aligned}&\tan\theta\sin\theta \\ &= \frac{2t}{1 - t^2} \times \frac{2t}{1 + t^2} \\ &= \frac{4t^2}{1 - t^4}\end{aligned}$$

**Question 4b**

$$\begin{aligned}&\cos 2\theta \\ &= 2\cos^2\theta - 1 \\ &= \frac{2(1 - t^2)^2}{(1 + t^2)^2} - 1 \\ &= \frac{2(1 - 2t^2 + t^4) - (1 + 2t^2 + t^4)}{(1 + t^2)^2} \\ &= \frac{1 - 6t^2 + t^4}{(1 + t^2)^2}\end{aligned}$$

**Question 5a**

$$\begin{aligned}&\frac{2}{1 - \cos x} \\ &= \frac{2}{1 - \frac{1 - t^2}{1 + t^2}} \\ &= \frac{2 + 2t^2}{1 + t^2 - 1 + t^2} \\ &= \frac{2(1 + t^2)}{2t^2} \\ &= \frac{1 + t^2}{t^2} \\ &= \frac{1 + \tan^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)} \\ &= \frac{\sec^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)} \\ &= \frac{1}{\cos^2\left(\frac{x}{2}\right)} \times \frac{\cos^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)} \\ &= \frac{1}{\sin^2\left(\frac{x}{2}\right)} \\ &= \operatorname{cosec}^2\left(\frac{x}{2}\right)\end{aligned}$$

**Question 5b**

$$\begin{aligned}&\frac{1 + \cos x + \sin x}{1 - \cos x + \sin x} \\ &= \frac{1 + \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2}}{1 - \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2}} \\ &= \frac{1 + t^2 + 1 - t^2 + 2t}{1 + t^2 - 1 + t^2 + 2t} \\ &= \frac{2 + 2t}{2t + 2t^2} \\ &= \frac{2(1 + t)}{2t(1 + t)} \\ &= \frac{1}{t} \\ &= \frac{1}{\tan\left(\frac{x}{2}\right)} \\ &= \cot\left(\frac{x}{2}\right)\end{aligned}$$

**Question 8a**

$\sin x - 2\cos x = R(\sin x \cos\alpha - \cos x \sin\alpha)$

$R\cos\alpha = 1 \quad (1) \quad (\cos\alpha > 0)$

$-R\sin\alpha = -2 \quad (2) \quad (\sin\alpha > 0)$

$\therefore \alpha$ is in the 1st quadrant

$(1)^2 + (2)^2$: $R^2(\cos^2\alpha + \sin^2\alpha) = 1^2 + 2^2$

$R^2 = 5$

$R = \sqrt{5}$

$(2) \div (1)$: $\dfrac{-R\sin\alpha}{R\cos\alpha} = \dfrac{-2}{1}$

$\tan\alpha = 2$

$\alpha = 1.107149$

$\therefore \sin x - 2\cos x = \sqrt{5}\sin(x - 1.107149)$

$\therefore \sqrt{5}\sin(x - 1.107149) = 2$

$\sin(x - 1.107149) = \dfrac{2}{\sqrt{5}}$

$x - 1.107149 = 1.107149, \pi - 1.107149$

$x = 2.214297, \pi$

$= 2.21$ (2 dp), $\pi$

$\sin\dfrac{\pi}{4} = \dfrac{2t}{1+t^2}$

$\dfrac{1}{\sqrt{2}} = \dfrac{2t}{1+t^2}$

$1 + t^2 = 2\sqrt{2}t$

$t^2 - 2\sqrt{2}t + 1 = 0$

$t = \dfrac{2\sqrt{2} - \sqrt{(-2\sqrt{2})^2 - 4(1)(1)}}{2\times 1}$

only use the negative since $\sin\dfrac{3\pi}{4} = \dfrac{1}{\sqrt{2}}$

giving $t = \tan\dfrac{3\pi}{8}$ which is larger

$= \dfrac{2\sqrt{2} - \sqrt{4}}{2}$

$= \dfrac{2\sqrt{2} - 2}{2}$

$= \sqrt{2} - 1$

**Question 7**

let $t = \tan\theta$

$\dfrac{1+\cos 2\theta}{1-\cos 2\theta} = \dfrac{\left(1 + \dfrac{1-t^2}{1+t^2}\right)}{\left(1 - \dfrac{1-t^2}{1+t^2}\right)}$

$= \dfrac{1 + t^2 + 1 - t^2}{1 + t^2 - 1 + t^2}$

$= \dfrac{2}{2t^2}$

$= \dfrac{1}{t^2}$

$= \cot^2\theta$

**Question 8a**

$\sin x - 2\cos x = R(\sin x \cos\alpha - \cos x \sin\alpha)$

$R\cos\alpha = 1 \quad (1) \quad (\cos\alpha > 0)$

$-R\sin\alpha = -2 \quad (2) \quad (\sin\alpha > 0)$

$\therefore \alpha$ is in the 1st quadrant

$(1)^2 + (2)^2$: $R^2(\cos^2\alpha + \sin^2\alpha) = 1^2 + 2^2$

$R^2 = 5$

$R = \sqrt{5}$

$(2) \div (1)$: $\dfrac{-R\sin\alpha}{R\cos\alpha} = \dfrac{-2}{1}$

$\tan\alpha = 2$

$\alpha = 1.107149$

$\therefore \sin x - 2\cos x = \sqrt{5}\sin(x - 1.107149)$

$\therefore \sqrt{5}\sin(x - 1.107149) = 2$

$\sin(x - 1.107149) = \dfrac{2}{\sqrt{5}}$

$x - 1.107149 = 1.107149, \pi - 1.107149$

$x = 2.214297, \pi$

$= 2.21$ (2 dp), $\pi$

**Question 8b**

$\dfrac{2t}{1+t^2} - 2\dfrac{1-t^2}{1+t^2} = 2$

$2t - 2 + 2t^2 = 2 + 2t^2$

$2t - 2 = 2$

$t = 2$

$\tan\dfrac{A}{2} = 2$

$\dfrac{A}{2} = 1.107149$

$A = 2.214297$

Check $x = \pi$:

$\sin\pi - 2\cos\pi = 0 - 2(-1) = 2$ ✓

$\therefore x = 2.214297, \pi$

**Question 9**

$\cos x - 2\sin x = 1$

$\dfrac{1-t^2}{1+t^2} - 2\left(\dfrac{2t}{1+t^2}\right) = 1$

$1 - t^2 - 4t = 1 + t^2$

$2t^2 + 4t = 0$

$2t(t+2) = 0$

$t = 0 \quad t + 2 = 0$

$\tan\dfrac{x}{2} = 0 \quad \tan\dfrac{x}{2} = -2$

$\dfrac{x}{2} = 0 \quad \dfrac{x}{2} = -\tan^{-1}(2)$

$x = 0 \quad x = -2\tan^{-1}(2)$

$= -2.214297$

Check $\theta = -\pi$:

$\cos(-\pi) - 2\sin(-\pi) = -1 - 2(0) = -1 \neq 1$ ✗

Check $\theta = \pi$:

$\cos(\pi) - 2\sin(\pi) = -1 - 2(0) = -1 \neq 1$ ✗

$\therefore x = 0, -2.21$ (2 dp)

**Question 10**

$$\sqrt{\left(\sqrt{3}\right)^2+1^2}=2$$

$$\sqrt{3}\sin x-\cos x$$
$$=2\left(\sin x\times\frac{\sqrt{3}}{2}-\cos x\times\frac{1}{2}\right)$$
$$=2\left(\sin x\cos\frac{\pi}{6}-\cos x\sin\frac{\pi}{6}\right)$$
$$=2\sin\left(x-\frac{\pi}{6}\right)$$
$$\therefore 2\sin\left(x-\frac{\pi}{6}\right)=1$$
$$\sin\left(x-\frac{\pi}{6}\right)=\frac{1}{2}$$
$$x-\frac{\pi}{6}=\frac{\pi}{6},\frac{5\pi}{6}$$
$$x=\frac{\pi}{3},\pi$$

The only solutions in the domain $[0,2\pi]$ are $x=\frac{\pi}{3}$ and $x=\pi$.

**Alternative solution using *t* formulas:**

$$\sqrt{3}\sin x-\cos x=1$$
$$\sqrt{3}\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}=1$$
$$2\sqrt{3}t-1+t^2=1+t^2$$
$$2\sqrt{3}t=2$$
$$t=\frac{1}{\sqrt{3}}$$
$$\tan\frac{x}{2}=\frac{1}{\sqrt{3}}$$
$$\frac{x}{2}=\frac{\pi}{6}$$
$$x=\frac{\pi}{3}$$

Check $x=\pi$:

$\sqrt{3}\sin\pi-\cos\pi=0-(-1)=1$ ✓

The only solutions in the domain $[0,2\pi]$ are $x=\frac{\pi}{3}$ and $x=\pi$.

**Question 11**

$$\sqrt{1^2+1^2}=\sqrt{2}$$

$$\sin x+\cos x$$
$$=\sqrt{2}\left(\sin x\times\frac{1}{\sqrt{2}}+\cos x\times\frac{1}{\sqrt{2}}\right)$$
$$=\sqrt{2}\left(\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}\right)$$
$$=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$$
$$\therefore\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{2}$$
$$\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$$
$$x+\frac{\pi}{4}=\pi-\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right),2\pi$$
$$x+0.785398=2.780226,6.644552$$
$$x=1.994827\ldots,5.859154$$
$$x=2.00,5.86\ (2\text{ dp})$$

The only solutions in the domain $[0,2\pi]$ are $x=2.00$ and $x=5.86$, correct to 2 decimal places.

**Question 11 continued**

**Alternative solution using *t* formulas:**

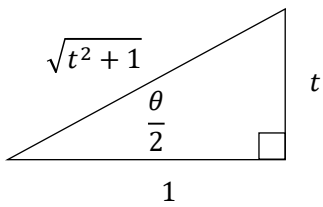
$$\sin x+\cos x=\frac{1}{2}$$
$$\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}=\frac{1}{2}$$
$$4t+2-2t^2=1+t^2$$
$$3t^2-4t-1=0$$
$$t=\frac{4\pm\sqrt{(-4)^2-4(3)(-1)}}{2\times3}$$
$$\tan\frac{x}{2}=\frac{4\pm\sqrt{28}}{6}$$
$$\frac{x}{2}=\tan^{-1}\left(\frac{4+\sqrt{28}}{6}\right),\pi$$
$$x=2\tan^{-1}\left(\frac{4+\sqrt{28}}{6}\right),2\pi$$
$$=1.994827\ldots,5.859154\ldots$$

Check $x=\pi$:

$\sin\pi+\cos\pi=0+(-1)=-1\neq0.5$ ✗

The only solutions in the domain $[0,2\pi]$ are $x=2.00$ and $x=5.86$, correct to 2 decimal places.

**Question 12**




$$\frac{2t}{1+t^2}-\frac{t}{\sqrt{1+t^2}}=0$$
$$\frac{2t}{1+t^2}=\frac{t}{\sqrt{1+t^2}}$$
$$2t=t\sqrt{1+t^2}$$
$$4t^2=t^2+t^4$$
$$t^4-3t^2=0$$
$$t^2(t^2-3)=0$$
$$t=0\quad t=\pm\sqrt{3}$$
$$\tan\frac{\theta}{2}=0\qquad\tan\frac{\theta}{2}=\pm\sqrt{3}$$
$$\frac{\theta}{2}=0\qquad\frac{\theta}{2}=\pm\frac{\pi}{3}$$
$$\theta=0\qquad\theta=\pm\frac{2\pi}{3}$$

Check $\theta=-\pi$:

$\sin(-\pi)-\sin\left(-\frac{\pi}{2}\right)=0-(-1)=1\neq0$ ✗

Check $\theta=\pi$:

$\sin(\pi)-\sin\left(\frac{\pi}{2}\right)=0-(1)=-1\neq0$ ✗

The solutions are $\theta=-\frac{2\pi}{3},0,\frac{2\pi}{3}$.

**Question 13**

$$\sec x + \tan x$$
$$= \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}$$
$$= \frac{(t^2+2t+1)}{1-t^2}$$
$$= \frac{(t+1)^2}{(1+t)(1-t)}$$
$$= \frac{1+t}{1-t} \quad \text{since } t \neq -1$$
$$= \frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}$$
$$= \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$$
$$\therefore a = \frac{1}{2}, b = \frac{\pi}{4}.$$

**Question 14**

$$3\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) = 3$$
$$3\left(\frac{1-t^2}{1+t^2}\right) - \frac{2t}{1+t^2} = 3 \quad \text{where } t = \tan\left(\frac{x}{4}\right)$$
$$3 - 3t^2 - 2t = 3 + 3t^2$$
$$6t^2 + 2t = 0$$
$$2t(3t+1) = 0$$
$$t = 0 \quad \text{or} \quad 3t + 1 = 0$$
$$\tan\left(\frac{x}{4}\right) = 0 \qquad \tan\left(\frac{x}{4}\right) = -\frac{1}{3}$$
$$\frac{x}{4} = 0 \qquad \text{no solution for } 0 \leq \frac{x}{4} \leq \frac{\pi}{2}$$
$$x = 0$$

Check $\theta = \pi$:

$$3\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = 3(0) - (1) = -1 \neq 3 \text{ ✗}$$

$\therefore x = 0$ is the only solution.

**Question 15a**

$$m\cos x - \sin x = 1$$
$$m\left(\frac{1-t^2}{1+t^2}\right) - \frac{2t}{1+t^2} = 1$$
$$m - mt^2 - 2t = 1 + t^2$$
$$m - mt^2 = t^2 + 2t + 1$$
$$m(1-t^2) = (t+1)^2$$
$$m(1-t)(1+t) = (t+1)^2$$
$$m(1-t) = t + 1 \qquad \text{since } t + 1 \neq 0 \text{ for } 0$$
$$m - mt = 1 + t$$
$$t + mt = m - 1$$
$$t(m+1) = m - 1$$
$$t = \frac{m-1}{m+1}$$

**Question 15b**

$$\tan\frac{x}{2} = \frac{\tan(\tan^{-1} m) - \tan\frac{\pi}{4}}{1 + \tan(\tan^{-1} m) \times \tan\frac{\pi}{4}}$$
$$= \tan\left(\tan^{-1} m - \frac{\pi}{4}\right)$$
$$\frac{x}{2} = \tan^{-1} m - \frac{\pi}{4}$$
$$x = 2\tan^{-1} m - \frac{\pi}{2}$$

**Question 16**

$$2\sin x - \cos x = 0$$
$$2\left(\frac{2t}{1+t^2}\right) - \frac{1-t^2}{1+t^2} = 0$$
$$4t - 1 + t^2 = 0$$
$$t^2 + 4t - 1 = 0$$
$$\therefore \tan\frac{\alpha}{2} \times \tan\frac{\beta}{2} = \frac{c}{a}$$
$$= \frac{-1}{1}$$
$$\therefore \tan\frac{\alpha}{2} = -\frac{1}{\tan\frac{\beta}{2}}$$

**Question 17a**

$$\sqrt{\frac{1-\cos x}{1+\sin x}}$$
$$= \sqrt{\frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2}}}$$
$$= \sqrt{\frac{1+t^2-1+t^2}{1+t^2+2t}}$$
$$= \sqrt{\frac{2t^2}{(t+1)^2}}$$
$$= \sqrt{\left(\frac{\sqrt{2}t}{t+1}\right)^2}$$
$$= \left|\frac{\sqrt{2}t}{t+1}\right|$$

**Question 17b**

$$\sqrt{\frac{1-\cos x}{1+\sin x}} = \frac{1}{\sqrt{2}}$$
$$\left|\frac{\sqrt{2}t}{t+1}\right| = \frac{1}{\sqrt{2}}$$
$$\left|\frac{t}{t+1}\right| = \frac{1}{2}$$
$$\frac{t}{t+1} = \pm\frac{1}{2}$$
$$\frac{t+1-1}{t+1} = \pm\frac{1}{2}$$
$$1 - \frac{1}{t+1} = \pm\frac{1}{2}$$
$$\frac{1}{t+1} = 1 \pm \frac{1}{2}$$
$$= \frac{1}{2}, \frac{3}{2}$$
$$t + 1 = \frac{2}{3}, 2$$
$$t = -\frac{1}{3}, 1$$
$$\tan\frac{x}{2} = -\frac{1}{3} \qquad \tan\frac{x}{2} = 1$$
$$\frac{x}{2} = \pi - \tan^{-1}\left(\frac{1}{3}\right) \qquad \frac{x}{2} = \frac{\pi}{4}$$
$$x = 2\pi - 2\tan^{-1}\left(\frac{1}{3}\right) \qquad x = \frac{\pi}{2}$$
$$= 5.639684$$
$$\therefore x = \frac{\pi}{2}, 5.64 \text{ (2 dp)}$$

**Question 1**

$$\int \frac{dx}{1+\sin x} \qquad \boxed{t=\tan\frac{x}{2},\quad dx=\frac{2dt}{1+t^2}}$$

$$=\int \frac{1}{1+\frac{2t}{1+t^2}}\times\frac{2dt}{1+t^2}$$
$$=2\int \frac{1}{1+t^2+2t}\,dt$$
$$=2\int \frac{1}{(t+1)^2}\,dt$$
$$=2\int (t+1)^{-2}\,dt$$
$$=-\frac{2}{t+1}+c$$
$$=-\frac{2}{\tan\frac{x}{2}+1}+c$$

**Question 2**

$$\int \frac{5}{12+13\cos x}\,dx \qquad \boxed{t=\tan\frac{x}{2},\quad dx=\frac{2dt}{1+t^2}}$$

$$=\int \frac{5}{12+13\left(\frac{1-t^2}{1+t^2}\right)}\times\frac{2dt}{1+t^2}$$
$$=10\int \frac{1}{12+12t^2+13-13t^2}\,dt$$
$$=10\int \frac{1}{25-t^2}\,dt$$
$$=\int\left(\frac{1}{5-t}+\frac{1}{5+t}\right)dt$$
$$=-\ln|5-t|+\ln|5+t|+c$$
$$=\ln\left|\frac{5+\tan\frac{x}{2}}{5-\tan\frac{x}{2}}\right|+c$$

**Question 3**

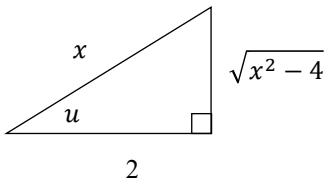
$$\int x^3\sqrt{1-x^2}\,dx \qquad \boxed{x=\sin\theta,\quad dx=\cos\theta\,d\theta}$$

$$=\int \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$$
$$=\int \sin^3\theta\sqrt{\cos^2\theta}\cos\theta\,d\theta$$
$$=\int \sin^3\theta\cos^2\theta\,d\theta$$
$$=\int \sin\theta\sin^2\theta\cos^2\theta\,d\theta$$
$$=\int \sin\theta\left(1-\cos^2\theta\right)\cos^2\theta\,d\theta$$
$$=-\int(-\sin\theta)(\cos^2\theta-\cos^4\theta)\,d\theta$$
$$=\frac{\cos^5\theta}{5}-\frac{\cos^3\theta}{3}+c \qquad \boxed{\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}}$$
$$=\frac{\sqrt{(1-x^2)^5}}{5}-\frac{\sqrt{(1-x^2)^3}}{3}+c$$

**Question 4**

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx \qquad \boxed{x=2\sec u,\quad dx=2\sec u\tan u\,du}$$

$$=\int \frac{1}{2\sec u\sqrt{4\sec^2 u-4}}\times 2\sec u\tan u\,du$$
$$=\int \frac{\tan u}{\sqrt{4\tan^2 u}}\,du$$
$$=\int \frac{\tan u}{2\tan u}\,du$$
$$=\frac{1}{2}\int du$$
$$=\frac{u}{2}+c$$
$$=\frac{1}{2}\tan^{-1}\frac{\sqrt{x^2-4}}{2}+c$$




Alternatively:

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx \qquad \boxed{u^2=x^2-4,\quad 2u\,du=2x\,dx,\quad dx=\frac{u\,du}{x}}$$

$$=\int \frac{1}{xu}\times\left(u\frac{du}{x}\right)$$
$$=\int \frac{1}{x^2}\,du$$
$$=\int \frac{1}{u^2+4}\,du$$
$$=\frac{1}{2}\tan^{-1}\frac{u}{2}+c$$
$$=\frac{1}{2}\tan^{-1}\frac{\sqrt{x^2-4}}{2}+c$$

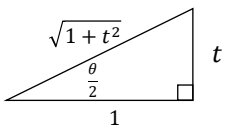
Hint: Integrands in the form $\frac{1}{x\sqrt{x^2-c}}$ can be solved using a $u^2$ substitution.

**Question 5a**

$$t=\tan\frac{\theta}{2}$$
$$\frac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{\theta}{2}$$
$$=\frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$$
$$=\frac{1}{2}\left(1+t^2\right)$$

**Question 5b**

$$\tan\frac{\theta}{2}=\frac{t}{1}$$
$$\cos\frac{\theta}{2}=\frac{1}{\sqrt{1+t^2}}$$
$$\sin\frac{\theta}{2}=\frac{t}{\sqrt{1+t^2}}$$
$$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$=2\times\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}}$$
$$=\frac{2t}{1+t^2}$$



**Question 5c**

$$\int \operatorname{cosec}\theta \; d\theta$$

$$= \int \frac{1+t^2}{2t} \times \frac{2dt}{1+t^2} \quad \text{from (a)}$$

$$= \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$= \ln\left|\tan\frac{\theta}{2}\right| + c$$

**Question 6**

$$\int \frac{1}{1-\cos x}\,dx$$

$$= \int \frac{1}{1-\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{1+t^2-1+t^2}\,dt$$

$$= 2\int \frac{dt}{2t^2}$$

$$= \int t^{-2}\,dt$$

$$= -\frac{1}{t} + c$$

$$= -\cot\frac{x}{2} + c$$

**Question 7**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$$

$$t = \tan\frac{\theta}{2}$$
$$dx = \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{1}{2-\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{1}{2+2t^2-1+t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{1+3t^2}\,dt$$

$$= \frac{2}{\sqrt{3}}\int_0^1 \frac{\sqrt{3}}{1+3t^2}\,dt$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}t\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}\right)$$

$$= \frac{2\sqrt{3}\pi}{9}$$

**Question 8**

$$\int \sqrt{x+4}\,dx$$

$$x = 4\tan^2 u$$
$$dx = 8\tan u \sec^2 u\,du$$

$$= \int \sqrt{4\tan^2 u + 4} \times 8\tan u \sec^2 u\,du$$

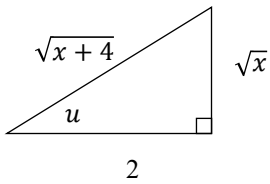
$$= 8\int \tan u \sec^2 u \sqrt{4\sec^2 u}\,du$$

$$= 16\int \tan u \sec^3 u\,du$$

$$= 16\int \tan u \sec u (\sec u)^2\,du$$

$$= \frac{16\sec^3 u}{3} + c$$

$$= \frac{2\sqrt{(x+4)^3}}{3} + c$$




**Question 9**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx$$

$$t = \tan\frac{x}{2}$$
$$dx = \frac{2dt}{1+t^2}$$

$$= \int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) + 5} \times \frac{2dt}{1+t^2}$$

$$= \int_{\frac{1}{\sqrt{3}}}^1 \frac{2}{6t - 4 + 4t^2 + 5 + 5t^2}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^1 \frac{2}{9t^2 + 6t + 1}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^1 \frac{2}{(3t+1)^2}\,dt$$

$$= 2\left[\frac{(3t+1)^{-1}}{-1\times 3}\right]_{\frac{1}{\sqrt{3}}}^1$$

$$= -\frac{2}{3}\left[\frac{1}{3t+1}\right]_{\frac{1}{\sqrt{3}}}^1$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{1}{\sqrt{3}+1}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{\sqrt{3}-1}{2}\right)$$

$$= -\frac{2}{3}\frac{(3-2\sqrt{3})}{4}$$

$$= \frac{2\sqrt{3}-3}{6}$$

**Question 10**

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx \qquad \boxed{\begin{array}{l} x = \sin^2\theta \\ dx = 2\sin\theta\cos\theta\,d\theta \end{array}}$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \times 2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\,d\theta$$

$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0-0)\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

**Question 11**

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta}$$

$$= \int_1^{\sqrt{3}} \frac{1+t^2}{2t} \times \frac{2dt}{1+t^2}$$

$$= \int_1^{\sqrt{3}} \frac{dt}{t}$$

$$= \left[\ln t\right]_1^{\sqrt{3}}$$

$$= \ln\sqrt{3} - \ln 1$$

$$= \ln 3^{\frac{1}{2}}$$

$$= \frac{1}{2}\ln 3$$

**Question 12**

$$\int \operatorname{cosec} x\,dx \qquad \boxed{\begin{array}{c} t = \tan\frac{x}{2} \\ dx = \frac{2dt}{1+t^2} \end{array}}$$

$$= \int \frac{1+t^2}{2t} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$= \ln\left|\tan\frac{x}{2}\right| + c$$

**Question 13**

$$\int \frac{1}{\sin x + \cos x}\,dx \qquad \boxed{\begin{array}{c} t = \tan\frac{x}{2} \\ dx = \frac{2dt}{1+t^2} \end{array}}$$

$$= \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int \frac{dt}{2t+1-t^2}$$

$$= 2\int \frac{dt}{2-(t-1)^2}$$

$$= 2\int \frac{dt}{(\sqrt{2}+t-1)(\sqrt{2}-t+1)}$$

$$= \frac{1}{\sqrt{2}}\int \left(\frac{1}{t-1+\sqrt{2}} + \frac{1}{1+\sqrt{2}-t}\right) dt$$

$$= \frac{1}{\sqrt{2}}\left(\ln\left|t-1+\sqrt{2}\right| - \ln\left|1+\sqrt{2}-t\right|\right) + c$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{t-1+\sqrt{2}}{1+\sqrt{2}-t}\right| + c$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{1+\sqrt{2}-\tan\frac{x}{2}}\right| + c$$

**Question 14a**

$$\operatorname{cosec} x + \cot x$$

$$= \frac{1+t^2}{2t} + \frac{1-t^2}{2t}$$

$$= \frac{2}{2t}$$

$$= \frac{1}{t}$$

$$= \cot\frac{x}{2}$$

**Question 14b**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{cosec} x + \cot x)\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\frac{x}{2}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\,dx$$

$$= 2\left[\ln\left(\sin\frac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= 2\left(\ln\left(\sin\frac{\pi}{4}\right) - \ln\left(\sin\frac{\pi}{6}\right)\right)$$

$$= 2\left(\ln\left(\frac{1}{\sqrt{2}}\right) - \ln\left(\frac{1}{2}\right)\right)$$

$$= 2\left(-\ln\left(\sqrt{2}\right) + \ln(2)\right)$$

$$= 2\left(-\frac{1}{2}\ln 2 + \ln(2)\right)$$

$$= 2\left(\frac{1}{2}\ln 2\right)$$

$$= \ln 2$$

**Question 15**

$$\int \frac{\sec^2 x}{1-\tan^2 x}\,dx$$
$$= \int \frac{1+\tan^2 x}{1-\tan^2 x}\,dx$$
$$= \int \sec 2x\,dx$$
$$= \frac{1}{2}\ln\left|\tan 2x + \sec 2x\right| + c$$

**Question 16**

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx \qquad \boxed{\begin{array}{c} x = 2\sin\theta \\ dx = 2\cos\theta\,d\theta \end{array}}$$
$$= \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} \times 2\cos\theta\,d\theta$$
$$= \int \frac{4\sin^2\theta}{2\cos\theta} \times 2\cos\theta\,d\theta$$
$$= 4\int \sin^2\theta\,d\theta$$
$$= 4\int \frac{1}{2}(1-\cos 2\theta)\,d\theta$$
$$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + c$$
$$= 2\theta - 2\sin\theta\cos\theta + c$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + c$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + c$$

**Question 17**

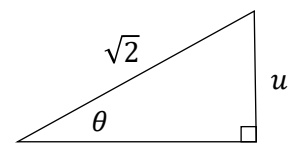
$$\int \frac{3+\cos x}{2-\cos x}\,dx \qquad \boxed{\begin{array}{c} t = \tan\dfrac{x}{2} \\ dx = \dfrac{2dt}{1+t^2} \end{array}}$$
$$= \int \frac{3+\frac{1-t^2}{1+t^2}}{2-\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int \frac{3+3t^2+1-t^2}{2+2t^2-1+t^2} \times \frac{dt}{1+t^2}$$
$$= 2\int \frac{2t^2+4}{(3t^2+1)(1+t^2)}\,dt$$
$$= 2\int \left(\frac{5}{3t^2+1} - \frac{1}{1+t^2}\right)dt$$
$$= \frac{10}{\sqrt{3}}\int \frac{\sqrt{3}}{3t^2+1}\,dt - 2\int \frac{1}{1+t^2}\,dt$$
$$= \frac{10}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}t\right) - 2\tan^{-1} t + c$$
$$= \frac{10}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{x}{2}\right) - x + c$$

**Question 18**

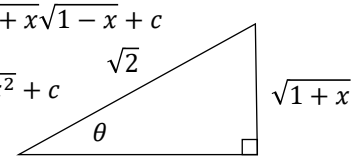
$$\int \sqrt{2x-x^2}\,dx$$
$$= \int \sqrt{-(x^2-2x+1-1)}\,dx$$
$$= \int \sqrt{1-(x-1)^2}\,dx \qquad \boxed{\begin{array}{c} x-1 = \sin\theta \\ dx = \cos\theta\,d\theta \end{array}}$$
$$= \int \sqrt{1-\sin^2\theta} \times \cos\theta\,d\theta$$
$$= \int \cos^2\theta\,d\theta$$
$$= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$
$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + c$$
$$= \frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4} + c$$
$$= \frac{\sin^{-1}(x-1)}{2} + \frac{(x-1)\sqrt{1-(x-1)^2}}{2} + c$$
$$= \frac{\sin^{-1}(x-1)}{2} + \frac{(x-1)\sqrt{2x-x^2}}{2} + c$$

**Question 19**

$$\int \sqrt{\frac{1-x}{1+x}}\,dx \qquad \boxed{\begin{array}{c} u^2 = 1+x \;\rightarrow x = u^2-1 \\ 2u\,du = dx \end{array}}$$
$$= \int \sqrt{\frac{1-(u^2-1)}{u^2}}\,2u\,du$$
$$= 2\int \sqrt{\frac{2-u^2}{u^2}}\,u\,du$$
$$= 2\int \sqrt{2-u^2}\,du \qquad \boxed{\begin{array}{c} u = \sqrt{2}\sin\theta \\ du = \sqrt{2}\cos\theta\,d\theta \end{array}}$$
$$= 2\int \sqrt{2-2\sin^2\theta} \times \sqrt{2}\cos\theta\,d\theta$$
$$= 2\sqrt{2}\int \sqrt{2\cos^2\theta} \times \cos\theta\,d\theta$$
$$= 4\int \cos^2\theta\,d\theta$$
$$= 2\int (1+\cos 2\theta)\,d\theta$$
$$= 2\theta + \sin 2\theta + c$$
$$= 2\theta + 2\sin\theta\cos\theta + c$$
$$= 2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + 2\times\frac{u}{\sqrt{2}}\times\frac{\sqrt{2-u^2}}{\sqrt{2}} + c$$
$$= 2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + u\sqrt{2-u^2} + c$$
$$= 2\sin^{-1}\left(\frac{\sqrt{1+x}}{\sqrt{2}}\right) + \sqrt{1+x}\sqrt{1-x} + c$$
$$= 2\sin^{-1}\left(\frac{\sqrt{1+x}}{\sqrt{2}}\right) + \sqrt{1+x}\sqrt{1-x} + c$$
$$= 2\sin^{-1}\left(\tfrac{\sqrt{1+x}}{\sqrt{2}}\right) + \sqrt{1-x^2} + c$$




$\sqrt{2-u^2}$




$\sqrt{2-(1+x)}$
$= \sqrt{1-x}$

**Question 20**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(1+\sin x)^3}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{\left(1+\sin\left(\frac{\pi}{2}-x\right)\right)^3}$$

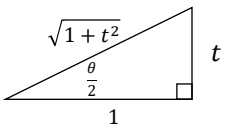
$$= \int_0^{\frac{\pi}{2}} \frac{dx}{(1+\cos x)^3}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{\left(2\cos^2\left(\frac{x}{2}\right)\right)^3}$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} \sec^6\left(\frac{x}{2}\right) dx \qquad \boxed{\begin{array}{c} t = \tan\frac{x}{2} \\ dx = \frac{2dt}{1+t^2} \end{array}}$$

$$= \frac{1}{8}\int_0^1 \left(\sqrt{1+t^2}\right)^6 \times \frac{2dt}{1+t^2}$$

$$= \frac{1}{4}\int_0^1 \left(1+t^2\right)^2 dt$$




$$= \frac{1}{4}\int_0^1 (1+2t^2+t^4)\, dt$$

$$= \frac{1}{4}\left[t + \frac{2t^3}{3} + \frac{t^5}{5}\right]_0^1$$

$$= \frac{1}{4}\left(\left(1+\frac{2}{3}+\frac{1}{5}\right) - (0+0+0)\right)$$

$$= \frac{7}{15}$$

**Question 21a**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\frac{1}{2}\sin x} \qquad \boxed{\begin{array}{c} t = \tan\frac{x}{2} \\ dx = \frac{2dt}{1+t^2} \end{array}}$$

$$= \int_0^1 \frac{1}{1+\frac{t}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{dt}{1+t^2+t}$$

$$= 2\int_0^1 \frac{dt}{t^2+t+\frac{1}{4}+\frac{3}{4}}$$

$$= 2\int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{4}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$$

$$= \frac{4}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{4}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$= \frac{2\pi}{3\sqrt{3}}$$

**Question 21b**

$$\int_0^a f(2a-x)\, dx \qquad \boxed{\begin{array}{l} u = 2a-x \\ du = -dx \\ dx = -du \end{array}}$$

$$= \int_{2a}^a f(u)\,(-du)$$

$$= \int_a^{2a} f(u)\, du$$

$$= \int_a^{2a} f(x)\, dx$$

$$\int_0^{2a} f(x)\, dx$$

$$= \int_0^a f(x)\, dx + \int_a^{2a} f(x)\, dx$$

$$= \int_0^a f(x)\, dx + \int_0^a f(2a-x)\, dx$$

$$= \int_0^a [f(x)+f(2a-x)]\, dx$$

**Question 21c**

$$\int_0^{\pi} \frac{x\, dx}{1+\frac{1}{2}\sin x}$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{x}{1+\frac{1}{2}\sin x} + \frac{\pi-x}{1+\frac{1}{2}\sin(\pi-x)}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{x}{1+\frac{1}{2}\sin x} + \frac{\pi-x}{1+\frac{1}{2}\sin x}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{1+\frac{1}{2}\sin x}\right) dx$$

$$= \pi\int_0^{\frac{\pi}{2}} \frac{dx}{1+\frac{1}{2}\sin x}$$

$$= \pi\left(\frac{2\pi}{3\sqrt{3}}\right) \text{ from } (i)$$

$$= \frac{2\pi^2}{3\sqrt{3}}$$

**Question 1**

$$\int x^2 \ln x \, dx$$

$u = \ln x \quad \frac{dv}{dx} = x^2$

$\frac{du}{dx} = \frac{1}{x} \quad v = \frac{x^3}{3}$

$$= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \times \frac{1}{x} dx$$
$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2 \, dx$$
$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$$
$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + c$$

**Question 2**

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$u = \cos x \quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = -\sin x \quad v = e^x$

$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$u = \sin x \quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = \cos x \quad v = e^x$

$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \left(\left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx\right)$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \left[e^x \sin x\right]_0^{\frac{\pi}{2}}$$
$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \frac{1}{2}\left(\left(0 - 1\right) + \left(e^{\frac{\pi}{2}} - 0\right)\right)$$
$$= \frac{e^{\frac{\pi}{2}} - 1}{2}$$

*Alternative using tabular IBP*

| S | D | I |
|---|---|---|
| + | $\cos x$ | $e^x$ |
| − | $-\sin x$ | $e^x$ |
| + | $-\cos x$ | $e^x$ |

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = \left[e^x \cos x + e^x \sin x\right]_0^{\frac{\pi}{2}}$$
$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = e^{\frac{\pi}{2}}(0 + 1) - 1(1 + 0)$$
$$\therefore \int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \frac{e^{\frac{\pi}{2}} - 1}{2}$$

**Question 3**

$$\int x^3 \sin x^2 \, dx$$

$u = \frac{x^2}{2} \quad \frac{dv}{dx} = 2x \sin x^2$

$\frac{du}{dx} = x \quad v = -\cos x^2$

$$= -\frac{x^2 \cos x^2}{2} + \int x \cos x^2 \, dx$$
$$= -\frac{x^2 \cos x^2}{2} + \frac{1}{2}\int 2x \cos x^2 \, dx$$
$$= -\frac{x^2 \cos x^2}{2} + \frac{\sin x^2}{2} + c$$

**Question 4**

$$\int \ln x \, dx$$

$u = \ln x \quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x} \quad v = x$

$$= x \ln x - \int x \times \frac{1}{x} dx$$
$$= x \ln x - \int dx$$
$$= x \ln x - x + c$$

**Question 5**

$$\int x e^{2x} \, dx$$

$u = x \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1 \quad v = \frac{1}{2}e^{2x}$

$$= \frac{x e^{2x}}{2} - \frac{1}{2}\int e^{2x} \, dx$$
$$= \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + c$$

**Question 6**

$$\int_0^{\pi} x \cos x \, dx$$

$u = x \quad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1 \quad v = \sin x$

$$= \left[x \sin x\right]_0^{\pi} - \int_0^{\pi} \sin x \, dx$$
$$= 0 - 0 + \left[\cos x\right]_0^{\pi}$$
$$= -1 - 1$$
$$= -2$$

**Question 7**

$$\int_0^2 t e^{-t} \, dt$$

$u = t \quad \frac{dv}{dt} = e^{-t}$

$\frac{du}{dt} = 1 \quad v = -e^{-t}$

$$= -\left[t e^{-t}\right]_0^2 + \int_0^2 e^{-t} \, dt$$
$$= -\frac{2}{e^2} - \left[e^{-t}\right]_0^2$$
$$= -\frac{2}{e^2} - \frac{1}{e^2} + 1$$
$$= 1 - \frac{3}{e^2}$$

**Question 8**

$$\int_0^1 \tan^{-1} x\, dx$$

$$\begin{array}{ll} u = \tan^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x^2} & v = x \end{array}$$

$$= \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$$
$$= \frac{\pi}{4} - 0 - \frac{1}{2}\left[\ln\left|1+x^2\right|\right]_0^1$$
$$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0)$$
$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$

**Question 9**

$$\int \ln(1+x)\, dx$$

$$\begin{array}{ll} u = \ln(1+x) & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x} & v = x \end{array}$$

$$= x\ln(1+x) - \int \frac{x}{1+x}\, dx$$
$$= x\ln(1+x) - \int \frac{1+x-1}{1+x}\, dx$$
$$= x\ln(1+x) - \int \left(1 - \frac{1}{1+x}\right) dx$$
$$= x\ln(1+x) - x + \ln|1+x| + c$$
$$= (x+1)\ln(1+x) - x + c$$

**Question 10**

$$\int x2^x\, dx$$

$$\begin{array}{ll} u = x & \dfrac{dv}{dx} = 2^x \\ \dfrac{du}{dx} = 1 & v = \dfrac{2^x}{\ln 2} \end{array}$$

$$= \frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\, dx$$
$$= \frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\left(\frac{2^x}{\ln 2}\right) + c$$
$$= \frac{x2^x}{\ln 2} - \frac{2^x}{\ln^2 2} + c$$

**Question 11**

$$\int \sin^{-1} x\, dx$$

$$\begin{array}{ll} u = \sin^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}} & v = x \end{array}$$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$
$$= x\sin^{-1} x + \frac{1}{2}\int (-2x)\left(1-x^2\right)^{-\frac{1}{2}}\, dx$$
$$= x\sin^{-1} x + \frac{1}{2}\times 2\left(1-x^2\right)^{\frac{1}{2}} + c$$
$$= x\sin^{-1} x + \sqrt{1-x^2} + c$$

**Question 12**

$$\int_4^{e+3} \ln|x-3|\, dx$$

$$\begin{array}{ll} u = \ln|x-3| & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{x-3} & v = x \end{array}$$

$$= \left[x\ln|x-3|\right]_4^{e+3} - \int_4^{e+3} \frac{x}{x-3}\, dx$$
$$= \big((e+3) - 0\big) - \int_4^{e+3} \frac{x-3+3}{x-3}\, dx$$
$$= e + 3 - \int_4^{e+3} \left(1 + \frac{3}{x-3}\right) dx$$
$$= e + 3 - \left[x + 3\ln|x-3|\right]_4^{e+3}$$
$$= e + 3 - \Big(\big((e+3)+3\big) - (4+0)\Big)$$
$$= 1$$

**Question 13**

$$\int \tan^{-1} x\, dx$$

$$\begin{array}{ll} u = \tan^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x^2} & v = x \end{array}$$

$$= x\tan^{-1} x - \int \frac{x}{1+x^2}\, dx$$
$$= x\tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right| + c$$

**Question 14**

$$\int \ln|\cos x|\, \operatorname{cosec}^2 x\, dx$$

$$\begin{array}{ll} u = \ln|\cos x| & \dfrac{dv}{dx} = \operatorname{cosec}^2 x \\ \dfrac{du}{dx} = -\tan x & v = -\cot x \end{array}$$

$$= -\ln|\cos x|\cot x - \int \tan x \cot x\, dx$$
$$= -\ln|\cos x|\cot x - \int dx$$
$$= -\ln|\cos x|\cot x - x + c$$

**Question 15**

$$I = \int 2^x \sin x \, dx$$

$$\boxed{\begin{array}{ll} u = \sin x & \dfrac{dv}{dx} = 2^x \\ \dfrac{du}{dx} = \cos x & v = \dfrac{2^x}{\ln 2} \end{array}}$$

$$= \frac{2^x \sin x}{\ln 2} - \frac{1}{\ln 2}\int 2^x \cos x \, dx$$

$$\boxed{\begin{array}{ll} u = \cos x & \dfrac{dv}{dx} = 2^x \\ \dfrac{du}{dx} = -\sin x & v = \dfrac{2^x}{\ln 2} \end{array}}$$

$$= \frac{2^x \sin x}{\ln 2} - \frac{1}{\ln 2}\left[\frac{2^x \cos x}{\ln 2} + \frac{1}{\ln 2}\int 2^x \sin x \, dx\right]$$

$$= \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2} - \frac{1}{\ln^2 2}\int 2^x \sin x \, dx$$

$$\therefore \left(1 + \frac{1}{\ln^2 2}\right) I = \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2} + c$$

$$I = \frac{\ln^2 2}{\ln^2 2 + 1}\left(\frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2}\right) + c$$

$$= \frac{2^x(\ln 2 \sin x - \cos x)}{\ln^2 2 + 1} + c$$

*Alternative using tabular IBP*

| S | D | I |
|---|---|---|
| + | $\sin x$ | $2^x$ |
| − | $\cos x$ | $\dfrac{2^x}{\ln 2}$ |
| + | $-\sin x$ | $\dfrac{2^x}{(\ln 2)^2}$ |

$$\int 2^x \sin x \, dx = \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2}$$

$$\therefore \left(1 + \frac{1}{\ln^2 2}\right) I = \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2} + c$$

$$I = \frac{\ln^2 2}{\ln^2 2 + 1}\left(\frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2}\right) + c$$

$$= \frac{2^x(\ln 2 \sin x - \cos x)}{\ln^2 2 + 1} + c$$

**Question 16**

a $\int x^2\sqrt{x-1}\, dx$

$$\boxed{\begin{array}{ll} u = x^2 & \dfrac{dv}{dx} = (x-1)^{\frac{1}{2}} \\ \dfrac{du}{dx} = 2x & v = \dfrac{2}{3}(x-1)^{\frac{3}{2}} \end{array}}$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\int x(x-1)^{\frac{3}{2}}\, dx$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\left[\frac{2x(x-1)^{\frac{5}{2}}}{5} - \frac{2}{5}\int (x-1)^{\frac{5}{2}}\, dx\right]$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{8x(x-1)^{\frac{5}{2}}}{15} + \frac{16}{105}(x-1)^{\frac{7}{2}} + c$$

$$= \frac{2x^2\sqrt{(x-1)^3}}{3} - \frac{8x\sqrt{(x-1)^5}}{15} + \frac{16\sqrt{(x-1)^7}}{105} + c$$

b $\int x^2\sqrt{x-1}\, dx$

$$\boxed{\begin{array}{l} u^2 = x - 1 \\ 2u\, du = dx \end{array}}$$

$$= \int x^2 u \times 2u \, du$$

$$= 2\int (u^2+1)^2 u^2 \, du$$

$$= 2\int (u^6 + 2u^4 + u^2)\, du$$

$$= \frac{2u^7}{7} + \frac{4u^5}{5} + \frac{2u^3}{3} + c$$

$$= \frac{2\sqrt{(x-1)^7}}{7} + \frac{4\sqrt{(x-1)^5}}{5} + \frac{2\sqrt{(x-1)^3}}{3} + c$$

**Question 17**

a $\int \dfrac{\ln x - 2}{(\ln x - 1)^2}\, dx$

$$\begin{array}{ll} u = x(\ln x - 2) & \dfrac{dv}{dx} = \dfrac{1}{x}(\ln x - 1)^{-2} \\ \dfrac{du}{dx} = x\left(\dfrac{1}{x}\right) + (\ln x - 2)(1) & v = -(\ln x - 1)^{-1} \\ \quad = \ln x - 1 & \quad = -\dfrac{1}{\ln x - 1} \end{array}$$

$$= -\frac{x(\ln x - 2)}{\ln x - 1} + \int \frac{\ln x - 1}{\ln x - 1}\, dx$$
$$= -\frac{x(\ln x - 2)}{\ln x - 1} + \int dx$$
$$= \frac{2x - x\ln x}{\ln x - 1} + x + c$$
$$= \frac{2x - x\ln x}{\ln x - 1} + \frac{x\ln x - x}{\ln x - 1} + c$$
$$= \frac{x}{\ln x - 1} + c$$

b $\therefore \int \dfrac{\ln x - 2}{(\ln x - 1)^2}\, dx = \int \dfrac{\ln x - 1 - 1}{(\ln x - 1)^2} dx$

$$= \int \frac{(\ln x - 1)(1) - (x)\left(\frac{1}{x}\right)}{(\ln x - 1)^2} dx$$
$$= \int \frac{d}{dx}\left(\frac{x}{\ln x - 1}\right) dx$$
$$= \frac{x}{\ln x - 1} + c$$

**Question 18**

$\int x^2 \ln x\, dx$

$$\begin{array}{ll} u = x^2 & \dfrac{dv}{dx} = \ln x \\ \dfrac{du}{dx} = 2x & v = x\ln x - x \end{array}$$

$$= x^2(x\ln x - x) - 2\int x(x\ln x - x)\, dx$$
$$= x^3\ln x - x^3 - 2\int (x^2\ln x - x^2)\, dx$$
$$= x^3\ln x - x^3 - 2\int x^2\ln x\, dx + 2\int x^2 dx$$
$$\therefore 3\int x^2\ln x\, dx = x^3\ln x - x^3 + \frac{2x^3}{3} + c$$
$$\int x^2\ln x\, dx = \frac{x^3\ln x}{3} - \frac{x^3}{9} + c$$

**Question 19**

$I = \int x\sin x\cos x\, dx$

$$\begin{array}{ll} u = x\sin x & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = \sin x + x\cos x & v = \sin x \end{array}$$

$$= x\sin^2 x - \int (\sin^2 x + x\sin x\cos x)\, dx$$
$$= x\sin^2 x - \int \sin^2 x\, dx - \int x\sin x\cos x\, dx$$
$$\therefore 2I = x\sin^2 x - \frac{1}{2}\int (1 - \cos 2x)\, dx$$
$$I = \frac{x\sin^2 x}{2} - \frac{1}{4}\left(x - \frac{1}{2}\sin 2x\right) + c$$
$$= \frac{x\sin^2 x}{2} - \frac{x}{4} + \frac{\sin 2x}{8} + c$$

**Question 20**

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

| | |
|---|---|
| $u = xe^x$ | $\frac{dv}{dx} = \frac{1}{(1+x)^2}$ |
| $\frac{du}{dx} = xe^x + e^x$ $= e^x(x+1)$ | $v = -\frac{1}{1+x}$ |

$$= -\frac{xe^x}{1+x} + \int e^x\,dx$$
$$= -\frac{xe^x}{1+x} + e^x + c$$
$$= \frac{-xe^x + e^x + xe^x}{1+x} + c$$
$$= \frac{e^x}{1+x} + c$$

*Alternatively*

$$\int \frac{xe^x}{(1+x)^2}\,dx$$
$$= \int \frac{(1+x)e^x - e^x}{(1+x)^2}\,dx$$
$$= \int \frac{d}{dx}\left(\frac{e^x}{1+x}\right)dx$$
$$= \frac{e^x}{1+x} + c$$

**Question 21**

Let $P(n)$ represent the proposition.

$P(0)$ is true since $\int_0^\infty x^0 e^{-x}\,dx = \int_0^\infty e^{-x}\,dx = -\left[e^{-x}\right]_0^\infty = -(0-1) = 1 = 1!$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\int_0^\infty x^k e^{-x}\,dx = k!$

RTP $P(k+1)$ $\int_0^\infty x^{k+1} e^{-x}\,dx = (k+1)!$

LHS $= \int_0^\infty x^{k+1} e^{-x}\,dx$

| | |
|---|---|
| $u = x^{k+1}$ | $\frac{dv}{dx} = e^{-x}$ |
| $\frac{du}{dx} = (k+1)x^k$ | $v = -e^{-x}$ |

$$= -\left[x^{k+1}e^{-x}\right]_0^\infty + (k+1)\int_0^\infty x^k e^{-x}\,dx$$
$$= -(0-0) + (k+1)\times k! \quad \text{from } P(k)$$
$$= (k+1)!$$
$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction.

**Question 1**

$$I_n = \int x^n e^{2x}\,dx$$

| | |
|---|---|
| $u = x^n$ | $\frac{dv}{dx} = e^{2x}$ |
| $\frac{du}{dx} = nx^{n-1}$ | $v = \frac{1}{2}e^{2x}$ |

$$= \frac{x^n e^{2x}}{2} - \frac{n}{2}\int x^{n-1} e^{2x}\,dx$$

$$= \frac{x^n e^{2x}}{2} - \frac{n}{2} I_{n-1}$$

**Question 2**

$$I_n = \int \sin^n x\,dx$$

| | |
|---|---|
| $u = \sin^{n-1} x$ | $\frac{dv}{dx} = \sin x$ |
| $\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x$ | $v = -\cos x$ |

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos x \cos x\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1 - \sin^2 x)dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$\therefore I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$$

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

**Question 3**

$$I_n + I_{n-2}$$

$$= \int (\cot^n x + \cot^{n-2} x)\,dx$$

$$= \int \cot^{n-2} x\,(\cot^2 x + 1)\,dx$$

$$= \int \operatorname{cosec}^2 x \cot^{n-2} x\;dx$$

$$= -\frac{1}{n-1}\cot^{n-1} x$$

$$\therefore I_n = -\frac{1}{n-1}\cot^{n-1} x - I_{n-2}$$

**Question 4**

$$I_0 = \int x^0 e^{2x}\,dx$$

$$= \int e^{2x}\,dx$$

$$= \frac{1}{2}e^{2x} + c$$

$$\int x^2 e^{2x}\,dx = I_2$$

$$= \frac{x^2 e^{2x}}{2} - \frac{2}{2} I_{2-1}$$

$$= \frac{x^2 e^{2x}}{2} - \left(\frac{xe^{2x}}{2} - \frac{1}{2} I_0\right)$$

$$= \frac{x^2 e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{1}{4}e^{2x} + c$$

**Question 5**

$$\int_0^1 x^3\left(x^2-1\right)^5\,dx$$
$$= I_3$$
$$= \frac{3-1}{3+11} I_1$$
$$= \frac{1}{7}\int_0^1 x\left(x^2-1\right)^5\,dx$$
$$= \frac{1}{14}\int_0^1 2x\left(x^2-1\right)^5\,dx$$
$$= \frac{1}{14}\left[\frac{\left(x^2-1\right)^6}{6}\right]_0^1$$
$$= \frac{1}{14}\left(0-\frac{1}{6}\right)$$
$$= -\frac{1}{84}$$

**Question 6a**

$$I_n + I_{n-1}$$
$$= \int_0^{\frac{\pi}{4}} \tan^{2n}\theta\,d\theta + \int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta\,(1+\tan^2\theta)\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \sec^2\theta\,(\tan\theta)^{2n-2}\,d\theta$$
$$= \left[\frac{\tan^{2n-1}\theta}{2n-1}\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2n-1} - 0$$
$$\therefore I_n = \frac{1}{2n-1} - I_{n-1}$$

**Question 6b**

$$I_3 = \frac{1}{2(3)-1} - I_2$$
$$= \frac{1}{5} - \left(\frac{1}{2(2)-1} - I_1\right)$$
$$= \frac{1}{5} - \frac{1}{3} + I_1$$
$$= -\frac{2}{15} + \int_0^{\frac{\pi}{4}} \tan^2\theta\,d\theta$$
$$= -\frac{2}{15} + \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - 1\right)d\theta$$
$$= -\frac{2}{15} + \left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$$
$$= -\frac{2}{15} + \left(\left(1-\frac{\pi}{4}\right) - (0-0)\right)$$
$$= \frac{13}{5} - \frac{\pi}{4}$$

**Question 7**

$$I_n = \int_1^{e^2} (\ln x)^n \, dx$$

| | |
|---|---|
| $u = (\ln x)^n$ | $\frac{dv}{dx} = 1$ |
| $\frac{du}{dx} = n(\ln x)^{n-1} \times \frac{1}{x}$ | $v = x$ |

$$= \left[ x(\ln x)^n \right]_1^{e^2} - n \int_1^{e^2} (\ln x)^{n-1} \, dx$$

$$\therefore I_n = e^2 \times (2)^n - 0 - nI_{n-1}$$

$$I_n = e^2 2^n - nI_{n-1}$$

**Question 8a**

$$I_n = \int_0^x \sec^n t \, dt$$

| | |
|---|---|
| $u = \sec^{n-2} t$ | $\frac{dv}{dt} = \sec^2 t$ |
| $\frac{du}{dt} = (n-2)\sec^{n-3} t \sec t \tan t$ <br> $= (n-2)\sec^{n-2} t \tan t$ | $v = \tan t$ |

$$= \left[ \sec^{n-2} t \tan t \right]_0^x - (n-2) \int \sec^{n-2} t \tan^2 t \, dt$$

$$= \sec^{n-2} x \tan x - 0 - (n-2) \int \sec^{n-2} t \left( \sec^2 t - 1 \right) dt$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n t \, dt + (n-2) \int \sec^{n-2} t \, dt$$

$$\therefore I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

**Question 8b**

$$\int_0^{\frac{\pi}{3}} \sec^4 t \, dt$$

$$= I_4$$

$$= \frac{\sec^2 \frac{\pi}{3} \tan \frac{\pi}{3}}{3} + \frac{2}{3} I_2$$

$$= 2^2 \times \frac{\sqrt{3}}{3} + \frac{2}{3} \left( \frac{1 \times \tan \frac{\pi}{3}}{1} + 0 \right)$$

$$= \frac{4\sqrt{3}}{3} + \frac{2}{3} \left( \sqrt{3} \right)$$

$$= 2\sqrt{3}$$

**Question 9**

$$I_n = \int x \ln^n x \, dx$$

| | |
|---|---|
| $u = (\ln x)^n$ | $\frac{dv}{dx} = x$ |
| $\frac{du}{dx} = \frac{n(\ln x)^{n-1}}{x}$ | $v = \frac{x^2}{2}$ |

$$= \frac{x^2 (\ln x)^n}{2} - \frac{n}{2} \int x (\ln x)^{n-1} \, dx$$

$$\therefore I_n = \frac{x^2 (\ln x)^n}{2} - \frac{n}{2} I_{n-1}$$

$$I_0 = \int x \, dx = \frac{x^2}{2} + c$$

$$I_2 = \frac{x^2 (\ln x)^2}{2} - \frac{2}{2} I_1$$

$$= \frac{x^2 (\ln x)^2}{2} - \left( \frac{x^2 \ln x}{2} - \frac{1}{2} I_0 \right)$$

$$= \frac{x^2 (\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{1}{2} \left( \frac{x^2}{2} + c \right)$$

$$= \frac{x^2 (\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + c$$

**Question 10**

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$$

$$\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = nx^{n-1} & v = \sin x \end{array}$$

$$= \left[ x^n \sin x \right]_0^{\frac{\pi}{2}} - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$\begin{array}{ll} u = x^{n-1} & \dfrac{dv}{dx} = \sin x \\ \dfrac{du}{dx} = (n-1)x^{n-2} & v = -\cos x \end{array}$$

$$= \left(\frac{\pi}{2}\right)^n + n \left\{ \left[ x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} - (n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx \right\}$$

$$= \left(\frac{\pi}{2}\right)^n + n(0-0) - n(n-1) I_{n-2}$$

$$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$$

*Alternative solution using tabular IBP*

$$I_n = \left[ x^n \sin x + nx^{n-1} \cos x \right]_0^{\frac{\pi}{2}} - n(n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$$

$$= \left( \left( \left(\frac{\pi}{2}\right)^n (1) + 0 \right) - (0+0) \right) - n(n-1) I_{n-2}$$

$$= \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$$

| S | D | I |
|---|---|---|
| + | $x^n$ | $\cos x$ |
| − | $nx^{n-1}$ | $\sin x$ |
| + | $n(n-1)x^{n-2}$ | $-\cos x$ |

**Question 11**

$$I_n = \int x^n \sqrt{2x+1} \, dx$$

$$\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = (2x+1)^{\frac{1}{2}} \\ \dfrac{du}{dx} = nx^{n-1} & v = \dfrac{1}{3}(2x+1)^{\frac{3}{2}} \end{array}$$

$$= \frac{x^n (2x+1)^{\frac{3}{2}}}{3} - \frac{n}{3} \int x^{n-1} (2x+1)^{\frac{3}{2}} \, dx$$

$$= \frac{x^n \sqrt{(2x+1)^3}}{3} - \frac{n}{3} \int (2x+1) x^{n-1} (2x+1)^{\frac{1}{2}} \, dx$$

$$= \frac{x^n \sqrt{(2x+1)^3}}{3} - \frac{n}{3} \int 2x \times x^{n-1} (2x+1)^{\frac{1}{2}} \, dx - \frac{n}{3} \int x^{n-1} (2x+1)^{\frac{1}{2}} \, dx$$

$$= \frac{x^n \sqrt{(2x+1)^3}}{3} - \frac{2n}{3} \int x^n \sqrt{2x+1} \, dx - \frac{n}{3} \int x^{n-1} \sqrt{2x+1} \, dx$$

$$\therefore I_n = \frac{x^n \sqrt{(2x+1)^3}}{3} - \frac{2n}{3} I_n - \frac{n}{3} I_{n-1}$$

$$\frac{2n+3}{3} I_n = \frac{x^n \sqrt{(2x+1)^3}}{3} - \frac{n}{3} I_{n-1}$$

$$\therefore I_n = \frac{x^n \sqrt{(2x+1)^3}}{2n+3} - \frac{n}{2n+3} I_{n-1}$$

**Question 12**

$$I_{n-1} = \int x^{-(n-1)} e^{2x}\, dx$$

| | |
|---|---|
| $u = x^{-(n-1)}$ | $\dfrac{dv}{dx} = e^{2x}$ |
| $\dfrac{du}{dx} = -(n-1)x^{-n}$ | $v = \dfrac{1}{2}e^{2x}$ |

$$= \frac{e^{2x}}{2x^{n-1}} + \frac{n-1}{2}\int x^{-n} e^{2x}\, dx$$

$$I_{n-1} = \frac{e^{2x}}{2x^{n-1}} + \frac{n-1}{2} I_n$$

$$\frac{n-1}{2} I_n = -\frac{e^{2x}}{2x^{n-1}} + I_{n-1}$$

$$I_n = -\frac{e^{2x}}{(n-1)x^{n-1}} + \frac{2}{n-1} I_{n-1}$$

**Question 13a**

$$I_{m,n} = \int x^m (\ln x)^n\, dx$$

| | |
|---|---|
| $u = (\ln x)^n$ | $\dfrac{dv}{dx} = x^m$ |
| $\dfrac{du}{dx} = n(\ln x)^{n-1} \times \dfrac{1}{x}$ | $v = \dfrac{x^{m+1}}{m+1}$ |

$$= \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\, dx$$

$$\therefore I_{m,n} = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

**Question 13b**

$$\int x^2 (\ln x)^2$$

$$= I_{2,2}$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2}{3} I_{2,1}$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2}{3}\left(\frac{x^3 \ln x}{3} - \frac{1}{3} I_{2,0}\right)$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2x^3 \ln x}{9} + \frac{2}{9}\int x^2\, dx$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2x^3 \ln x}{9} + \frac{2}{9}\left(\frac{x^3}{3}\right) + c$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2x^3 \ln x}{9} + \frac{2x^3}{27} + c$$

**Question 14**

$$I_n = \int \frac{\sin 3x}{x^n}\, dx$$

| | |
|---|---|
| $u = \sin 3x$ | $\dfrac{dv}{dx} = x^{-n}$ |
| $\dfrac{du}{dx} = 3\cos 3x$ | $v = \dfrac{x^{-n+1}}{-n+1}$ |
| | $= -\dfrac{1}{(n-1)x^{n-1}}$ |

$$= -\frac{\sin 3x}{(n-1)x^{n-1}} + \frac{3}{n-1}\int \frac{\cos 3x}{x^{n-1}}\, dx$$

$$= -\frac{\sin 3x}{(n-1)x^{n-1}} + \frac{3}{n-1} J_{n-1}$$

**Question 15**

$$I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx$$

$$\begin{array}{ll} u = \frac{1}{2}x^{n-1} & \frac{dv}{dx} = 2x(x^2+1)^{-2} \\ \frac{du}{dx} = \frac{1}{2}(n-1)x^{n-2} & v = -(x^2+1)^{-1} \end{array}$$

$$= -\frac{1}{2}\left[\frac{x^{n-1}}{x^2+1}\right]_0^1 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{x^2+1}\,dx$$

$$= -\frac{1}{4} - 0 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}(x^2+1)}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}\int_0^1 \frac{x^n}{(x^2+1)^2}\,dx + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}I_n + \frac{n-1}{2}I_{n-2}$$

$$\therefore \frac{3-n}{2}I_n = -\frac{1}{4} + \frac{n-1}{2}I_{n-2}$$

$$I_n = \frac{1}{2(n-3)} - \frac{n-1}{n-3}I_{n-2}$$

**Question 16**

$$I_m = \int_0^1 x^m(x^2-1)^5\,dx$$

$$\begin{array}{ll} u = \frac{1}{2}x^{m-1} & \frac{dv}{dx} = 2x(x^2-1)^5 \\ \frac{du}{dx} = \frac{m-1}{2}x^{m-2} & v = \frac{1}{6}(x^2-1)^6 \end{array}$$

$$= \frac{1}{12}\left[x^{m-1}(x^2-1)^6\right]_0^1 - \frac{(m-1)}{12}\int_0^1 x^{m-2}(x^2-1)^6\,dx$$

$$= 0 - \frac{m-1}{12}\int_0^1 x^{m-2}(x^2-1)(x^2-1)^5\,dx$$

$$= -\frac{m-1}{12}\int_0^1 x^m(x^2-1)^5\,dx + \frac{m-1}{12}\int_0^1 x^{m-2}(x^2-1)^5\,dx$$

$$\therefore \frac{m+11}{12}I_m = \frac{m-1}{12}I_{m-2}$$

$$\therefore I_m = \frac{m-1}{m+11}I_{m-2}$$

**Question 17**

$$I_n = \int_{-3}^0 x^n\sqrt{x+3}\,dx$$

$$\begin{array}{ll} u = x^n & \frac{dv}{dx} = (x+3)^{\frac{1}{2}} \\ \frac{du}{dx} = nx^{n-1} & v = \frac{2}{3}(x+3)^{\frac{3}{2}} \end{array}$$

$$= \frac{2}{3}\left[x^n(x+3)^{\frac{3}{2}}\right]_{-3}^0 - \frac{2n}{3}\int_{-3}^0 x^{n-1}(x+3)^{\frac{3}{2}}\,dx$$

$$= 0 - \frac{2n}{3}\int_{-3}^0 x^{n-1}(x+3)(x+3)^{\frac{1}{2}}\,dx$$

$$= -\frac{2n}{3}\int_{-3}^0 x^n\sqrt{x+3}\,dx - 2n\int_{-3}^0 x^{n-1}\sqrt{x+3}\,dx$$

$$\therefore I_n = -\frac{2n}{3}I_n - 2nI_{n-1}$$

$$\frac{2n+3}{3}I_n = -2nI_{n-1}$$

$$I_n = -\frac{6n}{2n+3}I_{n-1}$$

**Question 18**

$$I_n = \int \frac{dx}{\sin^n x}$$

| | |
|---|---|
| $u = \operatorname{cosec}^{n-2} x$ | $\frac{dv}{dx} = \operatorname{cosec}^2 x$ |
| $\frac{du}{dx} = (n-2)\operatorname{cosec}^{n-3} x\,(-\operatorname{cosec} x \cot x)$ | $v = -\cot x$ |

$$= \int \operatorname{cosec}^n x\,dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\,dx$$

$$= -\frac{\cos x}{\sin x} \times \frac{1}{\sin^{n-2} x} - (n-2)\int \operatorname{cosec}^{n-2} x \left(\operatorname{cosec}^2 x - 1\right) dx$$

$$= -\frac{\cos x}{\sin^{n-1} x} - (n-2)\int \operatorname{cosec}^n x\,dx + (n-2)\int \operatorname{cosec}^{n-2} x\,dx$$

$$\therefore I_n = -\frac{\cos x}{\sin^{n-1} x} - (n-2)I_n + (n-2)I_{n-2}$$

$$(n-1)I_n = -\frac{\cos x}{\sin^{n-1} x} + (n-2)I_{n-2}$$

$$\therefore I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}$$

**Question 19a**

$$I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2}\,dx$$

| | |
|---|---|
| $u = \frac{1}{2}x^{2n}$ | $\frac{dv}{dx} = 2xe^{x^2}$ |
| $\frac{du}{dx} = nx^{2n-1}$ | $v = e^{x^2}$ |

$$= \frac{1}{2}\left[x^{2n} e^{x^2}\right]_0^1 - n\int_0^1 x^{2n-1} e^{x^2}\,dx$$

$$= \frac{1}{2}(e - 0) - nI_{2n-1}$$

$$= \frac{e}{2} - nI_{2n-1}$$

**Question 19b**

$$I_5 = \frac{e}{2} - 2I_3$$

$$= \frac{e}{2} - 2\left(\frac{e}{2} - I_1\right)$$

$$= -\frac{e}{2} + 2\int_0^1 xe^{x^2}\,dx$$

$$= -\frac{e}{2} + \left[e^{x^2}\right]_0^1$$

$$= -\frac{e}{2} + e - 1$$

$$= \frac{e}{2} - 1$$

**Question 20**

$$I_n = \int_0^1 \sqrt{x}(1-x)^n \, dx$$

$$\begin{array}{ll} u = (1-x)^n & \dfrac{dv}{dx} = x^{\frac{1}{2}} \\ \dfrac{du}{dx} = -n(1-x)^{n-1} & v = \dfrac{2}{3}x^{\frac{3}{2}} \end{array}$$

$$\begin{aligned}
&= \frac{2}{3}\left[x^{\frac{3}{2}}(1-x)^n\right]_0^1 + \frac{2n}{3}\int_0^1 x^{\frac{3}{2}}(1-x)^{n-1}\,dx \\
&= \frac{2}{3}(0-0) + \frac{2n}{3}\int_0^1 x\sqrt{x}(1-x)^{n-1}\,dx \\
&= -\frac{2n}{3}\int_0^1 (1-x-1)\sqrt{x}(1-x)^{n-1}\,dx \\
&= -\frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^n\,dx + \frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^{n-1}\,dx \\
&= -\frac{2n}{3}I_n + \frac{2n}{3}I_{n-1} \\
\frac{2n+3}{3}I_n &= \frac{2n}{3}I_{n-1} \\
I_n &= \frac{2n}{2n+3}I_{n-1} \\
I_3 &= \frac{2(3)}{2(3)+3}I_2 \\
&= \frac{2}{3}\left(\frac{2(2)}{2(2)+3}\right)I_1 \\
&= \frac{8}{21}\left(\frac{2(1)}{2(1)+3}\right)I_0 \\
&= \frac{16}{105}\int_0^1 x^{\frac{1}{2}}\,dx \\
&= \frac{16}{105}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 \\
&= \frac{32}{315}
\end{aligned}$$

**Question 21a**

$$I_n = \int_0^1 \left(1-x^2\right)^n dx$$

$$\begin{array}{ll} u = \left(1-x^2\right)^n & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = n\left(1-x^2\right)^{n-1}(-2x) & v = x \end{array}$$

$$\begin{aligned}
&= \left[x\left(1-x^2\right)^n\right]_0^1 + 2n\int_0^1 x^2\left(1-x^2\right)^{n-1}dx \\
&= 0 - 0 + 2nJ_{n-1} \\
&= 2nJ_{n-1}
\end{aligned}$$

**Question 21b**

$$\begin{aligned}
\therefore I_n &= -2n\int_0^1 \left(1-x^2-1\right)\left(1-x^2\right)^{n-1}dx \\
&= -2n\int_0^1 \left(1-x^2\right)^n dx + 2n\int_0^1 \left(1-x^2\right)^{n-1}dx \\
\mathrm{I}_n &= -2nI_n + 2nI_{n-1} \\
(2n+1)I_n &= 2nI_{n-1} \\
I_n &= \frac{2n}{2n+1}I_{n-1}
\end{aligned}$$

**Question 21c**

$$\begin{aligned}
I_n - I_{n+1} &= \int_0^1 (1-x^2)^n \, dx - \int_0^1 (1-x^2)^{n+1} \, dx \\
&= \int_0^1 \left[(1-x^2)^n - (1-x^2)^{n+1}\right] dx \\
&= \int_0^1 \left[(1-x^2)^n (1-1+x^2)\right] dx \\
&= \int_0^1 x^2 (1-x^2)^n \, dx \\
&= J_n \\
\therefore J_n &= I_n - 2(n+1)J_n \quad \text{from (a)} \\
(2n+3)J_n &= I_n \\
J_n &= \frac{1}{2n+3} I_n
\end{aligned}$$

**Question 21d**

$$\begin{aligned}
J_n &= \frac{1}{2n+3}(2nJ_{n-1}) \\
&= \frac{2n}{2n+3} J_{n-1}
\end{aligned}$$

**Question 22a**

$$\int \frac{dx}{5-4\cos x} \qquad \boxed{\begin{array}{c} t = \tan\dfrac{x}{2} \\ dx = \dfrac{2dt}{1+t^2} \end{array}}$$

$$\begin{aligned}
&= \int \frac{1}{5 - \dfrac{4(1-t^2)}{1+t^2}} \times \frac{2dt}{1+t^2} \\
&= 2\int \frac{dt}{5+5t^2-4+4t^2} \\
&= 2\int \frac{dt}{9t^2+1} \\
&= \frac{2}{3}\int \frac{3}{(3t)^2+1} dt \\
&= \frac{2}{3}\tan^{-1} 3t + c \\
&= \frac{2}{3}\tan^{-1}\left(3\tan\frac{x}{2}\right) + c
\end{aligned}$$

**Question 22b**

$$\begin{aligned}
&\int_0^{\pi} \frac{\cos x}{5-4\cos x} dx \\
&= \int_0^{\pi} \frac{-\frac{1}{4}(5-4\cos x) + \frac{5}{4}}{5-4\cos x} dx \\
&= -\frac{1}{4}\int_0^{\pi} dx + \frac{5}{4}\int_0^{\pi} \frac{dx}{5-4\cos x} \\
&= -\frac{1}{4}\Big[x\Big]_0^{\pi} + \frac{5}{4}\left(\frac{\pi}{3}\right) \\
&= -\frac{\pi}{4} + \frac{5\pi}{12} \\
&= \frac{\pi}{6}
\end{aligned}$$

**Question 22c**

$$
\begin{aligned}
&u_{n+1} + u_{n-1} - \frac{5}{2}u_n \\
&= \int_0^{\pi} \frac{\cos\big((n+1)x\big) + \cos((n-1)x) - \frac{5}{2}\cos nx}{5 - 4\cos x}\,dx \\
&= \int_0^{\pi} \frac{\cos nx\cos x - \sin nx\sin x + \cos nx\cos x + \sin nx\sin x - \frac{5}{2}\cos nx}{5 - 4\cos x}\,dx \\
&= \int_0^{\pi} \frac{2\cos nx\cos x - \frac{5}{2}\cos nx}{5 - 4\cos x}\,dx \\
&= \frac{1}{2}\int_0^{\pi} \frac{\cos nx\,(4\cos x - 5)}{5 - 4\cos x}\,dx \\
&= -\frac{1}{2}\int_0^{\pi} \cos nx\,dx \\
&= -\frac{1}{2}\left[\frac{1}{n}\sin nx\right]_0^{\pi} \\
&= -\frac{1}{2n}(0 - 0) \\
&= 0
\end{aligned}
$$

**Question 23a**

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \frac{dx}{1 + 4x^2} \\
&= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{2}{1 + (2x)^2}\,dx \\
&= \frac{1}{2}\left[\tan^{-1} 2x\right]_0^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) \\
&= \frac{\pi}{8}
\end{aligned}
$$

**Question 23b**

$$
\begin{aligned}
I_n &= \int_0^{\frac{1}{2}} \frac{dx}{(1 + 4x^2)^n} \\
&= \int_0^{\frac{1}{2}} \big(1 + 4x^2\big)^{-n}\,dx \qquad \boxed{\begin{array}{ll} u = \big(1 + 4x^2\big)^{-n} & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = -n\big(1 + 4x^2\big)^{-(n+1)}(8x) & v = x \end{array}} \\
&= \left[\frac{x}{(1 + 4x^2)^n}\right]_0^{\frac{1}{2}} + 2n\int_0^{\frac{1}{2}} \frac{4x^2}{(1 + 4x^2)^{n+1}}\,dx \\
&= \frac{1}{2^{n+1}} - 0 + 2n\int_0^{\frac{1}{2}} \frac{1 + 4x^2 - 1}{(1 + 4x^2)^{n+1}}\,dx \\
&= \frac{1}{2^{n+1}} + 2n\int_0^{\frac{1}{2}} \frac{1}{(1 + 4x^2)^n}\,dx - 2n\int_0^{\frac{1}{2}} \frac{1}{(1 + 4x^2)^{n+1}}\,dx \\
I_n &= \frac{1}{2^{n+1}} + 2nI_n - 2nI_{n+1} \\
(1 - 2n)I_n &= -2nI_{n+1} + \frac{1}{2^{n+1}} \\
I_n &= \frac{2nI_{n+1}}{2n - 1} + \frac{1}{2^{n+1}(1 - 2n)}
\end{aligned}
$$

**Question 23c**

$$I_n = \frac{2nI_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)}$$

$$I_n - \frac{1}{2^{n+1}(1-2n)} = \frac{2nI_{n+1}}{2n-1}$$

$$I_{n+1} = \frac{2n-1}{2n} I_n + \frac{1}{n2^{n+2}}$$

$$I_3 = \frac{2(2)-1}{2(2)} I_2 + \frac{1}{(2)2^{(2)+2}}$$

$$= \frac{3}{4} I_2 + \frac{1}{32}$$

$$= \frac{3}{4}\left(\frac{2(1)-1}{2(1)} I_1 + \frac{1}{(1)2^{(1)+2}}\right) + \frac{1}{32}$$

$$= \frac{3}{4}\left(\frac{1}{2} \times \frac{\pi}{8} + \frac{1}{8}\right) + \frac{1}{32}$$

$$= \frac{3}{4}\left(\frac{\pi+2}{16}\right) + \frac{1}{32}$$

$$= \frac{3\pi+8}{64}$$

**Question 24a**

$$\left(1-t^2\right)^{\frac{n-3}{2}} - \left(1-t^2\right)^{\frac{n-1}{2}}$$

$$= \left(1-t^2\right)^{\frac{n-3}{2}}\left(1-\left(1-t^2\right)\right)$$

$$= t^2\left(1-t^2\right)^{\frac{n-3}{2}}$$

**Question 24b**

$$I_n = \int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}} dt$$

| | |
|---|---|
| $u = \left(1-t^2\right)^{\frac{n-1}{2}}$ | $\frac{dv}{dt} = 1$ |
| $\frac{du}{dt} = \frac{n-1}{2}\left(1-t^2\right)^{\frac{n-3}{2}}(-2t)$ | $v = t$ |

$$= \left[t\left(1-t^2\right)^{\frac{n-1}{2}}\right]_0^1 + (n-1)\int_0^1 t^2\left(1-t^2\right)^{\frac{(n-3)}{2}} dt$$

$$= 0 + (n-1)\int_0^1 \left(\left(1-t^2\right)^{\frac{n-3}{2}} - \left(1-t^2\right)^{\frac{n-1}{2}}\right) dt$$

$$= (n-1)\int_0^1 \left(1-t^2\right)^{\frac{n-3}{2}} dt - (n-1)\int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}} dt$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = (n-1)I_{n-2}$$

**Question 24c**

Let $P(n)$ represent the proposition.

$P(1)$ is true since

$$\begin{aligned} J_1 &= I_1 I_0 \\ &= \int_0^1 dt \times \int_0^1 \left(1-t^2\right)^{-\frac{1}{2}} dt \\ &= \Big[t\Big]_0^1 \times \Big[\sin^{-1} t\Big]_0^1 \\ &= (1-0)\times\left(\frac{\pi}{2}-0\right) \\ &= \frac{\pi}{2} \end{aligned}$$

If $P(k)$ is true for arbitrary $k \geq 1$ then

$J_k = kI_k I_{k-1} = \dfrac{\pi}{2}$

RTP: $P(k+1)$ $J_{k+1} = (k+1)I_{k+1}I_k = \frac{\pi}{2}$

$(k+1)I_{k+1}I_k$

$= kI_{k-1}I_k$ from (ii)

$= \dfrac{\pi}{2}$ from $P(k)$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

**Question 24d**

$0 \leq 1-t^2 \leq 1$ for $0 \leq t \leq 1$

$$\therefore\ 0 \leq \left(1-t^2\right)^{\frac{n-1}{2}} \leq \left(1-t^2\right)^{\frac{n-2}{2}}$$

$$\therefore\ 0 < \int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}} dt < \int_0^1 \left(1-t^2\right)^{\frac{n-2}{2}} dt$$

$0 < I_n < I_{n-1}$

**Question 24e**

$$\begin{aligned} J_n = nI_n I_{n-1} &= \frac{\pi}{2} \\ I_n I_{n-1} &= \frac{\pi}{2n} \\ I_n^2 &< \frac{\pi}{2n} \text{ from (iv)} \\ I_n &< \sqrt{\frac{\pi}{2n}} \end{aligned}$$

$$\begin{aligned} J_{n+1} = (n+1)I_{n+1}I_n &= \frac{\pi}{2} \\ I_{n+1}I_n &= \frac{\pi}{2(n+1)} \\ I_n^2 &> \frac{\pi}{2n} \text{ from (iv)} \\ \therefore \sqrt{\frac{\pi}{2(n+1)}} &< I_n^2 \end{aligned}$$

$$\therefore \sqrt{\frac{\pi}{2(n+1)}} < I_n < \sqrt{\frac{\pi}{2n}}$$

**Question 25a**

$$\begin{aligned}(2k)!! &= (2k)(2k-2)(2k-4)\ldots 2 \\ &= 2(k)2(k-1)2(k-2)\ldots 2(1) \\ &= 2^k k!\end{aligned}$$

**Question 25b**

$$\begin{aligned}&(2k+1)!!\,2^k k! \\ &= [(2k+1)(2k-1)(2k-3)\ldots 1][2(k)2(k-1)2(k-2)\ldots 2(1)] \\ &= [(2k+1)(2k-1)(2k-3)\ldots 1][(2k)(2k-2)(2k-4)\ldots 2] \\ &= (2k+1)(2k)(2k-1)(2k-2)(2k-3)(2k-4)\ldots 2\cdot 1 \\ &= (2k+1)!\end{aligned}$$

$$\therefore (2k+1)!! = \frac{(2k+1)!}{2^k k!}$$

**Question 25c**

$$\begin{aligned}I_{2n} &= \int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx \\ &= -\left[\cos x \sin^{2n-1} x\right]_0^{\frac{\pi}{2}} + (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n-2} x \cos^2 x\,dx \\ &= -(0-0) + (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n-2} x\left(1-\sin^2 x\right)dx \\ &= (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n-2} x\,dx - (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx \\ I_{2n} &= (2n-1)I_{2n-2} - (2n-1)I_{2n} \\ 2nI_{2n} &= (2n-1)I_{2n-2} \\ I_{2n} &= \frac{2n-1}{2n} I_{2n-2} \\ &= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2} I_{2n-4} \\ &= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\frac{2n-5}{2n-4} I_{2n-6} \\ &= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\frac{2n-5}{2n-4}\times\ldots\times\frac{1}{2} I_0 \\ &= \frac{(2n-1)!!}{(2n)!!}\int_0^{\frac{\pi}{2}} dx \\ &= \frac{\frac{(2n)!}{2^n n!}}{2^n n!}\times\left[x\right]_0^{\frac{\pi}{2}} \\ &= \frac{(2n)!}{2^{2n}(n!)^2}\left(\frac{\pi}{2}-0\right) \\ &= \frac{\pi(2n)!}{2^{2n+1}(n!)^2}\end{aligned}$$

**Question 26**

$$I_{2n} = \int_{-1}^{1} \left(1-x^2\right)^n dx$$

$$\begin{array}{ll} u = \left(1-x^2\right)^n & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = n\left(1-x^2\right)^{n-1}(-2x) & v = x \end{array}$$

$$\begin{aligned}
&= \left[x\left(1-x^2\right)^n\right]_{-1}^{1} + 2n\int_{-1}^{1} x^2\left(1-x^2\right)^{n-1} dx \\
&= 0 - 2n\int_{-1}^{1} \left(1-x^2-1\right)\left(1-x^2\right)^{n-1} dx \\
&= -2n\int_{-1}^{1} \left(1-x^2\right)^n dx + 2n\int_{-1}^{1} \left(1-x^2\right)^{n-1} dx \\
I_{2n} &= -2nI_{2n} + 2nI_{2n-2} \\
(2n+1)I_{2n} &= 2nI_{2n-2} \\
I_{2n} &= \frac{2n}{2n+1} I_{2n-2} \\
&= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} I_{2n-4} \\
&= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} I_{2n-6} \\
&= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} \times \ldots \times \frac{2}{3} I_0 \\
&= \frac{2n}{2n+1} \times \frac{2n}{2n} \times \frac{2n-2}{2n-1} \times \frac{2n-2}{2n-2} \times \frac{2n-4}{2n-3} \times \frac{2n-4}{2n-4} \times \ldots \times \frac{2}{3} \times \frac{2}{2} \times \int_{-1}^{1} dx \\
&= \frac{\left(2(n)2(n-1)2(n-2)\ldots 2(1)\right)^2}{(2n+1)!} \Big[x\Big]_{-1}^{1} \\
&= \frac{2^{2n}(n!)^2}{(2n+1)!}\left(1-(-1)\right) \\
&= \frac{2^{2n+1}(n!)^2}{(2n+1)!}
\end{aligned}$$

**Question 1**

$$\int \frac{dx}{x^2+6x+10}$$
$$= \int \frac{dx}{(x+3)^2+1^2}$$
$$= \tan^{-1}(x+3)+c$$

**Question 2**

$$\int \frac{1}{x^2+2x+2}\,dx$$
$$= \int \frac{1}{(x+1)^2+1^2}\,dx$$
$$= \tan^{-1}(x+1)+c$$

**Question 3a**

$$I_n = \int_1^e (\ln t)^n\,dt$$

$u = (\ln t)^n \qquad \frac{dv}{dt} = 1$

$\frac{du}{dt} = \frac{n(\ln t)^{n-1}}{t} \qquad v = t$

$$= \left[t(\ln t)^n\right]_1^e - n\int (\ln t)^{n-1}\,dt$$
$$= e - n\int (\ln t)^{n-1}\,dt$$
$$= e - nI_{n-1}$$

**Question 3b**

$$I_3 = e - 3I_2$$
$$= e - 3(e - 2I_1)$$
$$= -2e + 6(e - I_0)$$
$$= 4e - 6\int_1^e dt$$
$$= 4e - 6\left[t\right]_1^e$$
$$= 4e - 6(e-1)$$
$$= -2e + 6$$

**Question 4**

$$\int x\tan^{-1}x\,dx$$

$u = \tan^{-1}x \qquad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2}$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,dx$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c$$

**Question 5**

$$\int \sin x\cos(\cos x)\,dx$$
$$= -\int -\sin x\cos(\cos x)\,dx$$
$$= -\sin(\cos x) + c$$

**Question 6**

$$\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{16-(3+x)^2}}\,dx$$
$$= \sin^{-1}\left(\frac{x+3}{4}\right) + C$$

**Answer D**

**Question 7**

$$\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$$

$u = e^x + 1$
$du = e^x dx$

$$= \int_2^{e+1} \frac{du}{u^2}$$
$$= \int_2^{e+1} u^{-2}\,du$$
$$= -\left[\frac{1}{u}\right]_2^{e+1}$$
$$= -\left[\frac{1}{u}\right]_2^{e+1}$$
$$= -\left(\frac{1}{e+1} - \frac{1}{2}\right)$$
$$= \frac{1}{2} - \frac{1}{e+1}$$

**Question 8**

$$\int \frac{1}{x\ln x}\,dx$$
$$= \int \frac{1/x}{\ln x}\,dx$$
$$= \ln\left|\ln x\right| + c$$

**Question 9**

$$\int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}\,dx$$
$$= -\int_0^1 \left(-\frac{1}{\sqrt{1-x^2}}\right)\times(\cos^{-1}x)^1\,dx$$
$$= -\frac{1}{2}\left[(\cos^{-1}x)^2\right]_0^1$$
$$= -\frac{1}{2}\left(0 - \left(\frac{\pi}{2}\right)^2\right)$$
$$= \frac{\pi^2}{8}$$

**Answer C**

**Question 10**

$$\int_0^{\frac{\pi}{3}} \sec^3 x\tan x\,dx$$
$$= \int_0^{\frac{\pi}{3}} \sec x\tan x\,(\sec x)^2\,dx$$
$$= \left[\frac{\sec^3 x}{3}\right]_0^{\frac{\pi}{3}}$$
$$= \frac{8}{3} - \frac{1}{3}$$
$$= \frac{7}{3}$$

**Question 11**

The identity is true for all values of $n$ since

$$\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$$

**Answer D**

**Question 12**

$$\int \frac{dx}{(x-1)(x+2)}$$
$$= \frac{1}{3}\int \left(\frac{1}{x-1} - \frac{1}{x+2}\right) dx$$
$$= \frac{1}{3}(\ln|x-1| - \ln|x+2|) + c$$
$$= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + c$$

**Question 13**

$$\int_0^1 xe^{-x^2}\,dx$$
$$= -\frac{1}{2}\int_0^1 (-2x)e^{-x^2}\,dx$$
$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$
$$= -\frac{1}{2}(e^{-1} - 1)$$
$$= \frac{1}{2} - \frac{1}{2e}$$

**Question 14a**

$$a = \frac{10}{(-1)^2 + 4} = 2$$

equating coefficients of $x^2$:

$$a + b = 0 \quad \Rightarrow \quad b = -2$$

equating coefficients of $x$:

$$b + c = 0 \quad \Rightarrow \quad c = 2$$

**Question 14b**

$$\int \left(\frac{2}{x+1} + \frac{-2x+2}{x^2+4}\right) dx$$
$$= \int \left(\frac{2}{x+1} - \frac{2x}{x^2+4} + \frac{2}{x^2+4}\right) dx$$
$$= 2\ln|x+1| - \ln|x^2+4| + \tan^{-1}\frac{x}{2} + c$$

**Question 15**

$$\int \frac{t^2-1}{t^3}\,dt$$
$$= \int \left(\frac{1}{t} - t^{-3}\right) dt$$
$$= \ln|t| - \frac{t^{-2}}{-2} + c$$
$$= \ln|t| + \frac{1}{2t^2} + c$$

**Question 16**

$$\int \frac{dx}{\sqrt{6-x-x^2}}$$
$$= \int \frac{dx}{\sqrt{-\left(x^2 + x + \frac{1}{4} - \frac{25}{4}\right)}}$$
$$= \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(\frac{1}{2}+x\right)^2}}$$
$$= \sin^{-1}\left(\frac{x+\frac{1}{2}}{5/2}\right) + c$$
$$= \sin^{-1}\left(\frac{2x+1}{5}\right) + c$$

**Question 17**

$$\int x^2 \log_e(3x)\,dx$$

$u = \log_e 3x \quad \frac{dv}{dx} = x^2$

$\frac{du}{dx} = \frac{1}{x} \quad v = \frac{x^3}{3}$

$$= \frac{x^3}{3}\log_e 3x - \frac{1}{3}\int x^2\,dx$$
$$= \frac{x^3}{3}\log_e 3x - \frac{x^3}{9} + c$$

**Question 18**

$$\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$
$$= \int \cos\theta(\sin\theta)^{-4}\,d\theta$$
$$= \frac{(\sin\theta)^{-3}}{-3} + c$$
$$= -\frac{1}{3\sin^3\theta} + c$$

**Question 19**

$$\int \frac{dx}{\sqrt{2-4x-2x^2}}$$
$$= \int \frac{dx}{\sqrt{2(1-2x-x^2)}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{2-(1+x)^2}}$$
$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

**Question 20a**

$$\frac{\cos 2x - \cos 4x}{\sin 4x + \sin 2x}$$
$$= \frac{2\sin 3x \sin x}{2\sin 3x \cos x}$$
$$= \frac{\sin x}{\cos x} \quad \text{since } \sin 3x \neq 0$$
$$= \tan x$$

**Question 20b**

$$\sin 3x + \sin x$$
$$= 2\sin 2x \cos x$$
$$= 2(2\sin x \cos x)\cos x$$
$$= 4\sin x \cos^2 x$$

**Question 20c**

$2\cos 2x\cos x$

$= 2\left(2\cos^2 x - 1\right)\cos x$

$= 4\cos^3 x - 2\cos x$

**Question 20d**

$2\tan x\cos 3x$

$= \dfrac{2\sin x\cos 3x}{\cos x}$

$= \dfrac{\sin 4x - \sin 2x}{\cos x}$

$= \sec x\,(\sin 4x - \sin 2x)$

**Question 21**

$\int x\cos x\,dx$ $\quad \begin{array}{ll} u = x & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = 1 & v = \sin x \end{array}$

$= x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + c$

**Question 22**

$\int \dfrac{x+1}{x-2}\,dx$

$= \int \dfrac{x-2+3}{x-2}\,dx$

$= \int \left(1 + \dfrac{3}{x-2}\right)dx$

$= x + 3\ln|x-2| + c$

**Question 23**

$\int \dfrac{x}{\sqrt{16-x^2}}\,dx$

$= -\dfrac{1}{2}\int (-2x)\left(16-x^2\right)^{-\frac{1}{2}}\,dx$

$= -\dfrac{1}{2}\times 2\left(16-x^2\right)^{\frac{1}{2}} + c$

$= -\sqrt{16-x^2} + c$

**Answer B**

**Question 24**

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$ $\quad t = \tan\dfrac{x}{2}$, $dx = \dfrac{2dt}{1+t^2}$

$= \displaystyle\int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$

$= 2\displaystyle\int_0^1 \frac{1}{1+t^2+2t}\,dt$

$= 2\displaystyle\int_0^1 (t+1)^{-2}\,dt$

$= -2\left[\dfrac{1}{t+1}\right]_0^1$

$= -2\left(\dfrac{1}{2} - 1\right)$

$= 1$

**Question 25**

$\displaystyle\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$ $\quad u = e^x$, $du = e^x dx$

$= \displaystyle\int \frac{e^x}{\sqrt{1-u^2}} \times \frac{du}{e^x}$

$= \displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \sin^{-1} u + c$

$= \sin^{-1} e^x + c$

**Question 26**

$\displaystyle\int \frac{4x^3 - 2x^2 + 1}{2x-1}\,dx$

$= \displaystyle\int \frac{2x^2(2x-1)+1}{2x-1}\,dx$

$= \displaystyle\int \left(2x^2 + \frac{1}{2x-1}\right)dx$

$= \dfrac{2}{3}x^3 + \dfrac{1}{2}\ln|2x-1| + c$

**Question 27**

$\displaystyle\int \frac{dx}{\sqrt{3-4x-4x^2}}$

$= \displaystyle\int \frac{dx}{\sqrt{-(4x^2+4x+1-4)}}$

$= \displaystyle\int \frac{dx}{\sqrt{2^2-(2x+1)^2}}$

$= \dfrac{1}{2}\displaystyle\int \frac{2\,dx}{\sqrt{2^2-(2x+1)^2}}$

$= \dfrac{1}{2}\sin^{-1}\left(\dfrac{2x+1}{2}\right) + c$

**Question 28**

$\cos^4 x\sin^5 x$ is the product of an even and an odd function which is odd.

$\int_{-a}^{a} f(x)\,dx = 0$ for odd functions

$\therefore \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\sin^5 x\,dx = 0$

**Question 29**

$\displaystyle\int_0^1 \tan^{-1} x\,dx$ $\quad \begin{array}{ll} u = \tan^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x^2} & v = x \end{array}$

$= \left[x\tan^{-1} x\right]_0^1 - \displaystyle\int_0^1 \frac{x}{1+x^2}\,dx$

$= \dfrac{\pi}{4} - \dfrac{1}{2}\left[\ln\left|1+x^2\right|\right]_0^1$

$= \dfrac{\pi}{4} - \dfrac{1}{2}(\ln 2 - 0)$

$= \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$

**Question 30**

$$\int \frac{dx}{x^2-6x+13}$$

$$= \int \frac{dx}{(x-3)^2+2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c$$

**Question 31**

$$\sin 130° + \sin 10°$$

$$= \sin(70°+60°) + \sin(70°-60°)$$

$$= 2\sin 70°\cos 60°$$

$$= 2\sin 70° \times \frac{1}{2}$$

$$= \sin 70°$$

**Alt:**

$$\sin 130° + \sin 10°$$

$$= 2\sin\left(\frac{130+10}{2}\right)\cos\left(\frac{130-10}{2}\right)$$

$$= 2\sin 70°\cos 60°$$

$$= 2\sin 70° \times \frac{1}{2}$$

$$= \sin 70°$$

**Question 32**

$$f(x) = \sin 3x\cos x - \cos 2x \sin 2x$$

$$= \frac{1}{2}\left[\sin 4x + \sin 2x\right] - \frac{1}{2}\sin 4x$$

$$= \frac{1}{2}\sin 2x$$

The amplitude is $\frac{1}{2}$ and the period is $\frac{2\pi}{2} = \pi$.

**Question 33**

$$\frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{\left(1-\frac{1-t^2}{1+t^2}\right)}{\frac{2t}{1+t^2}}$$

$$= \frac{1+t^2-1+t^2}{2t}$$

$$= \frac{2t^2}{2t}$$

$$= t$$

**Answer B**

**Question 34**

$$\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}dx$$

$$= \int_1^4 \frac{x^3}{u^{\frac{3}{2}}} \times \frac{du}{2x}$$

$$= \frac{1}{2}\int_1^4 (u-1)u^{-\frac{3}{2}}du$$

$$= \frac{1}{2}\int_1^4 \left(u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\right)du$$

$$= \frac{1}{2}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]_1^4$$

$$= \frac{1}{2}\left((4+1)-(2+2)\right)$$

$$= \frac{1}{2}$$

**Question 35**

$$\int x\sqrt{x}\left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right)dx$$

$$= \int \left(x^2 + x^3\right)dx$$

$$= \frac{x^3}{3} + \frac{x^4}{4} + c$$

**Question 36**

$$\int_0^{\log_3 2} 3^{x\log_3 2}\,dx$$

$$= \int_0^{\log_3 2} 3^{\log_3 2^x}\,dx$$

$$= \int_0^{\log_3 2} 2^x\,dx$$

$$= \frac{1}{\ln 2}\left[2^x\right]_0^{\log_3 2}$$

$$= \frac{1}{\ln 2}\left(2^{\log_3 2} - 1\right)$$

**Question 37**

$$\frac{\sin B - \sin A}{\cos A - \cos B}$$

$$= -\frac{\sin A - \sin B}{\cos A - \cos B}$$

$$= -\frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$= \cot\left(\frac{A+B}{2}\right)$$

**Question 38**

$$\int_0^{\frac{\pi}{6}} \frac{d\theta}{9-8\cos^2\theta}$$

$$\begin{aligned} t &= \tan\theta \\ \frac{dt}{d\theta} &= \sec^2\theta \\ &= 1+t^2 \\ d\theta &= \frac{dt}{1+t^2} \end{aligned}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9-\frac{8}{\sec^2\theta}} \times \frac{dt}{1+t^2}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9-\frac{8}{1+t^2}} \times \frac{dt}{1+t^2}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9+9t^2-8} dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9t^2} dt$$

$$= \frac{1}{3}\int_0^{\frac{1}{\sqrt{3}}} \frac{3}{1+(3t)^2} dt$$

$$= \frac{1}{3}\left[\tan^{-1} 3t\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{1}{3}\left(\frac{\pi}{3}-0\right)$$

$$= \frac{\pi}{9}$$

**Question 39**

$$\int \frac{e^{2x}}{e^x+1} dx$$

$$\begin{aligned} u &= e^x \\ du &= e^x dx \\ dx &= \frac{du}{u} \end{aligned}$$

$$= \int \frac{u^2}{u+1} \times \frac{du}{u}$$

$$= \int \frac{u}{u+1} du$$

$$= \int \left(\frac{u+1}{u+1} - \frac{1}{u+1}\right) du$$

$$= \int \left(1 - \frac{1}{u+1}\right) du$$

$$= u - \ln(u+1) + c$$

$$= e^x - \ln|e^x+1| + c$$

**Question 40a**

$$a = \frac{8-2(-1)}{4+(-1)^2} = 2$$

equating coefficients of $x^2$: $a+b=0 \quad \Rightarrow \quad b=-2$

equating constants: $4a+c=8 \quad \Rightarrow \quad c=0$

$\therefore a=2, b=-2, c=0$

**Question 40b**

$$\int_0^4 \frac{8-2x}{(1+x)(4+x^2)} dx$$

$$= \int_0^4 \left(\frac{2}{1+x} - \frac{2x}{4+x^2}\right) dx$$

$$= \left[2\ln(1+x) - \ln\left(4+x^2\right)\right]_0^4$$

$$= (2\ln 5 - \ln 20) - (2\ln 1 - \ln 4)$$

$$= \ln 25 - \ln 20 + \ln 4$$

$$= \ln \frac{25\times 4}{20}$$

$$= \ln 5$$

**Question 41**

The 5 can be absorbed into the values of $A$ and $B$, so is not needed in either denominator as a factor. The easiest way to get the required denominator of $(x+h)^2$ is to have a constant over $(x+h)^2$ and another constant over $x+h$.

**Answer C**

**Question 42**

$$\int_{-1}^1 \frac{1}{1+e^{-x}} \times \frac{e^x}{e^x} dx$$

$$= \int_{-1}^1 \frac{e^x}{e^x+1} dx$$

$$= \left[\ln(e^x+1)\right]_{-1}^1$$

$$= \ln(e+1) - \ln(e^{-1}+1)$$

$$= \ln(e+1) - \ln\left(\frac{1+e}{e}\right)$$

$$= \ln\left(\frac{e+1}{(1+e)/e}\right)$$

$$= \ln e$$

$$= 1$$

**Answer B**

**Question 43**

$$\begin{aligned} 2\cos\theta + 3\sin\theta &= -2 \\ 2\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right) &= -2 \\ 2-2t^2+6t &= -2-2t^2 \\ 6t &= -4 \\ t &= -\frac{2}{3} \\ \tan\frac{\theta}{2} &= -\frac{2}{3} \\ \frac{\theta}{2} &= \pi - \tan^{-1}\left(\frac{2}{3}\right) \\ \theta &= 2\left(\pi - \tan^{-1}\left(\frac{2}{3}\right)\right) \\ &= 5.107180 \end{aligned}$$

Check $\theta = \pi$:

$2\cos\pi + 3\sin\pi = -2 + 0 = -2$ ✓

$\therefore t = 5.11\ (2\ dp), \pi$

**Question 44a**

$$I_n = \int_0^1 \sqrt{x}(1-x)^n\,dx$$

$$\begin{array}{|ll|}\hline u=(1-x)^n & \dfrac{dv}{dx}=x^{\frac{1}{2}} \\ \dfrac{du}{dx}=-n(1-x)^{n-1} & v=\dfrac{2}{3}x^{\frac{3}{2}} \\ \hline\end{array}$$

$$= \frac{2}{3}\left[x^{\frac{3}{2}}(1-x)^n\right]_0^1 + \frac{2n}{3}\int_0^1 x^{\frac{3}{2}}(1-x)^{n-1}\,dx$$

$$= 0 + \frac{2n}{3}\int_0^1 \sqrt{x}\cdot x(1-x)^{n-1}\,dx$$

$$= -\frac{2n}{3}\int_0^1 \sqrt{x}(1-x-1)(1-x)^{n-1}\,dx$$

$$= -\frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^n\,dx + \frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^{n-1}\,dx$$

$$\therefore\ I_n = -\frac{2n}{3}I_n + \frac{2n}{3}I_{n-1}$$

$$\frac{2n+3}{3}I_n = \frac{2n}{3}I_{n-1}$$

$$I_n = \left(\frac{2n}{2n+3}\right)I_{n-1}$$

**Question 44b**

$$I_3 = \left(\frac{2(3)}{2(3)+3}\right)I_2$$

$$= \frac{2}{3}\left(\frac{2(2)}{2(2)+3}\right)I_1$$

$$= \frac{8}{21}\left(\frac{2(1)}{2(1)+3}\right)I_0$$

$$= \frac{16}{105}\int_0^1 \sqrt{x}\,dx$$

$$= \frac{16}{105}\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^1$$

$$= \frac{32}{315}$$

**Question 45**

$$\sin 3x + \sin x + \sin 2x$$

$$= \sin(2x+x) + \sin(2x-x) + \sin 2x$$

$$= 2\sin 2x\cos x + \sin 2x$$

$$= \sin 2x\,(2\cos x + 1)$$

**Question 46a**

$$c = \frac{16}{2^2+4} = 2$$

equating coefficients of $x^2$:

$-a + c = 0 \quad \Rightarrow \quad a = 2$

equating constants:

$2b + 4c = 16 \quad \Rightarrow \quad b = 4$

$a = 2, b = 4, c = 2$

**Question 46b**

$$\int \frac{16}{(x^2+4)(2-x)}\,dx$$

$$= \int \left(\frac{2x+4}{x^2+4} + \frac{2}{2-x}\right)dx$$

$$= \int \left(\frac{2x}{x^2+4} + \frac{4}{x^2+4} + \frac{2}{2-x}\right)dx$$

$$= \ln\left|x^2+4\right| + 2\tan^{-1}\left(\frac{x}{2}\right) - 2\ln|2-x| + c$$

**Question 47**

$$\int_0^1 \sin^{-1}x\,dx$$

$$\begin{array}{|ll|}\hline u=\sin^{-1}x & \dfrac{dv}{dx}=1 \\ \dfrac{du}{dx}=\dfrac{1}{\sqrt{1-x^2}} & v=x \\ \hline\end{array}$$

$$= \left[x\sin^{-1}x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{2} - 0 + \frac{1}{2}\int_0^1 (-2x)\left(1-x^2\right)^{-\frac{1}{2}}\,dx$$

$$= \frac{\pi}{2} + \frac{1}{2}\times 2\left[\left(1-x^2\right)^{\frac{1}{2}}\right]_0^1$$

$$= \frac{\pi}{2} - 1$$

**Question 48**

$$\int_0^1 \frac{x}{(x+1)(2x+1)}\,dx$$

$$= \int_0^1 \left(\frac{1}{x+1} - \frac{1}{2x+1}\right)dx$$

$$= \left[\ln|x+1| - \frac{1}{2}\ln|2x+1|\right]_0^1$$

$$= \left(\ln 2 - \frac{1}{2}\ln 3\right) - \left(\ln 1 - \frac{1}{2}\ln 1\right)$$

$$= \ln\frac{2}{\sqrt{3}}$$

**Question 49**

$$\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$$

$$\begin{array}{|ll|}\hline u=x & \dfrac{dv}{dx}=\sec^2 x \\ \dfrac{du}{dx}=1 & v=\tan x \\ \hline\end{array}$$

$$= \left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,dx$$

$$= \frac{\pi}{4} - 0 + \left[\ln|\cos x|\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \ln\left|\frac{1}{\sqrt{2}}\right|$$

$$= \frac{\pi}{4} + \ln 1 - \ln\sqrt{2}$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

**Answer A**

**Question 50**

$$\int (\tan^3 2x + \tan 2x)\,dx$$
$$= \int \tan 2x\,(\tan^2 2x + 1)\,dx$$
$$= \frac{1}{2}\int 2\sec^2 2x\,(\tan 2x)^1\,dx$$
$$= \frac{\tan^2 2x}{2\times 2} + c$$
$$= \frac{1}{4}\tan^2 2x + c$$

**Answer C**

**Question 51**

$$\cos x - 3\sin x + 3 = 0$$
$$\frac{1-t^2}{1+t^2} - 3\left(\frac{2t}{1+t^2}\right) + 3 = 0$$
$$1 - t^2 - 6t + 3 + 3t^2 = 0$$
$$2t^2 - 6t + 4 = 0$$
$$t^2 - 3t + 2 = 0$$
$$(t-1)(t-2) = 0$$
$$t = 1, 2$$
$$\tan\frac{x}{2} = 1 \qquad \tan\frac{x}{2} = 2$$
$$\frac{x}{2} = \frac{\pi}{4} \qquad \frac{x}{2} = \tan^{-1}(2)$$
$$x = \frac{\pi}{2} \qquad x = 2.214297$$

Check $\theta = \pi$:

$\cos\pi - 3\sin\pi + 3 = -1 + 0 + 3 = 2 \neq 0$ ✗

$\therefore x = \frac{\pi}{2}, 2.21$ (2 dp)

**Question 52**

$$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin\theta}\,d\theta \qquad \boxed{\begin{array}{c} t = \tan\frac{\theta}{2} \\ d\theta = \frac{2dt}{1+t^2} \end{array}}$$
$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1 - \frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+t^2-2t}\,dt$$
$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(t-1)^2}\,dt$$
$$= -2\left[\frac{1}{t-1}\right]_0^{\frac{1}{\sqrt{3}}}$$
$$= -2\left(\frac{1}{\frac{1}{\sqrt{3}} - 1} + 1\right)$$
$$= -2\left(\frac{\sqrt{3}}{1-\sqrt{3}} + \frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$$
$$= -2\left(\frac{1}{1-\sqrt{3}}\right) \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$
$$= -\frac{2(1+\sqrt{3})}{1-3}$$
$$= \sqrt{3} + 1$$

**Question 53**

$$\int \sqrt{\frac{5-x}{5+x}}\,dx$$
$$= \int \sqrt{\frac{5-x}{5+x}} \times \sqrt{\frac{5-x}{5-x}}\,dx$$
$$= \int \frac{5-x}{\sqrt{25-x^2}}\,dx$$
$$= \int \frac{5}{\sqrt{25-x^2}}\,dx + \frac{1}{2}\int (-2x)\left(25-x^2\right)^{-\frac{1}{2}}\,dx$$
$$= 5\sin^{-1}\left(\frac{x}{5}\right) + \frac{1}{2}\times 2\left(25-x^2\right)^{\frac{1}{2}} + c$$
$$= 5\sin^{-1}\left(\frac{x}{5}\right) + \sqrt{25-x^2} + c$$

**Question 54a**

$$A = \frac{10}{1+(-3)^2} = 1$$

equating coefficients of $x^2$:

$A + B = 0 \quad \Rightarrow \quad B = -1$

equating constants:

$A + 3C = 10 \quad \Rightarrow \quad C = 3$

$A = 1, B = -1, C = 3$

**Question 54b**

$$\int \frac{10}{3+\tan\theta}\,d\theta \qquad \boxed{\begin{array}{c} t = \tan\theta \\ d\theta = \frac{dt}{1+t^2} \end{array}}$$
$$= \int \frac{10}{3+t} \times \frac{dt}{1+t^2}$$
$$= \int \frac{10}{(3+t)(1+t^2)}\,dt$$
$$= \int \left(\frac{1}{3+t} - \frac{t-3}{1+t^2}\right)dt$$
$$= \int \left(\frac{1}{3+t} - \frac{t}{1+t^2} + \frac{3}{1+t^2}\right)dt$$
$$= \ln|3+t| - \frac{1}{2}\ln\left|1+t^2\right| + 3\tan^{-1} t + c$$
$$= \ln|3+\tan\theta| - \frac{1}{2}\ln\left|1+\tan^2\theta\right| + 3\tan^{-1}(\tan\theta) + c$$
$$= \ln|\tan\theta + 3| - \frac{1}{2}\ln\left|\sec^2\theta\right| + 3\theta + c$$

**Question 55a**

$$I_0 = \int_0^1 \frac{1}{\sqrt{1+x}}\,dx$$
$$= \int_0^1 (1+x)^{-\frac{1}{2}}\,dx$$
$$= \left[\frac{(1+x)^{\frac{1}{2}}}{1/2}\right]_0^1$$
$$= 2\left(\sqrt{2}-1\right)$$
$$= 2\sqrt{2} - 2$$

**Question 55b**

$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}dx$

| | |
|---|---|
| $u = x^n$ | $\frac{dv}{dx} = \frac{1}{\sqrt{1+x}}$ |
| $\frac{du}{dx} = nx^{n-1}$ | $v = 2(1+x)^{\frac{1}{2}}$ |

$= 2\left[x^n(1+x)^{\frac{1}{2}}\right]_0^1 - 2n\int_0^1 x^{n-1}(1+x)^{\frac{1}{2}}dx$

$= 2\sqrt{2} - 0 - 2n\int_0^1 x^{n-1}\sqrt{1+x}\,dx$

$= 2\sqrt{2} - 2n(I_n + I_{n-1})$ given

$\therefore (2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$

$I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$

**Question 55c**

$I_1 = \frac{2\sqrt{2} - 2(1)I_0}{2(1)+1}$

$= \frac{2\sqrt{2} - 2\left(2\sqrt{2} - 2\right)}{3}$

$= \frac{4 - 2\sqrt{2}}{3}$

$I_2 = \frac{2\sqrt{2} - 2(2)I_1}{2(2)+1}$

$= \frac{2\sqrt{2} - 4\left(\frac{4-2\sqrt{2}}{3}\right)}{5}$

$= \frac{6\sqrt{2} - 16 + 8\sqrt{2}}{15}$

$= \frac{14\sqrt{2} - 16}{15}$

**Question 56**

$\int_1^2 \frac{dx}{x(1+x^2)}$

$= \int_1^2 \left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$

$= \left[\ln|x| - \frac{1}{2}\ln\left|1+x^2\right|\right]_1^2$

$= \left(\ln 2 - \frac{1}{2}\ln 5\right) - \left(\ln 1 - \frac{1}{2}\ln 2\right)$

$= \frac{3}{2}\ln 2 - \frac{1}{2}\ln 5$

**Question 57**

$\int \frac{x}{\sqrt{1-x}}dx$

$u^2 = 1 - x$

$2u\,du = -dx$

$dx = -2u\,du$

$= \int \frac{(1-u^2)}{u})(-2u\,du)$

$= 2\int (u^2 - 1)\,du$

$= \frac{2u^3}{3} - 2u + C$

$= \frac{2\sqrt{(1-x)^3}}{3} - 2\sqrt{1-x} + c$

**Question 58**

$I = \int e^{-2x}\cos x\,dx$

| | |
|---|---|
| $u = \cos x$ | $\frac{dv}{dx} = e^{-2x}$ |
| $\frac{du}{dx} = -\sin x$ | $v = -\frac{1}{2}e^{-2x}$ |

$I = -\frac{1}{2}e^{-2x}\cos x - \frac{1}{2}\int e^{-2x}\sin x\,dx$

| | |
|---|---|
| $u = \sin x$ | $\frac{dv}{dx} = e^{-2x}$ |
| $\frac{du}{dx} = \cos x$ | $v = -\frac{1}{2}e^{-2x}$ |

$= -\frac{1}{2}e^{-2x}\cos x$

$-\frac{1}{2}\left(-\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\int e^{-2x}\cos x\,dx\right)$

$= -\frac{1}{2}e^{-2x}\cos x + \frac{1}{4}e^{-2x}\sin x - \frac{1}{4}I + c$

$\frac{5}{4}I = -\frac{1}{2}e^{-2x}\cos x + \frac{1}{4}e^{-2x}\sin x + c$

$I = \frac{e^{-2x}(\sin x - 2\cos x)}{5} + c$

**Question 59**

for $0 < x < 1 \quad x^n > x^{n+1}$

$\therefore \frac{1}{1+x^n} < \frac{1}{1+x^{n+1}}$ $\therefore$ statement I is true

$\sin x$ and $\cos x$ are symmetrical about $\frac{\pi}{4}$,

so $\int_0^{\frac{\pi}{2}} \sin x\,dx = \int_0^{\frac{\pi}{2}} \cos x\,dx$ $\therefore$ statement II is true

**Answer D**

**Question 60**

$\int_0^1 \cos^{-1} x\,dx$

| | |
|---|---|
| $u = \cos^{-1} x$ | $\frac{dv}{dx} = 1$ |
| $\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$ | $v = x$ |

$= \left[x\cos^{-1} x\right]_0^1 + \int_0^1 \frac{x}{\sqrt{1-x^2}}dx$

$= 0 - 0 - \frac{1}{2}\int_0^1 (-2x)\left(1-x^2\right)^{-\frac{1}{2}}dx$

$= -\frac{1}{2}\times 2\left[\left(1-x^2\right)^{\frac{1}{2}}\right]_0^1$

$= -(0-1)$

$= 1$

**Question 61a**

$I_1 = \int_0^1 \frac{x}{\sqrt{1-x}}dx$

$u^2 = 1 - x$

$2u\,du = -dx$

$dx = -2u\,du$

$= \int_1^0 \frac{1-u^2}{u} \times (-2u\,du)$

$= 2\int_0^1 (1-u^2)\,du$

$= 2\left[u - \frac{u^3}{3}\right]_0^1$

$= 2\left(1 - \frac{1}{3}\right)$

$= \frac{4}{3}$

**Question 61b**

$$
\begin{aligned}
&I_{n-1} - I_n \\
&= \int_0^1 \frac{x^{n-1}}{\sqrt{1-x}}\,dx - \int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx \\
&= \int_0^1 \frac{x^{n-1}(1-x)}{\sqrt{1-x}}\,dx \\
&= \int_0^1 x^{n-1}\sqrt{1-x}\,dx
\end{aligned}
$$

**Question 61c**

$$
\begin{aligned}
&I_{n-1} - I_n \\
&= \int_0^1 x^{n-1}\sqrt{1-x}\,dx
\end{aligned}
$$

$$
\begin{array}{ll}
u = \sqrt{1-x} & \dfrac{dv}{dx} = x^{n-1} \\
\dfrac{du}{dx} = -\dfrac{1}{2}(1-x)^{-\frac{1}{2}} & v = \dfrac{x^n}{n} \\
\quad = -\dfrac{1}{2\sqrt{1-x}} &
\end{array}
$$

$$
\begin{aligned}
&= \frac{1}{n}\left[x^n\sqrt{1-x}\right]_0^1 + \frac{1}{2n}\int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx \\
&I_{n-1} - I_n = \frac{1}{2n} I_n \\
&2nI_{n-1} - 2nI_n = I_n \\
&(2n+1)I_n = 2nI_{n-1} \\
&I_n = \frac{2n}{2n+1} I_{n-1}
\end{aligned}
$$

**Question 62**

$$
\int \sec x\,dx \qquad \boxed{\begin{array}{l} t = \tan\dfrac{x}{2} \\ dx = \dfrac{2dt}{1+t^2} \end{array}}
$$

$$
\begin{aligned}
&= \int \frac{1+t^2}{1-t^2}\left(\frac{2dt}{1+t^2}\right) \\
&= 2\int \frac{1}{1-t^2}\,dt \\
&= \int \left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt \\
&= \ln|1+t| - \ln|1-t| + c \\
&= \ln\left|\frac{1+t}{1-t}\right| + c
\end{aligned}
$$

**Answer B**

**Question 63**

$$
\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx \qquad \boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = \tan^2 x \\ & \quad = \sec^2 x - 1 \\ \dfrac{du}{dx} = 1 & v = \tan x - x \end{array}}
$$

$$
\begin{aligned}
&= \left[x(\tan x - x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (\tan x - x)\,dx \\
&= \left(\frac{\pi}{4}\left(1 - \frac{\pi}{4}\right)\right) - 0 - \left[-\ln|\cos x| - \frac{x^2}{2}\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{4}\left(1 - \frac{\pi}{4}\right) - \left(\left(-\ln\frac{1}{\sqrt{2}} - \frac{\pi^2}{32}\right) - (0-0)\right) \\
&= \frac{\pi}{4} - \frac{\pi^2}{16} + \frac{\pi^2}{32} - \frac{1}{2}\ln 2 \\
&= \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\ln 2
\end{aligned}
$$

**Question 64**

$$
\begin{aligned}
&\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx \\
&= 2\int \left(\frac{1}{2}x^{-\frac{1}{2}}\right)\cos\sqrt{x}\,dx \\
&= 2\sin\sqrt{x} + c
\end{aligned}
$$

**Answer B**

**Question 65a**

$$
B = \frac{3(1)^2 + 3(1) - 2}{(1+1)^2} = 1
$$

equating coefficients of $x^2$:

$$
\begin{aligned}
&A + B = 3 \quad \Rightarrow \quad A = 2 \\
&A = 2, B = 1
\end{aligned}
$$

**Question 65b**

$$
\begin{aligned}
&\int \frac{3x^2 + 3x - 2}{(x-1)(x+1)^2}\,dx \\
&= \int \left(\frac{2}{x+1} + \frac{1}{x-1} + \frac{1}{(x+1)^2}\right)dx \\
&= 2\ln|x+1| + \ln|x-1| - \frac{1}{x+1} + c \\
&= \ln\left|(x+1)^2(x-1)\right| - \frac{1}{x+1} + c
\end{aligned}
$$

**Question 66**

$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx \qquad \begin{array}{l} x = 2\cos\theta \\ \frac{dx}{d\theta} = -2\sin\theta \\ dx = -2\sin\theta\,d\theta \end{array}$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{1}{4\cos^2\theta\sqrt{4-4\cos^2\theta}}(-2\sin\theta\,d\theta)$$
$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos^2\theta\sin\theta} \times \sin\theta\,d\theta$$
$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2\theta\,d\theta$$
$$= \frac{1}{4}\Big[\tan\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \frac{1}{4}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{4}\left(\frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{2\sqrt{3}}$$
$$= \frac{\sqrt{3}}{6}$$

**Question 67a**

$$\frac{x^2-x+1}{x^2+2x+1} = \frac{1(x^2+2x+1) - 3(x+1) + 3}{x^2+2x+1}$$
$$= 1 - \frac{3}{x+1} + \frac{3}{(x+1)^2}$$

$A = 1, B = -3, C = 3$

**Question 67b**

$$\int \frac{x^2-x+1}{(x+1)^2}\,dx$$
$$= \int \left(1 - \frac{3}{x+1} + \frac{3}{(x+1)^2}\right)dx$$
$$= x - 3\ln|x+1| - \frac{3}{x+1} + c$$

**Question 68**

$$\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx \qquad \begin{array}{ll} u = x & \frac{dv}{dx} = \sec^2 x \\ \frac{du}{dx} = 1 & v = \tan x \end{array}$$

$$= \Big[x\tan x\Big]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan x\,dx$$
$$= \frac{\pi\sqrt{3}}{3} + \Big[\ln|\cos x|\Big]_0^{\frac{\pi}{3}}$$
$$= \frac{\pi\sqrt{3}}{3} + \ln\frac{1}{2} - \ln 1$$
$$= \frac{\pi\sqrt{3}}{3} - \ln 2$$

**Question 69a**

$$I_n = \int_0^{\pi} x^n \sin x\,dx \qquad \begin{array}{ll} u = x^n & \frac{dv}{dx} = \sin x \\ \frac{du}{dx} = nx^{n-1} & v = -\cos x \end{array}$$

$$= -\Big[x^n\cos x\Big]_0^{\pi} + n\int_0^{\pi} x^{n-1}\cos x\,dx$$
$$= -(-\pi^n - 0) + n\int_0^{\pi} x^{n-1}\cos x\,dx$$

$$\begin{array}{ll} u = x^{n-1} & \frac{dv}{dx} = \cos x \\ \frac{du}{dx} = (n-1)x^{n-2} & v = \sin x \end{array}$$

$$= \pi^n + n\left\{\Big[x^{n-1}\sin x\Big]_0^{\pi} - (n-1)\int x^{n-2}\sin x\,dx\right\}$$
$$I_n = \pi^n + n\{0 - (n-1)I_{n-2}$$
$$= \pi^n - n(n-1)I_{n-2}$$

**Question 69b**

$$\int_0^{\pi} x^4\sin x\,dx$$
$$= I_4$$
$$= \pi^4 - 4(3)I_2$$
$$= \pi^4 - 12\left(\pi^2 - 2(1)I_0\right)$$
$$= \pi^4 - 12\pi^2 + 24\int_0^{\pi}\sin x\,dx$$
$$= \pi^4 - 12\pi^2 + 24\Big[-\cos x\Big]_0^{\pi}$$
$$= \pi^4 - 12\pi^2 + 24(1+1)$$
$$= \pi^4 - 12\pi^2 + 48$$

**Question 70**

$$I - J$$
$$= \int_0^{\ln 2} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$
$$= \Big[\ln|e^x + e^{-x}|\Big]_0^{\ln 2}$$
$$= \ln\left(2 + \frac{1}{2}\right) - \ln(1+1)$$
$$= \ln\left(\frac{5}{4}\right)$$

**Answer D**

**Question 71**

$$f(t) + g(t)$$
$$= 3\sin 4t + 3\sin\left(4t + \frac{\pi}{3}\right)$$
$$= 3\left(\sin 4t + \sin\left(4t + \frac{\pi}{3}\right)\right)$$
$$= 3\left(\sin\left(4t + \frac{\pi}{6} - \frac{\pi}{6}\right) + \sin\left(4t + \frac{\pi}{6} + \frac{\pi}{6}\right)\right)$$
$$= 3\left(2\sin\left(4t + \frac{\pi}{6}\right)\cos\frac{\pi}{6}\right)$$
$$= 6\sin\left(4t + \frac{\pi}{6}\right) \times \frac{\sqrt{3}}{2}$$
$$= 3\sqrt{3}\sin\left(4t + \frac{\pi}{6}\right)$$

**Question 72a**

i $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$$\begin{array}{|ll|}\hline u = \sin^{n-1} x & \dfrac{dv}{dx} = \sin x \\ \dfrac{du}{dx} = (n-1)\sin^{n-2} x \cos x & v = -\cos x \\ \hline\end{array}$$

$$= -\left[\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\, dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,(1-\sin^2 x)\, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

$\therefore I_n = (n-1)I_{n-2} - (n-1)I_n$

$nI_n = (n-1)I_{n-2}$

$I_n = \dfrac{n-1}{n} I_{n-2}$

**Question 72b**

$\int_0^{\frac{\pi}{2}} \sin^5 x\, dx$

$= I_5$

$= \frac{4}{5} I_3$

$= \frac{4}{5} \times \frac{2}{3} I_1$

$= \frac{8}{15} \int_0^{\frac{\pi}{2}} \sin x\, dx$

$= \frac{8}{15}\left[-\cos x\right]_0^{\frac{\pi}{2}}$

$= \frac{8}{15}(0+1)$

$= \frac{8}{15}$

**Question 73**

$\int \sin^3 x\, dx$

$= \int \sin^2 x \sin x\, dx$

$= \int (1-\cos^2 x)\sin x\, dx$

$= \int \sin x\, dx - \int \sin x \cos^2 x\, dx$

$= -\cos x + \dfrac{\cos^3 x}{3} + c$

**Answer B**

**Question 74**

$$\begin{aligned} \sin 4x + \sin 3x + \sin 2x &= 0 \\ \sin 4x + \sin 2x + \sin 3x &= 0 \\ \sin(3x+x) + \sin(3x-x) + \sin 3x &= 0 \\ 2\sin 3x \cos x + \sin 3x &= 0 \\ \sin 3x\,(2\cos x + 1) &= 0 \\ \sin 3x = 0 \quad 2\cos x + 1 &= 0 \\ 3x = 0, \pi, 2\pi, 3\pi \quad \cos x &= -\frac{1}{2} \\ x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi \qquad x &= \pi - \frac{\pi}{3} \\ &= \frac{2\pi}{3} \end{aligned}$$

$\therefore x = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi$

**Question 75**

$\int x\sqrt{x-1}\, dx$ $\qquad \begin{array}{|c|}\hline u^2 = x-1 \\ 2u\, du = dx \\ \hline\end{array}$

$= \int (u^2+1)u \times 2u\, du$

$= 2\int (u^4 + u^2)\, du$

$= \dfrac{2u^5}{5} + \dfrac{2u^3}{3} + c$

$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + c$

**Question 76**

$\int \dfrac{x}{\sqrt{2-x^2}} dx$ $\qquad \begin{array}{|c|}\hline x = \sqrt{2}\sin\theta \\ dx = \sqrt{2}\cos\theta\, d\theta \\ \hline\end{array}$

$= \int \dfrac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}} \times \sqrt{2}\cos\theta\, d\theta$

$= \int \dfrac{\sqrt{2}\sin\theta}{\sqrt{2}\times\sqrt{1-\sin^2\theta}} \times \sqrt{2}\cos\theta\, d\theta$

$= \sqrt{2}\int \dfrac{\sin\theta}{\cos\theta} \times \cos\theta\, d\theta$

$= \sqrt{2}\int \sin\theta\, d\theta$

$= -\sqrt{2}\cos\theta + c$

$= -\sqrt{2}\sqrt{1-\sin^2\theta} + c$

$= -\sqrt{2}\times\sqrt{1-\dfrac{x^2}{2}} + c$

$= -\sqrt{2-x^2} + c$

**Question 77**

$\int \dfrac{\cos^3 x + \sin^3 x}{\cos x + \sin x} dx$

$= \int \dfrac{(\cos x + \sin x)(\cos^2 - \cos x \sin x + \sin^2 x)}{\cos x + \sin x} dx$

$= \int (1 - \cos x \sin x)\, dx$

$= \int \left(1 - \frac{1}{2}\sin 2x\right) dx$

$= x + \frac{1}{4}\cos 2x + c$

**Answer A**

**Question 78**

$$\int \frac{\sin 2x + \sin x}{\cos^2 x}\,dx$$
$$= \int \left(\frac{2\sin x \cos x}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$$
$$= \int (2\tan x + \sin x\,(\cos x)^{-2})\,dx$$
$$= -2\ln|\cos x| + \frac{1}{\cos x} + c$$
$$= -2\ln|\cos x| + \sec x + c$$

**Question 79**

$$\int e^x \sin x\,dx \qquad \begin{array}{ll} u = \sin x & \frac{dv}{dx} = e^x \\ \frac{du}{dx} = \cos x & v = e^x \end{array}$$
$$= e^x \sin x$$
$$- \int e^x \cos x\,dx \qquad \begin{array}{ll} u = \cos x & \frac{dv}{dx} = e^x \\ \frac{du}{dx} = -\sin x & v = e^x \end{array}$$
$$= e^x \sin x - \left\{e^x \cos x + \int e^x \sin x\,dx\right\}$$
$$\therefore 2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$$
$$\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2}$$

**Question 80**

$$\int \sin 2x \sin 4x\,dx$$
$$= \frac{1}{2}\int \Big[\cos(2x - 4x) - \cos(2x + 4x)\Big]\,dx$$
$$= \frac{1}{2}\int \Big[\cos(-2x) - \cos 6x\Big]\,dx$$
$$= \frac{1}{2}\int \Big[\cos 2x - \cos 6x\Big]\,dx$$
$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{6}\sin 6x\right] + c$$
$$= \frac{\sin 2x}{4} - \frac{\sin 6x}{12} + c$$

**Question 81**

$$\int \frac{\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)}{\cos\left(x - \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)}\,dx$$
$$= \int \frac{2\sin x \cos\frac{\pi}{4}}{2\cos x \cos\frac{\pi}{4}}\,dx$$
$$= \int \frac{\sin x}{\cos x}\,dx$$
$$= -\ln|\cos x| + c$$

**Question 82**

$$\int \frac{x}{\sqrt{x+5}}\,dx \qquad \begin{array}{l} u^2 = x + 5 \\ 2u\,du = dx \end{array}$$
$$= \int \frac{u^2 - 5}{u} \times 2u\,du$$
$$= 2\int (u^2 - 5)\,du$$
$$= \frac{2u^3}{3} - 10u + C$$
$$= \frac{2\sqrt{(x+5)^3}}{3} - 10\sqrt{x+5} + c$$
$$= \frac{2(x+5)\sqrt{x+5}}{3} - 10\sqrt{x+5} + c$$
$$= \frac{2}{3}\sqrt{x+5}(x + 5 - 15) + c$$
$$= \frac{2}{3}(x - 10)\sqrt{x+5} + c$$

**Answer D**

**Question 83**

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1 + \cos x + \sin x}$$
$$= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{1 + \cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)}\,dx$$
$$= \int_o^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{1 + \sin x + \cos x}\,dx$$
$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x + \sin x} - \int_0^{\frac{\pi}{2}} \frac{x\,dx}{1 + \cos x + \sin x}$$
$$\therefore 2\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1 + \cos x + \sin x} = \frac{\pi}{2}\ln 2$$
$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1 + \cos x + \sin x} = \frac{\pi}{4}\ln 2$$

**Question 84**

$$\int \frac{dx}{x^3\sqrt{x^2 - 4}} \qquad \begin{array}{l} x = 2\sec\theta \\ dx = 2\sec\theta\tan\theta\,d\theta \end{array}$$
$$= \int \frac{1}{8\sec^3\theta\sqrt{4\sec^2\theta - 4}} \times 2\sec\theta\tan\theta\,d\theta$$
$$= \int \frac{2\sec\theta\tan\theta}{8\sec^3\theta\,2\tan\theta}\,d\theta$$
$$= \frac{1}{8}\int \cos^2\theta\,d\theta$$
$$= \frac{1}{8}\int \left(\frac{1}{2}(1 + \cos 2\theta)\right) d\theta$$
$$= \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + c$$
$$= \frac{1}{16}\sec^{-1}\frac{x}{2} + \frac{1}{16} \times \frac{\sqrt{x^2 - 4}}{x} \times \frac{2}{x} + c$$
$$= \frac{1}{16}\cos^{-1}\frac{2}{x} + \frac{\sqrt{x^2 - 4}}{8x^2} + c$$

**Question 85a**

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx \qquad \boxed{\begin{array}{l} u = \pi - x \\ du = -dx \\ dx = -du \end{array}}$$

$$= \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}(-du)$$

$$= \int_0^{\pi} \frac{(\pi-u)(\sin u)}{1+(-\cos u)^2}\,du$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$$

**Question 85b**

$$\therefore \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$$

$$= \int_0^{\pi} \left(\frac{\pi \sin x}{1+\cos^2 x} - \frac{x \sin x}{1+\cos^2 x}\right) du$$

$$2\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx \qquad \boxed{\begin{array}{l} v = \cos x \\ \dfrac{dv}{dx} = -\sin x \\ dx = -\dfrac{dv}{\sin x} \end{array}}$$

$$= \pi \int_1^{-1} \frac{\sin x}{1+v^2} \times \left(-\frac{dv}{\sin x}\right)$$

$$= \pi \int_{-1}^{1} \frac{1}{1+v^2}\,dv$$

$$= \pi \left[\tan^{-1} v\right]_{-1}^{1}$$

$$= \pi \left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi^2}{2}$$

$$\therefore \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$$

**Question 86**

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}\,dx$$

$$\therefore 2\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$= \left[x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$$

**Answer D**

**Question 87a**

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$$

$$= \int_{\frac{\pi}{4}}^{0} \ln\left(1+\tan\left(\frac{\pi}{4}-u\right)\right)(-du)$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\tan\frac{\pi}{4}-\tan u}{1+\tan\frac{\pi}{4}\tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan u}{1+\tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\tan u+1-\tan u}{1+\tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right) dx$$

**Question 87b**

$$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\ln 2 - \ln(1+\tan x)\right) dx$$

$$2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,dx$$

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{1}{2}\ln 2\left[x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\ln 2 \times \frac{\pi}{4}$$

$$= \frac{\pi \ln 2}{8}$$

**Question 88**

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\,dx$$

$$= \int (e^x - e^{-x})(e^x + e^{-x})^{-2}\,dx$$

$$= -\frac{1}{e^x + e^{-x}} + \boldsymbol{c}$$

**Question 89a**

$$I_n - I_{n-1}$$

$$= \int_0^{\frac{\pi}{6}} (\sin^{2n}\theta \sec\theta - \sin^{2(n-1)}\sec\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sin^{2n-2}\theta \sec\theta\,(\sin^2\theta - 1)\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sin^{2n-2}\theta \sec\theta\,(-\cos^2\theta)\,d\theta$$

$$= -\int_0^{\frac{\pi}{6}} \cos\theta \sin^{2n-2}\theta\,d\theta$$

$$= -\left[\frac{\sin^{2n-1}\theta}{2n-1}\right]_0^{\frac{\pi}{6}}$$

$$= -\frac{\left(\frac{1}{2}\right)^{2n-1}}{2n-1} - 0$$

$$= -\frac{1}{2^{2n-1}(2n-1)}$$

**Question 89b**

$$\therefore \sum_{k=1}^{n} (I_k - I_{k-1}) = \sum_{k=1}^{n} \frac{-1}{2^{2k-1}(2k-1)}$$

$$(I_1 - I_0) + (I_2 - I_1) + (I_3 - I_2) + \ldots + (I_n - I_{n-1})$$

$$= -\sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)}$$

$$I_n - I_0 = -\sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)}$$

$$I_n = I_0 - \sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)}$$

$$= \int_0^{\frac{\pi}{6}} \sec\theta\,d\theta - \sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)}$$

$$= \left[\ln|\sec\theta + \tan\theta|\right]_0^{\frac{\pi}{6}} - \sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)}$$

$$= \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \ln\left|1+0\right| - \sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)}$$

$$= \ln\sqrt{3} - \sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)}$$

$$= \frac{1}{2}\ln 3 - \sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)}$$

**Question 90a**

$$I_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot x\,dx$$

$$= \left[\ln|\sin x|\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \ln\frac{1}{\sqrt{2}} - \ln\frac{1}{2}$$

$$= \ln 1 - \ln\sqrt{2} - \ln 1 + \ln 2$$

$$= 0 - \frac{1}{2}\ln 2 - 0 + \ln 2$$

$$= \frac{1}{2}\ln 2$$

**Question 90b**

$$I_{n-2} + I_n$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cot^{n-2}x + \cot^n x)\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^{n-2}x(1+\cot^2 x)\,dx$$

$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (-\operatorname{cosec}^2 x)\cot^{n-2}x\,dx$$

$$= -\left[\frac{\cot^{n-1}x}{n-1}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= -\frac{1}{n-1}\left(1 - \left(\sqrt{3}\right)^{n-1}\right)$$

$$= \frac{\left(\sqrt{3}\right)^{n-1} - 1}{n-1}$$

$$= \frac{1}{n-1}\left(3^{\frac{1}{2}(n-1)} - 1\right)$$

**Question 90c**

$$I_5 = \frac{1}{5-1}\left(3^{\frac{1}{2}(5-1)} - 1\right) - I_3$$
$$= 2 - \left(\frac{1}{3-1}\left(3^{\frac{1}{2}(3-1)} - 1\right) - I_1\right)$$
$$= 1 + \frac{1}{2}\ln 2$$

**Question 91**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} \qquad \boxed{\begin{aligned} t &= \tan\frac{\theta}{2} \\ d\theta &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int_0^1 \frac{1}{4\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) + 6} \times \frac{2dt}{1+t^2}$$
$$= 2\int_0^1 \frac{1}{8t - 2 + 2t^2 + 6 + 6t^2}\,dt$$
$$= 2\int_0^1 \frac{1}{8t^2 + 8t + 4}\,dt$$
$$= \int_0^1 \frac{1}{4t^2 + 4t + 2}\,dt$$
$$= \frac{1}{2}\int_0^1 \frac{2}{(2t+1)^2 + 1^2}\,dt$$
$$= \frac{1}{2}\left[\tan^{-1}(2t+1)\right]_0^1$$
$$= \frac{1}{2}(\tan^{-1}3 - \tan^{-1}1)$$
$$= \frac{1}{2}\tan^{-1}(\tan(\tan^{-1}3 - \tan^{-1}1))$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{\tan(\tan^{-1}3) - \tan(\tan^{-1}1)}{1 + \tan(\tan^{-1}3)\tan(\tan^{-1}1)}\right)$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{3-1}{1+(3)(1)}\right)$$
$$= \frac{1}{2}\tan^{-1}\frac{1}{2}$$

**Question 92**

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$$

$$\boxed{\begin{aligned} x &= \frac{1-u}{1+u} \\ \frac{dx}{du} &= \frac{(1+u)(-1) - (1-u)(1)}{(1+u)^2} \\ &= -\frac{2}{(1+u)^2} \\ dx &= -\frac{2du}{(1+u)^2} \end{aligned}}$$

$$= \int_1^0 \frac{\ln\left(1 + \frac{1-u}{1+u}\right)}{1 + \frac{(1-u)^2}{(1+u)^2}} \times \left(-\frac{2du}{(1+u)^2}\right)$$
$$= 2\int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{1 + 2u + u^2 + 1 - 2u + u^2}\,du$$
$$= \int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{u^2+1}\,du$$
$$= \int_0^1 \left(\frac{\ln 2}{u^2+1} - \frac{\ln(1+u)}{u^2+1}\right)du$$
$$\therefore I = \ln 2\int_0^1 \frac{1}{u^2+1}\,du - I$$
$$2I = \ln 2\left[\tan^{-1}u\right]_0^1$$
$$I = \frac{\ln 2}{2}\left(\frac{\pi}{4} - 0\right)$$
$$= \frac{\pi\ln 2}{8}$$

**Question 93a**

$$\frac{5-5x^2}{(1+2x)(1+x^2)} = \frac{a}{1+2x} + \frac{bx+c}{1+x^2}$$
$$a = \frac{5 - 5\left(-\frac{1}{2}\right)^2}{1 + \left(-\frac{1}{2}\right)^2} = 3$$

equating coefficients of $x^2$: $a + 2b = -5 \quad \Rightarrow \quad b = -4$
equating constants: $a + c = 5 \quad \Rightarrow \quad c = 2$

$$\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx$$
$$= \int_0^1 \left(\frac{3}{1+2x} - \frac{4x-2}{1+x^2}\right)dx$$
$$= \int_0^1 \left(\frac{3}{1+2x} - \frac{4x}{1+x^2} + \frac{2}{1+x^2}\right)dx$$
$$= \left[\frac{3}{2}\ln|1+2x| - 2\ln\left|1+x^2\right| + 2\tan^{-1}x\right]_0^1$$
$$= \left(\frac{3}{2}\ln 3 - 2\ln 2 + \frac{\pi}{2}\right) - \left(\frac{3}{2}\ln 1 - 2\ln 1 + 0\right)$$
$$= \frac{1}{2}(\ln 3^3 - \ln 2^4 + \pi)$$
$$= \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$$

**Question 93b**

$$\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1+2\sin 2x+\cos 2x}\,dx$$

$$\begin{aligned} t &= \tan x \\ \frac{dt}{dx} &= \sec^2 x \\ dx &= \frac{dt}{\sec^2 x} \\ &= \frac{dt}{1+\tan^2 x} \\ &= \frac{dt}{1+t^2} \end{aligned}$$

$$= \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+\frac{4t}{1+t^2}+\frac{1-t^2}{1+t^2}} \times \frac{dt}{1+t^2}$$
$$= \int_0^1 \frac{1-t^2}{1+t^2+4t+1-t^2} \times \frac{dt}{1+t^2}$$
$$= \int_0^1 \frac{1-t^2}{2(1+2t)(1+t^2)}\,dt$$
$$= \frac{1}{10}\int_0^1 \frac{5-5t^2}{(1+2t)(1+t^2)}\,dt$$
$$= \frac{1}{10} \times \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$$
$$= \frac{1}{20}\left(\pi + \ln\frac{27}{16}\right)$$

**Question 94a**

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$$

$$\begin{aligned} u &= x^n & \frac{dv}{dx} &= \cos x \\ \frac{du}{dx} &= nx^{n-1} & v &= \sin x \end{aligned}$$

$$= \left[x^n \sin x\right]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$$
$$= \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$$

$$\begin{aligned} u &= x^{n-1} & \frac{dv}{dx} &= \sin x \\ \frac{du}{dx} &= (n-1)x^{n-2} & v &= -\cos x \end{aligned}$$

$$= \left(\frac{\pi}{2}\right)^n - n\left\{\left[-x^{n-1}\cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x\,dx\right\}$$
$$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n\{0 + (n-1)I_{n-2}\}$$
$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

**Question 94b**

$$A = \int_0^{\frac{\pi}{2}} x^4 \cos x\,dx$$
$$= I_4$$
$$= \left(\frac{\pi}{2}\right)^4 - 4\times 3\times I_2$$
$$= \frac{\pi^4}{16} - 12\left(\left(\frac{\pi}{2}\right)^2 - 2\times 1\times I_0\right)$$
$$= \frac{\pi^4}{16} - 3\pi^2 + 24\int_0^{\frac{\pi}{2}} \cos x\,dx$$
$$= \frac{\pi^4}{16} - 3\pi^2 + 24\left[\sin x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi^4}{16} - 3\pi^2 + 24$$

**Question 95**

$I_n$

$$= \int_0^1 \frac{dx}{(1+x^2)^n}$$

$$\begin{aligned} u &= (1+x^2)^{-n} & \frac{dv}{dx} &= 1 \\ \frac{du}{dx} &= -n(1+x^2)^{-n-1}(2x) & v &= x \end{aligned}$$

$$= \left[\frac{x}{(1+x^2)^n}\right]_0^1 + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$
$$= \frac{1}{2^n} - 0 + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$
$$= \frac{1}{2^n} + 2n\int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}\,dx$$
$$= \frac{1}{2^n} + 2n\int_0^1 \frac{1}{(1+x^2)^n}\,dx - 2n\int_0^1 \frac{1}{(1+x^2)^{n+1}}\,dx$$

$$\therefore I_n = \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$$
$$(1-2n)I_n = \frac{1}{2^n} - 2nI_{n+1}$$
$$I_n = \frac{1}{2^n(1-2n)} + \frac{2n}{2n-1}I_{n+1}$$

**Question 96a**

$$I_n = \int_0^1 x^n\sqrt{1-x^3}\,dx$$

| | |
|---|---|
| $u = x^{n-2}$ | $\frac{dv}{dx} = x^2(1-x^3)^{\frac{1}{2}}$ |
| $\frac{du}{dx} = (n-2)x^{n-3}$ | $v = -\frac{2}{9}(1-x^3)^{\frac{3}{2}}$ |

$$= -\frac{2}{9}\left[x^{n-2}(1-x^3)^{\frac{3}{2}}\right]_0^1$$
$$+\frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)^{\frac{3}{2}}\,dx$$
$$= 0 + \frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)\sqrt{1-x^3}\,dx$$
$$= \frac{2}{9}(n$$
$$-2)\int_0^1 \left(x^{n-3}\sqrt{1-x^3} - x^n\sqrt{1-x^3}\right)dx$$
$$I_n = \frac{2n-4}{9}I_{n-3} - \frac{2n-4}{9}I_n$$
$$\frac{2n+5}{9}I_n = \frac{2n-4}{9}I_{n-3}$$
$$I_n = \frac{2n-4}{2n+5}I_{n-3}$$

**Question 96b**

$$I_8 = \frac{2(8)-4}{2(8)+5}I_5$$
$$= \frac{4}{7}\left(\frac{2(5)-4}{2(5)+5}\right)I_2$$
$$= \frac{8}{35}\int_0^1 x^2\sqrt{1-x^3}\,dx$$
$$= \frac{8}{35}\int_1^0 x^2\times u\times\left(-\frac{2u\,du}{3x^2}\right)$$
$$= \frac{16}{105}\int_0^1 u^2\,du$$
$$= \frac{16}{105}\left[\frac{u^3}{3}\right]_0^1$$
$$= \frac{16}{315}$$

**Question 97a**

$$2\sin x\sin\big((2k+1)x\big)$$
$$= \cos\big((2k+1)x - x\big) - \cos\big((2k+1)x + x\big)$$
$$= \cos(2kx + x - x) - \cos(2kx + x + x)$$
$$= \cos(2kx) - \cos(2(k+1)x)$$

**Question 97b**

Let $P(n)$ represent the proposition

$P(1)$ is true since LHS $= \sin x$; RHS $= \frac{1-\cos(2x)}{2\sin x} = \frac{1-(1-2\sin^2 x)}{2\sin x} = \frac{2\sin^2 x}{2\sin x} = \sin x$

If $P(k)$ is true for some arbitrary $k \geq 1$ then
$\sin x + \sin 3x + \sin 5x + \ldots + \sin\big((2k-1)x\big)$
$= \frac{1-\cos(2kx)}{2\sin x}$

$RTP\ P(k+1)$:
$\sin x + \sin 3x + \sin 5x + \ldots + \sin\big((2k-1)x\big)$
$+ \sin\big((2k+1)x\big) = \frac{1-\cos(2(k+1)x)}{2\sin x}$

$LHS$
$= \frac{1-\cos(2kx)}{2\sin x} + \sin\big((2k+1)x\big)$ from $P(k)$
$= \frac{1-\cos(2kx)+2\sin x\sin\big((2k+1)x\big)}{2\sin x}$
$= \frac{1-\cos(2kx)+\cos(2kx)-\cos(2(k+1)x)}{2\sin x}$
$= \frac{1-\cos(2(k+1)x)}{2\sin x}$
$= RHS$

$\therefore P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for $n \geq 1$ by induction.

**Question 98a**

$$(1+t^2)^{n-1} + t^2(1+t^2)^{n-1}$$
$$= (1+t^2)^{n-1}(1+t^2)$$
$$= (1+t^2)^{n-1}$$

**Question 98b**

$$= \int_0^x (1+t^2)^n\,dt$$

| | |
|---|---|
| $u = (1+t^2)^n$ | $\frac{dv}{dt} = 1$ |
| $\frac{du}{dx} = n(1+t^2)^{(n-1)(2t)}$ | $v = t$ |

$$= \left[t(1+t^2)^n\right]_0^x - 2n\int t^2(1+t^2)^{n-1}\,dt$$
$$= x(1+x^2)^n$$
$$-2n\int\left((1+t^2)^n - (1+t^2)^{n-1}\right)dt$$
$$\therefore\ I_n = x(1+x^2)^n - 2nI_n + 2n\,I_{n-1}$$
$$(2n+1)I_n = x(1+x^2)^n + 2nI_{n-1}$$
$$I_n = \frac{x(1+x^2)^n}{2n+1} + \frac{2n}{2n+1}I_{n-1}$$

**Question 99a**

$$I_n = \int_{e^{-1}}^{1} (1+\log_e x)^n \, dx$$

$$\boxed{\begin{array}{ll} u = (1+\ln x)^n & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = n(1+\ln x)^{n-1} \times \dfrac{1}{x} & v = x \end{array}}$$

$$= \left[ x(1+\ln x)^n \right]_{e^{-1}}^{1} - n \int_{e^{-1}}^{1} (1+\ln x)^{n-1} \, dx$$
$$= 1 - 0 - nI_{n-1}$$
$$= 1 - nI_{n-1}$$

**Question 99b**

$$J_n = \int_{e^{-1}}^{1} (\log_e x)(1+\log_e x)^n \, dx$$
$$= \int_{e^{-1}}^{1} (1+\log_e x - 1)(1+\log_e x)^n \, dx$$

$$= \int_{e^{-1}}^{1} (1+\log_e x)^{n+1} \, dx$$
$$- \int_{e^{-1}}^{1} (1+\log_e x)^n \, dx$$
$$= I_{n+1} - I_n$$
$$= 1 - (n+1)I_n - I_n$$
$$= 1 - (n+2)I_n$$

**Question 99c**

$$J_3 = 1 - (3+2)I_3$$
$$= 1 - 5(1 - 3I_2)$$
$$= -4 + 15(1 - 2I_1)$$
$$= 11 - 30(1 - I_0)$$
$$= -19 + 30 \int_{e^{-1}}^{1} dx$$
$$= -19 + 30 \left[ x \right]_{e^{-1}}^{1}$$
$$= -19 + 30 \left( 1 - \frac{1}{e} \right)$$
$$= 11 - \frac{30}{e}$$

**Question 100**

$$\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} \, dx$$
$$= \int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} \times \frac{\sin 3x}{\sin 3x} \, dx$$
$$= \int \frac{\sin 3x \, (\cos 5x + \cos 4x)}{\sin 3x - 2\sin 3x \cos 3x} \, dx$$
$$= \int \frac{2\sin\frac{3x}{2}\cos\frac{3x}{2}\left(2\cos\frac{9x}{2}\cos\frac{x}{2}\right)}{\sin 3x - \sin 6x} \, dx$$
$$= \int \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}} \, dx$$
$$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2} \, dx$$
$$= -\int (\cos 2x + \cos x) \, dx$$
$$= -\frac{1}{2}\sin 2x - \sin x + c$$