WARNING

This textbook chapter covers the **2024 syllabus** which is first taught from Term 4 2026 and will **first be tested at the 2027 Extension 2 HSC**.

If you are sitting the HSC in 2026 you are studying the 2017 syllabus. You can use this textbook chapter, but it will cover some content that you will be tested on in the Extension 1 HSC, not in the Extension 2 HSC. This includes:

- using the product to sum and difference identities in Lesson 4.7 and the revision exercise
- using the t-formulas in Lesson 4.9 and part of Lesson 4.10 and the revision exercise.

This textbook chapter does cover all of the 2017 content on integration, as new content has been added for the 2024 syllabus, but nothing has been removed.

WARNING

HSC Mathematics Extension 2

Steve Howard

Under construction - currently only including Chapter 4 Further integration

A textbook for the Mathematics Extension 2 (2024) NSW Syllabus

Last updated: July 2026
Check howardmathematics.com for the latest version

This version first published in 2026-2027 based on versions first published in 2019-2020 by Steve Howard - Howard and Howard Education

This digital textbook is continually changed and updated. For the most up-to-date version please go to howardmathematics.com.

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About the Author

Steve is a Mathematics Teacher, having taught at Cowra High School for 30 years, and now mainly working from home creating Mathematics resources but also teaching extension courses online with Mount View High School. He has also taught gifted and talented students online through xsel and Aurora College and is a Senior Marker for the Extension 2 HSC.

He has a particular passion for Mathematics Extension 2. This textbook reflects many years of developing teaching resources and approaches for the course. The textbook is completely free - see howardmathematics.com for the most up-to-date version. The version of the textbook for the 2024 syllabus combines the original textbook with the 1000 Revision Questions, expanded and adjusted as needed.

Steve writes commercial online professional development courses for teachers covering the Senior Mathematics courses through Teacher Training Australia (TTA). The online courses for teachers cover this material in much greater depth and include thousands of recorded examples. For the courses currently offered see stevehoward.tta.edu.au. He also writes the Trial HSCs and Year 11 Final Exam papers for TTA, also available at the same website.

Steve loves finding more efficient techniques for solving mathematical questions, by trawling through other teachers' solutions or making up his own approaches when there must be a better way. Many of the approaches you see here are unique, with emphases on both understanding the work, plus little tricks to help students succeed in exams.

Steve did 4 Unit Mathematics as a student way back in 1988, gaining a mark of 198/200 and training as an actuary. Working in an office in the city wasn’t for him, so he went back to uni to retrain as a teacher, then headed to the country, where he lives in an owner-built mud house, with chickens, goats and parrots (which you will sometimes hear in his recordings)!!

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Introduction

Welcome! I hope all students and teachers find this book useful and enjoyable in your journey in Extension 2 Maths. This textbook will be continually changed and updated over time. For a free download of the latest version, please visit howardmathematics.com.

This textbook is primarily a passion project. It is supported indirectly through school copyright licensing and by increasing traffic to my commercial online professional development courses for teachers through TTA.

Some of the features of this textbook that students and teachers may find useful:

- The questions are in the style of past HSC questions where possible, so students are learning the right type of questions throughout the course.
- The questions are graded in difficulty so students can work at their own level.
- Each exercise starts with Basic questions that match the examples, so students can ease themselves into the new concepts if they need to. More confident students can skip these first questions and start with Medium and then Challenging.
- Each chapter has 100 revision questions, largely based on the matching exercise from the 1000 Revision Questions from the 2017 syllabus.
- All questions have fully worked solutions.
- There are lots of diagrams that help students understand concepts that would normally only be dealt with algebraically.
- There are many hints and tips to help students answer questions more efficiently, while also highlighting common techniques and recurring tricks used in HSC exam questions.

The textbook should be set out over roughly 44 lessons - the final number will be determined by how the material is best bundled into lessons. The aim is for students to be able to finish the course as soon as possible, so that there are months available for revision before the HSC to consolidate and master the content.

If you find any mistakes or have any ideas that would make the textbook better, please contact me via email below.

Cheers
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Thanks to all the teachers and students who have let me know errors or improvements for the textbook or 1000 Revision Questions, which have flowed through to this updated textbook.

Chapter 4
Further integration

Integration is a great Extension 2 topic, as the question often gives no hint as to how it should be approached, so you need to develop a deep understanding of the dozens of different techniques to know how to solve the integral. The solution is then relatively short, while often also relying on accurate algebra skills and using formulas from the reference sheet correctly, so students can show their ability without taking up too much exam time. Some of the integration skills we cover in this chapter are important in *Applications of calculus to mechanics* later in the course, which is largely applied integration.

During the last 6 years *Further integration* has been the second least used topic, unlike for the syllabus that went before where it was always neck and neck with Complex Numbers for the number one spot. *Further integration* is largely the same as MEX-C1 Further Integration in the 2017 syllabus, with the addition of trigonometric product to sum identities and t-formulas, which move here from Year 11 Extension 1. With the addition of this material, expect the share of the marks for *Further integration* to increase. I have covered the product to sum identities and t-formulas later in the chapter, so we can get stuck into integration right from the start of the chapter, but you prefer to do this content first.

Lessons

4.1 Completing the square & manipulating integrands

4.2 The reverse chain rule & u substitutions

4.3 Other u substitutions

4.4 Splitting the numerator

4.5 Partial fractions

4.6 Definite integrals

4.7 Product to sum and difference identities

4.8 Powers of trigonometric functions

4.9 Expressions and equations using the t-formulas

4.10 Trigonometric substitutions

4.11 Integration by parts

4.12 Recurrence relations

Chapter 4 Further integration - 100 revision questions

4.1 Completing the square & manipulating integrands

In Lesson 1 we cover:

- Integration in Extension 2
- Completing the square on a quadratic denominator
- Changing an integrand into an appropriate form using algebraic manipulation

We start by looking at how integration is used in Extension 2 integration and take a deeper look at some background information that can help in more unusual questions. We then look at a variety of definite and indefinite integrals where we need to complete the square on the denominator so that we can use the standard integrals from Extension 1. We finish the lesson by looking at integrals where we need to manipulate the integrand into a different form before we can use the standard integrals.

This lesson mainly covers the 9^{th} and 11^{th} dot points from *Further integration*, with questions on the 15^{th} and 16^{th} dot points included in the exercises.

Integration in Extension 2

Extension 2 integration involves a massive number of different techniques, some you have seen before in Advanced and Extension 1, and others which are new. As well as the new techniques you will learn, integration in Extension 2 is also harder as you will often be given a bare question with no hints as to how it should be solved – you generally won't be given the substitution if one is needed, or any other hint if a different technique is needed, and you may have to use more than one integration technique to complete the solution.

Integration questions occur throughout the HSC – most marks are easy to gain for those who have practised. There are occasionally harder questions scattered throughout the paper, not just in Q16. Focus on the small differences between integrands that require quite different approaches and learn the tricks that appear through the chapter.

There are online integral calculators that can help you if you are doing a question without worked solutions. Search for 'integral calculator' and end up at sites like integral-calculator.com – type in the integrand and it will give you the final solution plus working if you select it. Just keep in mind that it may use different techniques to what you would like, and the final answer may be equivalent to what you want but in a different form. This is especially true with trigonometric functions and inverse trigonometric functions.

At the end of this document is a summary of the topic, including many standard integrals that you would benefit from memorising.

One of the results of having so many techniques to learn is we often focus on how to do each type without really understanding what we are doing and why we are doing it.

The trick to successfully being able to solve the greatest variety of integration questions is to deeply understand that integration is the inverse operation of differentiation.

This might seem to be an obvious statement, but it is something we often forget. At its simplest it means that we can differentiate the primitive function to arrive back at the integrand.

Indefinite and definite integrals need to be understood in different ways.

Consider the indefinite integral, or primitive function, $\int x^2 dx$ – what does it mean? We are trying to find the family of functions whose derivative is x^2. The solution is $\frac{1}{3}x^3 + c$, which means that if we differentiate any of the infinite numbers of functions in the form $\frac{1}{3}x^3 + c$ the answer is x^2.

Now look at the definite integral $\int_1^3 x^2\, dx$ – how is this different? At its simplest we can say that it is the area bounded by the x–axis, the curve $y = x^2$ and the lines $x = 1$ and $x = 3$. If this curve was beneath the x–axis we would have to think of the definite integral as the negative of this area. We can find the definite integral by evaluating <u>any</u> primitive of x^2 at $x = 3$ and subtracting the value of the <u>same</u> primitive at $x = 1$. So, we find the difference between $\frac{1}{3}x^3 + c$ when $x = 3$ and $x = 1$. Since we can use any primitive, we usually let $c = 0$ for convenience without thinking about it, but we would get the same result if we used $\frac{1}{3}x^3 - 6$ twice, or any other primitive twice.

So indefinite integrals are a way of finding the family of functions that can be differentiated to arrive at the integrand, while definite integrals are most easily understood as the signed area between the curve and the x–axis between the upper and lower limits.

Standard integrals

Standard integrals are common primitive functions which form the basis of our integration. They all come from Advanced and Extension 1 – there are no new standard integrals in the Extension 2 course.

To find any standard integral we simply reverse the differentiation rules. For example, when $y = \sin f(x)$ is differentiated we find $y' = f'(x)\cos f(x)$, so the primitive of $f'(x)\cos f(x)$ is $\sin f(x) + c$. This gives us the standard integral

$$\int f'(x)\cos f(x)\, dx = \sin f(x) + c.$$

The reference sheet has a list of the only standard integrals we need to know for the HSC. They appear in two places on the reference sheet – there is a short list on the Extension 1 page and a longer list on the Advanced page.

Advanced

$$\int f'(x)\,[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

This is one version of the reverse chain rule – a massively underrated tool for Extension 2 integration.

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

or use this version in terms of $f(x)$:

$$\int f'(x)\,a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

We will use the Advanced standard integrals throughout the chapter.

Extension 1

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

or use these versions in terms of $f(x)$:

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

Careful – in this version of the standard integral we need 'a' in the numerator and there is no $\frac{1}{a}$ on the RHS.

Careful – in this version of the standard integral we don't have 'a' in the numerator, so we need $\frac{1}{a}$ on the RHS.

When we use integrals whose primitive is the form $\tan^{-1}\left[\frac{f(x)}{a}\right]$ we will use the version shown in the box to the right as it is more useful, not the version from the reference sheet.

We will use the Extension 1 standard integrals later this lesson when we complete the square in the denominator of the integrand.

There are other standard integrals that you might see in other resources, but they are NOT required in the HSC.

$$\int f'(x)\text{cosec}^2 f(x)\ dx = -\cot f(x) + c$$

$$\int f'(x) \sec f(x) \tan f(x)\, dx = \sec f(x) + c$$

$$\int f'(x)\, \text{cosec}\, f(x) \cot f(x)\, dx = -\text{cosec}\, f(x) + c$$

$$\int f'(x) \sec f(x)\ dx = \ln\left|\sec f(x) + \tan f(x)\right| + c$$

$$\int f'(x)\text{cosec}\, f(x)\, dx = -\ln\left|\text{cosec}\, f(x) + \cot f(x)\right| + c$$

These are lesser used trig standard integrals. See the mnemonics at the end of the chapter.

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}} dx = \ln\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + c$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}} dx = \ln\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + c$$

These standard integrals come from a much earlier syllabus, and you shouldn't see them again.

You can safely ignore the seven standard integrals on this page for the HSC, although we will use them in some exercises in this textbook to increase our understanding.

We have just seen a lot of differentiation and integration rules for trig functions to remember – see the summary at the end of this chapter for some mnemonics to help you remember them.

Completing the square on a quadratic denominator

Some integrands need to be manipulated into a different form before we can use the standard integrals. We will start by looking at some requiring completing the square in the denominator so that we can use the Extension 1 standard integrals. These are common HSC questions.

Example 1

Find $\displaystyle\int \frac{dx}{x^2+10x+29}$.

$$= \int \frac{dx}{x^2+10x+25+4}$$

$$= \int \frac{dx}{(x+5)^2+2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+5}{2}\right)+c$$

Only perform one step of working on each line – it is very easy to make a simple mistake when completing the square within an integral.

When we use $\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a}+c$, don't forget $\frac{1}{a}$ on the RHS – another common mistake.

Example 2

Find $\int_{-3}^{2} \frac{1}{\sqrt{16-6x-x^2}}\,dx$.

$$= \int_{-3}^{2} \frac{1}{\sqrt{-(x^2+6x-16)}}\,dx$$

One way to avoid making a minus mistake is to take out a factor of minus one when the leading term is negative.

$$= \int_{-3}^{2} \frac{1}{\sqrt{-(x^2+6x+9-25)}}\,dx$$

$$= \int_{-3}^{2} \frac{1}{\sqrt{5^2-(x+3)^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x+3}{5}\right)\right]_{-3}^{2}$$

When we use $\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$, we do not need $\frac{1}{a}$ on the RHS.

$$= \sin^{-1} 1 - \sin^{-1} 0$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Now let's have a look at some integrands where we need to complete the square using a function of x.

Example 3

Find $\int_{0}^{\ln 4} \frac{3e^x}{e^{2x}-2e^x+10}\,dx$.

$$= \int_{0}^{\ln 4} \frac{3e^x}{(e^x-1)^2+3^2}\,dx$$

$$= \left[\tan^{-1}\left(\frac{e^x-1}{3}\right)\right]_{0}^{\ln 4}$$

Here we have $\int \frac{a \times f'(x)}{a^2+[f(x)]^2}\,dx$, so we do not need $\frac{1}{a}$ as a was included in the numerator.

$$= \tan^{-1}\left(\frac{4-1}{3}\right) - \tan^{-1}\left(\frac{1-1}{3}\right)$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

Example 4

Find $\displaystyle\int \frac{\cos 2x}{\sqrt{8-2\sin 2x-\sin^2 2x}}\,dx$.

$$= \int \frac{\cos 2x}{\sqrt{-(\sin^2 2x + 2\sin 2x - 8)}}\,dx$$

$$= \int \frac{\cos 2x}{\sqrt{9-(\sin^2 2x + 2\sin 2x + 1)}}\,dx$$

$$= \frac{1}{2}\int \frac{2\cos 2x}{\sqrt{3^2-[\sin 2x + 1]^2}}\,dx$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{\sin 2x + 1}{3}\right) + c$$

Here we use $\displaystyle\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$.

Changing an integrand into an appropriate form using algebraic manipulation

Many integrands need to be manipulated into a different form before we can use out standard integrals. In the next example we start by rearranging the integrand using two successive trigonometric identities, then simplify before using standard integrals.

Example 5

Find $\displaystyle\int \frac{1+\sin 2x}{\cos x + \sin x}\,dx$.

When we have different angles (x and $2x$) our first step is usually to convert the trig ratios with the higher argument (angle) to the smaller argument.

$$= \int \frac{1+2\sin x\cos x}{\sin x + \cos x}\,dx$$

At the moment we are at a dead end, but if we use a Pythagorean identity, we can simplify the integrand.

$$= \int \frac{\sin^2 x + 2\sin x\cos x + \cos^2 x}{\sin x + \cos x}\,dx$$

$$= \int \frac{(\sin x + \cos x)^2}{\sin x + \cos x}\,dx$$

$$= \int (\sin x + \cos x)\,dx$$

$$= -\cos x + \sin x + c$$

When we have a fraction with powers of e in the numerator and denominator, we will generally try to rearrange the integrand so that the numerator is the derivative of the denominator. To do this, we must have any powers of e in the numerator with the same powers as those in the denominator. The denominator can also have a constant term which disappears with differentiation. If one term in the denominator is the odd–one–out, then we need to divide the numerator and denominator by that power of e so that that term becomes a constant. In the next example e^x is the odd one out, so we start by dividing the numerator and denominator by e^x.

Example 6

Find $\int \frac{e^{2x}}{e^{2x} - 3e^x} dx$.

$$= \int \frac{e^x}{e^x - 3} dx$$

$$= \ln\left|e^x - 3\right| + c$$

Basic

1 Evaluate $\displaystyle\int_{-7}^{1} \frac{dx}{x^2+6x+25}$.

2 Evaluate $\displaystyle\int_{0}^{\frac{3\sqrt{3}}{4}} \frac{1}{\sqrt{9-4x^2}}\,dx$.

3 Find $\displaystyle\int \frac{1}{\sqrt{20-8x-x^2}}\,dx$.

4 Find $\displaystyle\int \frac{dx}{12x^2+12x+78}$.

5 Find $\displaystyle\int \frac{x^2+2x}{x^2+2x+5}\,dx$.

6 Find $\displaystyle\int \frac{e^{-x}}{e^{-x}-1}\,dx$.

Medium

7 Find $\displaystyle\int \frac{1}{x^2-3x+3}\,dx$.

8 Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{5-2t+t^2}\,dt$.

9 Find $\displaystyle\int \frac{2}{4x^2-4x+17}\,dx$.

10 Find $\displaystyle\int \frac{2x}{x^4+2x^2+5}\,dx$.

11 Find $\displaystyle\int \frac{2}{e^x+1}\,dx$.

12 Find $\displaystyle -\int \frac{2x}{\sqrt{3-2x^2-x^4}}\,dx$.

13 Find $\displaystyle\int \frac{2\cos x\,(\sin x+1)}{(\sin x+3)(\sin x-1)}\,dx$.

14 Find $\displaystyle\int \frac{2\cos x\,(\sin x+1)}{(\sin x+3)(\sin x-1)}\,dx$.

15 Find $\displaystyle\int \frac{dx}{(x^2+4)\tan^{-1}\left(\frac{x}{2}\right)}$.

16 It is given that $\dfrac{dy}{dx} = \dfrac{\cos x}{1+\sin^2 x}$ and $y=1$ when $x=\dfrac{\pi}{2}$. Find y as a function of x.

Challenging

17 Find $\displaystyle\int_{-\frac{\pi}{2}}^{0} \frac{\cos\theta}{\sin^2\theta + 2\sin\theta + 4}\,d\theta$.

18 Find $\displaystyle\int \frac{e^{3x}}{\sqrt{7 + 12e^{3x} - 4e^{6x}}}\,dx$.

19 Find $\displaystyle\int \frac{\cos x - \sin x}{2 + \sin 2x}\,dx$.

20 Find $\displaystyle\int \frac{\sin^3 x}{\cos x - 1}\,dx$.

21 Show that $\displaystyle\int \frac{e^{\log_2 x}}{x}\,dx = \ln 2\, x^{\frac{1}{\ln 2}} + c$.

22 Find $\displaystyle\int \frac{\sin 2x + \cos 2x}{\cos x}\,dx$.

23 Find $\displaystyle\int \frac{dx}{\sec x - 1}$.

24 Find $\displaystyle\int (2x \operatorname{cosec} x - x^2 \cot x \operatorname{cosec} x)\,dx$.

25 Given α, β are the roots of $ax^2 + bx + c = 0$, prove that:

$$\int \frac{\cos x\,(2a\sin x + b)}{(\sin x - \alpha)(\sin x - \beta)}\,dx = a\ln\left|a\sin^2 x + b\sin x + c\right| + k,$$

where k is a constant.

4.2 The reverse chain rule & u substitutions

In Lesson 2 we cover:

- The reverse chain rule, u substitutions and u squared substitutions

The integrals we look at this lesson can be solved using either the reverse chain rule or a suitable u or u^2 substitution. Using the reverse chain rule is always much more efficient, but keep in mind that depending on the wording of the question, a u or u^2 substitution must be used.

This lesson mainly covers the 6th dot point from *Further integration*, with questions on the 11th and 15th dot points included in the exercises.

The reverse chain rule, u substitutions and u squared substitutions

Many integration questions in Extension 1 involve u substitutions, and in Extension 1 you must be given the substitution to use. You may also be given the choice of whether you can use an alternative method such as the reverse chain rule. It depends whether the examiners are wanting to specifically check if you know how to solve integrals using substitution.

In Extension 2 we could get exactly the same integral, with either no guidance as to how we should solve it or being directed to 'use a suitable substitution' without being told what it should be. We either need to work out which substitution to use, or if the integrand is in the correct form, we can use the reverse chain rule if the wording of the question allows. If you learn to recognise integrands which are suitable for the reverse chain rule you can save yourself lots of effort and time, which benefit you later in the exam.

The reverse chain rule is generally much quicker than using substitution, involving less algebra and with less room for simple algebraic errors. It is a vastly underused technique in Extension 2.

One version of the reverse chain rule is the first standard integral on the Advanced page of the reference sheet, where we have a function of x raised to a power other than -1.

$$\int f'(x)\,[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c \quad \text{where } n \neq -1$$

A more generalised form of the reverse chain rule is

$$\int f'(x)\, g\big(f(x)\big) dx = G\big(f(x)\big) + c$$

where $G(x)$ is the primitive function of $g(x)$.

Note that an integrand can be solved using the reverse chain rule if it involves the product of a composite function [a function of a function such as $g\big(f(x)\big)$] and a multiple of the derivative of the inner function [$f'(x)$]. We may need to multiply the integral by a fudge factor to get it in the correct form, or rearrange the integrand, for example by rewriting it index form or using reciprocal trig ratios.

As well as being a much more efficient alternative to most integrals that would otherwise involve substitution, mastering the reverse chain rule also makes integration by parts and recurrence relations much easier when we get to them later in the chapter.

To help understand the reverse chain rule, consider the integral

$$\int x\left(2x^2+3\right)^3 dx.$$

Before we solve this integral using the reverse chain rule, let's differentiate the composite function $\left(2x^2+3\right)^4$, where we have the inner function of $2x^2+3$ raised to the power of 4.

$$\begin{aligned}\frac{d}{dx}\left[\left(2x^2+3\right)^4\right] &= 4\left(2x^2+3\right)^3(4x)\\ &= 16x\left(2x^2+3\right)^3\end{aligned}$$

So, when we integrate $16x\left(2x^2+3\right)^3$ we will get $\left(2x^2+3\right)^4+c$, or more importantly for our use of the reverse chain rule, when we integrate $4x\left(2x^2+3\right)^3$ we will get $\frac{\left(2x^2+3\right)^4}{4}+c$.

- In this second form we have the product of the composite function, $\left(2x^2+3\right)^3$, and the derivative of its inner function, $4x$.
- To find the primitive of $4x\left(2x^2+3\right)^3$, we leave the inner function alone and take the primitive of the outer function – in this case the primitive of cubing a function is to increase the power by 1 to become 4 then divide by 4.

The solution to the integral at the top of the page is shown below.

$$\int x\left(2x^2+3\right)^3 dx$$

The derivative of $2x^2+3$ is $4x$, so change x to $4x$, and place a fudge factor of $\frac{1}{4}$ at the front so the first two lines are equal.

$$=\frac{1}{4}\int 4x\left(2x^2+3\right)^3 dx$$

Leave the fudge factor at the front, take the primitive of the outer function, leaving the inner function alone.

$$=\frac{1}{4}\times\frac{\left(2x^2+3\right)^4}{4}+c$$

Simplify the primitive.

$$=\frac{\left(2x^2+3\right)^4}{16}+c$$

We will now look at integrals that can be solved using either the reverse chain rule or by a suitable substitution – we will solve each question using both methods.

For most substitutions we will let u equal the inner function of the composite function, then differentiate both sides to find an expression for du.

When the composite function is the square root of the inner function then we let u^2 equal the inner function (the radicand), as the algebra is easier because we work with integral powers rather than fractional powers

When we differentiate both sides, the LHS becomes $2u\,du$, which is an application of the chain rule.

Again, a reminder that you should master both methods – the reverse chain rule is more efficient in the questions where the integrand is in the correct form, but you may be required to solve the integral using a substitution, plus we will see integrals next lesson which can only be solved using substitution.

It is helpful to think of the functions we use in *Further integration* in five groups: Exponential functions, Trigonometric functions, Algebraic functions, Inverse trigonometric functions, and Logarithmic functions. We will use a mix of the five groups as the inner and outer functions.

In the examples below we solve the question using the reverse chain rule on the left and using a u or u^2 substitution on the right of the dividing line.

Example 1

Find $\displaystyle\int_0^1 x^3(x^4-2)^5\,dx$.

$$\int_0^1 x^3(x^4-2)^5\,dx$$

$$=\frac{1}{4}\int_0^1 4x^3(x^4-2)^5\,dx$$

$$=\frac{1}{4}\left[\frac{(x^4-2)^6}{6}\right]_0^1$$

$$=\frac{1}{4}\left(\frac{1}{6}-\left(\frac{64}{6}\right)\right)$$

$$=-\frac{63}{24}$$

$$\int_0^1 x^3(x^4-2)^5\,dx$$

$$\boxed{\begin{aligned} u &= x^4-2 \\ du &= 4x^3dx \end{aligned}}$$

$$=\frac{1}{4}\int_0^1 (x^4-2)^5\times 4x^3dx$$

$$=\frac{1}{4}\int_{-2}^{-1} u^5\,du$$

Don't forget to change the limits when the variable changes

$$=\frac{1}{4}\left[\frac{u^6}{6}\right]_{-2}^{-1}$$

$$=\frac{1}{24}(1-64)$$

$$=-\frac{63}{24}$$

Note that the solution using the reverse chain rule required only four lines, while the solution using the u substitution was seven lines long – two lines to find an equation for du and dx, plus another five lines for the solution.

Common mistakes when using the substitution method for definite integrals include forgetting to change the limits so that they are in terms of u, or just replacing dx with du, rather than differentiating the equation linking u and x to find the equation linking du and dx.

We can see that using the reverse chain rule has many advantages, so always use it when the question allows.

Example 2

Find $\int_1^2 \frac{x}{\sqrt{2x^2-1}}\,dx$.

$$\int_1^2 \frac{x}{\sqrt{2x^2-1}}\,dx$$

$$= \frac{1}{4}\int_1^2 4x\left(2x^2-1\right)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{4}\times 2\left[\left(2x^2-1\right)^{\frac{1}{2}}\right]_1^2$$

$$= \frac{1}{2}\left(\sqrt{7}-1\right)$$

$$\boxed{\begin{aligned} u^2 &= 2x^2-1 \\ 2u\,du &= 4xdx \end{aligned}}$$

$$\int_1^2 \frac{x}{\sqrt{2x^2-1}}\,dx$$

$$= \frac{1}{4}\int_1^2 \frac{1}{\sqrt{2x^2-1}}\times 4xdx$$

$$= \frac{1}{4}\int_1^{\sqrt{7}} \frac{1}{u}\times 2u\,du$$

$$= \frac{1}{2}\int_1^{\sqrt{7}} du$$

$$= \frac{1}{2}\Big[u\Big]_1^{\sqrt{7}}$$

$$= \frac{1}{2}\left(\sqrt{7}-1\right)$$

Example 3

Find $\int e^{3x}\sqrt{e^{3x}+1}\,dx$.

$$\int e^{3x}\sqrt{e^{3x}+1}\,dx$$

$$= \frac{1}{3}\int 3e^{3x}\left(e^{3x}+1\right)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\times\frac{2}{3}\left(e^{3x}+1\right)^{\frac{3}{2}}+c$$

$$= \frac{2\sqrt{\left(e^{3x}+1\right)^3}}{9}+c$$

$$\boxed{\begin{aligned} u^2 &= e^{3x}+1 \\ 2u\,du &= 3e^{3x}dx \end{aligned}}$$

$$\int e^{3x}\sqrt{e^{3x}+1}\,dx$$

$$= \frac{1}{3}\int \sqrt{e^{3x}+1}\times 3e^{3x}dx$$

$$= \frac{1}{3}\int u\,2u\,du$$

$$= \frac{2}{3}\int u^2\,du$$

$$= \frac{2}{3}\times\frac{u^3}{3}+c$$

$$= \frac{2\sqrt{\left(e^{3x}+1\right)^3}}{9}+c$$

Example 4

Find $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$.

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$$

$$= \frac{1}{2}\int_0^{\sqrt{\pi}} 2x\sin(x^2)\,dx$$

$$= -\frac{1}{2}\Big[\cos x^2\Big]_0^{\sqrt{\pi}}$$

$$= -\frac{1}{2}\Big(\cos\pi - \cos 0\Big)$$

$$= -\frac{1}{2}(-1-1)$$

$$= 1$$

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx \qquad \boxed{\begin{aligned} u &= x^2 \\ du &= 2xdx \end{aligned}}$$

$$= \frac{1}{2}\int_0^{\sqrt{\pi}} \sin(x^2)\,2xdx$$

$$= \frac{1}{2}\int_0^{\pi} \sin u\,du$$

$$= -\frac{1}{2}\Big[\cos u\Big]_0^{\pi}$$

$$= -\frac{1}{2}\Big(\cos\pi - \cos 0\Big)$$

$$= -\frac{1}{2}(-1-1)$$

$$= 1$$

Example 5

Find $\int \sqrt{\cos 2x}\sin 2x\,dx$.

$$\int \sqrt{\cos 2x}\sin 2x\,dx$$

$$= -\frac{1}{2}\int (-2\sin 2x)(\cos 2x)^{\frac{1}{2}}\,dx$$

$$= -\frac{1}{2}\times\frac{2}{3}(\cos 2x)^{\frac{3}{2}} + c$$

$$= -\frac{\sqrt{\cos^3 2x}}{3} + c$$

$$\int \sqrt{\cos 2x}\sin 2x\,dx \qquad \boxed{\begin{aligned} u^2 &= \cos 2x \\ 2u\,du &= -2\sin 2x\,dx \end{aligned}}$$

$$= -\frac{1}{2}\int \sqrt{\cos 2x}\times(-2\sin 2x\,dx)$$

$$= -\frac{1}{2}\int u\times 2udu$$

$$= -\int u^2\,du$$

$$= -\frac{u^3}{3} + c$$

$$= -\frac{\sqrt{\cos^3 2x}}{3} + c$$

Example 6

Find $\displaystyle\int e^x \sin(e^x)\,dx$.

$$\int e^x \sin(e^x)\,dx$$
$$= -\cos(e^x) + c$$

$$\int e^x \sin(e^x)\,dx \qquad \boxed{\begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned}}$$

$$= \int \sin(e^x) \times e^x dx$$

$$= \int \sin u\,du$$

$$= -\cos u + c$$

$$= -\cos e^x + c$$

Example 7

Find $\displaystyle\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}dx$.

$$\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}dx$$

$$= \int \frac{1}{\sqrt{1-x^2}} \times e^{\sin^{-1} x}\,dx$$

$$= e^{\sin^{-1} x} + c$$

$$\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}dx \qquad \boxed{\begin{aligned} u &= \sin^{-1} x \\ du &= \frac{dx}{\sqrt{1-x^2}} \end{aligned}}$$

$$= \int e^{\sin^{-1} x} \times \frac{dx}{\sqrt{1-x^2}}$$

$$= \int e^u\,du$$

$$= e^u + c$$

$$= e^{\sin^{-1} x} + c$$

Basic

1 Find $\int x^2(x^3+4)^5\,dx$.

2 Evaluate $\int_1^2 \frac{x^2}{\sqrt{2x^3-1}}\,dx$.

3 Find $\int e^{2x}\sqrt{e^{2x}-1}\,dx$.

4 Find $\int x\cos x^2\,dx$.

5 Evaluate $\int_0^{\frac{\pi}{4}} \tan^3 x\sec^2 x\,dx$.

6 Find $\int \frac{\cos x}{\sin^5 x}\,dx$.

Medium

7 Find $\int \sin^{\frac{3}{2}} 2x\cos 2x\,dx$.

8 Find $\int \sqrt{\sin 2x}\cos 2x\,dx$.

9 Find $\int e^x\cos e^x\,dx$.

10 Find $\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx$.

11 Evaluate $\int_1^{e^{\frac{\pi}{2}}} \frac{\cos(\ln x)}{x}\,dx$

12 Find $\int (3x^2+2x)\sqrt{x^3+x^2}\,dx$.

13 Find $\int x^2e^{x^3}\,dx$.

14 Find $\int \frac{\sin(\tan^{-1}x)}{1+x^2}\,dx$.

15 Find $\int (e^{t^2}+16)te^{t^2}\,dt$.

16 Find $\int \frac{\operatorname{cosec} x\cot x}{1+\operatorname{cosec}^2 x}\,dx$.

17 Find $\int e^xe^{e^x}\,dx$.

18 Find $\int \frac{\ln^3 x}{x}\,dx$.

19 Find $\int \sqrt{\frac{2+x}{2-x}}\,dx$.

20 Find $\int \frac{\sin(\tan\theta)}{\cos^2\theta}\,d\theta$.

Challenging

21 Find $\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$

22 Find $\int \frac{x\sin\left(\sqrt{2x^2-1}\right)}{\sqrt{2x^2-1}}\,dx$.

23 Find $\int \frac{e^{\sqrt{\sin x}}}{\sec x\sqrt{\sin x}}\,dx$.

24 Find $\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$.

25 Find $\int_1^{e^{\frac{\pi}{2}}} \frac{\sin(\ln|x|)}{x}\,dx$.

26 Evaluate $\int_{\sqrt{2}}^{\sqrt{3}} \frac{2e^{\frac{2}{x^2}}}{x^3}\,dx$.

27 Find $\int \frac{xe^{\sqrt{2x^2-1}}\sin\left(e^{\sqrt{2x^2-1}}\right)}{\sqrt{2x^2-1}}\,dx$.

28 Indefinite integrals with trigonometric integrands can have many correct solutions.
Prove that $\int \sin^3 2x\cos 2x\,dx$ is equal to each of the following expressions.

a $\frac{\sin^4 2x}{8}+c$

b $\frac{\cos^4 2x}{8}-\frac{\cos 4x}{8}+c$

c $\frac{\cos^4 2x}{8}+\frac{\sin^2 2x}{4}+c$

d $\frac{\cos^4 2x}{8}-\frac{\cos^2 2x}{4}+c$

e $-\frac{\cos 4x}{16}+\frac{\cos 8x}{64}+c$

f $-\frac{\cos 4x}{16}-\frac{\sin^2 4x}{16}+c$

4.3 Other u substitutions

In Lesson 3 we cover:

- Other u substitutions and u squared substitutions

The integrals we look at this lesson can be solved using a suitable u or u^2 substitution but are not in the correct form to use the reverse chain rule. These questions could also occur in Extension 1, where the substitution would be given, so the trick in Extension 2 is to recognise the best substitution to make.

This lesson mainly covers the 6^{th} dot point from *Further integration*, with questions on the 11^{th}, 15^{th} and 16^{th} dot points included in the exercises.

Other u and u^2 substitutions

We now look at integrands which are not suitable for the reverse chain rule, so that we must use a u or u^2 substitution. Again, the substitution may or may not be given.

In general, simpler integrands are easier to integrate, so we aim to choose a substitution that makes the integrand simpler. This will generally occur when we let u or u^2 equal the inner function of the composite function, though in more complicated examples this will not be the case. The only way to determine the easiest method for unusual examples is lots of practice.

Later in the chapter we will look at trig substitutions, which are only needed in a small number of cases and often involve more lines of working.

In the next example we cannot use the reverse chain rule as we have x^3, rather than x, in front of the square root.

Example 1

Find $\int x^3\sqrt{x^2-1}\,dx$.

$$\int x^3\sqrt{x^2-1}\,dx$$

$$\boxed{\begin{aligned} u^2 &= x^2-1 \\ 2u\,du &= 2x\,dx \\ x^2 &= u^2+1 \end{aligned}}$$

The outer function of the composite function is a square root, so we let u^2 equal the inner function. Making x^2 the subject is optional but helps simplify the integrand.

$$= \frac{1}{2}\int x^2\sqrt{x^2-1}\times 2xdx$$

$$= \frac{1}{2}\int (u^2+1)u\times 2u\,du$$

$$= \int (u^4+u^2)\,du$$

I always write brackets here, but the integral sign and du act as grouping symbols, so the brackets are not needed.

$$= \frac{u^5}{5}+\frac{u^3}{3}+c$$

$$= \frac{(x^2-1)^{\frac{5}{2}}}{5}+\frac{(x^2-1)^{\frac{3}{2}}}{3}$$

I normally skip this step but notice how the denominator of the powers tell us to take the square roots in the next step, while the numerators tell us the power required inside the square root.

$$= \frac{\sqrt{(x^2-1)^5}}{5}+\frac{\sqrt{(x^2-1)^3}}{3}+c$$

Example 2

Evaluate $\displaystyle\int_1^3 \frac{\sqrt{x}}{1+x}\,dx$.

$$\int_1^3 \frac{\sqrt{x}}{1+x}\,dx \qquad \boxed{\begin{aligned} u^2 &= x \\ 2u\,du &= dx \end{aligned}}$$

$$= \int_1^{\sqrt{3}} \frac{u}{1+u^2} \times 2udu$$

$$= 2\int_1^{\sqrt{3}} \frac{u^2}{1+u^2}\,du$$

$$= 2\int_1^{\sqrt{3}} \frac{1+u^2-1}{1+u^2}\,du$$

$$= 2\int_1^{\sqrt{3}} \left(1 - \frac{1}{1+u^2}\right) du$$

$$= 2\left[u - \tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= 2\left(\left(\sqrt{3} - \frac{\pi}{3}\right) - \left(1 - \frac{\pi}{4}\right)\right)$$

$$= 2\left(\sqrt{3} - \frac{\pi}{3} - 1 + \frac{\pi}{4}\right)$$

$$= 2\left(\sqrt{3} - 1 - \frac{\pi}{12}\right)$$

$$= 2\sqrt{3} - 2 - \frac{\pi}{6}$$

Example 3

Find $\int \frac{x^7}{\sqrt{1-x^4}}\,dx$.

$$\int \frac{x^7}{\sqrt{1-x^4}}\,dx$$

$$\boxed{\begin{aligned} u^2 &= 1-x^4 \\ 2u\,du &= -4x^3dx \end{aligned}}$$

$$= -\frac{1}{4}\int \frac{x^4}{\sqrt{1-x^4}} \times (-4x^3)dx$$

$$= -\frac{1}{4}\int \frac{1-u^2}{u} \times 2u\,du$$

$$= -\frac{1}{2}\int \left(1-u^2\right) du$$

$$= -\frac{1}{2}\left(u - \frac{u^3}{3}\right) + c$$

$$= \frac{\sqrt{(1-x^4)^3}}{6} - \frac{\sqrt{1-x^4}}{2} + c$$

The next example is a classic integral. It is a great question to use in class, as we can solve it in so many ways. It can be a very poor exam question without some guidance, as many students would waste time trying a variety of substitutions that appear to lead to dead ends, or use a variety of other substitutions which could soak up lots of time getting to the answer. The final answer varies depending on the method used, although the answers are equivalent. If we graph them in GeoGebra or similar we can see that they only differ by the value of the constant.

Some relatively simple solutions involve letting $u^2 = 1 + x$ or $u^2 = 1 - x$, which still soak up lots of time. Later in the chapter we will solve this same integral using the substitution $u^2 = x + 1$ with a trig substitution, and separately just with a trig substitution, which is not as bad as the u^2 substitutions mentioned.

We do have a much simpler solution that does not involve substitutions but rather involves rearranging the integrand so that we can use standard integrals. Since the integrand involves the square root of a rational function, we can rationalise the numerator (not the denominator), allowing us to split the fraction in two, leading to two standard integrals.

Example 4

Find $\int \sqrt{\dfrac{1-x}{1+x}}\,dx$.

$$= \int \sqrt{\frac{1-x}{1+x}} \times \sqrt{\frac{1-x}{1-x}}\,dx$$

We will end up with the square root of a perfect square in the numerator.

$$= \int \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{dx}{\sqrt{1-x^2}}dx + \frac{1}{2}\int (-2x)\left(1-x^2\right)^{-\frac{1}{2}}dx$$

We can now split the integral and deal with each part separately.

$$= \sin^{-1}x + \frac{1}{2} \times 2\left(1-x^2\right)^{\frac{1}{2}} + c$$

$$= \sin^{-1}x + \sqrt{1-x^2} + c$$

Some substitutions are much less obvious – the only way to be able to solve questions like these is lots of practice! This integral looks very similar to the last example, but we do need to start with a simple substitution. Since we are not replacing the whole radicand in the substitution, we use u rather than u^2.

Example 5

Find $\int x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$.

$$\int x\sqrt{\frac{1-x^2}{1+x^2}}\,dx \qquad \boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int \sqrt{\frac{1-x^2}{1+x^2}} \times 2x\,dx$$

$$= \frac{1}{2}\int \sqrt{\frac{1-u}{1+u}}\,du$$

$$= \frac{1}{2}\int \sqrt{\frac{1-u}{1+u}} \times \sqrt{\frac{1-u}{1-u}}\,du$$

$$= \frac{1}{2}\int \frac{1-u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du - \frac{1}{2}\int \frac{u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du + \frac{1}{4}\int (-2u)\left(1-u^2\right)^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\sin^{-1}u + \frac{1}{4} \times 2\left(1-u^2\right)^{\frac{1}{2}} + c$$

$$= \frac{1}{2}\sin^{-1}u + \frac{1}{2}\sqrt{1-u^2} + c$$

$$= \frac{\sin^{-1}x^2}{2} + \frac{\sqrt{1-x^4}}{2} + c$$

Basic

1 Find $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$.

2 Find $\int x^3\sqrt{x^2+1}\,dx$.

3 Find $\int x^5\sqrt{x^2+1}\,dx$.

4 Evaluate $\int_0^1 x^3\sqrt{1-x^2}\,dx$.

5 Evaluate $\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\,dx$.

6 Evaluate $\int_0^3 x\sqrt{x+1}\,dx$.

Medium

7 Find $\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$.

8 Evaluate $\int \frac{1}{2+\sqrt{x}}\,dx$.

9 Find $\int x^5\sqrt{2-x^3}\,dx$.

10 Find $\int \frac{2x^3}{(x^2-1)^2}\,dx$.

11 Find $\int \frac{x^3}{(x^2+1)^2}\,dx$.

12 Find $\int \frac{e^{2x}}{e^x+1}\,dx$.

13 Find the area under the curve $y = \frac{1}{x\sqrt{2x-1}}$ from $x = 1$ to $x = 2$.

Challenging

14 Show that $\int \frac{x}{\sqrt{x+5}}\,dx = \frac{2}{3}(x-10)\sqrt{x+5} + c$

15 The velocity of a particle, t seconds after it starts moving along the $x-$axis, is given by $\dot{x} = t\sqrt{\frac{1+2t^2}{1-2t^2}}$ metres per second. Determine the distance of the particle from its initial position after 0.5 seconds. Answer correct to the nearest millimetre.

16 Evaluate $\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\,dx$.

17 Show that $\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx = -x - 2\sqrt{x} - 2\ln\left|1-\sqrt{x}\right| + c$ using the following substitutions.

a $u^2 = x$

b $u = 1 - \sqrt{x}$

18 Find $\int \frac{dx}{\sqrt{e^{2x}-1}}$ by letting $u = \frac{1}{e^x}$.

19 Using the substitution $u = -x$, or otherwise, evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx$.

20 Use integration to prove $\int \sqrt{\frac{x}{x+1}}\,dx = \sqrt{x^2+x} - \ln(\sqrt{x} + \sqrt{x+1}) + c$.

You may assume $\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}\,dx = \ln\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right|$.

4.4 Splitting the numerator

In Lesson 4 we cover:

- Splitting the numerator
- Partial fractions by inspection

The integrands that we look at in this lesson are rational functions - the numerator and denominator are both polynomials. The integrals can be solved by rearranging the integrand into the sum or difference of simpler fractions.

We start by splitting the numerator, where we rearrange the numerator into multiples of the denominator or its derivative. We then look at partial fractions by inspection, where we split the fraction up without needing the formal methods, which we look at next lesson.

This lesson mainly covers the 7th, 8th and 10th dot points from *Further integration*, with questions on the 9th, 11th, 15th and 16th dot points included in the exercises.

Splitting the numerator

Rational functions are fractions where the numerator and denominator are both polynomials. Examples include:

$$\frac{x^2+2x+3}{x-1} \qquad \frac{x+2}{x+1} \qquad \frac{4x}{x^2+4x-6}$$

Splitting the numerator is a method of rewriting the numerator as multiples of either the denominator or the derivative of the denominator, so that we can then use standard integrals.

You may have used this method for dividing polynomials in Year 11, where it is a much better alternative to long division. We can also use this method for functions that would become rational functions after making a u substitution.

The degree of the numerator and the degree of the denominator determine whether we rearrange the numerator as multiples of the denominator, or multiples of the derivative of the denominator.

In all cases we need to end up with a rational function where the degree of the numerator is less than the degree of the denominator before we can use any of the standard integrals.

Case 1: The degree of the numerator is greater than or equal to that of the denominator

If the numerator has degree that is greater than or equal to the degree of the denominator, then we rewrite the numerator as multiples of the denominator, leaving us with the sum of a polynomial and a rational fraction.

Example 1

Find $\int \frac{x^2+2x+3}{x-1}dx$

$$= \int \frac{x(x-1)+3(x-1)+6}{x-1}dx$$

$$= \int \left(x+3+\frac{6}{x-1}\right)dx$$

$$= \frac{x^2}{2}+3x+6\ln\left|x-1\right|+c$$

Don't forget the absolute value signs – if the argument inside the absolute values could take a negative value, then you will lose marks in the HSC if it is missing.

Example 2

Evaluate $\int_{-3}^{-2} \frac{x+2}{x+1}\,dx$.

> The limits have been chosen here so that you cannot solve this definite integral if you don't use absolute values signs correctly.

$$= \int_{-3}^{-2} \frac{x+1+1}{x+1}\,dx$$

$$= \int_{-3}^{-2} \left(1 + \frac{1}{x+1}\right) dx$$

$$= \Big[x + \ln\left|x+1\right|\Big]_{-3}^{-2}$$

> Without the absolute values signs we could not evaluate the primitive functions here.

$$= \left(-2 + \ln\left|-1\right|\right) - \left(-3 + \ln\left|-2\right|\right)$$

$$= -2 + 0 + 3 - \ln 2$$

$$= 1 - \ln 2$$

Example 3

Find $\int_{1}^{\sqrt{3}} \frac{x^2+5}{x^2+1}\,dx$.

$$= \int_{1}^{\sqrt{3}} \frac{x^2+1+4}{x^2+1}\,dx$$

$$= \int_{1}^{\sqrt{3}} \left(1 + \frac{4}{x^2+1}\right) dx$$

$$= \Big[x + 4\tan^{-1} x\Big]_{1}^{\sqrt{3}}$$

$$= \left(\sqrt{3} + 4\left(\frac{\pi}{3}\right)\right) - \left(1 + 4\left(\frac{\pi}{4}\right)\right)$$

$$= \sqrt{3} - 1 + \frac{7\pi}{3}$$

The same techniques work for functions that would be rational functions after a u substitution.

Example 4

Find $\displaystyle\int \frac{e^{2x}+e^{x}}{e^{2x}-e^{x}}\,dx$.

$$= \int \frac{e^{2x}-e^{x}+2e^{x}}{e^{2x}-e^{x}}\,dx$$

$$= \int \left(1+\frac{2e^{x}}{e^{2x}-e^{x}}\right)dx$$

$$= \int \left(1+\frac{2e^{-x}}{1-e^{-x}}\right)dx$$

$$= x + 2\ln\left|1-e^{-x}\right| + c$$

Case 2: The degree of the numerator is one less than that of the denominator

If the degree of the numerator is one less than the degree of the denominator, then we rewrite the numerator as multiples of the derivative of the denominator, leaving us with the sum of a polynomial and a rational function. The primitive will include a logarithmic function.

Example 5

Find $\displaystyle\int \frac{4x}{x^2+4x+13}\,dx$.

$$= \int \frac{2(2x+4)-8}{x^2+4x+4+9}\,dx$$

$$= 2\int \frac{2x+4}{x^2+4x+13}\,dx - 8\int \frac{1}{(x+2)^2+3^2}\,dx$$

$$= 2\ln\left|x^2+4x+13\right| - \frac{8}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c$$

Since $x^2+4x+13$ is always positive we would not lose marks in the HSC if we did not include absolute values.

Partial fractions by inspection

Partial fractions involve breaking a rational function into the sum or difference of two or more simpler fractions - the partial fractions - which we then solve using standard integrals. It only works for rational functions whose denominator can be factorised - think about cross multiplying to realise why.

We can create partial fractions by inspection for easier examples. Next lesson we will look at three formal methods for harder examples but never forget that it can be easier to break up the rational function by inspection.

Example 6

Find $\int \frac{3}{x(x+1)}\,dx$.

$$= 3\int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= 3\ln|x| - 3\ln|x+1| + c$$

$$= 3\ln\left|\frac{x}{x+1}\right| + c$$

We will try putting 1 over each factor, and subtracting the two fractions, which will get rid of the x in the numerator when we cross multiply. We then need to place a fudge factor out the front, so that when it multiplies by the new fractions, we get an answer equivalent to the original question. In this case $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}$ so, we need a fudge factor of 3 out the front.

Example 7

Show that $\int_2^3 \frac{1}{x(x^2+1)}\,dx = \ln\left(\frac{3\sqrt{2}}{4}\right)$.

We will try putting 1 over each factor, and subtracting the two fractions, but cross multiplying we would be left with an x^2 we don't want and be subtracting an x. We can get around these problems if we make the second denominator x. Cross multiplying in our heads we see that no fudge factor is needed.

$$\text{LHS} = \int_2^3 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$$

$$= \int_2^3 \left(\frac{1}{x} - \frac{1}{2} \times \frac{2x}{x^2+1}\right) dx$$

$$= \left[\ln|x| - \frac{1}{2}\ln\left|x^2+1\right|\right]_2^3$$

$$= \left(\ln 3 - \frac{1}{2}\ln 10\right) - \left(\ln 2 - \frac{1}{2}\ln 5\right)$$

$$= \ln 3 - \ln\sqrt{10} - \ln 2 + \ln\sqrt{5}$$

$$= \ln\left(\frac{3\sqrt{5}}{2\sqrt{10}}\right)$$

$$= \ln\left(\frac{3}{2\sqrt{2}}\right)$$

$$= \ln\left(\frac{3\sqrt{2}}{4}\right)$$

Basic

1 Find $\int \frac{x^2-2x-2}{x+1}dx.$

2 Evaluate $\int_1^3 \frac{2x-3}{x+1}dx.$

3 Find $\int \frac{2x^2+19}{x^2+9}dx.$

4 Find $\int \frac{3x+2}{x^2+6x+10}dx$.

5 Find $\int \frac{2}{x(x-1)}dx$.

6 Find $\int \frac{2}{x(x^2+4)}dx.$

Medium

7 Evaluate $\int_{-2}^{2} \frac{2x}{x^2+4x+20}dx.$

8 Find $\int \frac{e^{2x}+e^x}{e^{2x}+1}dx$.

9 Find $\int \frac{x^4+1}{x^2+2}dx.$

10 Show that $\int \frac{x^3}{x^2+2}dx = 2-\ln 3$.

11 Find $\int \frac{x^6+3x^2-1}{x^3+1}dx.$

12 Find $\int \frac{x^2-4x+2}{(x-2)^3}dx.$

Challenging

13 Show that $\int_0^1 \frac{e^{2x}-e^x}{e^x+1}dx = \ln\left(\frac{4e^e}{[e(e+1)]^2}\right)$.

14 The area under the curve $y = \frac{e^{2x}}{\sqrt{e^{2x}+1}}$ from $x=0$ to $x=\ln 2$ is rotated about the x−axis. What is the volume of the solid of revolution formed? Answer correct to 2 decimal places.

15 Find $\int \frac{\cos x}{\sin^2 x - 3\sin x + 2}dx.$

16 Find $\int \frac{8x}{1+e^2x}dx.$

17 Find $\int \frac{\sec x \tan x}{\sec x + \sec^2 x} dx$.

18 Show that $\int \frac{\tan^2 x + \tan x + \sec^2 x - 2}{\tan x - 1} dx = \ln\left|\frac{\tan x - 1}{\cos x}\right| + 2x + c$.

19 Let $I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)} dx$.

a Show that $I = \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)} du$.

b Hence, find the value of I.

20 Find $\int \frac{\sin x}{\sin x + \cos x + e^x} dx$.

4.5 Partial fractions

In Lesson 5 we cover:

- Partial fractions using formal methods
  - Equating coefficients
  - Elimination by substitution
  - The cover-up method

In this lesson we look at splitting rational functions into partial fractions using formal methods rather than by inspection as we did last lesson. We look at three different methods of splitting a fraction into partial fractions, which can be simply integrated.

This lesson mainly covers the 7th and 8th dot points from *Further integration*, with questions on the 16th dot point included in the exercises.

Partial fractions using formal methods

Last lesson we looked at simple examples where we could split the rational function up by inspection, while in this lesson we look at formal methods needed for harder examples. Never forget that breaking up the rational function by inspection can be a valid alternative to the formal methods - make sure you show enough working though if the question is worth more than one mark. We will work through examples, with important notes beside each solution.

There are three methods we will investigate. All students should learn the first method, as it will work for all questions, even if a little more slowly than the other methods. More capable students should also learn the second and third methods as they each allow more efficient solutions to some types of questions. In exams you can choose whichever method you like.

Most exam questions will tell you the original integrand (a rational function) and the form of the partial fractions to use. Use the hints in the following examples so that you can answer those questions where the form for the partial fractions is not given.

Equating coefficients

The first method works for all questions but takes longer than the other two methods. We start by cross-multiplying the partial fractions on the RHS to create a single fraction, then equating the coefficients of x^2, x and the constant in the numerators. In more complicated questions the other two methods need to use this method to find the value of one or more variables.

Example 1

a Express $\dfrac{2x-7}{(x+1)(x-2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x-2}$ by equating coefficients.

$$\frac{2x-7}{(x+1)(x-2)}=\frac{a}{x+1}+\frac{b}{x-2}$$

$$\frac{2x-7}{(x+1)(x-2)}=\frac{a(x-2)+b(x+1)}{(x+1)(x-2)}$$

Cross-multiply on the RHS.

$$2x-7=(a+b)x-2a+b$$

Equate the numerators and separate the RHS into multiples of x and the constant term.

$$\therefore 2=a+b \qquad (1)$$

$$-7=-2a+b \qquad (2)$$

$$(1)-(2) \quad 9=3a \quad \rightarrow \quad a=3$$

$$\text{sub in } (1) \quad 2=3+b \quad \rightarrow \quad b=-1$$

Equate the coefficients of x and the constant terms to create simultaneous equations, then solve them to find a and b.

$$\therefore \frac{2x-7}{(x+1)(x-2)}=\frac{3}{x+1}-\frac{1}{x-2}$$

Rewrite the original fraction in the required form as asked in the question.

b Hence find $\displaystyle\int \frac{2x-7}{(x+1)(x-2)}\,dx$.

$$=\int\left(\frac{3}{x+1}-\frac{1}{x-2}\right)dx$$

$$=3\ln\left|x+1\right|-\ln\left|x-2\right|+c$$

Since $x+1$ and $x-2$ could be negative we must use absolute values.

If the question didn't give the required form, we could work it out as follows:

- The rational function has the numerator with a degree one less than the denominator.
- The factors in the denominator are $x+1$ and $x-2$, which give us the denominators in the two partial fractions.
- Both denominators are linear functions, so we put a constant function in the numerator – each has a degree one less than the denominator, just like the original rational function.

Elimination by substitution

The second method works quickly to find variables in partial fractions with linear denominators, but for the variables in other partial fractions we need to equate one or more of the coefficients as we did in Method 1.

We start by cross-multiplying the partial fractions on the RHS, but do NOT expand the terms.

We then find the value of the variables one at a time by substituting values for x that eliminate all but one of the variables.

If we used this method for Example 1 it would have saved us one line of working, so it is not that much more efficient.

Hints for the following example:

- When the original denominator has a squared factor the simplest form of the partial fractions involves a constant over a single power of that factor and a constant over the factor squared:

$$\frac{2x-3}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}$$

- Alternatively, we could expand the perfect square and use a linear numerator:

$$\frac{2x-3}{x^2(x-1)} = \frac{ax+b}{x^2} + \frac{c}{x-1}$$

- When there is a repeated factor then the substitution method cannot find all variables - equating coefficients will be needed for at least one variable – ‘a’ in this example as it is over the single x. This will also occur for the cover-up method we will use in a moment, which is an abbreviated form of the elimination by substitution method.

Example 2

a Express $\dfrac{2x-3}{x^2(x-1)}$ in the form $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x-1}$ by elimination by substitution.

$$\frac{2x-3}{x^2(x-1)}=\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x-1}$$

$$(2x-3)=ax(x-1)+b(x-1)+cx^2$$

Cross-multiply on the RHS but do NOT expand.

Let $x=0$ $\therefore 2(0)-3=a(0)+b(0-1)+c(0)$

$$-3=-b$$

$$b=3$$

Let $x=1$ $\therefore 2(1)-3=a(0)+b(0)+c(1)$

$$-1=c$$

$$c=-1$$

Substitute values of x that will eliminate other variables. We start by letting $x=0$ so that we can find b, then let $x=1$ so that we can find c.

equating coefficients of x^2: $0=a+c$

$$0=a-1$$

$$a=1$$

There is no substitution that leaves a while eliminating b and c, so we have to equate coefficients. Since a is part of the coefficient of x^2 and of x we could equate either one successfully.

$$\therefore \frac{2x-3}{x^2(x-1)}=\frac{1}{x}+\frac{3}{x^2}-\frac{1}{x-1}$$

b Hence evaluate $\displaystyle\int_2^3 \frac{2x-3}{x^2(x-1)}\,dx$.

$$\int_2^3 \frac{2x-3}{x^2(x-1)}\,dx=\int_2^3\left(\frac{1}{x}+\frac{3}{x^2}-\frac{1}{x-1}\right)dx$$

$$=\left[\ln|x|-\frac{3}{x}-\ln|x-1|\right]_2^3$$

Since x and $x-1$ cannot be negative for $2\le x\le 3$ we do NOT need to use absolute values.

$$=\left(\ln 3-1-\ln 2\right)-\left(\ln 2-\frac{3}{2}-0\right)$$

$$=\ln 3-2\ln 2+\frac{1}{2}$$

$$=\ln\left(\frac{3}{4}\right)+\frac{1}{2}$$

The cover-up method

The third method is a shortcut for elimination by substitution, allowing us to quickly find variables in partial fractions with linear denominators.

Just like elimination by substitution, it will not find variables in partial fractions with non-linear denominators. HSC questions are generally chosen so that this method cannot be used by itself, as it reduces these questions to mindless technique without understanding.

To find a variable over a linear denominator we do the following:

- Ignore (cover up) everything on the RHS except the variable we are trying to evaluate.
- Find the zero of the denominator under this variable.
- On the LHS ignore (cover up) the factor matching the denominator for the variable, then substitute the zero into what remains of the rational function and evaluate.

Let's redo part (a) of Example 1, then try a harder example.

Example 3

Express $\dfrac{2x-7}{(x+1)(x-2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x-2}$ using the cover−up method.

$$a=\frac{2(-1)-7}{-1-2}=3$$

To find a, we find the zero of its denominator, which is $x=-1$, and substitute it into the LHS, ignoring the factor $x+1$.

$$b=\frac{2(2)-7}{2+1}=-1$$

To find b, we find the zero of its denominator, which is $x=2$, and substitute it into the LHS, ignoring the factor $x-2$.

$$\therefore \frac{2x-7}{(x+1)(x-2)}=\frac{3}{x+1}-\frac{1}{x-2}$$

Note that this took three lines of working, whereas equating coefficients in Example 1 took seven lines, making this a much more efficient method..

Let's look at standard HSC-style question where we can use the cover-up method to find some of the variables and need to equate coefficients to find the others.

Example 4

Given $\dfrac{2x+1}{(x^2+1)(x+1)} = \dfrac{ax+b}{x^2+1} + \dfrac{c}{x+1}$, find the values of a, b and c using the $\text{cover} - \text{up}$ method and equating coefficients.

$$\frac{2x+1}{(x^2+1)(x+1)} \equiv \frac{ax+b}{x^2+1} + \frac{c}{x+1}$$

$$c = \frac{2(-1)+1}{(-1)^2+1} = -\frac{1}{2}$$

cover-up method

$$\frac{2x+1}{(x^2+1)(x+1)} \equiv \frac{(ax+b)(x+1) - \frac{1}{2}(x^2+1)}{(x^2+1)(x+1)}$$

$$\equiv \frac{\left(a-\frac{1}{2}\right)x^2 + (a+b)x + \left(b-\frac{1}{2}\right)}{(x^2+1)(x+1)}$$

equating coefficients

$$\therefore a - \frac{1}{2} = 0$$

equating coefficients of x^2

$$\therefore a = \frac{1}{2}$$

$$b - \frac{1}{2} = 1$$

equating constant terms

$$b = \frac{3}{2}$$

$\therefore a = \frac{1}{2}, b = \frac{3}{2}$ and $c = -\frac{1}{2}$.

In the last example one of the factors, x^2+1, is an irreducible quadratic – it cannot be factorised within the real number system.

We can use the cover-up method to find the values of a and b if we find and use the complex zero of x^2+1, which is $x=i$. We cover up everything on the RHS except for $ax+b$, and cover up x^2+1 on the LHS, then substitute $x=i$ into both sides. After rewriting the RHS in Cartesian form we find a and b by equating the real and imaginary components.

Note that we are using complex numbers as a tool to make our calculations easier, but the final answers must be purely real values for a and b. We will still find c in the normal manner.

Example 5

Given $\dfrac{2x+1}{(x^2+1)(x+1)}$

$=\dfrac{ax+b}{x^2+1}+\dfrac{c}{x+1}$, find the values of a, b and c using the cover−up method.

$$a(i)+b=\frac{2(i)+1}{i+1}\times\frac{1-i}{1-i}$$

Substitute $x=i$ into both sides, then rationalise the denominator.

$$b+ai=\frac{2i+2+1-i}{1^2+1^2}$$

$$\therefore b+ai=\frac{3}{2}+\frac{1}{2}i$$

$$\therefore a=\frac{1}{2}, b=\frac{3}{2}$$

Equate the real and imaginary components.

$$c=\frac{2(-1)+1}{(-1)^2+1}$$

Use the cover-up method to find c.

$$=-\frac{1}{2}$$

$\therefore a=\dfrac{1}{2}, b=\dfrac{3}{2}$ and $c=-\dfrac{1}{2}$.

Basic

1a Express $\dfrac{2x+1}{(x+1)(x+2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x+2}$ by equating coefficients.

b Hence find $\displaystyle\int \frac{2x+1}{(x+1)(x+2)}\,dx$.

2a Express $\dfrac{2x+3}{x^2(x+1)}$ in the form $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x+1}$ by elimination by substitution.

b Hence evaluate $\displaystyle\int_2^3 \frac{2x+3}{x^2(x+1)}\,dx$.

3 Express $\dfrac{2x+1}{(x+1)(x+2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x+2}$ using the cover−up method.

4 Given $\dfrac{2x+3}{x^2(x+1)}=\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x+1}$, find the values of a, b and c using the cover−up method and equating coefficients.

5a Find a and b given that $\dfrac{1-x}{(x+1)(x+2)}\equiv\dfrac{a}{x+1}+\dfrac{b}{x+2}$.

b Hence find $\displaystyle\int \frac{1-x}{(x+1)(x+2)}\,dx$.

6 Rewrite $\dfrac{x+1}{x(x-1)^2}$ as the sum or difference of partial fractions.

7 Find $\displaystyle\int_0^1 \frac{x+2}{x^2+7x+12}\,dx$.

Medium

8 Express $\dfrac{5x-5}{(x+2)(x^2+1)}$ in the form $\dfrac{a}{x+2}+\dfrac{bx+c}{x^2+1}$ using the cover−up method and complex numbers, and hence evaluate $\displaystyle\int_3^4 \frac{5x-5}{(x+2)(x^2+1)}\,dx$.

9 It can be shown that $\dfrac{2}{x^3+x^2+x+1}=\dfrac{1}{x+1}-\dfrac{x}{x^2+1}+\dfrac{1}{x^2+1}$ (Do NOT prove this.)

Use this result to evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1}\,dx$.

10 It can be shown that $\dfrac{8(1-x)}{(2-x^2)(2-2x+x^2)}=\dfrac{4-2x}{2-2x+x^2}-\dfrac{2x}{2-x^2}$ (Do NOT prove this.)

Use this result to evaluate $\displaystyle\int_0^1 \frac{8(1-x)}{(2-x^2)(2-2x+x^2)}\,dx$.

11 Evaluate $\displaystyle\int_2^5 \frac{x-6}{x^2+3x-4}\,dx$.

12a Find a, b and c given that $\dfrac{-5x+2}{x^2(x-1)} \equiv \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x-1}$.

b Hence evaluate $\displaystyle\int_2^3 \frac{-5x+2}{x^2(x-1)}\,dx$.

13a Given that $\dfrac{16x-43}{(x-3)^2(x+2)}$ can be written as $\dfrac{16x-43}{(x-3)^2(x+2)} = \dfrac{a}{(x-3)^2} + \dfrac{b}{x-3} + \dfrac{c}{x+2}$.

where a, b and c are real numbers, find a, b and c.

b Hence find $\displaystyle\int \frac{16x-43}{(x-3)^2(x+2)}\,dx$.

14 Express $\dfrac{2x+3}{(x+1)(x^2+4)}$ in the form $\dfrac{a}{x+1} + \dfrac{bx+c}{x^2+4}$ and hence evaluate $\displaystyle\int_0^2 \frac{2x+3}{(x+1)(x^2+4)}\,dx$.

Challenging

15 Find $\displaystyle\int \frac{3x^2+8}{x(x^2+4)}\,dx$.

16 Find $\displaystyle\int \frac{2x^3-x^2-8x-2}{x(x-2)}\,dx$.

17 Find $\displaystyle\int \frac{e^x+1}{e^{2x}-5e^x+6}\,dx$.

18 Given that $\dfrac{3x^3+x^2+12x-12}{(x^2+4)(x^2+2x+4)} = \dfrac{x-3}{x^2+2x+4} + \dfrac{2x}{x^2+4}$, show that

$$\int_0^1 \frac{3x^3+x^2+12x-12}{(x^2+4)(x^2+2x+4)}\,dx = \ln\frac{5\sqrt{7}}{8} + \frac{4}{\sqrt{3}}\left(\frac{\pi}{6} - \tan^{-1}\frac{2}{\sqrt{3}}\right).$$

19 Use partial fractions to show that $\dfrac{3!}{x(x+1)(x+2)(x+3)} = \dfrac{1}{x} - \dfrac{3}{x+1} + \dfrac{3}{x+2} - \dfrac{1}{x+3}$.

20 It is given that $t^4+4 = (t^2+2t+2)(t^2-2t+2)$.

a Find A and B so that $\dfrac{16}{t^4+4} = \dfrac{A+2t}{t^2+2t+2} + \dfrac{B-2t}{t^2-2t+2}$.

b A particle is initially at the origin and moves to the right along the x−axis. Its velocity t seconds after it started moving is $\dfrac{16}{t^4+4}$ metres per second.
Show that its displacement after t seconds is

$$x = \ln\left(\frac{t^2+2t+2}{t^2-2t+2}\right) + 2\tan^{-1}(t+1) + 2\tan^{-1}(t-1) \text{ metres.}$$

c Prove that the particle will approach but never reach $x = 2\pi$ metres.

In Lesson 6 we cover:

- Simplifying definite integrals using symmetry
- Variable of integration (dummy variables)
- Simplifying definite integrals using translations and dilations
- Simplifying definite integrals for odd and even functions
- Simplifying definite integrals using horizontal reflections

In this lesson we look at a wide variety of techniques that help us evaluate definite integrals. The techniques we look at this lesson are rarely used one at a time in Extension 2 but can be required within longer solutions or tricky multiple-choice questions.

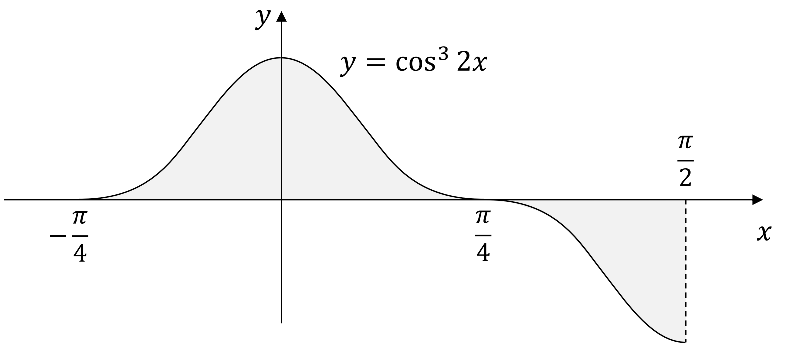
This lesson is massively important for harder questions but is not covered in the syllabus in any great detail. This content best fits with the 15^{th} and 16^{th} dot points from *Further integration*.

Simplifying definite integrals using symmetry

An easy method to simplify definite integrals involves identifying parts of the integral that are:

- equal, where we simplify the integral by doubling one of the parts
- of the same magnitude but opposite sign, where we cancel out the two parts.

Consider the graph of $y = \cos^3 2x$ in the domain $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right]$.



There are three equal areas in this domain, which could lead us to a variety of obvious and not so obvious relationships between its parts, that we can use to create a simpler integral.

Some obvious results include:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^3 2x \, dx = 2\int_0^{\frac{\pi}{4}} \cos^3 2x \, dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^3 2x \, dx = 0.$$

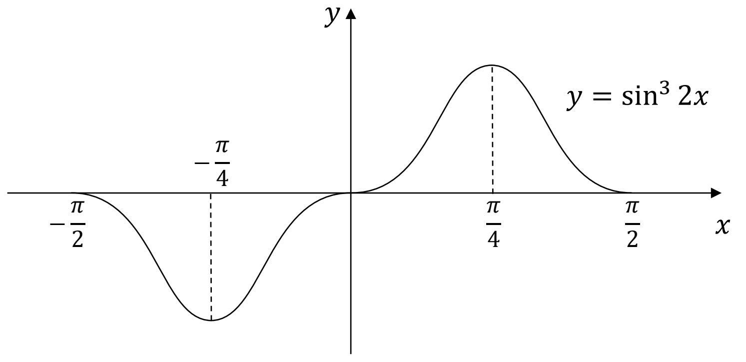
Less obvious results include:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 2x \, dx = \int_0^{\frac{\pi}{4}} \cos^3 2x \, dx \quad \text{and} \quad \int_{-\frac{\pi}{4}}^{0} \cos^3 2x \, dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^3 2x \, dx.$$

These types of relationships are particularly useful in simplifying definite integrals involving functions of sine and cosine. Since $\cos 2x$ has a period of π, we could add or subtract π to the upper and lower limits of any integral above without changing its value.

Example 1

Consider the graph of $y = \sin^3 2x$ for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



a Use the graph to arrange these definite integrals in ascending order.

$$\text{A: } \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^3 2x\, dx = \int_{-\frac{\pi}{4}}^{0} \sin^3 2x\, dx + \int_{0}^{\frac{\pi}{4}} \sin^3 2x\, dx = -\frac{1}{3} + \frac{1}{3} = 0$$

$$\text{B: } \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \sin^3 2x\, dx = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \sin^3 2x\, dx + \int_{-\frac{\pi}{4}}^{0} \sin^3 2x\, dx + \int_{0}^{\frac{\pi}{4}} \sin^3 2x\, dx = -\frac{1}{3} + 0 \text{ (from } a) = -\frac{1}{3}$$

$$\text{C: } \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 2x\, dx = \int_{0}^{\frac{\pi}{4}} \sin^3 2x\, dx = \frac{1}{3} \quad \text{since } \sin^3 2x \text{ is symmetrical about } x = \frac{\pi}{4}$$

In ascending order, we have: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \sin^3 2x\, dx,\quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^3 2x\, dx,\quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 2x\, dx.$

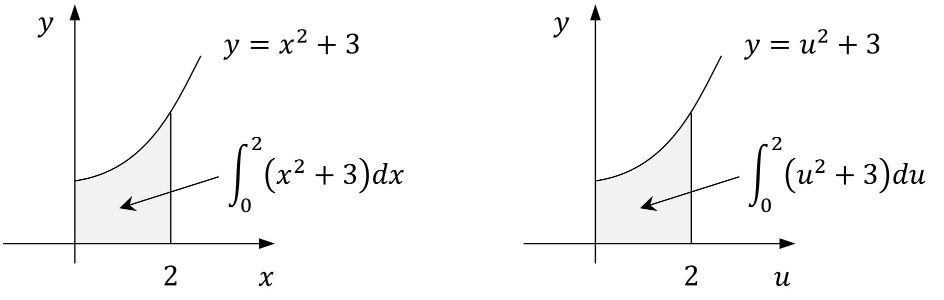
b What value of a in the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ maximises the value of $\int_{a}^{\frac{\pi}{2}} \sin^3 2x\, dx$?

$\int_{a}^{\frac{\pi}{2}} \sin^3 2x\, dx$ is maximised when we exclude as much of the domain where the graph of $y = \sin^3 2x$ is below the x–axis, and include as much of the domain where the graph of $y = \sin^3 2x$ is above the x–axis.

This occurs when $a = 0$.

Variable of integration (dummy variables)

The value of a definite integral is independent of the variable used − this is an important technique used in proofs, often called dummy variables. In the diagrams below we can easily see that $\int_0^2 (x^2+3)dx$ and $\int_0^2 (u^2+3)du$ must be identical.



This relationship only applies to definite integrals. If we compare $\int (x^2+3)\,dx$ and $\int (u^2+3)\,du$ we need to remember that not only do both sides end up with a function still in terms of the different variables, but they each include an unknown constant, which are not necessarily equal. For these indefinite integrals we end up with $\frac{x^3}{3} + 3x + c_1$ and $\frac{u^3}{3} + 3u + c_2$, which follow a similar pattern, but which are not equal - except by coincidence.

We will use symmetry and a u substitution, together with a change of variable in the next example. Here we are comparing integrals under two different functions, rather than comparing integrals under different sections of the same function.

Example 2

Show that $\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \cos^n 2x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$.

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \cos^n 2x \, dx$$

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2dx \\ dx &= \frac{du}{2} \end{aligned}}$$

$= 2\int_{\pi}^{\frac{5\pi}{4}} \cos^n 2x \, dx$ since $\cos^n 2x$ is symmetrical about $x = \pi$

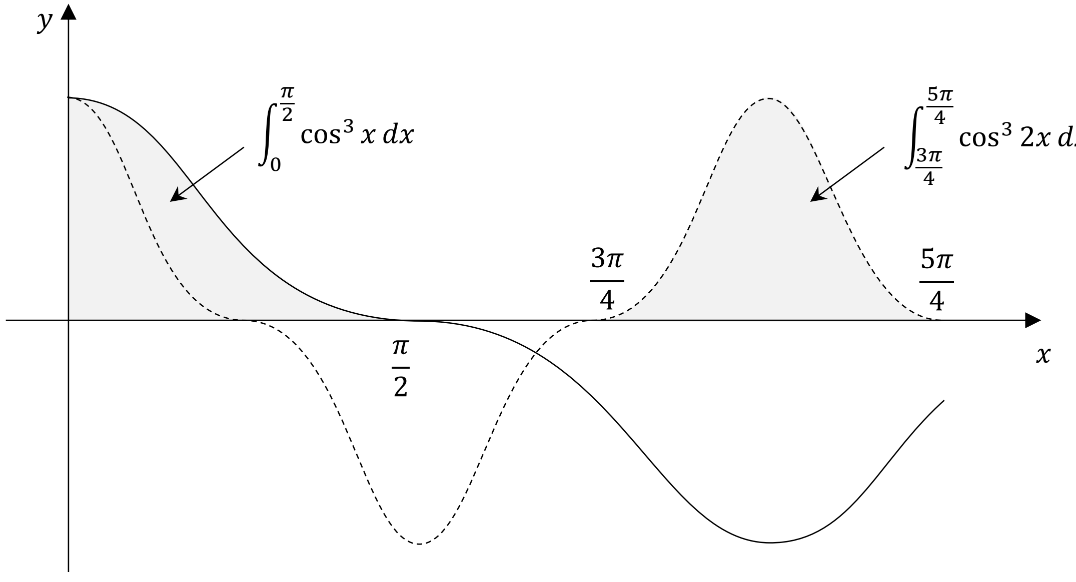
$$= 2\int_{2\pi}^{\frac{5\pi}{2}} \cos^n u \times \frac{du}{2}$$

$$= \int_{2\pi}^{\frac{5\pi}{2}} \cos^n u \, du$$

$= \int_0^{\frac{\pi}{2}} \cos^n u \, du$ since $\cos u$ has a period of 2π we can subtract 2π from both limits

$= \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ changing the variable

If $n = 3$ we can see the relationship between the definite integrals as the areas below.



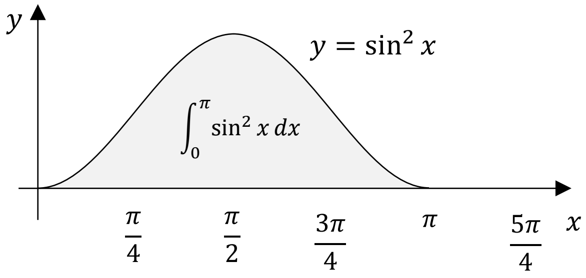
Simplifying definite integrals using translations and dilations

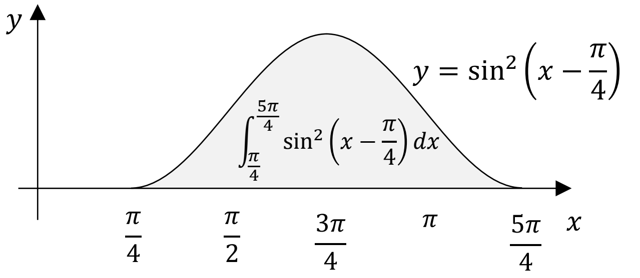
In Advanced we looked at how to quickly sketch transformations of a function, and we can use some of these skills to help simplify some definite integrals. We will look at horizontal reflections shortly, but let's consider horizontal translations and dilations. Again, we are comparing integrals under two different functions.

Horizontal translations

When a function is translated horizontally, it is not only the function that moves, but also the upper and lower limits and the area between the curve and x-axis.

For example, the area under the curve $y = \sin^2 x$ from $x = 0$ to π, must be the same as the area under the curve $y = \sin^2\left(x - \frac{\pi}{4}\right)$ from $x = \frac{\pi}{4}$ to $\frac{5\pi}{4}$, shown below, where the curve, the limits and the area all move $\frac{\pi}{4}$ units to the right, preserving the area.





We will normally be trying to simplify the function in the definite integral, so moving from right to left we see that:

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin^2\left(x - \frac{\pi}{4}\right) dx = \int_0^{\pi} \sin^2 x\, dx.$$

In general, using translations to simplify the function in an integral, we have:

$$\boxed{\int_a^b f(x - c)dx = \int_{a-c}^{b-c} f(x)dx\,.}$$

We can prove this rule by letting $u = x - c$ and then changing the variable.

Note that we added c units to the value inside the function, but subtracted c units from the limits.

Horizontal dilations

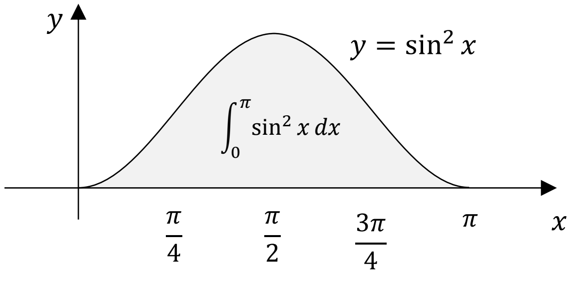
When the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is dilated horizontally by a factor of k:

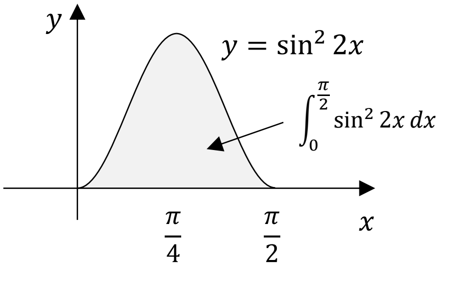
- the new function is $y = f(kx)$
- the new limits are $x = \frac{a}{k}$ and $x = \frac{b}{k}$
- the new area under the curve is $\frac{1}{k}$ times the old area, since the width has been multiplied by $\frac{1}{k}$ while the height is unaffected.

Using definite integrals we see:

$$\boxed{\int_a^b f(x)\,dx = k\int_{\frac{a}{k}}^{\frac{b}{k}} f(kx)\,dx}$$

For example, the area under the curve $y = \sin^2 x$ from $x = 0$ to π, must be twice the area under the curve $y = \sin^2 2x$ from $x = 0$ to $\frac{\pi}{2}$, shown below, where the curve and the limits have been compressed towards the y-axis by a factor of 2.





In general, using dilations to simplify the function in an integral, we have:

$$\boxed{\int_a^b f(kx)dx = \frac{1}{k}\int_{ka}^{kb} f(x)dx}$$

We can prove this rule by letting $u = kx$ and then changing the variable.

Note that we divided the value inside the function by k, but multiplied the limits by k. We also multiply the definite integral on the right by $\frac{1}{k}$ as its area increases by a factor of k.

Example 3

Given that $f(x)$ is symmetrical on the line $x = a$, prove the following results.

a $\displaystyle\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$$ since $f(x)$ is symmetrical about $x = a$

$$= 2\int_0^a f(x)\,dx$$

b $\displaystyle\int_{2a}^{3a} f(x)\,dx = \int_0^a f(x-a)\,dx.$

$$\int_{2a}^{3a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx$$ since $f(x)$ is symmetrical about $x = a$

$$= \int_0^a f(x-a)\,dx$$ translating a units to the right

c $\displaystyle\int_0^a f(2x)\,dx = \int_0^a f(x)\,dx$

$$\int_0^a f(2x)\,dx = \frac{1}{2}\int_0^{2a} f(x)\,dx$$ using a horizontal dilation of $f(x)$ by a factor of 2

$$= \int_0^a f(x)\,dx$$ from part (a)

Example 4

Show that the value of $\frac{1}{n}\int_0^{2na} e^{\frac{x}{2n}}\,dx + n\int_0^{\frac{a}{n}} e^{nx}\,dx$ for $a, n \in \mathbb{Z}^+$ is independent of the value of n.

$$\frac{1}{n}\int_0^{2na} e^{\frac{x}{2n}}\,dx + n\int_0^{\frac{a}{n}} e^{nx}\,dx = 2n \times \frac{1}{n}\int_0^{a} e^{x}\,dx + \frac{1}{n} \times n\int_0^{a} e^{x}\,dx$$

$$= 2\int_0^{a} e^{x}\,dx + \int_0^{a} e^{x}\,dx$$

$$= 3\int_0^{a} e^{x}\,dx$$

$$= 3\left[e^{x}\right]_0^{a}$$

$= 3(e^{a} - 1)$ which is independent of n.

Simplifying definite integrals for odd and even functions

In Advanced we looked at odd and even functions, where the integrals that are equal or cancel out are everything to the left and right of the y-axis. We will see these rules used in Extension 2, often when we have a combination of two odd and/or even functions, where we first need to determine whether the total function is odd or even. To help determine the rules, rather than memorising them, we can use simple power functions as shown below. Realise that the rules apply to all odd or even functions - the power functions are simply a memory aid.

Sum or difference of two functions $f(x) \pm g(x)$

- If both functions are odd their sum and difference are both odd.
  - For example, x^3 and x^5 are odd, and we see that $x^3 + x^5$ and $x^3 - x^5$ are also odd since all powers are odd.
- If both functions are even their sum and difference are both even.
  - For example, x^2 and x^4 are even, and we see that $x^2 + x^4$ and $x^2 - x^4$ are also even as all powers are even.
- If one function is odd and the other even, then their sum and difference are neither odd nor even.
  - For example, x^2 is even and x^3 is odd, and we see that $x^2 + x^3$ and $x^2 - x^3$ are neither odd nor even, as they have a mix of odd and even powers.

Product or quotient of two functions $f(x) \times g(x)$ and $\dfrac{f(x)}{g(x)}$

- If both functions are odd, or both functions are even, their product and quotient are even.
  - For example, x^4 and x^2 are even, as are $x^4 \times x^2 = x^6$ and $x^4 \div x^2 = x^2$. Similarly, x^5 and x^3 are odd, while $x^5 \times x^3 = x^8$ is even as is $x^5 \div x^3 = x^2$.
- If one function is odd and the other even their product or quotient is odd.
  - For example, x^4 is even and x^3 is odd, while $x^4 \times x^3 = x^7$ is odd as is $x^4 \div x^3 = x^1$.

Composite functions $f\left(g(x)\right)$

- If both functions are odd, then the composite function is odd.
  - For example, x^3 and x^5 are odd, as is $\left(x^3\right)^5 = x^{15}$.
- If one or both functions are even, then the composite function is even.
  - For example, x^4 is even and x^3 is odd, while $(x^4)^3 = x^{12}$ is even.

Equivalent definite integrals involving even functions

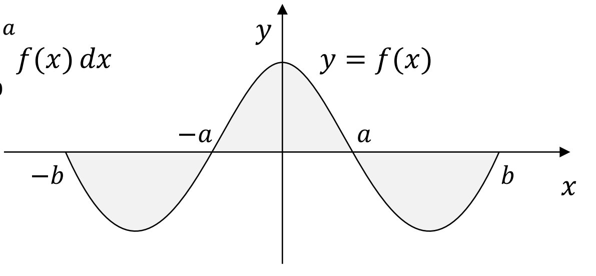
An even function is defined by $f(-x) = f(x)$, so the function has line symmetry on the y-axis.

Here are some of the rules that apply to even functions, with diagrams to help understanding.

1. Matching areas on opposite sides of the y-axis are equal.

$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx$ and $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$\int_{-b}^{-a} f(x)\,dx = \int_{a}^{b} f(x)\,dx$

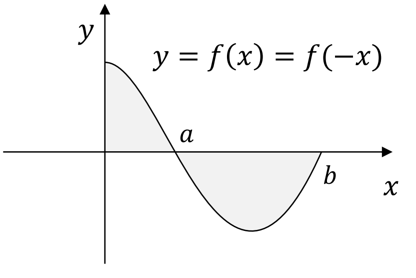


2. The function and its reflection on the y-axis coincide, so definite integrals between the same limits must be equal.

$\int_{0}^{a} f(-x)\,dx = \int_{0}^{a} f(x)\,dx$

$\int_{a}^{b} f(-x)\,dx = \int_{a}^{b} f(x)\,dx$

$\int_{a}^{b} \left(f(x) + f(-x)\right) dx = 2\int_{a}^{b} f(x)\,dx$



As an aside, the primitive of an even function is odd, so $F(-x) = -F(x)$.

Equivalent definite integrals involving odd functions

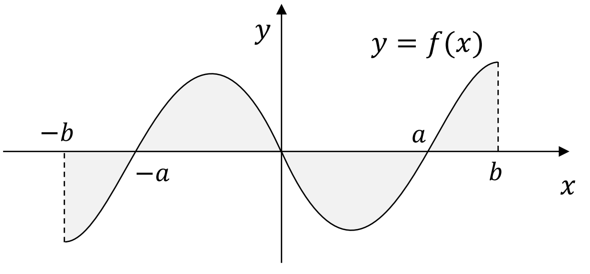
An odd function is defined by $f(-x) = -f(x)$, so the function has point symmetry at the origin.

Here are some of the rules that apply to odd functions, with diagrams to help understanding.

4. Matching areas on opposite sides of the y-axis are of equal magnitude but opposite sign.

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx \text{ and } \int_{-a}^{a} f(x)\,dx = 0$$

$$\int_{-b}^{-a} f(x)\,dx = -\int_{a}^{b} f(x)\,dx$$

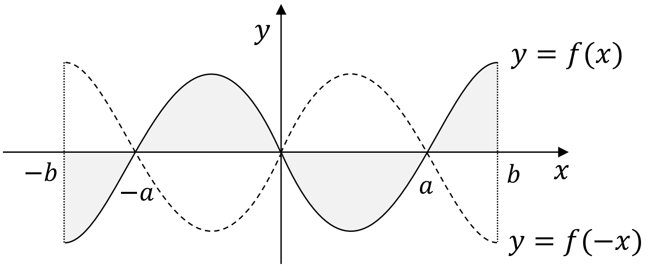


5. The function and its reflection on the y-axis are reflections on the x-axis, so definite integrals between the same limits must be multiplied by -1.

$$\int_{0}^{a} f(-x)\,dx = -\int_{0}^{a} f(x)\,dx$$

$$\int_{a}^{b} f(-x)\,dx = -\int_{a}^{b} f(x)\,dx$$

$$\int_{a}^{b} \big(f(x) + f(-x)\big)\,dx = 0$$



As an aside, the primitive of an odd function is even, so $F(-x) = F(x)$.

Example 5

State whether each of the following definite integrals in the form $I = \int_{-a}^{a} f(x)\,dx$ is equal to zero or $2J$, where $J = \int_{0}^{a} f(x)\,dx$. Justify your answer.

a $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin x \tan x\,dx$

$\sin x$ and $\tan x$ are both odd, so their product is even, $\therefore I = 2J$.

b $I = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (x^3 - \sin x)\,dx$

x^3 and $\sin x$ are both odd, so their difference is odd, $\therefore I = 0$.

c $I = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{x^4}{\sin x}\,dx$

x^4 is even and $\sin x$ is odd, so their quotient is odd, $\therefore I = 0$.

d $I = \int_{-1}^{1} e^{x^2}\,dx$

x^2 is even so e^{x^2} is even, $\therefore I = 2J$.

e $I = \int_{-3}^{3} \tan(x^3 + x)\,dx$

$\tan x$ and $x^3 + x$ are odd so the composite function is odd, $\therefore I = 0$.

Simplifying definite integrals using horizontal reflections

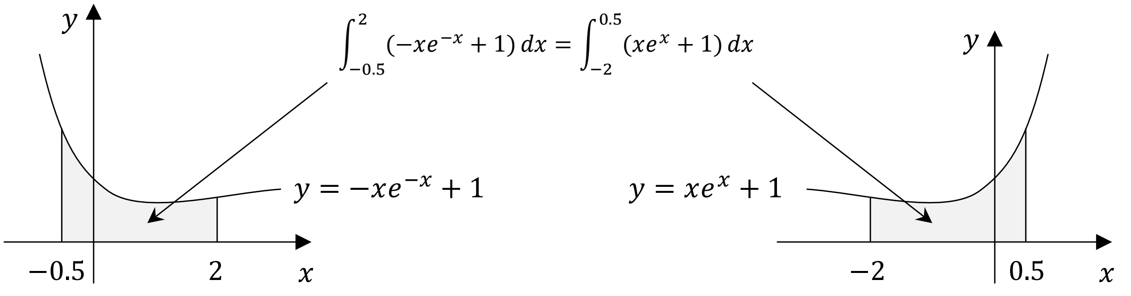
We now look at reflecting any function to create a second function and finding parts of the two different functions that are equal or cancel out. These results also apply to odd and even functions.

Reflecting definite integrals on the y-axis

When we reflect a curve and the area under it about the y-axis, we replace x with $-x$ in the function. Note that the upper and lower limits swap their relative order, so the negative of the old upper limit is now the lower limit, and vice versa.

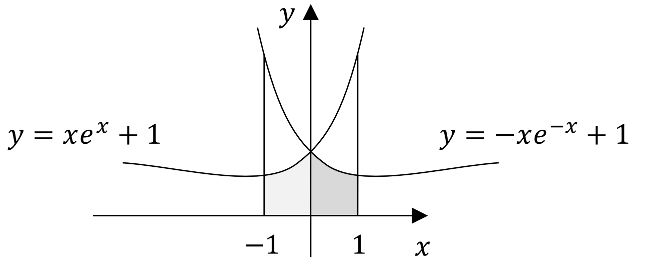
$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-x)\,dx$$

In the diagram below we have $f(x) = xe^x + 1, a = -2$ and $b = 0.5$.



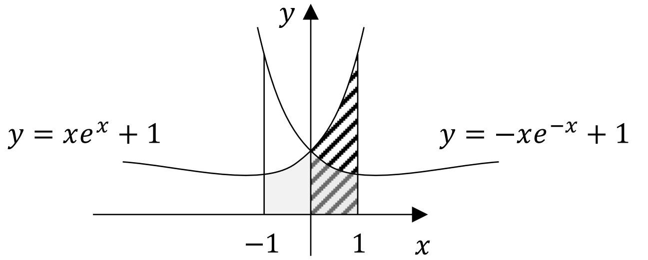
If the lower limit is zero, we find a common application: $\int_0^a f(x)\,dx = \int_{-a}^{0} f(-x)\,dx$.

In the diagram at right, we have $f(x) = xe^x + 1, a = 1$.



We can extend this result to find:

$$\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$$



A little harder to see in the diagram, but the two overlapping areas under the two curves between $x = 0$ and $x = 1$ are equal to the area under $y = xe^x + 1$ between $x = -1$ and $x = 1$.

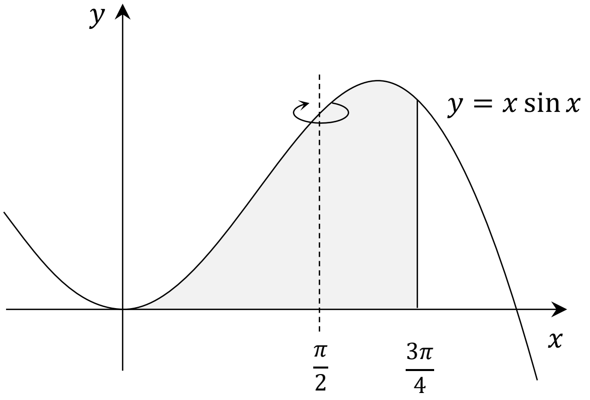
Reflecting definite integrals on $x = k$

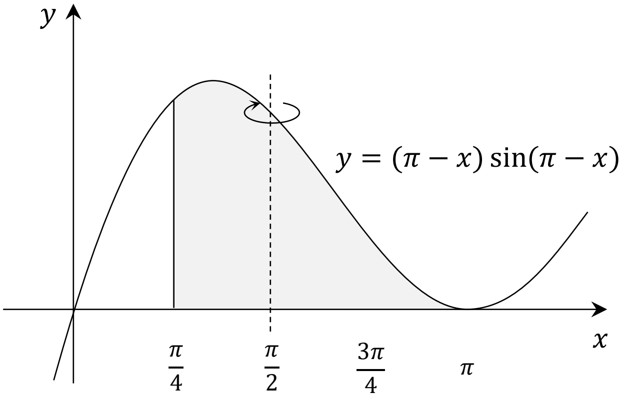
When we reflect the function $y = f(x)$ horizontally about the line $x = k$ the new function is $f(2k - x)$. You are NOT expected to know this, but it can come in handy. If it is needed in a question, you will be asked to prove it in part (a) then use it in part (b), or it will be given in the question.

If we also reflect the limits on $x = k$ then we create equivalent definite integrals:

$$\int_a^b f(x)\,dx = \int_{2k-b}^{2k-a} f(2k - x)\,dx$$

Consider the area between the curve $y = x\sin x$ and the x-axis between the lines $x = 0$ and $x = \frac{3\pi}{4}$, shown in the diagram below left. We will reflect it on the line $x = \frac{\pi}{2}$.





The equation of the new function is $y = \left(2\left(\frac{\pi}{2}\right) - x\right)\sin\left(2\left(\frac{\pi}{2}\right) - x\right) = (\pi - x)\sin(\pi - x)$.

The limits are also reflected about $x = \frac{\pi}{2}$.

- the new upper limit is $\frac{\pi}{2} + \left(\frac{\pi}{2} - \text{old lower limit}\right) = \frac{\pi}{2} + \left(\frac{\pi}{2} - 0\right) = 2\left(\frac{\pi}{2}\right) - 0 = \pi$
- the new lower limit is $\frac{\pi}{2} + \left(\frac{\pi}{2} - \text{old upper limit}\right) = \frac{\pi}{2} + \left(\frac{\pi}{2} - \frac{3\pi}{4}\right) = 2\left(\frac{\pi}{2}\right) - \frac{3\pi}{4} = \frac{\pi}{4}$.

From the diagram above we can see that $\displaystyle\int_{\frac{\pi}{4}}^{\pi} (\pi - x)\sin(\pi - x)\,dx = \int_0^{\frac{3\pi}{4}} x\sin x\,dx$.

We will see expressions like this when we look at integration by parts later in the chapter. We will normally use these rules in reverse, where we recognise a function in the form $f(2k - x)$ and change it to $f(x)$, adjusting the limits if needed.

Reflecting definite integrals on the midpoint of the limits, $\boldsymbol{x = \frac{a+b}{2}}$

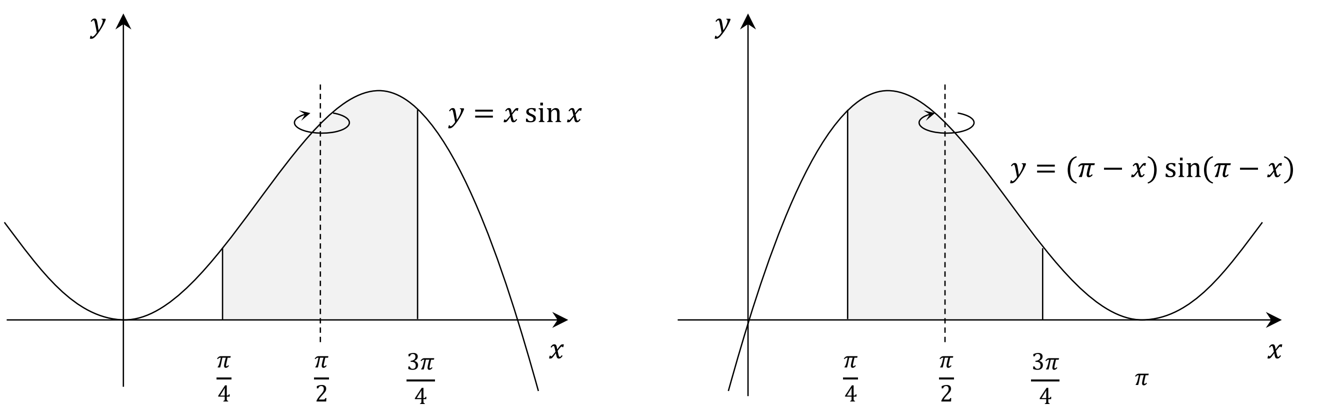
Generally, when we reflect a function on the line $x = k$ we choose a value of k which is halfway between the two limits as it creates more useful results - we reflect on the line $x = \frac{a+b}{2}$. This leads us to the relationship which is informally known as King's rule, which is more useful when the lower limit is zero, as we will see over the page.

When we reflect the function $y = f(x)$ horizontally on the line $x = \frac{a+b}{2}$ the new function is $f\left(2\left(\frac{a+b}{2}\right) - x\right) = f(a + b - x)$. Again, you are NOT expected to know this, but it can come in handy. If it is needed in a question, you will be asked to prove it in part (a) then use it in part (b), or it will be given in the question.

$$\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx.$$

Since the limits are equidistant from $x = \frac{a+b}{2}$ they do not appear to change, which can cause confusion.

Using similar definite integrals as before, but from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$ so that they are equidistant from $x = \frac{\pi}{2}$, we see the following diagrams. Note that the limits appear not to have changed, as they actually swapped.



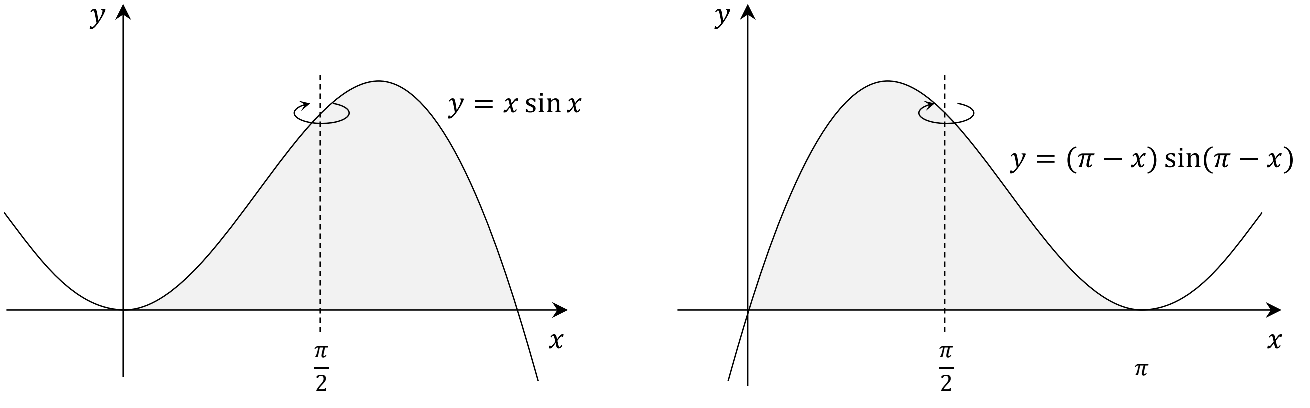
From the diagram above we can see that $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\pi - x)\sin(\pi - x)\,dx = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\sin x\,dx$.

Letting the lower limit equal zero, we see:

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

This is the reflection of $f(x)$ on $x = \frac{a}{2}$.

Using similar definite integrals as before, but from $x = 0$ to $x = \pi$ so that they are equidistant from $x = \frac{\pi}{2}$ and with $x = 0$ as the lower limit, we see the following diagrams.



From the diagram above we can see that $\int_0^{\pi} (\pi - x)\sin(\pi - x)\,dx = \int_0^{\pi} x\sin x\,dx$.

The most common application of this version of King's rule involves trig functions between 0 and $\frac{\pi}{2}$, where we let $a = \frac{\pi}{4}$. This leads to solutions using complementary angle properties, notably $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

In summary - keep the following rules in mind when you solve definite integrals. You will be asked to prove them if they are needed, but they are often an option in other questions.

$$\int_a^b f(x-c)dx = \int_{a-c}^{b-c} f(x)dx \qquad \int_a^b f(kx)dx = \frac{1}{k}\int_{ka}^{kb} f(x)dx$$

$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-x)\,dx \qquad \int_a^b f(x)\,dx = \int_{2k-b}^{2k-a} f(2k-x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \qquad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Let's use various reflection and translation properties in some examples.

Example 6

a Prove $\int_0^a f(a-x)dx = \int_0^a f(x)dx$.

$$\int_0^a f(a-x)dx \qquad \boxed{\begin{aligned} u &= a-x \\ du &= -dx \\ dx &= -du \end{aligned}}$$

$$= \int_a^0 f(u) \times (-du)$$

$$= \int_0^a f(u)du$$

$$= \int_0^a f(x)dx$$

b Hence show that $\int_0^\pi x\sin x\,dx = \pi$.

$$\int_0^\pi x\sin x\,dx = \int_0^\pi (\pi - x)\sin(\pi - x)\,dx$$

$$\int_0^\pi x\sin x\,dx = \int_0^\pi \pi\sin(\pi - x)\,dx - \int_0^\pi x\sin(\pi - x)\,dx$$

$$2\int_0^\pi x\sin x\,dx = \pi\int_0^\pi \sin x\,dx$$

$$\int_0^\pi x\sin x\,dx = \frac{\pi}{2}\int_0^\pi \sin x\,dx$$

$$= \frac{\pi}{2}\Big[-\cos x\Big]_0^\pi$$

$$= \frac{\pi}{2}\big(-(-1) - (-1)\big)$$

$$= \pi$$

Example 7

Given $\int_0^a f(a-x)dx = \int_0^a f(x)dx$, evaluate $\int_0^1 x(1-x)^7 x\, dx$.

$$\int_0^1 x(1-x)^7 x\, dx$$

$$= \int_0^1 (1-x)x^7\, dx$$

$$= \int_0^1 \left(x^7 - x^8\right) dx$$

$$= \left[\frac{x^8}{8} - \frac{x^9}{9}\right]_0^1$$

$$= \left(\frac{1}{8} - \frac{1}{9}\right) - \left(0 - 0\right)$$

$$= \frac{1}{72}$$

Example 8

Evaluate $\int_0^{\frac{\pi}{2}} \tan\left(x - \frac{\pi}{4}\right)\sec\left(x - \frac{\pi}{4}\right) dx$.

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x \sec x\, dx$ translating $\frac{\pi}{4}$ to the left

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x)\, dx$ where $f(x)$ is odd since $\tan x$ is odd and $\sec x$ is even

$= 0$

Example 9

The odd function $f(x)$ is defined for all real x.

a Show that $\int_0^a \big(f(a-x)-f(x-a)\big)\,dx = 2\int_0^a f(x)\,dx$ using reflections and translations.

$$\int_0^a \big(f(a-x)-f(x-a)\big)\,dx$$

$$= \int_0^a f(a-x)\,dx - \int_0^a f(x-a)\,dx$$

$$= \int_0^a f(x)\,dx - \int_{-a}^0 f(x)\,dx$$ reflecting (1) about $x=\frac{a}{2}$ and translating (2) a units to the left

$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$$ since $f(x)$ is odd

$$= 2\int_0^a f(x)\,dx$$

b Show that $\int_0^a \big(f(a-x)-f(x-a)\big)\,dx = 2\int_0^a f(x)\,dx$ using the substitution $u = x-a$.

$$\int_0^a \big(f(a-x)-f(x-a)\big)\,dx \qquad \boxed{\begin{array}{c} u = a-x \\ du = -dx \end{array}}$$

$$= \int_{-a}^0 \big(f(-u)-f(u)\big)\,du$$

$$= \int_0^a \big(f(u)-f(-u)\big)\,du$$

$$= \int_0^a \big(f(x)-f(-x)\big)\,dx$$ changing variables

$$= \int_0^a \big(f(x)+f(x)\big)\,dx$$ since $f(x)$ is odd

$$= 2\int_0^a f(x)\,dx$$

Basic

1 Prove $\int_0^a f(a-x)dx = \int_0^a f(x)dx$, and hence find $\int_0^{2\pi} x\cos x\,dx$.

2 Given $\int_0^a f(a-x)dx = \int_0^a f(x)dx$, evaluate $\int_0^2 x\sqrt{2-x}\,dx$.

3 Evaluate $\int_{-2}^{2} (x+x^3+x^5)(1+x^2+x^4)dx$.

4 Evaluate $\int_{-\pi}^{0} \sin\left(x+\frac{\pi}{2}\right)\cos\left(x+\frac{\pi}{2}\right)dx$.

5 Give a condition on a and b that guarantees $\int_a^b (x^3-x^2)\,dx$ is positive.

Medium

6 A function $f(x)$ has the property that $f(x)+f(a-x)=f(a)$.
Given $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ prove $\int_0^a f(x)\,dx = \frac{a}{2}f(a)$.

7 Without evaluating the integrals, which one of the following integrals is greater than zero?

A $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x^2}\cos x\,dx$ B $\int_{-\pi}^{\pi} x^3\cos x\,dx$ C $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sin^2 x-\cos^2 x)\,dx$ D $\int_{-1}^{1} \sin^{-1}(x^3)\,dx$

8 Which integral is equal to $\int_{-a}^{a} f(x)\,dx$ for all $f(x)$?

A $\int_0^a (f(x-a)-f(-x))\,dx$ B $\int_0^a (f(a+x)-f(x))\,dx$

C $\int_0^a (f(x)+f(-x))\,dx$ D $\int_0^a (f(x-a)-f(a-x))\,dx$

9 It is given that $f(x)$ is a non−zero even function and $g(x)$ is a non−zero odd function.
Which expression is equal to $\int_{-a}^{a} f(x)+g(x)\,dx$?

A $\int_0^a g(x)+g(-x)\,dx$ B $2\int_0^a g(x)+g(-x)\,dx$

C $\int_0^a f(x)+f(-x)\,dx$ D $2\int_0^a f(x)+f(-x)\,dx$

10 Prove $\int_a^{3a} f(x-a)\,dx = \int_{-a}^{a} f(a-x)\,dx$.

11 Which of these integrals has the smallest value?

A $\int_0^{\frac{\pi}{6}} \sin x\,dx$ B $\int_0^{\frac{\pi}{6}} \sin^2 x\,dx$ C $\int_0^{\frac{\pi}{6}} (1-\sin x)\,dx$ D $\int_0^{\frac{\pi}{6}} (1-\sin^2 x)\,dx$

12 Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx$.

13 Evaluate $\displaystyle\int_0^2 \left(1 + \sin\left(\pi(1-x)^3\right)\right) dx$.

14 For what values of a is $\displaystyle\int_0^2 \frac{dx}{1 - \sqrt{x-a}}$ defined?

15 Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2\left(x - \frac{\pi}{4}\right)}$.

16 Given that $xy = x + 1$, evaluate $\displaystyle\int_3^5 x\, dy$.

17 Evaluate $\displaystyle\int_{-1}^1 \frac{\tan^{-1} x}{1 + \sin^2 x}\, dx$.

18 Find the smallest value of $\displaystyle\int_0^1 \left(x^2 - a\right)^2 dx$ as a varies.

19 Given that $f(a+x) = f(a-x)$, prove each of the following results.

a $\displaystyle\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$

b $\displaystyle\int_0^a f(2x)\, dx = \int_0^a f(x)\, dx$

c $\displaystyle\int_0^a f(x+a)\, dx = \int_0^a f(x)\, dx$

d $\displaystyle\int_0^a f(x-a)\, dx = \int_{2a}^{3a} f(x)\, dx$

Challenging

20 Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\, dx$.

21 Given $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, which expression must be equal to $\displaystyle\int_0^a [f(a-x) + f(a+x)] dx$?

A $\displaystyle\int_0^a f(x)\, dx$ B $\displaystyle\int_0^{2a} f(x)\, dx$ C $\displaystyle 2\int_0^a f(x)\, dx$ D $\displaystyle\int_{-a}^a f(x)\, dx$

22 Evaluate $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$.

23a Prove $\displaystyle\int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}\,dx = \int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x}\right)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}\,dx$.

b Hence or otherwise evaluate $\displaystyle I = \int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x + \sin\frac{1}{x}\right)}\,dx$.

24 Given that $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \big(f(x)+f(-x)\big)\,dx$, evaluate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \sin^2 x}{1+e^x}\,dx$.

25a Prove $\displaystyle\int_{a}^{b} \frac{f(x)}{f(a+b-x)+f(x)}\,dx = \frac{b-a}{2}$.

b Hence evaluate $\displaystyle\int_{2}^{5} \frac{x^2}{\sqrt{29-x^2}+x}\,dx$, using the substitution $u = x^2$.

26 Given $\displaystyle\int_{a}^{b} f(x)\,dx = \frac{1}{2}\int_{a}^{b} \big(f(x)+f(a+b-x)\big)\,dx$, evaluate $\displaystyle\int_{3}^{7} \frac{\ln(x+2)}{\ln(24+10x-x^2)}\,dx$.

4.7 Product to sum and difference identities

In Lesson 7 we cover:

- Product to sum and difference identities
- Solving equations using the product to sum and difference identities
- Sum and difference to product identities (optional)
- Product to sum identities and integration

In this lesson we look at the product to sum and difference identities, content that has been moved from Year 11 Extension 1. Expect to see this content used in harder questions than they have been in previous Extension 1 HSCs.

We also briefly look at the sum and difference to product identities as an alternative when we need to change a sum or difference into a product, although they generally offer little advantage over the product to sum and difference identities. They are not in the syllabus, so feel free to skip them.

Note that the first part of this lesson doesn't actually use integration at all – we will be working with expressions and equations..

This lesson is massively important for harder questions but is not covered in the syllabus in any great detail. This content best fits with the 15$^{\text{th}}$ and 16$^{\text{th}}$ dot points from *Further integration*.

Product to sum and difference identities

When we are given the product of the sine or cosine of one argument and the sine or cosine of a different argument, we can convert the product into the sum or difference of sines or cosines using the product to sum and difference identities listed below. We can use them when the two arguments are the same, but it is usually easier to use the double angle formula.

We need to be able to convert back and forth between the two forms of expression as:

- The product of two trig ratios allows us to find zeros when solving equations or graphing more easily.
- The sum or difference of two trig ratios allows us to integrate and differentiate more easily.

These identities are applications of the double angle formula from the Extension 1 course.

The four product to sum rules are:

$$\cos A \cos B = \tfrac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big]$$

$$\sin A \sin B = \tfrac{1}{2}\Big[\cos(A-B) - \cos(A+B)\Big]$$

$$\sin A \cos B = \tfrac{1}{2}\Big[\sin(A+B) + \sin(A-B)\Big]$$

$$\cos A \sin B = \tfrac{1}{2}\Big[\sin(A+B) - \sin(A-B)\Big]$$

These identities ARE given on the reference sheet

Can we remember these identities without the reference sheet? Yes - but always check the reference sheet in an exam!

1. If the two ratios on the LHS are the same (both cos or both sin) then use cos on the RHS.
2. If the two ratios on the LHS are different (one cos and one sin) then use sin on the RHS.
3. If cos is on the RHS use the difference $(A-B)$ first and the sum $(A+B)$ second.
4. If sin is on the RHS use the sum $(A+B)$ first and the difference $(A-B)$ second.
5. If cos is the second ratio on the LHS then add the ratios on the RHS
6. If sin is the second ratio on the LHS then subtract the ratios on the RHS.

Now let's look at how one of the four rules is derived - the other three rules are proved in a similar way.

Prove $\cos A \cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$.

$$\text{RHS} = \frac{1}{2}\left[\cos A \cos B + \sin A \sin B + \cos A \cos B - \sin A \sin B\right]$$

$$= \frac{1}{2}\left[2\cos A \cos B\right]$$

$$= \cos A \cos B$$

$$\therefore \cos A \cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

Example 1

Write $\sin 2x \sin 4x$ as the difference of cosines.

$$\sin 2x \sin 4x = \frac{1}{2}\left[\cos(2x-4x)-\cos(2x+4x)\right]$$

$$= \frac{1}{2}\left(\cos(-2x)-\cos 6x\right)$$

$$= \frac{1}{2}(\cos 2x - \cos 6x) \quad \text{cosine is even}$$

- Both ratios on the LHS are the same so use cos on the RHS.
- When we use cos, we use the difference first and the sum second.
- Since sin is second on the LHS we subtract the ratios on the RHS.

Alternatively:

$$\sin 2x \sin 4x = \sin 4x \sin 2x$$

$$= \frac{1}{2}\left[\cos(4x-2x)-\cos(4x+2x)\right]$$

$$= \frac{1}{2}(\cos 2x - \cos 6x)$$

Note that it is much easier to rearrange the terms so the ratio with the larger angle is first

Example 2

Show that $\sin 52.5° \cos 7.5° = \dfrac{\sqrt{3}+\sqrt{2}}{4}$.

$$\begin{aligned}\text{LHS} &= \frac{1}{2}\left[\sin(52.5° + 7.5°) + \sin(52.5° - 7.5°)\right] \\ &= \frac{1}{2}(\sin 60° + \sin 45°) \\ &= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{2}\left(\frac{\sqrt{6}+2}{2\sqrt{2}}\right) \\ &= \frac{1}{2}\left(\frac{\sqrt{3}+\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{3}+\sqrt{2}}{4}\end{aligned}$$

- The ratios on the LHS are different so use sin on the RHS.
- When we use sin we use the sum first and the difference second.
- Since sin is second on the LHS we subtract the ratios on the RHS.

Example 3

a Write $\cos\theta - \cos 3\theta$ as the product of trigonometric functions.

$$\begin{aligned}\cos\theta - \cos 3\theta &= \cos(2\theta - \theta) - \cos(2\theta + \theta) \\ &= 2\sin 2\theta \sin\theta\end{aligned}$$

- We are using the identities backwards.
- When both ratios on the RHS are cos the two ratios on the LHS are the same.
- Since we subtract the ratios on the RHS the second one on the LHS must be cos, so they are both cos.
- The first angle is the difference, and the second angle is the sum, so we have 2θ then θ.
- We double it, as we don't have the $\frac{1}{2}$ from the identities.

b Hence show that $\dfrac{\cos\theta - \cos 3\theta}{2 - 2\cos^2\theta} = 2\cos\theta$.

$$\begin{aligned}\frac{\cos\theta - \cos 3\theta}{2 - 2\cos^2\theta} &= \frac{2\sin 2\theta \sin\theta}{2\sin^2\theta} \\ &= \frac{2\sin\theta\cos\theta\sin\theta}{\sin^2\theta} \\ &= 2\cos\theta\end{aligned}$$

Solving equations using the product to sum and difference identities

Let's use the identities to solve trig equations.

Example 4

Solve $\cos 3x - \cos 7x = \sin 5x$ for $0 \leq x \leq \dfrac{\pi}{2}$.

$$\begin{aligned} \cos 3x - \cos 7x &= \cos(5x - 2x) - \cos(5x + 2x) \\ &= 2\sin 5x \sin 2x \\ \therefore 2\sin 5x \sin 2x &= \sin 5x \\ \sin 5x\,(2\sin 2x - 1) &= 0 \end{aligned}$$

$$\sin 5x = 0 \quad \text{or} \quad 2\sin 2x - 1 = 0$$

$$5x = 0, \pi, 2\pi \qquad\qquad 2\sin 2x = 1$$

$$x = 0, \frac{\pi}{5}, \frac{2\pi}{5} \qquad\qquad \sin 2x = \frac{1}{2}$$

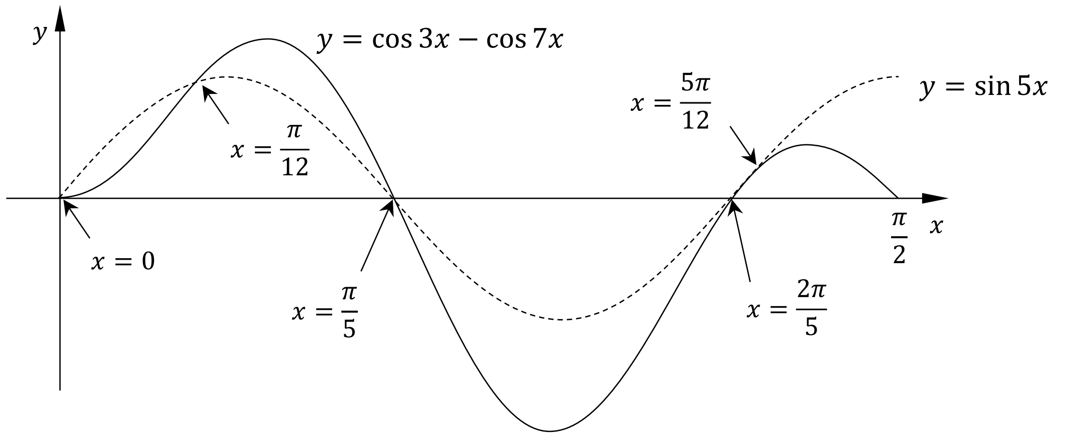
$$2x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$\therefore x = 0, \frac{\pi}{12}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{5\pi}{12}$$

Optional: It can help your understanding to graph both sides of the equation in your favourite graphing application (GeoGebra, Desmos etc) and see where they intersect, as shown below.

Note that since $\frac{2\pi}{5}$ and $\frac{5\pi}{12}$ are so close together, it would be easy to miss one of them if we just look at the graphs. Graphing $y = \cos 3x - \cos 7x - \sin 5x$ and looking for the x-intercepts can be safer.



Sum and difference to product identities (optional)

Although not mentioned in the syllabus, the sum to product identities are applications of the product to sum identities. You do not need to learn these rules, as the product to sum and difference identities solve related questions just as efficiently – they are just provided here so that you can be assured that you do NOT need to know them.

The four sum and difference to product identities are:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

These identities ARE NOT given on the reference sheet.

The identities can be determined by using the product to sum and difference identities and replacing A with $\frac{A+B}{2}$ and B with $\frac{A-B}{2}$, simplifying and rearranging.

If you are very good at remembering formulas, then you can try to memorise these identities, but it is safer to use the product to sum and difference identities from earlier in the lesson.

In the following examples we will solve each question two ways – first using the product to sum and difference identities and then using the sum and difference to product identities. As you will see, there is no advantage in using this second set of identities.

Example 5

a Prove that $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

$$\text{RHS} = 2 \times \frac{1}{2}\left[\cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right)\right]$$

$$= \cos\left(\frac{A+B-A+B}{2}\right) + \cos\left(\frac{A+B+A-B}{2}\right)$$

$$= \cos\left(\frac{2B}{2}\right) + \cos\left(\frac{2A}{2}\right)$$

$$= \cos A + \cos B$$

b Hence solve $\cos 4x + \cos 3x - \cos 2x - \cos x = 0$ for $0 \le x \le \pi$.

$$2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{x}{2}\right) - 2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) = 0$$

$$2\cos\left(\frac{x}{2}\right)\left[\cos\left(\frac{7x}{2}\right) - \cos\left(\frac{3x}{2}\right)\right] = 0$$

$$-2\cos\left(\frac{x}{2}\right)\left[\cos\left(\frac{5x}{2} - x\right) - \cos\left(\frac{5x}{2} + x\right)\right] = 0$$

$$-2\cos\left(\frac{x}{2}\right)\sin\left(\frac{5x}{2}\right)\sin\left(\frac{x}{2}\right) = 0$$

$$-\sin x \sin\left(\frac{5x}{2}\right) = 0$$

$$\therefore \sin x = 0 \quad or \quad \sin\left(\frac{5x}{2}\right) = 0$$

$$x = 0, \pi \qquad \frac{5x}{2} = 0, \pi, 2\pi$$

$$x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}$$

$$\therefore x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \pi$$

The pairs of angles $4x$ and $3x$, and $2x$ and x each have a difference of x, so we can leave them as these pairs. We could also rearrange the LHS to use $4x$ and $2x$, and $3x$ and x.

Example 6

Write $\cos 3x + \cos x$ as a function of $\cos x$ and show that $\cos 3x = 4\cos^3 x - 3\cos x$:

a using the product to sum and difference identities

$$\begin{aligned}
\cos 3x + \cos x &= \cos(2x + x) + \cos(2x - x) \\
&= 2\cos 2x \cos x \\
&= 2\left(2\cos^2 x - 1\right)\cos x \\
&= 4\cos^3 x - 2\cos x \\
\therefore \cos 3x &= 4\cos^3 x - 3\cos x
\end{aligned}$$

b using the sum and difference to product identities.

$$\begin{aligned}
\cos 3x + \cos x &= 2\cos\left(\frac{3x + x}{2}\right)\cos\left(\frac{3x - x}{2}\right) \\
&= 2\cos 2x \cos x \\
&= 2\left(2\cos^2 x - 1\right)\cos x \\
&= 4\cos^3 x - 2\cos x \\
\therefore \cos 3x &= 4\cos^3 x - 3\cos x
\end{aligned}$$

As we can see both methods took five lines, so there is little if any point learning the sum and difference to product identities.

Product to sum identities and integration

The product to sum identities can also be used to solve definite and indefinite integrals. Once the identities have been used, the integration is fairly simple.

Example 7

Show that $\int_0^{\frac{\pi}{6}} \cos 3x \sin 2x \, dx = \frac{3\sqrt{3}-4}{10}$.

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}\left[\sin(3x+2x) - \sin(3x-2x)\right] dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \left[\sin 5x - \sin x\right] dx$$

$$= \frac{1}{2}\left[-\frac{\cos 5x}{5} + \cos x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\left(-\frac{\cos\frac{5\pi}{6}}{5} + \cos\frac{\pi}{6}\right) - \left(-\frac{\cos 0}{5} + \cos 0\right)\right)$$

$$= \frac{1}{2}\left(\frac{\cos\frac{\pi}{6}}{5} + \cos\frac{\pi}{6} + \frac{1}{5} - 1\right)$$

$$= \frac{1}{2}\left(\frac{6}{5} \times \frac{\sqrt{3}}{2} - \frac{4}{5}\right)$$

$$= \frac{3\sqrt{3}-4}{10}$$

Example 8

Show that $\int \sin mx \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c$ for $m, n \in Z^+$ and $m \neq n$.

$$= \int \frac{1}{2}\Big[\cos(mx-nx) - \cos(mx+nx)\Big]\, dx$$

$$= \int \frac{1}{2}\Big[\cos(m-n)x - \cos(m+n)x\Big]\, dx$$

$$= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c$$

Example 9

The curve $y = \sin 2x + \sin 4x$ between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x-axis.

Find the volume of the solid of revolution created.

$$V = \pi \int_0^{\frac{\pi}{4}} (\sin 2x + \sin 4x)^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(\sin^2 2x + 2\sin 4x \sin 2x + \sin^2 4x\right) dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}(1-\cos 4x) + 2\left(\frac{1}{2}[\cos(4x-2x) - \cos(4x+2x)]\right) + \frac{1}{2}(1-\cos 8x)\right) dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(1 - \frac{1}{2}\cos 4x + \cos 2x - \cos 6x - \frac{1}{2}\cos 8x\right) dx$$

$$= \pi \left[x - \frac{\sin 8x}{16} - \frac{\sin 4x}{8} - \frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \pi\left(\left(\frac{\pi}{4} - 0 - 0 + \frac{1}{6} + \frac{1}{2}\right) - (0 - 0 - 0 - 0 + 0)\right)$$

$$= \frac{\pi(3\pi+8)}{12} \text{ units}^3$$

Basic

1 Write $\sin 4x \cos 3x$ as the sum of sines.

2 Show that $\cos 75° \cos 15° = \frac{1}{4}$.

3 Show that $\frac{\sin(A+B)+\sin(A-B)}{\cos(A-B)+\cos(A+B)} = \tan A$.

4 Show that $\cos 130° + \cos 10° = \cos 70°$.

5 Use the product to sum results to show that $\cos\frac{\pi}{12} - \cos\frac{5\pi}{12} = \cos\frac{\pi}{4}$.

6 Prove that $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$.

7a Find $\int \cos 3x \cos 2x\, dx$.

b Find $\int \sin 8x \cos 5x\, dx$.

8a Evaluate $\int_0^{\frac{\pi}{4}} \sin 3x \sin 4x\, dx$.

b Evaluate $\int_{\frac{\pi}{2}}^{\pi} \cos 5x \sin 3x\, dx$.

Medium

9 Solve $\sin 4x - \sin 2x = 0$ for $0 \le x \le \pi$, using:

a the double angle results

b the product to sum identities

10 Prove $\int \sin x \sin 2x\, dx = \frac{2\sin^3 x}{3} + c$:

a Using the double angle results and the reverse chain rule

b Using the product to sum identities.

c Which method is most efficient for this integral?

11 Find the values of A and B such that $\cos 6x \cos 3x - \sin 7x \sin 4x = \cos Ax \cos Bx$.

12 If $\sin(x-y) = M$ and $\sin(x+y) = N$ show that $\cos x \sin y = \frac{M+N}{2}$.

13 Show that $\cot A \cot B = \frac{\cos(A-B)+\cos(A+B)}{\cos(A-B)-\cos(A+B)}$.

14 Write $\sin 3x + \sin x$ as a function of $\sin x$.

15 Prove that $\tan\left(\frac{A+B}{2}\right) = \frac{\sin A + \sin B}{\cos A + \cos B}$.

16 Solve $\sin 3x + \cos 2x - \sin x = 0$ for $0 \leq x \leq 2\pi$.

17 Evaluate $\int_0^{\pi} \sin(mx)\cos(nx)\,dx$ where m and n are positive and even, with $m \neq n$.

18a Show that $\sin 2\theta + \sin 4\theta - \sin 6\theta = 4\sin\theta\sin 2\theta\sin 3\theta$.

b Hence solve $\sin 6\theta = \sin 2\theta + \sin 4\theta$ for $[0, \pi]$.

19 Solve $\sin 2x + \sin 3x + \cos 2x + \cos 3x = 0$ for $[0, 2\pi]$.

20a Show that $\dfrac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$ where $\cos 2x \neq -\dfrac{1}{2}$ and $\cos 3x \neq 0$.

b Hence solve $\sin x + \sin 3x + \sin 5x = \cos x + \cos 3x + \cos 5x$ for $0 \leq x \leq \pi$, given $\cos 2x \neq -\dfrac{1}{2}$.

Challenging

21 Solve $\cos\left(x + \dfrac{\pi}{6}\right)\sin\left(x - \dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{4}$ for $0 \leq x \leq 2\pi$.

22a Show that $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$.

b Hence solve $\sin^2 4x - \sin^2 2x = \sin 6x$ for $0 \leq x \leq \dfrac{\pi}{2}$.

23 Express $4\cos 2x \sin x + \cos 3x - 2\sin x + 1$ in the form $R\sin(3x + \alpha) + c$.

24a Prove that $\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right) + \cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right) = \sin A$.

b Hence solve $\sin\left(x + \dfrac{\pi}{6}\right)\cos\left(x - \dfrac{\pi}{6}\right) + \cos\left(x + \dfrac{\pi}{6}\right)\sin\left(x - \dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$ for $-\pi \leq x \leq \pi$.

25 Find $\int x\cos 4x^2 \cos 2x^2\,dx$.

4.8 Powers of trigonometric functions

In Lesson 8 we cover:

- Powers of trigonometric functions
  - Powers of sine and cosine
  - Powers of tangent and secant
  - Powers of cotangent and cosecant (briefly)

In this lesson we look at powers of trigonometric functions, either singly or in pairs. Integrands involving powers of trigonometric functions require the use of the Pythagorean identities and standard integrals, including the reverse chain rule. As we have many different approaches depending on the function(s) and the powers, this is a great way to increase your understanding of many aspects of integration.

This is an underutilised source of HSC questions for the early to middle sections of the exam, as they rarely occur!

This lesson mainly covers the 11^{th}, 15^{th} and 16^{th} dot points from *Further integration*.

Powers of trigonometric functions

Trigonometric functions can occur on their own, or in pairs, often with the same function. For example, sine and cosine are commonly paired, as are tangent and secant, due to their links through their derivatives and in the Pythagorean identities.

Cotangent and cosecant are paired but are rarely used in exams. The relationships are identical to those between tangent and secant, with negative signs included. We will only cover them briefly, then use them in exercises.

Powers of sine and cosine

To integrate powers of sine and cosine we use the following rules and standard integrals.

$$\sin^2 x + \cos^2 x = 1$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\int \sin x \, dx = -\cos x + c$$
$$\int \cos x \, dx = \sin x + c$$

Memorise these identities. Notice that the only difference is the sign.

It is best to practise many examples until you are able to determine the technique to be used. Alternatively, consider these suggestions.

- If the integrand is in the form $\sin x \cos^n x$ or $\sin^m x \cos x$ then use the reverse chain rule
- If the integrand is in the form $\sin^m x$ or $\cos^n x$, then break off $\sin^2 x$ or $\cos^2 x$ respectively, expand, then use the reverse chain rule
- If the integrand is in the form $\sin^m x \cos^n x$ we have different techniques required depending on the values of m and n.
  - if m or n is odd take out a factor of the trig function with the odd power - if they are both odd, it doesn't matter which one you choose - then use the Pythagorean identity to express the trig function that now has an even power in terms of the other trig function
  - if m and n are both even, then replace both functions using the half-angle identities: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

Personally, I prefer to do enough examples so that the correct method is obvious, rather than trying to remember more rules.

It is best to practise many examples until you are able to determine the technique to be used. Alternatively, consider these suggestions.

- If the integrand is in the form $\sin x\cos^n x$ or $\sin^m x\cos x$ then use the reverse chain rule
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- If the integrand is in the form $\sin^m x\cos^n x$ we have different techniques required depending on the values of m and n.
    - if m or n is odd take out a factor of the trig function with the odd power - if they are both odd, it doesn't matter which one you choose - then use the Pythagorean identity to express the trig function that now has an even power in terms of the other trig function
    - if m and n are both even, then replace both functions using the half-angle identities: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$.

Personally, I prefer to do enough examples so that the correct method is obvious, rather than trying to remember more rules.

Example 1

a Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \cos x\sin^3 x\,dx$

$$= \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left(\left(\frac{1}{2}\right)^4 - 0\right)$$

$$= \frac{1}{64}$$

b Find $\displaystyle\int \sin^3 x\ dx$.

$$= \int \sin^2 x\sin x\ dx$$

$$= \int (1-\cos^2 x)\sin x\,dx$$

$$= \int \sin x\,dx - \int \sin x\cos^2 x\,dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + c$$

Example 2

a Evaluate $\int_0^{\frac{\pi}{3}} \cos^2 x \sin^3 x \, dx$

$$= \int_0^{\frac{\pi}{3}} \sin x \cos^2 x \sin^2 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \cos^2 x \, (1 - \cos^2 x) \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \, (\cos^2 x - \cos^4 x) \, dx$$

$$= -\int_0^{\frac{\pi}{3}} (-\sin x)\left(\cos^2 x - \cos^4 x\right) dx$$

$$= -\left[\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right]_0^{\frac{\pi}{3}}$$

$$= -\left(\left(\frac{1}{24} - \frac{1}{160}\right) - \left(\frac{1}{3} - \frac{1}{5}\right)\right)$$

$$= \frac{47}{480}$$

b Find $\int \cos^2 x \sin^2 x \;\, dx$.

$$= \int \frac{1}{2}(1 + \cos 2x)\frac{1}{2}(1 - \cos 2x) dx$$

$$= \frac{1}{4}\int \left(1 - \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2}(1 + \cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{x}{8} - \frac{1}{32}\sin 4x + c$$

Powers of tangent and secant

Integration involving $\tan x$ and $\sec x$ can be more involved, as the two functions are linked by many different rules, which can cause confusion. Not all integrands involving tangent and/or secant can be integrated using the simple techniques below - some rely on integration by parts as we will see later in the chapter.

This complexity makes integration using tangent and secant great for Extension 2. We need to solve the integrals using the following Pythagorean identities and standard integrals.

$$\tan^2 x + 1 = \sec^2 x$$
$$\int \tan x \, dx = -\ln|\cos x| + c$$
$$\int \sec x \; dx = \ln|\sec x + \tan x| + c$$
$$\int \sec^2 x \, dx = \tan x + c$$
$$\int \sec x \tan x \, dx = \sec x + c$$

Memorise these rules - see the mnemonics at the end of the chapter.

Again, it is best to practise many examples until you are able to determine the technique to be used. Alternatively, consider these suggestions.

- If the integrand is in the form $\tan^m x \, \sec^2 x$ or $\sec x \tan x \sec^n x$ then use the reverse chain rule
- If the integrand is in the form $\tan^m x$ or $\sec^n x$, then break off $\tan^2 x$ or $\sec^2 x$ respectively, expand, then use the reverse chain rule - if you can - it won't always work
- If the integrand is in the form $\tan^m x \, \sec^n x$ we have different techniques required depending on the values of m and n.
  - if m is odd take out a factor of $\tan x \sec x$, then use the Pythagorean identity to express the remaining function in terms of $\sec x$ then use the reverse chain rule
  - if n is even then take out a factor of $\sec^2 x$, , then use the Pythagorean identity to express the remaining function in terms of $\tan x$ then use the reverse chain rule
  - for other cases there is no one method to follow, which makes these questions a challenge!

Example 3

a Find $\int \sec^2 x \tan^4 x \ dx$

$$= \frac{\tan^5 x}{5} + c$$

b Find $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^4 x \, dx$.

$$= \int_0^{\frac{\pi}{4}} \tan^2 x \tan^2 x \, dx$$
$$= \int_0^{\frac{\pi}{4}} \tan^2 x \, (\sec^2 x - 1) \, dx$$
$$= \int_0^{\frac{\pi}{4}} (\sec^2 x \, (\tan x)^2 - \tan^2 x) \, dx$$
$$= \int_0^{\frac{\pi}{4}} (\sec^2 x \, (\tan x)^2 - (\sec^2 x - 1)$$
$$= \left[\frac{\tan^3 x}{3} - \tan x + x\right]_0^{\frac{\pi}{4}}$$
$$= \left(\frac{1}{3} - 1 + \frac{\pi}{4}\right) - \left(0 - 0 + 0\right)$$
$$= \frac{3\pi - 8}{12}$$

Example 4

a Find $\int \tan^3 x \sec^3 x \, dx$

$$= \int \tan x \sec x \tan^2 x \sec^2 x \, dx$$
$$= \int \tan x \sec x \, (\sec^2 x - 1) \sec^2 x \, dx$$
$$= \int \tan x \sec x \, (\sec^4 x - \sec^2 x) \ dx$$
$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c$$

b Find $\int \tan^2 x \sec^4 x \, dx$.

$$= \int \tan^2 x \sec^4 x \, dx$$
$$= \int \sec^2 x \tan^2 x \, (1 + \tan^2 x) dx$$
$$= \int \sec^2 x \ (\tan^2 x + \tan^4 x) dx$$
$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + c$$

Example 5

Prove $\int \sec x \, dx = \ln|\tan x + \sec x| + c$.

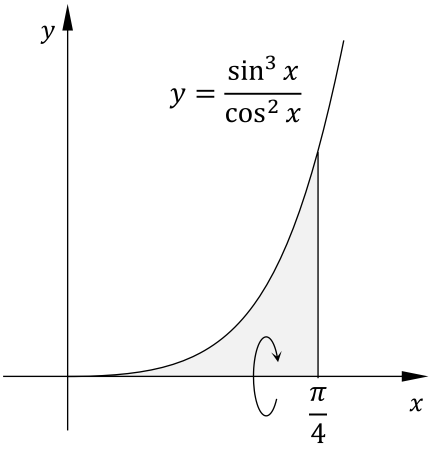
$$\int \sec x \, dx = \int \sec x \times \frac{\tan x + \sec x}{\sec x + \tan x} dx$$
$$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$$
$$= \ln|\tan x + \sec x| + c$$

If this question was asked in an exam, it would be scaffolded so that students could see the method required. Alternatively, we work backwards from the primitive to see that we need to change the integrand to be
$$\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}.$$

The next example is interesting as it looks like it should be included with the integrals including powers of sine and cosine, but as we can see, after a slight manipulation it involves powers of tangent and secant.

Example 6

The area between $y = \dfrac{\sin^2 x}{\cos^4 x}$ and the x−axis between $x = 0$ and $x = \dfrac{\pi}{4}$ is rotated about the x−axis. What is the volume of solid of revolution formed?



$$V = \pi \int_0^{\frac{\pi}{4}} \left(\frac{\sin^2 x}{\cos^4 x}\right)^2 dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos^8 x} dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \tan^4 x \sec^2 x \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \tan^4 x \left(1 + \tan^2 x\right) dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \, \left(\tan^4 x + \tan^6 x\right) dx$$

$$= \pi \left[\frac{\tan^5 x}{5} + \frac{\tan^7 x}{7}\right]_0^{\frac{\pi}{4}}$$

$$= \pi \left(\left(\frac{1}{5} + \frac{1}{7}\right) - (0 + 0)\right)$$

$$= \frac{12\pi}{35} \text{ units}^3$$

Powers of cotangent and cosecant

There are many similarities between the pair of functions $\cot x$ and $\operatorname{cosec} x$ and the pair of functions $\tan x$ and $\sec x$ that we have just looked at. There are no separate examples for this pair – have a look at the previous slides for the techniques to be used, replacing $\sec x$ with $\operatorname{cosec} x$ and $\tan x$ with $\cot x$, remembering that there will also be a minus sign involved.

Basic

1a Find $\int \sin x \cos^3 x \; dx$.

b Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 x \, dx$.

2a Find $\int \sec^4 x \tan^2 x \, dx$.

b Find $\int \sec^4 x \, dx$.

3 Prove $\int \operatorname{cosec} x \, dx = -\ln|\cot x + \operatorname{cosec} x| + c$.

4 Find $\int \tan x \sec^3 x \, dx$.

5 Find $\int \cos^4 x \, dx$.

6 Find $\int \tan^3 x \, dx$.

7 Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot^3 x \operatorname{cosec}^2 x \, dx$.

8 Find $\int \operatorname{cosec}^4 x \, dx$.

9 Find $\int \sqrt{\cos x} \sin^3 x \, dx$.

10 Find $\int \sin^2 x \cos^4 x \; dx$.

Medium

11 Find $\int \frac{\cos 2x}{\cos x} dx$.

12 Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan x}{\cos^2 x} dx$.

13 Find $\int (\cos^4 x - \sin^4 x) \, dx$.

14 Find $\int (\tan^4 x - \sec^4 x) \, dx$.

15 Find $\int \sec^6 x \, dx$.

16 The area between $y = \sec^2 x \tan^3 x$ and the $x-$axis between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the $x-$axis. What is the volume of solid of revolution formed?

Challenging

17 Find $\int \frac{(\cos x + \sin x)^3}{\sin 2x + 1} dx$.

18 Find $\int \frac{\sqrt{1 + \sin x}}{\sec x} dx$.

19 Find $\int \cot^2 x \, dx$.

20 Find $\int \frac{\sin^5 x}{\cos^4 x} dx$.

21 Find $\int \cot^3 x \, dx$.

22 Find $\int \cot^3 x \operatorname{cosec}^3 x \, dx$.

23 Show that $\int \sqrt{1 + \sin x} \, dx = 2 \sin \frac{x}{2} - 2 \cos \frac{x}{2} + c$.

4.9 Expressions and equations using the t-formulas

In Lesson 9 we cover:

- $t-$formulas
- Using $t-$formulas to simplify expressions
- Using $t-$formulas to solve equations

In this lesson we derive the t-formulas from the double angle formulas and use them to simplify expressions and solve equations. This content used to be included in Extension 1 and has nothing to do with integration. We will use t-formulas for integration next lesson.

This lesson covers the 5th dot point from *Further integration*.

t−formulas

Some proofs, equations and integrals are difficult to solve using standard trig techniques, but we can use the double angle results to find expressions that allow us to use algebraic approaches.

If we let $t = \tan\frac{A}{2}$, then we can write each of the trigonometric ratios in terms of t – these are called the t-formulas and are listed below.

$$\sin A = \frac{2t}{1+t^2}$$

$$\cos A = \frac{1-t^2}{1+t^2}$$

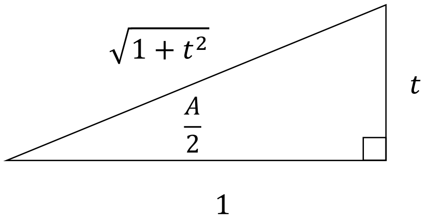
$$\tan A = \frac{2t}{1-t^2}$$

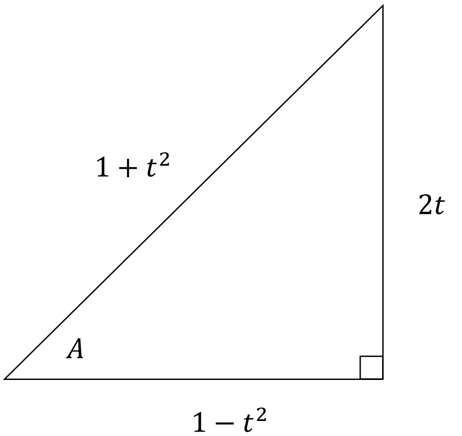
These identities ARE given on the reference sheet.

Note that tan is undefined for odd multiples of $\frac{\pi}{2}$ (odd multiples of $90°$), so $\frac{A}{2}$ cannot equal $\frac{\pi}{2}, \frac{3\pi}{2}$ etc (or $90°, 270°$). This means that A cannot equal $\pi, 3\pi$ etc (or $180°, 540°$). This becomes important later in the lesson when we use t-formulas to solve equations.

You can think of the t-formulas as the Swiss army knife of trigonometry - not always the best tool for the job, but useful for all sorts of little odd jobs.

When A is acute we can visualise $t = \tan\frac{A}{2}$ using the triangle below left and the t−formulas using the triangle below right. These diagrams become useful in solutions.





Example 1

Prove $\sin A = \dfrac{2t}{1+t^2}$, where $t = \tan\dfrac{A}{2}$.

$$\begin{aligned}
\sin A &= \sin\left[2\left(\frac{A}{2}\right)\right] \\
&= 2\sin\frac{A}{2}\cos\frac{A}{2} && \text{using the double angle result} \\
&= \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}} && \text{since } \cos^2\theta+\sin^2\theta = 1 \\
&= \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} && \text{dividing the numerator and denominator by } \cos^2\frac{A}{2} \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

We can prove $\cos A = \frac{1-t^2}{1+t^2}$ in a similar manner, while the proof of $\tan A = \frac{2t}{1-t^2}$ is simpler as we don't need to use the Pythagorean identity.

Using t−formulas to simplify expressions

In the next set of examples, we will use t−formulas to simplify expressions and find the exact value of trigonometric ratios.

Example 2

a Use t−formulas to prove that $\dfrac{1-\cos A}{\sin A} = \tan\dfrac{A}{2}$.

$$\begin{aligned}\frac{1-\cos A}{\sin A} &= \frac{1-\dfrac{1-t^2}{1+t^2}}{\dfrac{2t}{1+t^2}}\\ &= \frac{1+t^2-1+t^2}{2t}\\ &= \frac{2t^2}{2t}\\ &= t\\ &= \tan\frac{A}{2}\end{aligned}$$

Be careful when you are working with fractions within fractions – this is a common cause of simple mistakes in the HSC. Make your handwriting large and use plenty of space on the page!

We multiplied the numerator and denominator by $1+t^2$ to remove the fractions within fractions correctly.

b Hence show that $\tan 15° = 2-\sqrt{3}$.

$$\begin{aligned}\tan 15° &= \tan\left(\frac{30°}{2}\right)\\ &= \frac{1-\cos 30°}{\sin 30°} \qquad \text{from (a)}\\ &= \frac{1-\dfrac{\sqrt{3}}{2}}{\dfrac{1}{2}}\\ &= 2(1)-2\left(\frac{\sqrt{3}}{2}\right)\\ &= 2-\sqrt{3}\end{aligned}$$

Example 3

Prove that $\dfrac{2}{1+\cos x} = \sec^2\left(\dfrac{x}{2}\right)$.

$$\begin{aligned}\frac{2}{1+\cos x} &= \frac{2}{1+\dfrac{1-t^2}{1+t^2}}\\ &= \frac{2+2t^2}{1+t^2+1-t^2}\\ &= \frac{2(1+t^2)}{2}\\ &= 1+t^2\\ &= 1+\tan^2\left(\frac{x}{2}\right)\\ &= \sec^2\left(\frac{x}{2}\right)\end{aligned}$$

Let's prove that $\tan 15° = 2-\sqrt{3}$ directly from the $t-$ formulas.

Example 4

Use the formula $\sin\theta = \dfrac{2t}{1+t^2}$ to show that $\tan 15° = 2-\sqrt{3}$.

let $t = \tan 15°$

$$\sin 30° = \frac{2t}{1+t^2}$$

Note $\sin 150°$ is also $\frac{1}{2}$, so we will get two solutions to this equation, but only one of them answers the question.

$$\frac{1}{2} = \frac{2t}{1+t^2}$$

$$1+t^2 = 4t$$

$$t^2-4t+1=0$$

This quadratic has two solutions for t. The smaller one is the value of $\tan 15°$ and the larger one is the value of $\tan 75°$ since $\tan$ is an increasing function in the domain and $0° < 15° < 75° < 90°$, so we only use '$-$' rather than '$\pm$' in the quadratic formula.

$$\begin{aligned}t &= \frac{4-\sqrt{(-4)^2-4(1)(1)}}{2\times 1}\\ &= \frac{4-\sqrt{12}}{2}\\ &= \frac{4-2\sqrt{3}}{2}\\ &= 2-\sqrt{3}\end{aligned}$$

Using $t-$formulas to solve equations

In Extension 1 you will have used a variety of methods to solve trigonometric equations. In the next few examples, we will use $t-$formulas to solve trigonometric equations.

As we noted earlier in the lesson, when we use $t-$formulas, A cannot equal any odd multiple of π since $\tan\frac{A}{2}$ is undefined for these angles. This means that when we solve equations using $t-$formulas, we will only find solutions that are not odd multiples of π, and so we need to manually check whether any angles in the domain for which $\tan\frac{A}{2}$ is undefined also solve the equation.

Expect to lose marks in the Extension 2 HSC unless you specifically check whether π radians or $180°$ is also a solution or justify why it cannot be a solution.

We will generally look for solutions in the domain $[0, 2\pi]$, so we only need to manually check whether π also solves the equation. If the equation is solved over a different domain, then you will need to manually check all odd multiples of π within the domain.

In the next example we will solve the same equation using the auxiliary angle method from Extension 1 and then using $t-$formulas. The equation has been chosen since $x = \pi$ is a solution but it will not appear using $t-$formulas, only from the manual check.

While the auxiliary angle method shouldn't be required in Extension 2, it can still be used to solve some trig equations where $t-$formulas are used. I would expect that the standard expected in Extension 2 will be higher than in Extension 1, so we will follow the trend from recent Extension 1 HSCs and avoid using the shortcut method. I will use one method in the next example, and a different method in the exercises.

Example 5

a Solve $\sin x + \cos x = -1$ in the domain $[0, 2\pi]$ by first writing $\sin x + \cos x$ in the form $R\sin(x + \alpha)$ where α is acute.

$$\sin x + \cos x = R(\sin x \cos\alpha + \cos x \sin\alpha)$$
$$R\cos\alpha = 1 \quad (1) \quad (\cos\alpha > 0)$$
$$R\sin\alpha = 1 \quad (2) \quad (\sin\alpha > 0)$$
$\therefore$ α is in the 1st quadrant

Get into the habit of finding the values of R and α using simultaneous equations, rather than using the shortcuts.

$$(1)^2 + (2)^2: \; R^2(\cos^2\alpha + \sin^2\alpha) = 1^2 + 1^2$$
$$R^2 = 2$$
$$R = \sqrt{2}$$

$$(2) \div (1): \; \frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{1}$$
$$\tan\alpha = 1$$
$$\alpha = \frac{\pi}{4}$$
$$\therefore \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$
$$\therefore \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = -1$$
$$\sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$
$$x + \frac{\pi}{4} = \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$x = \pi, \frac{3\pi}{2}$$

b Solve $\sin x + \cos x = -1$ in the domain $[0, 2\pi]$ using t formulas.

$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = -1$$

The solutions to this equation with t will include all solutions to the original equation that are NOT odd multiples of π.

$$2t + 1 - t^2 = -1 - t^2$$
$$2t = -2$$
$$t = -1$$

$$\tan\frac{A}{2} = -1$$
$$\frac{A}{2} = \frac{3\pi}{4}$$

This is the only solution to the original equation that is NOT an odd multiple of π.

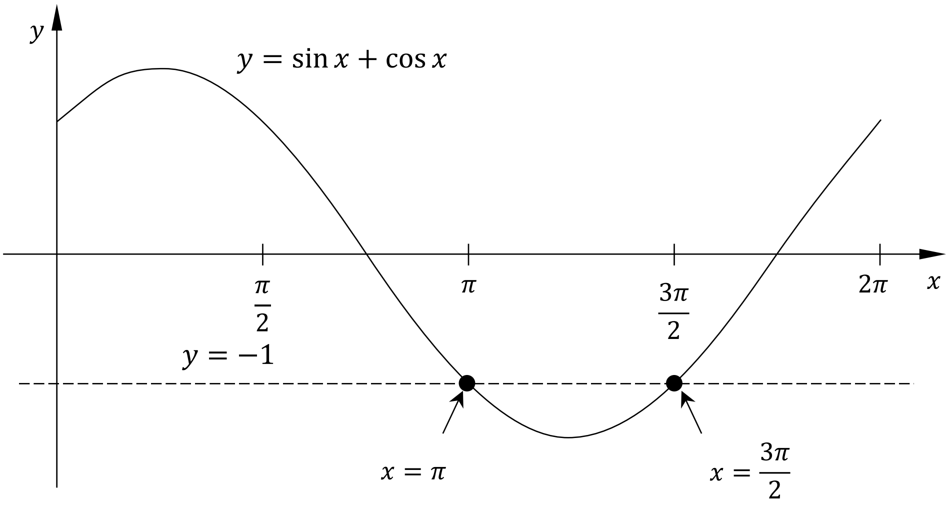
$$A = \frac{3\pi}{2}$$

Check $x = \pi$: $\sin\pi + \cos\pi = 0 + (-1) = -1$ ✓

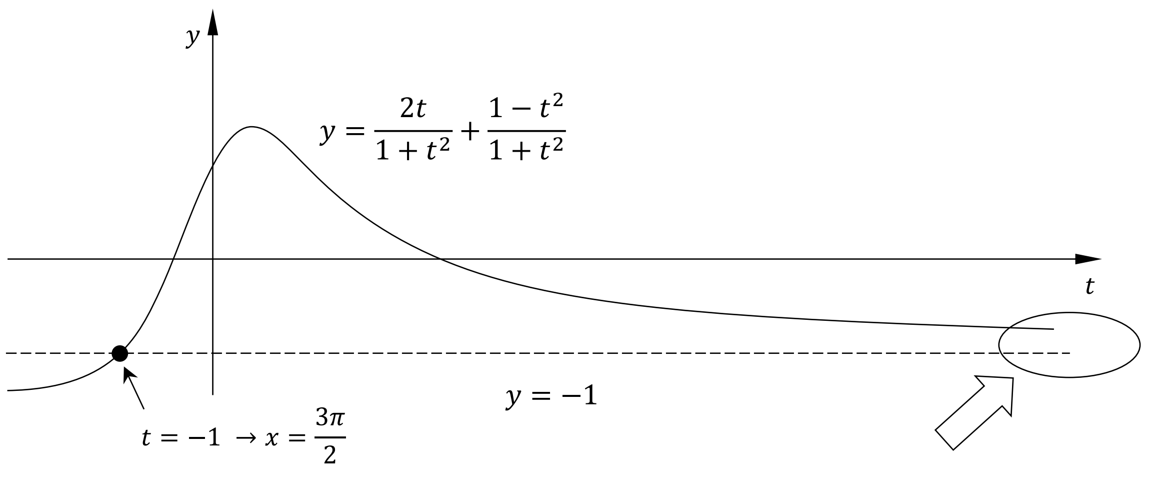
Always manually check whether any odd multiple of π in the domain solves the equation when using t – formulas.

$$\therefore x = \pi, \frac{3\pi}{2}$$

How can one solution just disappear? If we look at the graphs of $y = \sin x + \cos x$ and $y = -1$ within the domain $[0, 2\pi]$, shown below, we can see that they intersect when $x = \pi$ and $x = \frac{3\pi}{2}$, which matches the two solutions to the equation.



When we look at the graphs of $y = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$ and $y = -1$ with an unrestricted domain, shown below, we can see that there is only one point of intersection when $t = -1$, which leads to the solution $x = \frac{3\pi}{2}$.



What happened to the other solution? Even though there isn't a second point of intersection, we can see that as $y = -1$ is an asymptote to $y = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$, the curve and the line get closer as $t \to \infty$, in other words as $\tan\frac{A}{2} \to \infty$. This shows $\frac{A}{2} \to \frac{\pi}{2}$ and so $A \to \pi$. This is where the second solution of $x = \pi$ has 'disappeared'. If we extended the graph to the left, we would also see that the graph approaches the same asymptote as $t \to -\infty$, and if our domain included $-\pi$ then it would also be a solution to the equation.

Could we tell that $x = \pi$ is a solution without manually checking whether $x = \pi$ solves the equation? Yes, but it is much easier to just check whether $x = \pi$ solves the equation, plus what we look at below hasn't been in the last two syllabuses.

The alternative way to find if $x = \pi$ is a solution involves what happens to y as $t \to \infty$ - if it approaches the RHS of the equation then $x = \pi$ is also a solution.

In this case:

Do NOT use this method in an exam – it wastes a massive amount of time and may not earn you full marks.

$$y = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$$

$$= \frac{-t^2+2t+1}{t^2+1}$$

$$= \frac{-1+\frac{2}{t}+\frac{1}{t^2}}{1+\frac{1}{t^2}}$$ dividing the numerator and denominator by t^2

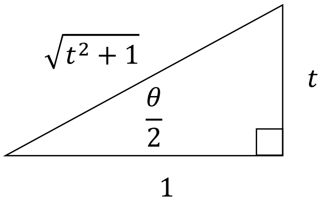
Now $\lim_{t\to\infty}\left(\frac{-1+\frac{2}{t}+\frac{1}{t^2}}{1+\frac{1}{t^2}}\right) = \left(\frac{-1+0+0}{1+0}\right) = \frac{-1}{1} = -1.$

Since this matches the RHS of the equation that means all odd multiples of π in the domain are also solutions – in this case it means that $x = \pi$ is also a solution to the equation.

Example 6

Solve $\cos\theta - \sin\frac{\theta}{2} = 0$ for $-\pi \le \theta \le \pi$.

To find $\sin\frac{\theta}{2}$ in terms of t, draw a quick diagram where $\tan\frac{\theta}{2} = \frac{t}{1}$.



$$\frac{1-t^2}{1+t^2} - \frac{t}{\sqrt{t^2+1}} = 0$$

$$\frac{1-t^2}{1+t^2} = \frac{t}{\sqrt{t^2+1}}$$

$$1-t^2 = t\sqrt{1+t^2}$$

$$1-2t^2+t^4 = t^2+t^4$$

Since we have squared both sides of the equation, we must check that we have not introduced false solutions by checking that any solutions solve $1-t^2 = t\sqrt{1+t^2}$.

$$3t^2 = 1$$

$$t^2 = \frac{1}{3}$$

$$t = \frac{1}{\sqrt{3}}$$

since $t = -\frac{1}{\sqrt{3}}$ does not solve $1-t^2 = t\sqrt{1+t^2}$

$$\tan\frac{\theta}{2} = \frac{1}{\sqrt{3}}$$

$$\frac{\theta}{2} = \frac{\pi}{6}$$

since $-\pi \le \theta \le \pi \rightarrow -\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{\pi}{2}$

$$\theta = \frac{\pi}{3}$$

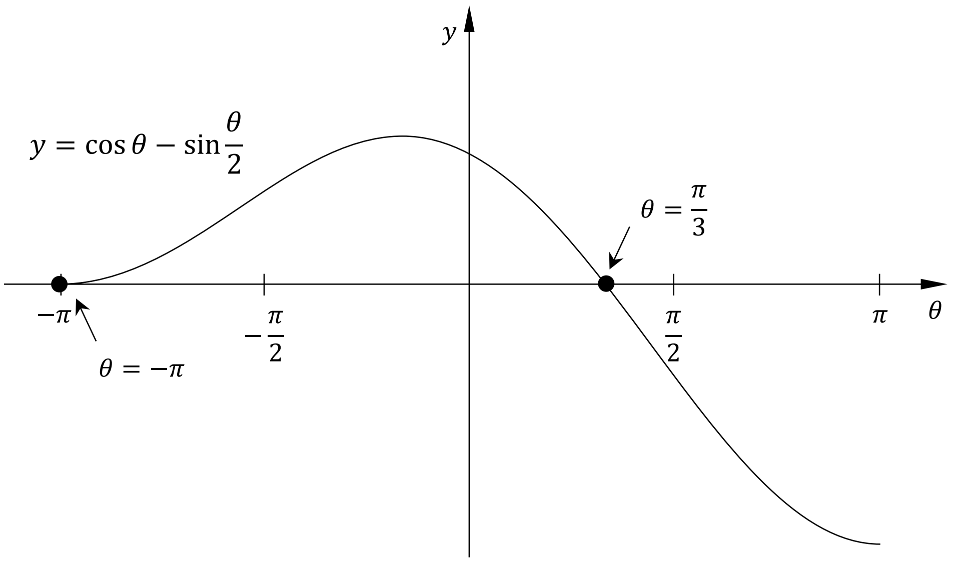
Check $\theta = -\pi$: $\cos(-\pi) - \sin\left(\frac{-\pi}{2}\right) = -1-(-1) = 0$ ✓

Check $\theta = \pi$: $\cos\pi - \sin\frac{\pi}{2} = -1-1 = -2 \ne 0$ ✗

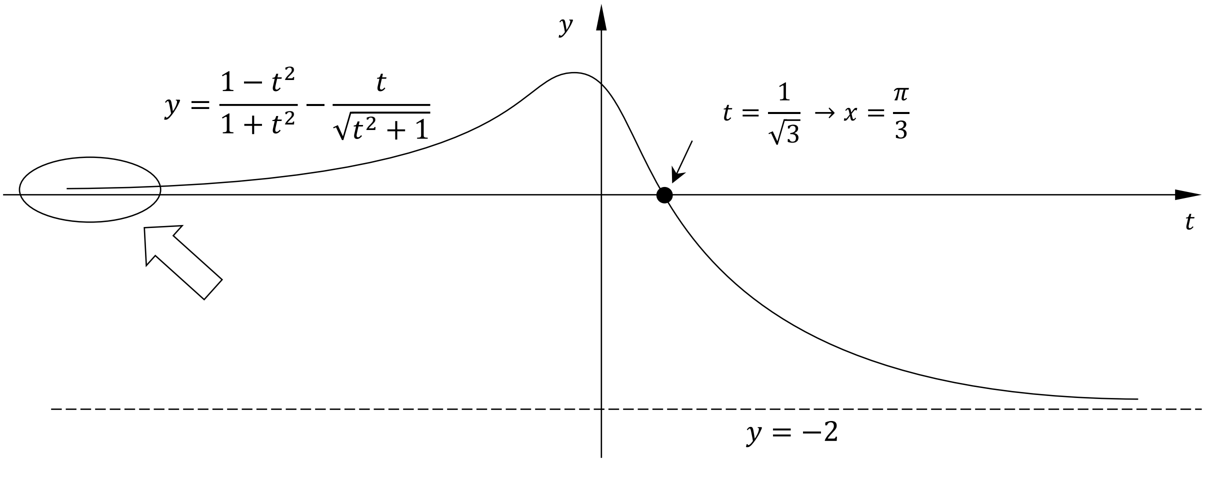
Always check all angles in the domain for which t is undefined, even if it will not lead to a solution.

The solutions are $\theta = -\pi, \frac{\pi}{3}$.

If we look at the graph of $y = \cos\theta - \sin\frac{\theta}{2}$, we can see that it intersects the θ-axis at two points in the domain, giving us the solutions of $x = -\pi, \frac{\pi}{3}$.



When we look at the graph of $y = \frac{1-t^2}{1+t^2} - \frac{t}{\sqrt{t^2+1}}$ within an unrestricted domain, shown below, we can see that there is one point of intersection when $t = \frac{1}{\sqrt{3}}$, which leads to the solutions of $x = \frac{\pi}{3}$.



There are two horizontal asymptotes - $y = 0$ as $t \to -\infty$ and $y = -2$ as $t \to \infty$. Since the RHS of the equation is 0, this means that since $y = \frac{1-t^2}{1+t^2} - \frac{t}{\sqrt{t^2+1}}$ approaches 0 as $t \to -\infty$, then $\frac{\theta}{2} \to -\frac{\pi}{2}$, so $\theta \to -\pi$. This gives us the second solution to the equation.

In these two examples, either $-\pi$ or $-\pi$ was a solution but that won't always be the case – these examples have been chosen to increase our understanding.

Basic

1 Prove $\cos A = \dfrac{1-t^2}{1+t^2}$, where $t = \tan\dfrac{A}{2}$.

2a Use $t-$formulas to prove that $\dfrac{1+\cos A}{\sin A} = \cot\dfrac{A}{2}$

b Hence show that $\cot 15° = 2+\sqrt{3}$.

3 Solve $2\tan\dfrac{x}{2} = \sqrt{3}\left(1-\tan^2\dfrac{x}{2}\right)$ for $0 \le x \le 2\pi$.

4 Prove the following results, where $t = \tan\dfrac{\theta}{2}$:

a $\tan\theta\sin\theta = \dfrac{4t^2}{1-t^4}$

b $\cos 2\theta = \dfrac{1-6t^2+t^4}{(1+t^2)^2}$

5 Prove the following results:

a $\dfrac{2}{1-\cos x} = \operatorname{cosec}^2\left(\dfrac{x}{2}\right)$

b $\dfrac{1+\cos x+\sin x}{1-\cos x+\sin x} = \cot\left(\dfrac{x}{2}\right)$

6a Use the formula $\tan\theta = \dfrac{2t}{1-t^2}$ to show that $\tan\dfrac{\pi}{8} = \sqrt{2}-1$.

b Use the formula $\sin\theta = \dfrac{2t}{1+t^2}$ to show that $\tan\dfrac{\pi}{8} = \sqrt{2}-1$.

Medium

7 Prove $\cot^2\theta = \dfrac{1+\cos 2\theta}{1-\cos 2\theta}$.

8a Solve $\sin x - 2\cos x = 2$ in the domain $[0, 2\pi]$ by first writing $\sin x - \cos x$ in the form $R\sin(x-\alpha)$ where α is acute, giving answers to 2 decimal places if needed.

b Solve $\sin x - \cos x = 1$ in the domain $[0, 2\pi]$ using t formulas, giving answers to 2 decimal places if needed.

9 Solve $\cos x - 2\sin x = 1$ in the domain $[-\pi, \pi]$ using $t-$formulas, giving answers to 2 decimal places if needed.

10 Prove that the only solutions to $\sqrt{3}\sin x - \cos x = 1$ for $[0, 2\pi]$ are $x = \dfrac{\pi}{3}$ and $x = \pi$.

11 Prove that the only solutions to $\sin x + \cos x = 0.5$ for $[0, 2\pi]$ are $x = 2.00$ and $x = 5.86$, correct to 2 decimal places.

12 Solve $\sin\theta - \sin\dfrac{\theta}{2} = 0$ for $-\pi \le \theta \le \pi$.

13 Given $\sec x + \tan x = \tan(ax+b)$, use the substitution $t = \tan\left(\dfrac{x}{2}\right)$ to find a and b when $t \ne -1$.

Challenging

14 Use $t-$formulas to solve $3\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) = 3$ for $0 \leq x \leq 2\pi$.

15 The equation $m\cos x - \sin x = 1$, where $m > 1$, has one acute solution.
(Do NOT prove this.)

a Show that $t = \dfrac{m-1}{m+1}$ where $t = \tan\dfrac{x}{2}$.

b Hence show that the single acute solution is $x = 2\tan^{-1} m - \dfrac{\pi}{2}$.

16 The two solutions to the equation $2\sin x - \cos x = 0$ in the domain $[0, 2\pi]$ are α and β.
Prove that $\tan\dfrac{\alpha}{2}$ is the negative reciprocal of $\tan\dfrac{\beta}{2}$.

17a Show that $\sqrt{\dfrac{1-\cos x}{1+\sin x}} = \left|\dfrac{\sqrt{2}t}{t+1}\right|$ where $t = \tan\dfrac{x}{2}$.

b Hence solve $\sqrt{\dfrac{1-\cos x}{1+\sin x}} = \dfrac{1}{\sqrt{2}}$ for $[0, 2\pi]$.

4.10 Trigonometric substitutions

In Lesson 10 we cover:

- Integrating with $t-$formulas
- Other trigonometric substitutions

In this lesson we look at more trigonometric integrands, this time looking at methods we keep in reserve for use when easier methods will not work. We look at how to recognise those similar integrands which are more easily solved using a u or u^2 substitution.

This lesson mainly covers the 6th dot point from *Further integration*, with questions on the 15th dot point included in the exercises.

Integrating with $t-$formulas

Integration using t-formulas is one of our methods of last resort, as many integrands involving trigonometric functions can be solved more easily using other methods.

Once you have substituted for t, the integrand can be rearranged into one of the standard integrals and solved. If you are solving an indefinite integral, remember to rewrite your answer in terms of the original variable so that you have answered the question.

Most questions end up using similar methods through the solution, so we will only go through a couple of examples. Take care during an exam, as simple errors often cost marks - incorrectly simplifying complex fractions (fractions within fractions) or not rewriting the final answer of an indefinite integral in terms of the original variable are common mistakes.

A note on finding dx or $d\theta$ as a function of t

When we have an integrand as a function of x, and use the substitution $t = \tan\frac{x}{2}$, we need to find an expression for dx in terms of t. This is always $dx = \frac{2dt}{1+t^2}$. Note that the RHS on the final line is almost the same as the t-result for $\sin A$ with d inserted. The t-formula for $\sin A$ is on the Reference Sheet.

A common question that arises is - can we just use this result, or do we need to derive it each time? For most questions you can just memorise and use this result. See the marking guidelines for Q11(f) in 2022 and Q11(d) in 2024, where that is exactly what has been done. Other marking guidelines use the full derivation, but I would expect that you can achieve full marks using the shortcut.

There is always the possibility that part (i) of a question could ask you to derive the result - see Q1(e) in 2005.

See the full derivation on the right, which uses the Pythagorean identity $\sec^2\frac{x}{2} = 1 + \tan^2\frac{x}{2}$.

$$
\begin{aligned}
t &= \tan\frac{x}{2} \\
\frac{dt}{dx} &= \frac{1}{2}\sec^2\frac{x}{2} \\
&= \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) \\
&= \frac{1}{2}(1 + t^2) \\
dx &= \frac{2dt}{1 + t^2}
\end{aligned}
$$

Example 1

Find $\int \frac{dx}{1+\cos x}$ using $t = \tan\frac{x}{2}$.

$$\int \frac{dx}{1+\cos x}$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{1+t^2+1-t^2}\,dt$$

Take care when simplifying nested fractions within integrals - this is a very common exam mistake.

$$= 2\int \frac{1}{2}\,dt$$

$$= t + c$$

$$= \tan\frac{x}{2} + c$$

Example 2

Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\cos x}$ using $t = \tan\frac{x}{2}$.

$$\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\cos x}$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int_0^1 \frac{1}{4+5\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{dt}{4+4t^2+5-5t^2}$$

$$= 2\int_0^1 \frac{dt}{9-t^2}$$

$$= \frac{1}{3}\int_0^1 \left(\frac{1}{3-t} + \frac{1}{3+t}\right) dt$$

$$= \frac{1}{3}\Big[-\ln(3-t) + \ln(3+t)\Big]_0^1$$

$$= \frac{1}{3}\big((-\ln 2 + \ln 4) - (-\ln 3 + \ln 3)\big)$$

$$= \frac{1}{3}\ln\frac{4}{2}$$

$$= \frac{1}{3}\ln 2$$

Trigonometric substitutions

Another method we can use when nothing else works is to use a trig substitution, though they are rarely needed. Often, we can use a u^2 substitution instead and save a lot of work, as trig substitutions are often quite long.

Trig substitutions can be used whenever the integrand involves a square root (including in the denominator) where the radicand is the sum or difference of two squares. This might involve completing the square to make the appropriate substitution obvious.

In the following examples we will see how the trig substitution gives us an expression that we can then simplify using a Pythagorean identity.

Example 3

Using the substitution $x = \sec\theta$, or otherwise, evaluate $\int x^3\sqrt{x^2-1}\,dx$ where $0 \le x \le \frac{\pi}{2}$.

$$\int x^3\sqrt{x^2-1}\,dx \qquad \boxed{\begin{aligned} x &= \sec\theta \\ dx &= \sec\theta\tan\theta\,d\theta \end{aligned}}$$

$$= \int \sec^3\theta\sqrt{\sec^2\theta - 1}\sec\theta\tan\theta\,d\theta$$

$$= \int \sec^3\theta\sqrt{\tan^2\theta}\sec\theta\tan\theta\,d\theta$$

The trig substitution chosen is $x = \sec\theta$, as then the radicand becomes the LHS of the Pythagorean identity $\sec^2\theta - 1 = \tan^2\theta$, which simplifies to $\tan\theta$.

$$= \int \sec^4\theta\tan^2\theta\,d\theta$$

$$= \int \sec^2\theta\sec^2\theta\tan^2\theta\,d\theta$$

$$= \int \sec^2\theta(\tan^2\theta + 1)\tan^2\theta\,d\theta$$

$$= \int \sec^2\theta\tan^4\theta\,d\theta + \int \sec^2\theta\tan^2\theta\,d\theta$$

$$= \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3} + c \qquad \boxed{\begin{aligned} \tan^2\theta &= \sec^2\theta - 1 \\ &= x^2 - 1 \\ \tan\theta &= \sqrt{x^2-1} \end{aligned}}$$

We take the positive root only, since $\tan x > 0$ in the domain

$$= \frac{\sqrt{(x^2-1)^5}}{5} + \frac{\sqrt{(x^2-1)^3}}{3} + c$$

Example 4

Use a suitable trigonometric substitution to find $\displaystyle\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$, where $x > 0$.

$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx \qquad \boxed{\begin{array}{c} x = 2\tan u \\ dx = 2\sec^2 u\,du \end{array}}$$

The trig substitution chosen is $x = 2\tan u$, as then the radicand becomes the LHS of the Pythagorean identity $4\left(1+\tan^2 u\right) = 4\sec^2 u$, which then simplifies to $2\sec u$.

$$= \int \frac{1}{4\tan^2 u\sqrt{4+4\tan^2 u}} \times 2\sec^2 u\,du$$

$$= \frac{1}{2}\int \frac{\sec^2 u}{\tan^2 u\sqrt{4\sec^2 u}}\,du$$

$$= \frac{1}{2}\int \frac{\sec^2 u}{\tan^2 u \times 2\sec u}\,du$$

$$= \frac{1}{4}\int \frac{\sec u}{\tan^2 u}\,du$$

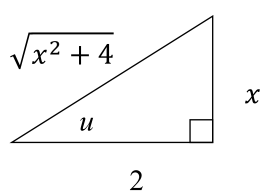
$$= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\,du$$

$$= \frac{1}{4}\int (\sin u)^{-2} \times \cos u\,du$$

$$= \frac{1}{4}\int \frac{d}{du}\left(-(\sin u)^{-1}\right)du$$

$$= -\frac{1}{4\sin u} + c$$

$$= -\frac{1}{4\left(\dfrac{x}{\sqrt{x^2+4}}\right)} + c$$



$$= -\frac{\sqrt{x^2+4}}{4x} + c$$

Integrals in the form $\displaystyle\int \frac{1}{x\sqrt{x^n - c}}\,dx$ $n = 1,2,3\ldots$ can be solved using $u^2 =$ radicand, avoiding trig substitutions. For example, in the integral $\displaystyle\int \frac{1}{x\sqrt{x-4}}\,dx$ we would let $u^2 = x - 4$.

Trigonometric substitutions (continued)

We now return to an indefinite integral we first encountered in Lesson 3 Example 4, where we found the integral by rationalising the numerator. That is the simplest method for this integral, with 5 lines of working.

In the next example we will use a trig substitution which takes 8 lines. We will also solve this same integral in the exercises starting with a u^2 substitution - this takes 14 lines including a second substitution.

Example 5

Using the substitution $x = \cos 2\theta$, or otherwise, evaluate $\int \sqrt{\frac{1-x}{1+x}}\, dx$.

$$\int \sqrt{\frac{1-x}{1+x}}\, dx \qquad \boxed{\begin{aligned} x &= \cos 2\theta \\ dx &= -2\sin 2\theta \, d\theta \end{aligned}}$$

$$= -2\int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \sin 2\theta \, d\theta$$

$$= -4\int \sqrt{\frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta - 1)}} \sin\theta\cos\theta \, d\theta$$

$$= -4\int \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \sin\theta\cos\theta \, d\theta$$

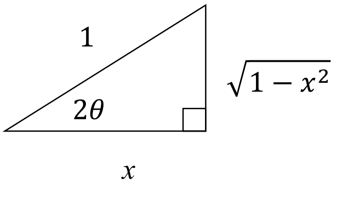
$$= -4\int \frac{\sin\theta}{\cos\theta} \times \sin\theta\cos\theta \, d\theta$$

$$= -4\int \sin^2\theta \, d\theta$$

$$= -4\int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= -2\theta + \sin 2\theta + c$$

$$= -\cos^{-1}(x) + \sqrt{1-x^2} + c$$



Basic

1 Find $\displaystyle\int \frac{dx}{1+\sin x}$ using the substitution $t = \tan\frac{x}{2}$.

2 Using the substitution $t = \tan\frac{x}{2}$, or otherwise, find $\displaystyle\int \frac{5}{12+13\cos x}\,dx$.

3 Using the substitution $x = \sin\theta$, or otherwise, find $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

4 Using the substitution $x = 2\sec\theta$, or otherwise, find $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$.

5 Let $t = \tan\frac{\theta}{2}$

a Show that $\dfrac{dt}{d\theta} = \dfrac{1}{2}\left(1+t^2\right)$

b Show that $\sin\theta = \dfrac{2t}{1+t^2}$

c Use the substitution $t = \tan\frac{\theta}{2}$ to find $\displaystyle\int \operatorname{cosec}\theta\ d\theta$.

6 Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, find $\displaystyle\int \frac{1}{1-\cos x}\,dx$.

Medium

7 Using the substitution $t = \tan\frac{\theta}{2}$ evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$.

8 Using the substitution $x = 4\tan^2 u$ find $\displaystyle\int \sqrt{x+4}\,dx$.

9 Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx$.

10 Using the substitution $x = \sin^2\theta$, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx$.

11 Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, evaluate $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta}$.

12 Prove $\displaystyle\int \operatorname{cosec} x\,dx = \ln\left|\tan\frac{x}{2}\right| + c$ using the substitution $t = \tan\frac{x}{2}$.

13 Find $\displaystyle\int \frac{1}{\sin x + \cos x}\,dx$ using the substitution $t = \tan\frac{x}{2}$.

14a Use the substitution $t = \tan\frac{x}{2}$ to show that $\operatorname{cosec} x + \cot x = \cot\frac{x}{2}$.

b Hence show that $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{cosec} x + \cot x)\, dx = \ln 2$.

15 Find $\int \frac{\sec^2 x}{1 - \tan^2 x} dx$.

Challenging

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16 Use a suitable trig substitution to find $\int \frac{x^2}{\sqrt{4 - x^2}} dx$.

17 Use a suitable trig substitution to find $\int \frac{3 + \cos x}{2 - \cos x} dx$.

18 Use a suitable trig substitution to find $\int \sqrt{2x - x^2}\, dx$.

19 Use the substitution $u^2 = 1 + x$ to find $\int \sqrt{\frac{1 - x}{1 + x}} dx$.

20 Use the fact that $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{(1 + \sin x)^3}$.

21a Show $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \frac{1}{2}\sin x} = \frac{2\pi}{3\sqrt{3}}$.

b Show $\int_0^{2a} f(x)\, dx = \int_0^a [f(x) + f(2a - x)]\, dx$.

c Hence evaluate $\int_0^{\pi} \frac{x\, dx}{1 + \frac{1}{2}\sin x}$.

4.11 Integration by parts

In Lesson 11 we cover:

- Integration by parts
- Integration by parts using DETAIL
- Integration by parts using the reverse chain rule
- Other applications of integration by parts
- Tabular integration by parts (optional)

In this lesson we look at integration by parts, a method that we generally only use to integrate an integrand that involves two different types of functions. It is sometimes used for two similar functions as we will see.

This lesson covers the 11th and 12th dot points from *Further integration*.

Integration by parts

Integration by parts (IBP) is used to find or evaluate integrals when other methods don't work. You can use it on any integral - you will just waste your time and effort. We will mainly use IBP on integrals which are the product of two different types of functions, such as $e^x \sin x$, which is the product of an exponential function and a trigonometric function, though there are a small number of other integrands where it is required. IBP works for both indefinite and definite integrals.

Integration by parts comes about by taking the product rule from differentiation, integrating both sides and rearranging.

$$\frac{d}{dx}(u \times v) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\therefore \int \frac{d}{dx}(u \times v)dx = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$u \times v = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$\boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

In words:

1. split the integrand into two functions
2. integrate one factor to obtain v, which then appears in both terms on the right-hand side
3. the other function is u and stays the same for the primitive function, uv, while is differentiated for the integrand, $v\frac{du}{dx}$.

This makes more sense when you see it used in the examples to follow.

We used integration to remove the derivative, but don't forget that when an indefinite integral 'disappears' it leaves a constant. The IBP formula should include $uv + c$ rather than just uv, but since there is still an indefinite integral in the formula which will have its own constant, we ignore it for the moment as it will be absorbed into the other constant at the end of the solution.

In some questions you will need to integrate by parts twice to get the answer. We will briefly look at tabular integration by parts, which is generally not recommended, but useful when we need to use IBP twice, though we can use it in any question.

Integration by parts using DETAIL

Once you split the original expression into the correct functions for u and $\frac{dv}{dx}$ then this work is straightforward. However, if you split the integrand incorrectly then the work becomes either much harder or impossible, and you have to start again.

The trick to IBP lies in correctly splitting the integrand into one function that is easy to integrate and a second function that is simpler when it is differentiated.

To split the integrand and make our solution work as easily as possible we have two choices.

(1) DETAIL − This rule of thumb helps us to choose which function is most easily integrated, which we make $\frac{dv}{dx}$, when we have two different types of functions. It works in the vast majority of questions in the HSC or from textbooks.

(2) Reverse chain rule − If one of the functions is a composite function we find $\frac{dv}{dx}$ as the product of the composite function and the derivative of the inner function. This can then be integrated using the reverse chain rule.

There are some other integrands where neither of these shortcuts work, but if we always try and find the most detailed function that can be integrated for $\frac{dv}{dx}$, we will split the function successfully.

DETAIL

Let's have a look at the DETAIL mnemonic, which focuses on choosing the function to be integrated.

D – Let $\frac{dv}{dx}$ equal the first function from this list to appear in the integrand

E – Exponential
T – Trigonometric
Easiest to Integrate, so these are the best functions for $\frac{dv}{dx}$

A – Algebraic
Easy to Integrate, so still good for $\frac{dv}{dx}$

I – Inverse Trig
L – Logarithmic
Hard to Integrate, so no good for $\frac{dv}{dx}$

For example, $\int x^2 \ln x \, dx$ has x^2 (an Algebraic function) and $\ln x$ (a Logarithmic function).

Since A comes before L in DETAIL we let $\frac{dv}{dx} = x^2$ while u equals the rest of the integrand, in this case $u = \ln x$.

DETAIL focuses on the function that is best to integrate, but it is also interesting to consider how well each function works as u. We want $\frac{du}{dx}$ to be much simpler than u.

- Logarithmic functions differentiate to become simple algebraic functions, so are the best choice for u if available.
- Inverse trig functions differentiate to become algebraic functions, so are also a good choice for u.
- Algebraic functions differentiate easily so are a good choice for u.
- Trig functions often cycle from sine to cosine, so while we can use them for u they will require us to integrate twice (see Example 2)
- Exponential functions don't get simpler when they are differentiated, plus we need to know extra rules, so are no good as u.

If we only have one function in the integrand we often let $\frac{dv}{dx} = 1$ (as this is easy to integrate) and let u equal the function.

In the exercises we will do one of the integrals using different choices for u and $\frac{dv}{dx}$. In one case, we will use DETAIL; in the other, we will go against it.

- when we follow DETAIL, the answer is straightforward
- when we go against DETAIL, we will still get the correct answer, but it will be a harder solution.

Alternatives to **DETAIL** are **LIATE** or **ILATE**, which focus on choosing the function for u rather than the function for $\frac{dv}{dx}$. In other countries **LIPET** is also used for choosing u, where P stands for Polynomials rather than Algebraic. Note that since we will probably never have an integrand with a logarithmic and an inverse trig function it doesn't matter whether LIATE or ILATE is used. Since exponential and trigonometric functions can both be integrated, we can swap the order of those terms and still achieve success.

Example 1

Find $\int x \ln x \, dx$.

Using DETAIL $\frac{dv}{dx} = x$ and $u = \ln x$

$$\int x \ln x \, dx$$

$u = \ln x$	$\frac{dv}{dx} = x$
$\frac{du}{dx} = \frac{1}{x}$	$v = \frac{x^2}{2}$

At the start it can help to draw these lines to see which pairs we need to multiply

$$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \times \frac{1}{x} dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

Using DETAIL $\frac{dv}{dx} = e^x$

Example 2

Evaluate $\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$.

Using DETAIL $\frac{dv}{dx} = e^x$ and $u = \sin x$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$u = \sin x$	$\frac{dv}{dx} = e^x$
$\frac{du}{dx} = \cos x$	$v = e^x$

$u = \cos x$	$\frac{dv}{dx} = e^x$
$\frac{du}{dx} = -\sin x$	$v = e^x$

$$= \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$= \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \left(\left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx\right)$$

We have to use Integration by Parts a second time. Since the u and v don't form part of our actual solution we can use the same variables again. Using DETAIL $\frac{dv}{dx} = e^x$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \left[e^x \cos x\right]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \frac{1}{2}\left(\left(e^{\frac{\pi}{2}} - 0\right) - \left(0 - 1\right)\right)$$

$$= \frac{e^{\frac{\pi}{2}} + 1}{2}$$

This is a very common integrand for IBP. e^x stays the same while sine and cosine cycle, requiring us to use IBP twice.

We will come back and solve these two integrals using tabular IBP shortly.

Integration by parts using the reverse chain rule

The vast majority of questions you come across will be solved using DETAIL, but there are more obscure questions where DETAIL will not work since both functions are of the same type, commonly with one of the functions being a composite function. Here we will use the reverse chain rule. While we could often simplify the integrand first using u substitution it would cause more confusion (you would need a different variable as you couldn't use u twice) and make for a longer solution. This method is also good for recurrence relations where we are not given the rule.

Let $\frac{dv}{dx}$ be the product of the composite function and the derivative of the inner function.

In Example 3 we will look at $\int x^3 e^{x^2}\,dx$.

- e^{x^2} is the composite function.
- The derivative of the inner function, x^2, is $2x$
- Let $\frac{dv}{dx} = 2xe^{x^2}$
- That leaves us with $\frac{x^2}{2}$ from the integrand, which gives us u.

Example 3

Find $\int x^3 e^{x^2}\,dx$.

Using RCR $\frac{dv}{dx} = 2xe^{x^2}$ and $u = \frac{x^2}{2}$

$$\boxed{\begin{array}{ll} u = \frac{x^2}{2} & \frac{dv}{dx} = 2xe^{x^2} \\ \frac{du}{dx} = x & v = e^{x^2} \end{array}}$$

$$\begin{aligned}
&\int x^3 e^{x^2}\,dx \\
&= \frac{x^2e^{x^2}}{2} - \int xe^{x^2}\,dx \\
&= \frac{x^2e^{x^2}}{2} - \frac{1}{2}\int 2xe^{x^2}\,dx \\
&= \frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2} + c
\end{aligned}$$

Other applications of integration by parts

We can also use IBP with the reverse chain rule to integrate integrands with two similar functions, although quite often it is a hidden example of reversing the quotient rule as we will see in the next example.

Example 4

Find $\displaystyle\int \frac{\ln x}{(1+\ln x)^2}\,dx$:

a Using IBP

$$\int \frac{\ln x}{(1+\ln x)^2}\,dx$$

$$\boxed{\begin{array}{ll} u = x\ln x & \dfrac{dv}{dx} = \dfrac{1}{x}(1+\ln x)^{-2} \\ \dfrac{du}{dx} = \ln x + 1 & v = -(1+\ln x)^{-1} \\ & \quad = -\dfrac{1}{1+\ln x} \end{array}}$$

$$= \int \frac{x\ln x}{x(1+\ln x)^2}\,dx$$

$$= -\frac{x\ln x}{1+\ln x} + \int \frac{\ln x + 1}{1+\ln x}\,dx$$

$$= -\frac{x\ln x}{1+\ln x} + x + c$$

$$= \frac{x}{1+\ln x} + c$$

b By first showing $\dfrac{\ln x}{(1+\ln x)^2} = \dfrac{(1+\ln x)(1)-(x)\left(\frac{1}{x}\right)}{(1+\ln x)^2}$.

$$\begin{aligned} \frac{\ln x}{(1+\ln x)^2} &= \frac{1+\ln x - 1}{(1+\ln x)^2} \\ &= \frac{(1+\ln x)(1)-(x)\left(\frac{1}{x}\right)}{(1+\ln x)^2} \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{\ln x}{(1+\ln x)^2}\,dx &= \int \frac{(1+\ln x)(1)-(x)\left(\frac{1}{x}\right)}{(1+\ln x)^2}\,dx \\ &= \int \frac{d}{dx}\left(\frac{x}{1+\ln x}\right)dx \\ &= \frac{x}{1+\ln x} + c \end{aligned}$$

There are some integrands that we would think should work using other methods, such as the next example, which is very similar to integrals in Lesson 8 where we used Pythagorean identities.

Example 5

Given $\int \sec x \, dx = \ln\left|\sec x + \tan x\right|$, use integration by parts to show that

$$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + c.$$

$$\int \sec^3 x \ dx$$

$$\boxed{\begin{array}{ll} u = \sec x & \dfrac{dv}{dx} = \sec^2 x \\ \dfrac{du}{dx} = \sec x \tan x & v = \tan x \end{array}}$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$$

$$= \sec x \tan x - \int \sec^3 x \ dx + \int \sec x \, dx$$

$$\therefore 2\int \sec^3 x \ dx = \sec x \tan x + \ln\left|\sec x + \tan x\right| + c$$

$$\int \sec^3 x \ dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + c$$

Tabular integration by parts (optional)

A little-known variation of integration by parts is tabular integration by parts. As with any unusual method, I would generally not recommend you use it in the HSC or assessments. The main problem occurs if you make an error - you may achieve no marks at all if the method is not covered by the marking scheme. This section is only included here as I often get asked about this method, and whether it is OK. Personally, I don’t use it.

This method is mainly useful for quickly checking an answer you have found using the normal setting out of IBP. The only integrals where I would consider its use would be integrals that require using IBP twice, such as we saw in Example 2, where it is much more efficient.

We start with a table which has three columns.

S	**D**	**I**

- The first column is S for sign. We will place $+$ on the first row, then alternate between $-$ and $+$ down the column as we add new rows.
- The second column is D for derivative. We will place the function for u in the first row and differentiate from row to row as we down the column.
- The third column is I for integrate. We will place the function for $\frac{dv}{dx}$ in the first row and integrate from row to row as we go down the column.

We choose the functions for $\frac{dv}{dx}$ and u as normal. If the integrand requires use of the reverse chain rule this method will quickly reach a dead end, and it is much easier to use ordinary IBP.

We continue to add rows to the table until the product of the two terms on the same line is an integrand we can find or evaluate easily. This happens when either we:

- differentiate u until it reaches 0, so the remaining integral is zero
- create a simple integral we can solve
- create an integral that matches the original (so that we can swing it back with the original)

We will also look at using tabular IBP for a recurrence relation next lesson.

Let's start by solving $\int x \ln x \, dx$ from Example 1, using tabular IBP. As we will see, this integral is not one where tabular IBP is more useful than the normal setting out.

We used DETAIL to find that $\frac{dv}{dx} = x$ and $u = \ln x$, so here is the first row of the table.

S	D	I
$+$	$\ln x$	x

The next row is completed by placing $-$ in the S column, differentiating $\ln x$ and finding the primitive of x.

S	D	I
$+$	$\ln x$	x
$-$	$\frac{1}{x}$	$\frac{x^2}{2}$

Here we see that $\frac{1}{x} \times \frac{x^2}{2} = \frac{x}{2}$, and we can easily find $\int \frac{x}{2} dx$.

So, what do we do with this table?

S	D	I
$+$	$\ln x$	x
$-$	$\frac{1}{x}$	$\frac{x^2}{2}$

1. Green - Multiply S and D on the first row by I on the next row. In other examples we will need to do this more than once.
2. Red - Multiply all three cells on the last line and take their primitive.

So, $\int x \ln x \, dx = + \ln x \times \frac{x^2}{2} - \int \left(\frac{1}{x} \times \frac{x^2}{2} \right) dx$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \times \frac{x^2}{2} + c$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

As I mentioned, this integral is better done using the normal setting out.

Let’s compare this example done both ways. We could skip a couple of steps in tabular IBP if we are careful.

$$\int x \ln x \, dx \qquad \boxed{\begin{array}{ll} u = \ln x & \dfrac{dv}{dx} = x \\ \dfrac{du}{dx} = \dfrac{1}{x} & v = \dfrac{x^2}{2} \end{array}}$$

$$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \times \frac{1}{x} dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

S	D	I
$+$	$\ln x$	x
$-$	$\dfrac{1}{x}$	$\dfrac{x^2}{2}$

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \, dx$$
$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

Now let’s look at an example where we have to complete more rows of the table until we can find an integral that is easy to integrate.

Example 6

Find $\int x^2 \sin x \, dx$ using tabular IBP.

Using DETAIL $\dfrac{dv}{dx} = \sin x$ and $u = x^2$

$$\int x^2 \sin x \, dx$$
$$= +(x^2)(-\cos x) - (2x)(-\sin x) + (2)(\cos x) + c$$
$$= (2 - x^2)\cos x + 2x \sin x + c$$

S	D	I
$+$	x^2	$\sin x$
$-$	$2x$	$-\cos x$
$+$	2	$-\sin x$
$-$	0	$\cos x$

We could stop at the third row as we can easily find $\int (-2\sin x)dx$, but going one more row makes things easier as $\int 0 \, dx$ is even simpler.

Let’s redo Example 2. This time we continue the table until we get an integral which is a multiple of the original integral, which we can then swing back to the other side.

Example 7

Evaluate $\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$ using tabular IBP.

Using DETAIL $\frac{dv}{dx} = e^x$ and $u = \sin x$

S	D	I
$+$	$\sin x$	e^x
$-$	$\cos x$	e^x
$+$	$-\sin x$	e^x

Continue until the product of the two terms is a multiple of the original integral.

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \left[e^x \sin x - e^x \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = e^{\frac{\pi}{2}}(1-0) - 1(0-1)$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \frac{e^{\frac{\pi}{2}} + 1}{2}$$

You can see that this is much shorter than the solution using the normal method.

This is why I would recommend that integrals which require using IBP twice are suitable for using tabular IBP.

Basic

Use integration by parts to find or evaluate the following integrals.

1 $\int x^2 \ln x \, dx$

2 $\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$

3 $\int x^3 \sin x^2 \, dx$

4 $\int \ln x \, dx$. Hint: let $u = \ln x$ and $\frac{dv}{dx} = 1$. You may wish to memorise this result.

5 $\int x e^{2x} \, dx$

6 $\int_0^{\pi} x \cos x \, dx$

7 $\int_0^2 t e^{-t} \, dt$

8 $\int_0^1 \tan^{-1} x \, dx$

Medium

Use integration by parts to find or evaluate the following integrals.

9 $\int \ln|1 + x| \, dx$

10 $\int x 2^x \, dx$

11 $\int \sin^{-1} x \, dx$

12 $\int_4^{e+3} \ln|x - 3| \, dx$

13 $\int \tan^{-1} x \, dx$

14 $\int \ln|\cos x| \operatorname{cosec}^2 x \, dx$

15 $\int 2^x \sin x \, dx$

Challenging

16 Find $\int x^2\sqrt{x-1}\,dx$:

a Using IBP

b Using a u^2 substitution

17 Find $\int \frac{\ln x - 2}{(\ln x - 1)^2}\,dx$:

a Using IBP

b By using the quotient rule to find a function whose derivative equals the integrand.

18 Using the result from Q4, repeat Q1 but ignore DETAIL and let $u = x^2$ and $\frac{dv}{dx} = \ln x$.

19 Find $\int x \sin x \cos x\,dx$.

20 Find $\int \frac{xe^x}{(1+x)^2}\,dx$.

4.12 Recurrence relations

In Lesson 12 we cover:

- Recurrence relations
- Recurrence relations using integration by parts
- Recurrence relations using other integration methods
- Finding recurrence relations when the relation is not given

In this lesson we derive and use recurrence relations that link integrals involving unknown powers. This is a common source of exam questions, particularly late in the paper, which can become a source of easy marks using the hints in this lesson.

This lesson covers the 14^{th} dot point from *Further integration* , with questions on the 15^{th} dot point included in the exercises.

Recurrence relations

Recurrence relations express an integral (where the integrand includes a function to the power of n) as a function of another integral whose integrand includes the same function to a different power of n. For example, we can prove that

$$\int_0^1 x^n\sqrt{1-x^2}\,dx = \left(\frac{n-1}{n+2}\right)\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx$$

The first integral includes the term x^n and is expressed as a relation of the integral including the term x^{n-2}. The power in the second integral is usually lower, as in this example, which gives us the common alternative name of reduction formula.

Typically, we use abbreviations such as I_n for the integrals, and will find a relationship between I_n and I_{n-1}, or I_n and I_{n-2}, so, we would write the reduction formula in the above example in the much simpler form

$$I_n = \left(\frac{n-1}{n+2}\right)I_{n-2}.$$

We can use letters other than I for the integral, such as U or A, or leave the integral written in complete form.

We often use the recurrence relations to evaluate integrals with higher powers by breaking them down into functions of a lower-powered integral that we can integrate.

Recurrence relations are normally considered to be of medium to higher difficulty, as it is very easy to choose the wrong method and have to start again. Success has traditionally required much practice and a deep knowledge of differentiation and integration. Part of the problem is that while most recurrence relations are found using integration by parts, others can only be solved using standard integration techniques. While the 2024 syllabus only mentions using IBP for recurrence relations, other methods could easily be covered as an application of other dot points, so make sure you cover these types as well.

There is a very easy way to choose the correct method, which converts the topic to a simple mechanical process. This is great for quickly gaining marks in an exam, but our understanding suffers! We will also try to attain a deeper understanding of the work.

We can also use tabular IBP for those recurrence relations where IBP applies but is only an interesting aside - I recommend you use the traditional setting out.

As a rule of thumb, look at the coefficient of the lower-powered integral on the RHS (I_{n-1} or I_{n-2}) and see if it involves n. Here are three examples:

A $I_n = \dfrac{e}{2} - nI_{n-1}$ B $I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$ C $I_n = \dfrac{1}{2n-1} - I_{n-1}$

Integration by parts

If the coefficient involves n (examples A and B above) then use IBP. The numerator of the coefficient also gives us an important rule of thumb - use it (or the multiple closest to the power in the original integral) as the power to use for u. This rule of thumb requires that the integral with the higher power, when written in index form (usually n), is the subject of the reduction formula, so on the LHS. If the integrand involves a power in the denominator we need to rearrange to make the other integral the subject for this rule of thumb to work.

Other methods

If the coefficient does not involve n (example C above) then we do not use integration by parts. It is normally easiest to move both integrals to the one side and simplify, then rearrange to find the given result.

Let's have a look at some integrals and recurrence relations to plan the easiest method.

Original integral	**Recurrence relation to be proved**
$I_n = \displaystyle\int_1^{e^2} (\log_e x)^n \, dx$	$I_n = e^2 2^n - nI_{n-1}$

- The coefficient of I_{n-1} involves n so use IBP.
- The numerator is n, so let $u = (\log_e x)^n$

$I_n = \displaystyle\int_0^{x} \sec^n t \, dt$	$I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

- The coefficient of I_{n-2} involves n so use IBP.
- The numerator is $n-2$, so let $u = \sec^{n-2} t$.

Original integral	**Recurrence relation to be proved**
$I_n = \int_0^1 x^{2n+1} e^{x^2}\,dx$	$I_n = \frac{e}{2} - nI_{n-1}$

- The coefficient of I_{n-1} involves n so use IBP.
- The numerator is n, and in this case we use a multiple of n, so $u = x^{2n}$.
- Where there are two functions in the integral, use the function that is already to the power of some function of n.

$I_n = \int_0^{\frac{\pi}{4}} \tan^{2n}\theta\,d\theta$	$I_n = \frac{1}{2n-1} - I_{n-1}$

- The coefficient of I_{n-1} is -1. As it does not involve n we don't use integration by parts.
- The easiest method is to rearrange the expression and first prove that

$$I_n + I_{n-1} = \frac{1}{2n-1}$$

$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$	$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

The coefficient of I_{n-2} $n(n-1)$ is the product of two consecutive terms, which indicates that we will integrate by parts twice:

- the first time using $u = x^n$
- the second time using $u = x^{n-1}$

Note that we use the higher term for the first IBP. Since we use IBP twice, this is a situation where tabular IBP can be useful.

Recurrence relations using integration by parts

Let's start with the most common questions you will encounter for recurrence relations, those where you do need to use IBP.

Example 1

If $I_n = \int x^n e^x \, dx$ prove that $I_n = x^n e^x - nI_{n-1}$.

The coefficient of I_{n-1} involves n, so use IBP. The numerator is n which gives us the power we want, so we let $u = x^n$.

$$I_n = \int x^n e^x \, dx \qquad \boxed{\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = nx^{n-1} & v = e^x \end{array}}$$

$$= x^n e^x - \int nx^{n-1} e^x \, dx$$

$$= x^n e^x - n\int x^{n-1} e^x \, dx$$

$$= x^n e^x - nI_{n-1}$$

We can also solve this using tabular IBP, but the normal method above is recommended.

$$I_n = \int x^n e^x \, dx$$

$$= +(x^n)(e^x) - \int (nx^{n-1})(e^x) \, dx$$

$$\therefore I_n = x^n e^x - n\int x^{n-1} e^x \, dx$$

$$= x^n e^x - nI_{n-1}$$

S	D	I
$+$	x^n	e^x
$-$	nx^{n-1}	e^x

Add rows until the product involving the differentiated term produces the required lower-powered integrand.

This is the last time we will discuss tabular IBP, though we will use it as an alternative solution in exercises requiring IBP twice.

Example 2

If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ prove that $I_n = \frac{n-1}{n} I_{n-2}$.

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$u = \cos^{n-1} x$	$\frac{dv}{dx} = \cos x$
$\frac{du}{dx} = -(n-1)\cos^{n-2} x \sin x$	$v = \sin x$

$$= \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \left(1 - \cos^2 x\right) dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$\therefore I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = (n-1)I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

The coefficient of I_{n-2} involves n, so use IBP. Looking at the final answer, we see a numerator of $n-1$ which gives us the power we want, so we let $u = \cos^{n-1} x$.

Example 3

In Example 1, we proved that if $I_n = \int x^n e^x \, dx$ then $I_n = x^n e^x - nI_{n-1}$.

Use this recurrence relation to evaluate $\int x^3 e^x \, dx$.

$$\int x^3 e^x \, dx$$

$= I_3$

$= x^3 e^x - 3I_2$ using the recurrence relation

$= x^3 e^x - 3\left(x^2 e^x - 2I_1\right)$

$= x^3 e^x - 3x^2 e^x + 6(x^1 e^x - 1I_0)$

$= x^3 e^x - 3x^2 e^x + 6xe^x - 6I_0$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6\int x^0 e^x \, dx$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6\int e^x \, dx$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$$

1. We usually start by evaluating I_0, as it will be the easiest integral to evaluate. Sometimes I_1 or rarely I_2 will be easy to integrate.
2. Determine which integral you are trying to find.
3. Use the reduction formula to express the integral you are trying to find in terms of I_0.
4. Evaluate.

Recurrence relations using other integration methods

We will also cover questions where we use methods other than IBP to find the recurrence relation. This could be included under the 15th and 16th dot points in the syllabus which relate to using multiple techniques of integration to solve theoretical and practical problems.

Example 4

If $I_n = \int \tan^n x \, dx$ prove that $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$.

The coefficient of I_{n-2} does not involve n, so move both integrals to the one side and simplify.

$$
\begin{aligned}
I_n + I_{n-2} &= \int \tan^n x \, dx + \int \tan^{n-2} x \, dx \\
&= \int \tan^{n-2} x \left(\tan^2 x + 1\right) dx \\
&= \int \tan^{n-2} x \sec^2 x \; dx \\
&= \int \frac{d}{dx}\left(\frac{\tan^{n-1} x}{n-1}\right) dx \\
&= \frac{1}{n-1}\tan^{n-1} x
\end{aligned}
$$

$$\therefore I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$$

Finding recurrence relations when the relation is not given

In most HSC questions, the recurrence relation will be given in the question, so we can use the tricks from this lesson to quickly choose whether to use IBP, and if so the correct value of u.

But how do we proceed if the recurrence relation is not given? This might occur if the examiners want to create a much more difficult question for use late in the paper.

In our shortcut that we have used so far this lesson we have focused on u, but if the recurrence relation is not known we shift our focus back to $\frac{dv}{dx}$, choosing it so that it is the most complicated function that we can easily integrate. This often uses IBP with the reverse chain rule that we used last lesson or is sometimes equal to 1. For trig functions we often need to split the powers so we can use the Pythagorean identities. It is important in all types that the n stays as a power of u, not becoming part of $\frac{dv}{dx}$.

Example 5

Find a recurrence relation for $I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx$, for $n = 0, 1, 2\,\ldots$, in terms of I_{n-2}.

$$I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx$$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^{n-1} & \frac{dv}{dx} = 2x\left(x^2+1\right)^{-2} \\ \frac{du}{dx} = \frac{1}{2}(n-1)x^{n-2} & v = -\left(x^2+1\right)^{-1} \end{array}} \leftarrow \text{RCR}$$

$$= -\frac{1}{2}\left[\frac{x^{n-1}}{x^2+1}\right]_0^1 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{x^2+1}\,dx$$

$$= -\frac{1}{4} - 0 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}\left(x^2+1\right)}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}\int_0^1 \frac{x^n}{(x^2+1)^2}\,dx + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}I_n + \frac{n-1}{2}I_{n-2}$$

$$\therefore \frac{3-n}{2}I_n = -\frac{1}{4} + \frac{n-1}{2}I_{n-2}$$

$$I_n = \frac{1}{2(n-3)} - \frac{n-1}{n-3}I_{n-2}$$

Basic

1 If $I_n = \int x^n e^{2x}\,dx$ prove that $I_n = \dfrac{x^n e^{2x}}{2} - \dfrac{n}{2} I_{n-1}$.

2 If $I_n = \int \sin^n x\,dx$ prove that $I_n = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$.

3 If $I_n = \int \cot^n x\,dx$ prove that $I_n = -\dfrac{1}{n-1}\cot^{n-1} x - I_{n-2}$.

4 Use the result from Question 1 to find $\int x^2 e^{2x}\,dx$.

5 It is given that when $I_n = \int_0^1 x^n \left(x^2 - 1\right)^5 dx$ for $n \geq 0$, that $I_n = \dfrac{n-1}{n+11} I_{n-2}$ for $n \geq 2$.

Use this recurrence relation to evaluate $\int_0^1 x^3\left(x^2 - 1\right)^5 dx$.

6 It is given that $I_n = \int_0^{\frac{\pi}{4}} \tan^{2n}\theta\,d\theta$ for $n \geq 0$.

a Show that $I_n = \dfrac{1}{2n-1} - I_{n-1}$ for $n \geq 1$.

b Hence calculate I_3.

7 It is given that $I_n = \int_1^{e^2} (\ln x)^n\,dx$ for $n \geq 0$. Show that $I_n = e^2 2^n - nI_{n-1}$.

Medium

8a Let $I_n = \int_0^x \sec^n t\,dt$ where $0 \leq x \leq \dfrac{\pi}{2}$. Show that $I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$.

b Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sec^4 t\,dt$.

9 Prove $\int x(\ln x)^n\,dx = \dfrac{x^2(\ln x)^n}{2} - \dfrac{n}{2}\int x(\ln x)^{n-1}\,dx$ and hence find $\int x\ln^2 x\,dx$.

10 If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$ prove that $I_n = \left(\dfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$.

11 If $I_n = \int x^n\sqrt{2x+1}\,dx$ prove that $I_n = \dfrac{x^n\sqrt{(2x+1)^3}}{2n+3} - \dfrac{n}{2n+3} I_{n-1}$.

12 If $I_n = \int x^{-n} e^{2x}\,dx$ prove $I_n = -\dfrac{e^{2x}}{(n-1)x^{n-1}} + \dfrac{2}{n-1} I_{n-1}$ for $n \geq 2$.

13a If $I_{m,n} = \int x^m (\ln x)^n \, dx$ prove $I_{m,n} = \dfrac{x^{m+1}(\ln x)^n}{m+1} - \dfrac{n}{m+1} I_{m,n-1}$

b Hence find $\int x^2 (\ln x)^2 \, dx$.

14 If $I_n = \int \dfrac{\sin 3x}{x^n} dx$ and $J_n = \int \dfrac{\cos 3x}{x^n} dx$ prove $I_n = -\dfrac{\sin 3x}{(n-1)x^{n-1}} + \dfrac{3}{n-1} J_{n-1}$ for $n \geq 1$.

Challenging

15 Find a recurrence relationship for $I_n = \int_0^1 \dfrac{x^n}{(x^2+1)^2} dx$ in terms of I_{n-2}, for $n > 3$.

16 Find a recurrence relationship for $I_m = \int_0^1 x^m \left(x^2 - 1\right)^5 dx$ in terms of I_{m-2}.

17 Find a recurrence relationship for $I_n = \int_{-3}^0 x^n \sqrt{x+3} \, dx$ in terms of I_{n-1}, for $n \geq 1$.

18 Find a recurrence relationship for $I_n = \int \dfrac{dx}{\sin^n x}$ in terms of I_{n-2}.

19a Given that $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2} \, dx$ where n is a positive integer, find a recurrence relation for I_{2n+1} in terms of I_{2n-1}.

b Hence evaluate I_5.

20 Consider the integral $I_n = \int_0^1 \sqrt{x}(1-x)^n \, dx$, for $n = 0, 1, 2, 3, \ldots$
By finding a recurrence relation for I_n in terms of I_{n-1}, evaluate I_3.

21 Let $I_n = \int_0^1 \left(1 - x^2\right)^n dx$ and $J_n = \int_0^1 x^2 \left(1 - x^2\right)^n dx$.

a Show that $I_n = 2nJ_{n-1}$.

b Hence show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$.

c Show that $J_n = I_n - I_{n+1}$ and hence deduce that $J_n = \dfrac{1}{2n+3} I_n$.

d Hence write down a reduction formula for J_n in terms of J_{n-1}.

22a Show that $\int \dfrac{dx}{5 - 4\cos x} = \dfrac{2}{3} \tan^{-1}\left(3 \tan \dfrac{x}{2}\right) + c$

b Given that $\int_0^\pi \dfrac{dx}{5 - 4\cos x} = \dfrac{\pi}{3}$ show that $\int_0^\pi \dfrac{\cos x}{5 - 4\cos x} dx = \dfrac{\pi}{6}$

c If $u_n = \int_0^\pi \dfrac{\cos nx}{5 - 4\cos x} dx$ show that $u_{n+1} + u_{n-1} - \dfrac{5}{2} u_n = 0$, for $n \geq 1$.

23 Let $I_n = \int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^n}$, where n is a positive integer.

a Find the value of I_1.

b Using integration by parts, show that $I_n = \frac{2nI_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)}$.

c Hence evaluate $I_3 = \int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^3}$.

24 Let $I_n = \int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}} dt$ for $n = 0, 1, 2, \dots$

a Show that $t^2\left(1-t^2\right)^{\frac{n-3}{2}} = \left(1-t^2\right)^{\frac{n-3}{2}} - \left(1-t^2\right)^{\frac{n-1}{2}}$.

b Using integration by parts, show that $nI_n = (n-1)I_{n-2}$ for $n = 2, 3, 4, \dots$

c Let $J_n = nI_nI_{n-1}$ for $n = 1, 2, 3, \dots$ By using the principle of mathematical induction, prove that $J_n = \frac{\pi}{2}$ for $n = 1, 2, 3, \dots$

d Briefly explain why $0 < I_n < I_{n-1}$ for $n = 1, 2, 3, \dots$

e Deduce that $\sqrt{\frac{\pi}{2(n+1)}} < I_n < \sqrt{\frac{\pi}{2n}}$ for $n = 1, 2, 3\dots$

25 Double factorials are defined as $n!! = \begin{cases} n(n-2)(n-4)\dots 2 & \text{for even integers } n \\ n(n-2)(n-4)\dots 1 & \text{for odd integers } n \end{cases}$

a Prove that for even integers $n = 2k$ that $(2k)!! = 2^k k!$

b Show $(2k+1)!!\,2^k k! = (2k+1)!$ and thus for odd integers $n = 2k+1$ that $(2k+1)!! = \frac{(2k+1)!}{2^k k!}$.

c Prove $I_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$.

26 Prove $I_{2n} = \int_{-1}^{1} \left(1-x^2\right)^n dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$.

Basic

1 Find $\int \frac{dx}{x^2+6x+10}$.

2 Find $\int \frac{1}{x^2+2x+2}\,dx$.

3 Let $I_n = \int_1^e (\ln t)^n\,dt$ for $n = 1, 2, 3, \ldots$

a Show that $I_n = e - nI_{n-1}$ for $n = 1, 2, 3, \ldots$

b Hence, or otherwise, find the exact value I_3.

4 Find $\int x\tan^{-1}x\,dx$.

5 Find $\int \sin x\cos(\cos x)dx$.

6 Which of the following is an expression for $\int \frac{1}{\sqrt{7-6x-x^2}}dx$?

A $\sin^{-1}\left(\frac{x-3}{2}\right)+c$ B $\sin^{-1}\left(\frac{x+3}{2}\right)+c$ C $\sin^{-1}\left(\frac{x-3}{4}\right)+c$ D $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

7 Using the substitution $u = e^x + 1$ or otherwise, evaluate $\int_0^1 \frac{e^x}{(1+e^x)^2}dx$.

8 Find $\int \frac{1}{x\ln x}dx$.

9 What is the value of $\int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}dx$?

A $\frac{\pi^2}{4}$ B $-\frac{\pi^2}{4}$ C $\frac{\pi^2}{8}$ D $-\frac{\pi^2}{8}$

10 Evaluate $\int_0^{\frac{\pi}{3}} \sec^3 x\tan x\,dx$.

11 If n is a non−negative integer, then for what values of n is $\int_0^1 x^n\,dx = \int_0^1 (1-x)^n\,dx$?

A no solution B non−zero n, only C n even, only D all values of n

12 Find $\int \frac{dx}{(x-1)(x+2)}$.

13 Evaluate $\int_0^1 xe^{-x^2}\,dx$.

14a Find real numbers a, b and c such that $\frac{10}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$.

b Hence find $\int \frac{10}{(x+1)(x^2+4)}dx$.

15 Find $\displaystyle\int \frac{t^2-1}{t^3}\,dt$.

16 Find $\displaystyle\int \frac{dx}{\sqrt{6-x-x^2}}$.

17 Find $\displaystyle\int x^2 \log_e(3x)\,dx$.

18 Find $\displaystyle\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$.

19 Find $\displaystyle\int \frac{dx}{\sqrt{2-4x-2x^2}}$.

20 Prove the following results:

a $\dfrac{\cos 2x-\cos 4x}{\sin 2x+\sin 4x}=\tan x$, for $\sin 3x \neq 0$.

b $\sin x+\sin 3x = 4\sin x\cos^2 x$.

c $\cos x+\cos 3x = 4\cos^3 x - 2\cos x$.

d $2\tan x\cos 3x = \sec x\,(\sin 4x - \sin 2x)$.

21 Find $\displaystyle\int x\cos x\,dx$.

22 Find $\displaystyle\int \frac{x+1}{x-2}\,dx$.

23 Which of the following is an expression for $\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$?

A $-2\sqrt{16-x^2}+c$ B $-\sqrt{16-x^2}+c$ C $\frac{1}{2}\sqrt{16-x^2}+c$ D $-\frac{1}{2}\sqrt{16-x^2}+c$

24 Using $t=\tan\frac{x}{2}$, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$.

25 Use the substitution $u=e^x$, or otherwise, find $\displaystyle\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$.

26 Find $\displaystyle\int \frac{4x^3-2x^2+1}{2x-1}\,dx$.

27 Find $\displaystyle\int \frac{dx}{\sqrt{3-4x-4x^2}}$.

28 Evaluate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\sin^5 x\,dx$.

29 Evaluate $\displaystyle\int_0^1 \tan^{-1} x\,dx$.

30 Find $\displaystyle\int \frac{dx}{x^2-6x+13}$.

31 Show that $\sin 130° + \sin 10° = \sin 70°$

32 Find the amplitude and period of $f(x) = \sin 3x \cos x - \cos 2x \sin 2x.$

33 Given the substitution $t = \tan\frac{\theta}{2}$, the expression $\frac{1-\cos\theta}{\sin\theta}$ can be written as which of the following.

A $2t$ B t C $\frac{1}{t}$ D 0

34 Use the substitution $u = 1 + x^2$ to evaluate $\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\,dx$.

35 Find $\int x\sqrt{x}\left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right)dx$.

36 Evaluate $\int_0^{\log_3 2} 3^{x\log_3 2}\,dx$.

Medium

37 Show that $\cot\left(\frac{A+B}{2}\right) = \frac{\sin B - \sin A}{\cos A - \cos B}$.

38 Evaluate $\int_0^{\frac{\pi}{6}} \frac{d\theta}{9 - 8\cos^2\theta}$ using the substitution $t = \tan\theta$.

39 Find $\int \frac{e^{2x}}{e^x + 1}\,dx$.

40a Find real numbers a, b and c such that $\frac{8-2x}{(1+x)(4+x^2)} = \frac{a}{1+x} + \frac{bx+c}{4+x^2}$.

b Hence evaluate in simplest form $\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\,dx$.

41 The algebraic fraction $\frac{x+1}{5(x+h)^2}$, where h is a non−zero real number can be written in partial fraction form, where A and B are real numbers, as

A $\frac{A}{x+h} + \frac{B}{x+h}$ B $\frac{A}{5x+h} + \frac{B}{(x+h)^2}$ C $\frac{A}{x+h} + \frac{B}{(x+h)^2}$ D $\frac{A}{5(x+h)} + \frac{B}{x+h}$

42 The value of $\int_{-1}^{1} \frac{1}{1+e^{-x}}\,dx$ is

A $\frac{1}{2}$ B 1 C $\ln(1+e)$ D $2\ln(1+e)$

43 Solve $2\cos\theta + 3\sin\theta = -2$ for $0 \le \theta \le 2\pi$ using the substitution $t = \tan\frac{\theta}{2}$.

44 Consider the integral $I_n = \int_0^1 \sqrt{x}(1-x)^n\,dx$, for $n = 0, 1, 2, 3, \ldots$

a Show that $I_n = \left(\dfrac{2n}{2n+3}\right) I_{n-1}$, for $n = 1, 2, 3$.

b Hence evaluate $I_3 = \int_0^1 \sqrt{x}(1-x)^3\,dx$.

45 Show that $\sin 3x + \sin 2x + \sin x = \sin 2x\,(2\cos x + 1)$.

46a Find a, b and c such that $\dfrac{16}{(x^2+4)(2-x)} = \dfrac{ax+b}{x^2+4} + \dfrac{c}{2-x}$.

b Find $\int \dfrac{16}{(x^2+4)(2-x)}\,dx$.

47 Evaluate $\int_0^1 \sin^{-1} x\,dx$.

48 Evaluate $\int_0^1 \dfrac{x}{(x+1)(2x+1)}\,dx$.

49 Which of the following equals $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$?

A $\dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$ B $\dfrac{\pi}{4} + \dfrac{1}{2}\ln 2$ C $\dfrac{\pi}{4} + 2\ln\sqrt{2}$ D $\dfrac{\pi}{4} - 2\ln\sqrt{2}$

50 Which of the following equals $\int (\tan^3 2x + \tan 2x)\,dx$?

A $\dfrac{1}{4}\tan^4 2x + \dfrac{1}{2}\sec^2 2x + c$ B $\tan^2 2x + c$ C $\dfrac{1}{4}\tan^2 2x + c$ D $\dfrac{1}{2}\tan^2 2x + c$

51 Solve $\cos x - 3\sin x + 3 = 0$ for $0 \le x \le 2\pi$ using the substitution $t = \tan\dfrac{x}{2}$.

52 Use the substitution $t = \tan\dfrac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \dfrac{1}{1-\sin\theta}\,d\theta$.

53 Find $\int \sqrt{\dfrac{5-x}{5+x}}\,dx$.

54a Find the real numbers A, B and C such that $\dfrac{10}{(3+x)(1+x^2)} = \dfrac{A}{3+x} + \dfrac{Bx+C}{1+x^2}$.

b Hence use the substitution $t = \tan\theta$ to find $\int \dfrac{10}{3+\tan\theta}\,d\theta$.

55 Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$, for $n \geq 0$.

a Show that $I_0 = 2\sqrt{2} - 2$.

b Given that $I_n + I_{n-1} = \int_0^1 x^{n-1}\sqrt{1+x}\,dx$, show that $I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$.

c Hence evaluate I_2 in exact form.

56 Evaluate $\int_1^2 \frac{dx}{x(1+x^2)}$.

57 Find $\int \frac{x}{\sqrt{1-x}}\,dx$.

58 Find $\int e^{-2x}\cos x\,dx$.

59 Consider the following two statements:

$$\mathbf{I}: \int_0^1 \frac{dx}{1+x^n} < \int_0^1 \frac{dx}{1+x^{n+1}} \quad (n > 1)$$

$$\mathbf{II}: \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx$$

Which of these statements is correct?

A Neither statement B Statement I only C Statement II only D Both statements

60 Evaluate $\int_0^1 \cos^{-1} x\,dx$.

61 Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx$.

a Show that $I_1 = \frac{4}{3}$.

b Show that $I_{n-1} - I_n = \int_0^1 x^{n-1}\sqrt{1-x}\,dx$.

c Use integration by parts on the result of part (b) to show that $I_n = \frac{2n}{2n+1}I_{n-1}$.

62 Use the substitution $t = \tan\frac{x}{2}$ to find which of the following are equal to $\int \sec x\,dx$.

A $\ln|(t+1)(t-1)| + c$ B $\ln\left|\frac{1+t}{1-t}\right| + c$ C $\ln\left|(1+t^2)(1-t^2)\right| + c$ D $\ln\left|\frac{t^2+1}{t^2-1}\right| + c$

63 Evaluate $\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx$.

64 Which of the following is equal to $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$?

A $\sin(\sqrt{x}) + c$ B $2\sin(\sqrt{x}) + c$ C $\frac{1}{\sin(\sqrt{x})} + c$ D $\frac{2}{\sin(\sqrt{x})} + c$

65a Find real numbers A and B such that $\dfrac{3x^2+3x-2}{(x-1)(x+1)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{1}{(x+1)^2}$.

b Hence find $\displaystyle\int \frac{3x^2+3x-2}{(x-1)(x+1)^2}\,dx$.

66 Use the substitution $x=2\cos\theta$ to evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$.

67a Find constants A, B and C such that $\dfrac{x^2-x+1}{(x+1)^2}=A+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$.

b Hence find $\displaystyle\int \frac{x^2-x+1}{(x+1)^2}\,dx$.

68 Show, using integration by parts, that $\displaystyle\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx=\frac{\pi\sqrt{3}}{3}-\ln 2$.

69 Let $I_n=\displaystyle\int_0^{\pi} x^n\sin x\,dx$, where $n=0,1,2,\ldots$

a Use integration by parts to show that $I_n=\pi^n-n(n-1)I_{n-2}$ for $n=2,3,4,..$

b Hence evaluate $\displaystyle\int_0^{\pi} x^4\sin x\,dx$.

70 If $I=\displaystyle\int_0^{\ln 2}\frac{e^x}{e^x+e^{-x}}\,dx$ and $J=\displaystyle\int_0^{\ln 2}\frac{e^{-x}}{e^x+e^{-x}}\,dx$, then the exact value of $I-J$ is:

A $\ln\left(\dfrac{5}{2}\right)$ B $\ln 2$ C $\ln 5$ D $\ln\left(\dfrac{5}{4}\right)$

71 Given $f(t)=3\sin(4t)$ and $g(t)=3\sin\left(4t+\dfrac{\pi}{3}\right)$, find $f(t)+g(t)$ in the form $R\sin(4t+\alpha)$.

72a If $I_n=\displaystyle\int_0^{\frac{\pi}{2}}\sin^n x\,dx$ show that $I_n=\dfrac{n-1}{n}I_{n-2}$.

b Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}}\sin^5 x\,dx$.

73 Find $\displaystyle\int \sin^3 x\,dx$

A $\dfrac{1}{4}\sin^4 x+c$ B $-\cos x+\dfrac{1}{3}\cos^3 x+c$ C $-\dfrac{1}{4}\sin^4 x+c$ D $-\cos x-\dfrac{1}{3}\cos^3 x+c$

74 Solve $\sin 4x+\sin 3x+\sin 2x=0$ for $0\le x\le\pi$.

75 Show that $\displaystyle\int x\sqrt{x-1}\,dx=\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$.

76 Find $\displaystyle\int \frac{x}{\sqrt{2-x^2}}\,dx$ using the substitution $x=\sqrt{2}\sin\theta$.

77 Which of the following is an expression for $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} dx$? You are given that $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

A $x + \frac{1}{4}\cos 2x + c$ B $x - \frac{1}{4}\cos 2x + c$ C $x + \frac{1}{2}\sin 2x + c$ D $x - \frac{1}{2}\sin 2x + c$

78 Find $\int \frac{\sin 2x + \sin x}{\cos^2 x} dx$.

79 Find $\int e^x \sin x \, dx$.

80 Prove $\int \sin 2x \sin 4x \, dx = \frac{\sin 2x}{4} - \frac{\sin 6x}{12} + c$.

81 Prove $\int \frac{\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)}{\cos\left(x - \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)} dx = -\ln|\cos x| + c$.

Challenging

82 $\int \frac{x}{\sqrt{x+5}} dx =$

A $2\sqrt{x+5} + c$ B $\frac{2}{3}\sqrt{(x+5)^3} + c$

C $\frac{2}{3}\left\{\sqrt{(x+5)^3} - 10\sqrt{x+5}\right\} + c$ D $\frac{2}{3}(x-10)\sqrt{x+5} + c$

83 By using the fact that $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x + \sin x} = \ln 2$, evaluate $\int_0^{\frac{\pi}{2}} \frac{x \, dx}{1 + \cos x + \sin x}$.

84 Prove $\int \frac{dx}{x^3\sqrt{x^2 - 4}} = \frac{1}{16}\cos^{-1}\left(\frac{2}{x}\right) + \frac{\sqrt{x^2 - 4}}{8x^2} + c$.

85 Using the substitution $u = \pi - x$:

a Show that $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx$.

b Hence deduce that $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$.

86 The value of $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$ is equal to:

A 0 B π C $\frac{\pi}{2}$ D $\frac{\pi}{4}$

87a Use the substitution $u = \frac{\pi}{4} - x$, to show $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x) \, dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan x}\right) dx$.

b Hence find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x) \, dx$.

88 Find $\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2} dx$.

89 Let $I_n = \int_0^{\frac{\pi}{6}} \sin^{2n}\theta \sec\theta \, d\theta$, for $n \geq 1$.

a Show that $I_n - I_{n-1} = -\dfrac{1}{2^{2n-1}(2n-1)}$.

b Hence show that $I_n = \dfrac{1}{2}\ln 3 - \displaystyle\sum_{k=1}^{n} \dfrac{1}{2^{2k-1}(2k-1)}$.

90 Given that $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x \, dx$, for $n = 1, 2, \ldots$

a Show that $I_1 = \dfrac{1}{2}\ln 2$.

b Show that $I_{n-2} + I_n = \dfrac{1}{n-1}\left(3^{\frac{1}{2}(n-1)} - 1\right)$, for $n = 2, 3, 4, \ldots$

c Find I_5.

91 Use the substitution $t = \tan\dfrac{\theta}{2}$ to show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$.

92 Show that $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$, using the substitution $x = \dfrac{1-u}{1+u}$.

93a Show that $\displaystyle\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx = \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$.

b Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1 + 2\sin 2x + \cos 2x}\,dx$ using the substitution $t = \tan x$.

94a If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$, show that for $n > 1$, $I_n = \left(\dfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$.

b Hence find the area of the finite region bounded by the curve $y = x^4 \cos x$ and the x–axis for $0 \leq x \leq \dfrac{\pi}{2}$.

95 Given $I_n = \displaystyle\int_0^1 \frac{dx}{(1+x^2)^n}$ for $n \geq 1$ show that $I_n = \dfrac{1}{2^n(1-2n)} + \dfrac{2n}{2n-1} I_{n+1}$.

96 Let $I_n = \int_0^1 x^n \sqrt{1-x^3}\,dx$, for $n \geq 2$.

a Show that $I_n = \dfrac{2n-4}{2n+5} I_{n-3}$ for $n \geq 5$.

b Hence find I_8.

97a Show that $2\sin x \sin\big((2k+1)x\big) = \cos(2kx) - \cos(2(k+1)x)$.

b Hence, use mathematical induction to prove for all integers $n \geq 1$, that
$\sin x + \sin 3x + \sin 5x + \ldots + \sin\big((2n-1)x\big) = \dfrac{1 - \cos(2nx)}{2\sin x}$.

98a Show that $\left(1+t^2\right)^{n-1}+t^2\left(1+t^2\right)^{n-1}=\left(1+t^2\right)^n$.

b Given $I_n=\int_0^x \left(1+t^2\right)^n \, dt$ for $n=1,2,3,\ldots,$ use integration by parts, and part (a) to show $I_n=\frac{1}{2n+1}\left(1+x^2\right)^n x+\frac{2n}{2n+1}I_{n-1}$.

99 For $n=0,1,2,3,\ldots,$ let $I_n=\int_{e^{-1}}^{1}(1+\log_e x)^n \, dx$ and $J_n=\int_{e^{-1}}^{1}(\log_e x)(1+\log_e x)^n \, dx$.

a Show that $I_n=1-nI_{n-1}$ for $n=1,2,3,\ldots$

b Show that $J_n=1-(n+2)I_n$ for $n=0,1,2,3,\ldots$

c Hence find the value of J_3 in simplest exact form.

100 By first multiplying the numerator and denominator of the integrand by $\sin 3x$, show that $\int \frac{\cos 5x+\cos 4x}{1-2\cos 3x} \, dx=-\frac{1}{2}\sin 2x-\sin x+c$.

Standard integrals - trigonometry

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\tan f(x)\,dx = \int \frac{f'(x)\sin f(x)}{\cos f(x)}\,dx = -\ln|\cos f(x)| + c$$

$$\int f'(x)\operatorname{cosec} f(x)\,dx = -\ln|\operatorname{cosec} f(x) + \cot f(x)| + c$$

$$\int f'(x)\sec f(x)\,dx = \ln|\sec f(x) + \tan f(x)| + c$$

$$\int f'(x)\cot f(x)\,dx = \int \frac{f'(x)\cos f(x)}{\sin f(x)}\,dx = \ln|\sin f(x)| + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)\operatorname{cosec}^2 f(x)\,dx = -\cot f(x) + c$$

$$\int f'(x)\sec f(x)\tan f(x)\,dx = \sec f(x) + c$$

$$\int f'(x)\operatorname{cosec} f(x)\cot f(x)\,dx = -\operatorname{cosec} f(x) + c$$

Trig identities

$$\sin^2 x + \cos^2 x = 1$$
$$\sin^2 x = 1 - \cos^2 x$$
$$\cos^2 x = 1 - \sin^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$
$$\tan^2 x = \sec^2 x - 1$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$
$$\cot^2 x = \operatorname{cosec}^2 x - 1$$

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Standard integrals - other

$$\int f'(x)\,[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c \quad \text{where } n \neq 1$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$$

$$\int f'(x)\,a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}\,dx = \ln\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + c$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}\,dx = \ln\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + c$$

Product to sum identities

$$\cos A\cos B = \tfrac{1}{2}\Big[\cos(A - B) + \cos(A + B)\Big]$$

$$\sin A\sin B = \tfrac{1}{2}\Big[\cos(A - B) - \cos(A + B)\Big]$$

$$\sin A\cos B = \tfrac{1}{2}\Big[\sin(A + B) + \sin(A - B)\Big]$$

$$\cos A\sin B = \tfrac{1}{2}\Big[\sin(A + B) - \sin(A - B)\Big]$$

t-formulas

$$t = \tan\frac{x}{2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(1 + t^2)$$

$$dx = \frac{2dt}{1 + t^2}$$

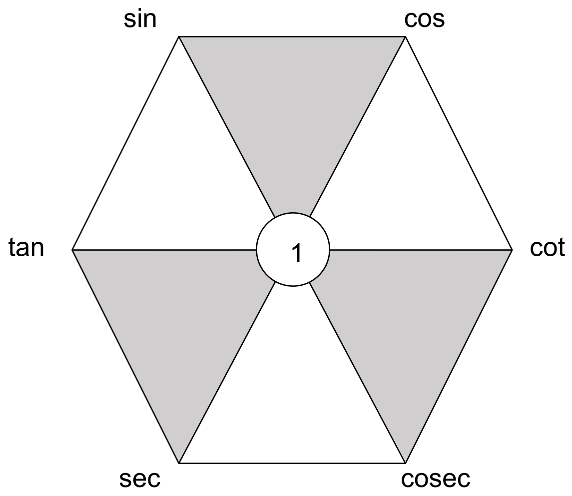
$$\sin x = \frac{2t}{1 + t^2}$$

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

$$\tan x = \frac{2t}{1 - t^2}$$

Co-function relations: The trigonometric functions cosine, cotangent, and cosecant on the right of the hexagon are co-functions of sine, tangent, and secant on the left respectively.

$\sin(90-\theta)=\cos\theta\,;\tan(90-\theta)=\cot\theta\,;\sec(90-\theta)=\operatorname{cosec}\theta$



Reciprocal identities: The two trigonometric functions of any diagonal are reciprocals of each other.

$$\operatorname{cosec}\theta=\frac{1}{\sin\theta}\,;\sec\theta=\frac{1}{\cos\theta}\,;\cot\theta=\frac{1}{tan\,\theta}$$

Product identities: Along the outside edges of the hexagon any trigonometric function equals the product of the functions of the adjacent vertices.

$$\tan\theta\times\cos\theta=\sin\theta\,;\sec\theta\times\cot\theta=\operatorname{cosec}\theta\ \text{ etc}$$

Quotient identities: along the outside edges of the hexagon, any trigonometric function equals the quotient of the next two trigonometric functions going either clockwise or counter clock wise.

$$\sin\theta=\tan\theta\div\sec\theta\,;\cot\theta\div\operatorname{cosec}\theta=\cos\theta\ \ etc$$

Pythagorean identities: For each shaded triangle, the upper left function squared plus the upper right function squared, equals the bottom function squared.

$$\sin^2\theta+\cos^2\theta=1\,;\tan^2\theta+1=\sec^2\theta\ \,;1+\cot^2\theta=\operatorname{cosec}^2\theta$$

To rearrange any identity, drop any term from the top line to the right end of the bottom line with a minus in front.

$$\sin^2\theta=1-\cos^2\theta\ \,;\tan^2\theta=\sec^2\theta-1\,;\operatorname{cosec}^2\theta\ -\cot^2\theta=1$$

Product to sum mnemonic

The fours Product to Sum Identities are very similar, so to save us looking at the Reference Sheet each time, try:

Factors — Difference — Sum

Factors are the same so use cos on RHS

$$\cos A\cos B=\frac{1}{2}\Big[\cos(A-B)+\cos(A+B)\Big]$$

$$\sin A\sin B=\frac{1}{2}\Big[\cos(A-B)-\cos(A+B)\Big]$$

Factors are different so use sin on RHS

$$\sin A\cos B=\frac{1}{2}\Big[\sin(A+B)+\sin(A-B)\Big]$$

$$\cos A\sin B=\frac{1}{2}\Big[\sin(A+B)-\sin(A-B)\Big]$$

If cos is the second factor add the ratios on the RHS, if sin then subtract

Sin or Cos on the RHS?

- If the factors use the same ratio then use cos on the RHS, if the factors use different ratios then use sin

Sum or Difference of Angles first on RHS?

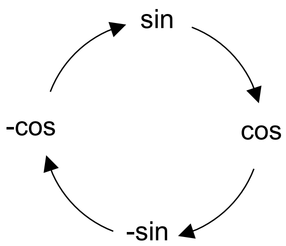
- If using cos then difference of angles is used first, if using sin then sum of angles is used first

Add or Subtract the Ratios on RHS?

- If cos is the second factor then add the ratios on RHS, if sin is the second factor then subtract

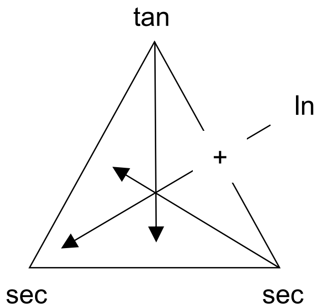
The arrows show differentiation - go against the arrows for primitives/integration. The bottom two triangles show rules that are more often used in integration.

When differentiating:
- any of the complementary ratios ($\cos\theta$, $\cot\theta$ or $\operatorname{cosec}\theta$) the sign changes.
- any ratio ending in 'c' ($\sec\theta$, $\operatorname{cosec}\theta$) gives two terms.

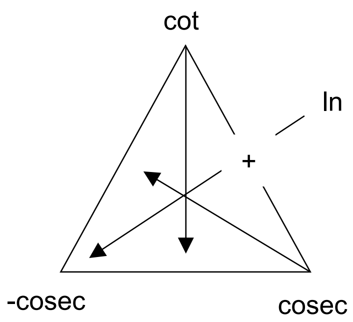


Function	**Derivative**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$-\sin x$	$-\cos x$
$-\cos x$	$\sin x$

In the diagrams below:
- an arrow pointing at a side means the derivative is the product of the vertices, so the derivative of $\sec x$ is $\sec x \tan x$
- the plus sign on the RH side means to take the ln of the sum of the vertices, so the derivative of $\ln|\tan x + \sec x|$ is $\sec x$



Function	**Derivative**
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\ln(\tan x + \sec x)$	$\sec x$



Function	**Derivative**
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln(\cot x + \operatorname{cosec} x)$	$-\operatorname{cosec} x$

Question 1

$$\int_{-7}^{1} \frac{dx}{x^2+6x+25}$$
$$= \int_{-7}^{1} \frac{dx}{x^2+6x+9+16}$$
$$= \int_{-7}^{1} \frac{dx}{(x+3)^2+4^2}$$
$$= \frac{1}{4}\left[\tan^{-1}\left(\frac{x+3}{4}\right)\right]_{-7}^{1}$$
$$= \frac{1}{4}\left(\tan^{-1}1 - \tan^{-1}(-1)\right)$$
$$= \frac{1}{4}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$
$$= \frac{\pi}{8}$$

Question 2

$$\int_{0}^{\frac{3\sqrt{3}}{4}} \frac{1}{\sqrt{9-4x^2}}dx$$
$$= \frac{1}{2}\int_{0}^{\frac{3\sqrt{3}}{4}} \frac{2}{\sqrt{3^2-(2x)^2}}dx$$
$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_{0}^{\frac{3\sqrt{3}}{4}}$$
$$= \frac{1}{2}\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}0\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{3} - 0\right)$$
$$= \frac{\pi}{6}$$

Question 3

$$\int \frac{dx}{\sqrt{20-8x-x^2}}dx$$
$$= \int \frac{dx}{\sqrt{-(x^2+8x-20)}}$$
$$= \int \frac{dx}{\sqrt{-\left((x+4)^2-36\right)}}$$
$$= \int \frac{dx}{\sqrt{6^2-(x+4)^2}}$$
$$= \sin^{-1}\left(\frac{x+4}{6}\right) + c$$

Question 4

$$\int \frac{dx}{12x^2+12x+78}$$
$$= \frac{1}{3}\int \frac{dx}{4x^2+4x+26}$$
$$= \frac{1}{6}\int \frac{2dx}{(2x+1)^2+5^2}$$
$$= \frac{1}{6}\times\frac{1}{5}\tan^{-1}\left(\frac{2x+1}{5}\right) + c$$
$$= \frac{1}{30}\tan^{-1}\left(\frac{2x+1}{5}\right) + c$$

Question 5

$$\int \frac{x^2+2x}{x^2+2x+5}dx$$
$$= \int \frac{x^2+2x+5-5}{x^2+2x+5}dx$$
$$= \int \left(1 - \frac{5}{(x+1)^2+2^2}\right)dx$$
$$= x - \frac{5}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$$

Question 6

$$\int \frac{e^{-x}}{e^{-x}-1}dx$$
$$= -\int \frac{-e^{-x}}{e^{-x}-1}dx$$
$$= -\ln\left|e^{-x}-1\right| + c$$

Question 7

$$\int \frac{1}{x^2-3x+3}dx$$
$$= \int \frac{1}{x^2-3x+\frac{9}{4}+\frac{3}{4}}dx$$
$$= \int \frac{1}{\left(x-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{3}{2}}{\sqrt{3}/2}\right) + c$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-3}{\sqrt{3}}\right) + c$$

Question 8

$$\int_{-1}^{1} \frac{1}{5-2t+t^2}dt$$
$$= \int_{-1}^{1} \frac{dt}{(t-1)^2+2^2}$$
$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{t-1}{2}\right)\right]_{-1}^{1}$$
$$= \frac{1}{2}\left(0-\left(-\frac{\pi}{4}\right)\right)$$
$$= \frac{\pi}{8}$$

Question 9

$$\int \frac{2}{4x^2-4x+17}\,dx$$
$$=\int \frac{2}{(2x-1)^2+4^2}\,dx$$
$$=\frac{1}{4}\tan^{-1}\left(\frac{2x-1}{4}\right)+c$$

Question 10

$$\int \frac{2x}{x^4+2x^2+5}\,dx$$
$$=\int \frac{2x}{(x^2+1)^2+2^2}\,dx$$
$$=\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{2}\right)+c$$

Question 11

$$\int \frac{2}{e^x+1}\,dx$$
$$=\int \frac{2e^{-x}}{1+e^{-x}}\,dx$$
$$=-2\int \frac{-e^{-x}}{1+e^{-x}}\,dx$$
$$=-2\ln\left|1+e^{-x}\right|+c$$

Question 12

$$-\int \frac{2x}{\sqrt{3-2x^2-x^4}}\,dx$$
$$=-\int \frac{2x}{\sqrt{4-(1+x^2)^2}}\,dx$$
$$=\cos^{-1}\left(\frac{x^2+1}{2}\right)+c \quad \text{or} \quad -\sin^{-1}\left(\frac{x^2+1}{2}\right)+c$$

Question 13

$$\int \frac{2\cos x\,(\sin x+1)}{(\sin x+3)(\sin x-1)}\,dx$$
$$=\int \frac{2\sin x\cos x+2\cos x}{\sin^2 x+2\sin x-3}\,dx$$
$$=\ln\left|\sin^2 x+2\sin x-3\right|+c$$

Question 14

$$\int \frac{2}{x\ln 7}\,dx$$
$$=\frac{2}{\ln 7}\int \frac{dx}{x}$$
$$=\frac{2}{\ln 7}\ln|x|+c$$
$$=\frac{2\ln|x|}{\ln 7}+c$$
$$=\frac{\ln\left(x^2\right)}{\ln 7}+c$$
$$=\log_7\left(x^2\right)+c$$

Question 15

$$\int \frac{dx}{(x^2+4)\tan^{-1}\left(\frac{x}{2}\right)}$$
$$=\frac{1}{2}\int \frac{\frac{2}{x^2+4}}{\tan^{-1}\left(\frac{x}{2}\right)}\,dx$$
$$=\frac{1}{2}\int \frac{\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{2}\right)\right)}{\tan^{-1}\left(\frac{x}{2}\right)}\,dx$$
$$=\frac{1}{2}\ln\left|\tan^{-1}\left(\frac{x}{2}\right)\right|+c$$

Question 16

$$\frac{dy}{dx}=\frac{\cos x}{1+\sin^2 x}$$
$$\int_1^y dy=\int_{\frac{\pi}{2}}^{x}\frac{\cos x}{1+\sin^2 x}\,dx$$
$$\Big[y\Big]_1^y=\int_{\frac{\pi}{2}}^{x}\frac{\frac{d}{dx}(\sin x)}{1+(\sin x)^2}\,dx$$
$$y-1=\left[\tan^{-1}\left(\sin x\right)\right]_{\frac{\pi}{2}}^{x}$$
$$=\tan^{-1}\left(\sin x\right)-\tan^{-1}\left(\sin\frac{\pi}{2}\right)$$
$$y=\tan^{-1}\left(\sin x\right)-\frac{\pi}{4}+1$$

Question 17

$$\int_{-\frac{\pi}{2}}^{0}\frac{\cos\theta}{\sin^2\theta+2\sin\theta+4}\,d\theta$$
$$=\int_{-\frac{\pi}{2}}^{0}\frac{\cos\theta}{(\sin\theta+1)^2+\sqrt{3}^2}\,d\theta$$
$$=\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{\sin\theta+1}{\sqrt{3}}\right)\right]_{-\frac{\pi}{2}}^{0}$$
$$=\frac{1}{\sqrt{3}}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}0\right)$$
$$=\frac{1}{\sqrt{3}}\left(\frac{\pi}{6}-0\right)$$
$$=\frac{\pi}{6\sqrt{3}} \qquad \left(\text{or } =\frac{\pi\sqrt{3}}{18}\right)$$

Question 18

$$\int \frac{e^{3x}}{\sqrt{7+12e^{3x}-4e^{6x}}}\,dx$$
$$=\int \frac{e^{3x}}{\sqrt{-(4e^{6x}-12e^{3x}-7)}}\,dx$$
$$=\int \frac{e^{3x}}{\sqrt{-((2e^{3x}-3)^2-4^2)}}\,dx$$
$$=\frac{1}{6}\int \frac{6e^{3x}}{\sqrt{4^2-(2e^{3x}-3)^2}}\,dx$$
$$=\frac{1}{6}\sin^{-1}\left(\frac{2e^{3x}-3}{4}\right)+c$$

Question 19

$$\int \frac{\cos x - \sin x}{2 + \sin 2x}\,dx$$
$$= \int \frac{\cos x - \sin x}{2 + 2\sin x \cos x}\,dx$$
$$= \int \frac{\cos x - \sin x}{\sin^2 x + 2\sin x \cos x + \cos^2 x + 1}\,dx$$
$$= \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2 + 1}\,dx$$
$$= \tan^{-1}\left(\sin x + \cos x\right) + c$$

Question 20

$$\int \frac{\sin^3 x}{\cos x - 1}\,dx$$
$$= \int \frac{\sin x \sin^2 x}{\cos x - 1}\,dx$$
$$= \int \frac{\sin x\,(1 - \cos^2 x)}{\cos x - 1}\,dx$$
$$= \int \frac{\sin x\,(1 + \cos x)(1 - \cos x)}{\cos x - 1}\,dx$$
$$= -\int (\sin x + \sin x \cos x)\,dx$$
$$= -\int \left(\sin x + \frac{1}{2}\sin 2x\right)dx$$
$$= \cos x + \frac{1}{4}\cos 2x + c$$

Question 21

$$\int \frac{e^{\log_2 x}}{x}\,dx$$
$$= \int \frac{e^{\left(\frac{\ln x}{\ln 2}\right)}}{x}\,dx$$
$$= \int \frac{\left(e^{\ln x}\right)^{\frac{1}{\ln 2}}}{x}\,dx$$
$$= \int \frac{x^{\frac{1}{\ln 2}}}{x}\,dx$$
$$= \int x^{\frac{1}{\ln 2} - 1}\,dx$$
$$= \ln 2\, x^{\frac{1}{\ln 2}} + c$$

Question 22

$$\int \frac{\sin 2x + \cos 2x}{\cos x}\,dx$$
$$= \int \frac{2\sin x \cos x + 2\cos^2 x - 1}{\cos x}\,dx$$
$$= \int (2\sin x + 2\cos x - \sec x)\,dx$$
$$= -2\cos x + 2\sin x - \ln\left|\tan x + \sec x\right| + c$$

Question 23

$$\int \frac{dx}{\sec x - 1}$$
$$= \int \frac{\cos x}{1 - \cos x}\,dx$$
$$= \int \frac{\cos x}{1 - \cos x} \times \frac{1 + \cos x}{1 + \cos x}\,dx$$
$$= \int \frac{\cos x + \cos^2 x}{1 - \cos^2 x}\,dx$$
$$= \int \frac{\cos x + \cos^2 x}{\sin^2 x}\,dx$$
$$= \int \cos x\,(\sin x)^{-2}\,dx + \int \cot^2 x\,dx$$
$$= -\frac{1}{\sin x} + \int \left(\operatorname{cosec}^2 x - 1\right)dx$$
$$= -\operatorname{cosec} x - \cot x - x + c$$

Question 24

$$\int (2x \operatorname{cosec} x - x^2 \cot x \operatorname{cosec} x)\,dx$$
$$= \int \left(\frac{2x}{\sin x} - \frac{x^2 \cos x}{\sin^2 x}\right)dx$$
$$= \int \frac{2x \sin x - x^2 \cos x}{\sin^2 x}\,dx$$
$$= \int \frac{d}{dx}\left(\frac{x^2}{\sin x}\right)dx$$
$$= \frac{x^2}{\sin x} + c$$

Alt:

$$\int (2x \operatorname{cosec} x - x^2 \cot x \operatorname{cosec} x)\,dx$$
$$= \int \left(\frac{d}{dx}\left(x^2\right)\operatorname{cosec} x + x^2 \frac{d}{dx}(\operatorname{cosec} x)\right)dx$$
$$= \int \frac{d}{dx}\left(x^2 \operatorname{cosec} x\right)dx$$
$$= x^2 \operatorname{cosec} x + c$$

25 $$\int \frac{\cos x\,(2a \sin x + b)}{(\sin x - \alpha)(\sin x - \beta)}\,dx$$
$$= \int \frac{\cos x\,(2a \sin x + b)}{\sin^2 x - (\alpha + \beta)\sin x + \alpha\beta}\,dx$$
$$= \int \frac{\cos x\,(2a \sin x + b)}{\sin^2 x + \frac{b}{a}\sin x + \frac{c}{a}}\,dx$$
$$= \int \frac{a \cos x\,(2a \sin x + b)}{a \sin^2 x + b \sin x + c}\,dx$$
$$= a\int \frac{2a \sin x \cos x + b \cos x}{a \sin^2 x + b \sin x + c}\,dx$$
$$= a \ln\left|a \sin^2 x + b \sin x + c\right| + k$$

Question 1

$$\int x^2(x^3+4)^5\,dx$$
$$=\frac{1}{3}\int_{-7}^{1} 3x^2(x^3+4)^5\,dx$$
$$=\frac{1}{3}\times\frac{(x^3+4)^6}{6}+c$$
$$=\frac{(x^3+4)^6}{18}+c$$

Alt:

$$\boxed{\begin{aligned} u &= x^3+4 \\ du &= 3x^2dx \end{aligned}}$$

$$\int x^2(x^3+4)^5\,dx$$
$$=\frac{1}{3}\int (x^3+4)^5\times 3x^2dx$$
$$=\frac{1}{3}\int u^5\,du$$
$$=\frac{1}{3}\times\frac{u^6}{6}+c$$
$$=\frac{(x^3+4)^6}{18}+c$$

Question 2

$$\int_1^2 \frac{x^2}{\sqrt{2x^3-1}}dx$$
$$=\frac{1}{6}\int_1^2 6x^2(2x^3-1)^{-\frac{1}{2}}\,dx$$
$$=\frac{1}{6}\times 2\left[(2x^3-1)^{\frac{1}{2}}\right]_1^2$$
$$=\frac{\sqrt{15}-1}{3}$$

Alt:

$$\boxed{\begin{aligned} u^2 &= 2x^3-1 \\ 2u\,du &= 6x^2dx \end{aligned}}$$

$$\int_1^2 \frac{x^2}{\sqrt{2x^3-1}}dx$$
$$=\frac{1}{6}\int_1^2 \frac{1}{\sqrt{2x^3-1}}\times 6x^2dx$$
$$=\frac{1}{6}\int_1^{\sqrt{15}} \frac{1}{u}\times 2u\,du$$
$$=\frac{1}{3}\int_1^{\sqrt{15}} du$$
$$=\frac{1}{3}\Big[u\Big]_1^{\sqrt{15}}$$
$$=\frac{\sqrt{15}-1}{3}$$

Question 3

$$\int e^{2x}\sqrt{e^{2x}-1}\,dx$$
$$=\frac{1}{2}\int 2e^{2x}(e^{2x}-1)^{\frac{1}{2}}\,dx$$
$$=\frac{1}{2}\times\frac{2}{3}(e^{2x}-1)^{\frac{3}{2}}+c$$
$$=\frac{\sqrt{(e^{2x}-1)^3}}{3}+c$$

Alt:

$$\boxed{\begin{aligned} u^2 &= e^{2x}-1 \\ 2u\,du &= 2e^{2x}dx \end{aligned}}$$

$$\int e^{2x}\sqrt{e^{2x}-1}\,dx$$
$$=\frac{1}{2}\int \sqrt{e^{2x}-1}\times 2e^{2x}dx$$
$$=\frac{1}{2}\int u\times 2u\,du$$
$$=\int u^2$$
$$=\frac{u^3}{3}+c$$
$$=\frac{\sqrt{(e^{2x}-1)^3}}{3}+c$$

Question 4

$$\int x\cos(x^2)\,dx$$
$$=\frac{1}{2}\int 2x\cos(x^2)\,dx$$
$$=\frac{1}{2}\sin(x^2)+c$$

Alt:

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2xdx \end{aligned}}$$

$$\int x\cos(x^2)\,dx$$
$$=\frac{1}{2}\int \cos(x^2)\times 2x\,dx$$
$$=\frac{1}{2}\int \cos u\,du$$
$$=\frac{1}{2}\sin u+c$$
$$=\frac{1}{2}\sin x^2+c$$

Question 5

$$\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx$$
$$= \int_0^{\frac{\pi}{4}} \sec^2 x \, (\tan x)^3 \, dx$$
$$= \left[\frac{\tan^4 x}{4}\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{4} - 0$$
$$= \frac{1}{4}$$

Alt:

$$\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx \qquad \boxed{\begin{aligned} u &= \tan x \\ du &= \sec^2 x \, dx \end{aligned}}$$
$$= \int_0^1 u^3 \, du$$
$$= \left[\frac{u^4}{4}\right]_0^1$$
$$= \frac{1}{4} - 0$$
$$= \frac{1}{4}$$

Question 6

$$\int \frac{\cos x}{\sin^5 x} \, dx$$
$$= \int \cos x \, (\sin x)^{-5} \, dx$$
$$= \frac{(\sin x)^{-4}}{-4} + c$$
$$= -\frac{1}{4\sin^4 x} + c$$

Alt:

$$\int \frac{\cos x}{\sin^5 x} \, dx \qquad \boxed{\begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned}}$$
$$= \int \frac{1}{\sin^5 x} \times \cos x \, dx$$
$$= \int u^{-5} \, du$$
$$= \frac{u^{-4}}{-4} + c$$
$$= -\frac{1}{4\sin^4 x} + c$$

Question 7

$$\int \sin^{\frac{3}{2}} 2x \cos 2x \, dx$$
$$= \frac{1}{2} \int 2\cos 2x \, (\sin 2x)^{\frac{3}{2}} \, dx$$
$$= \frac{1}{2} \times \frac{2}{5} (\sin 2x)^{\frac{5}{2}} + c$$
$$= \frac{\sin^{\frac{5}{2}} 2x}{5} + c$$

Alt:

$$\int \sin^{\frac{3}{2}} 2x \cos 2x \, dx \qquad \boxed{\begin{aligned} u &= \sin 2x \\ du &= 2\cos 2x \, dx \end{aligned}}$$
$$= \frac{1}{2} \int \sin^{\frac{3}{2}} 2x \times 2\cos 2x \, dx$$
$$= \frac{1}{2} \int u^{\frac{3}{2}} \, du$$
$$= \frac{1}{2} \times \frac{2}{5} u^{\frac{5}{2}} + c$$
$$= \frac{\sin^{\frac{5}{2}} 2x}{5} + c$$

Question 8

$$\int \sqrt{\sin 2x} \cos 2x \, dx$$
$$= \frac{1}{2} \int (2\cos 2x)(\sin 2x)^{\frac{1}{2}} \, dx$$
$$= \frac{1}{2} \times \frac{2}{3} (\sin 2x)^{\frac{3}{2}} + c$$
$$= \frac{\sqrt{\sin^3 2x}}{3} + c$$

Alt:

$$\int \sqrt{\sin 2x} \cos 2x \, dx \qquad \boxed{\begin{aligned} u^2 &= \sin 2x \\ 2u \, du &= 2\cos 2x \end{aligned}}$$
$$= \frac{1}{2} \int \sqrt{\sin 2x} \times 2\cos 2x$$
$$= \frac{1}{2} \int u \times 2u \, du$$
$$= \int u^2 \, du$$
$$= \frac{u^3}{3} + c$$
$$= \frac{\sqrt{\sin^3 2x}}{3} + c$$

Question 9

$$\int e^x \cos e^x \, dx$$
$$= \sin e^x + c$$

Alt:

$$\int e^x \cos e^x \, dx$$

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned}}$$

$$= \int \cos e^x \times e^x dx$$
$$= \int \cos u \, du$$
$$= \sin u + c$$
$$= \sin e^x + c$$

Question 10

$$\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}} dx$$
$$= -\int \frac{-1}{\sqrt{1-x^2}} \times e^{\cos^{-1} x} \, dx$$
$$= -e^{\cos^{-1} x} + c$$

Alt:

$$\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}} dx$$

$$\boxed{\begin{aligned} u &= \cos^{-1} x \\ du &= -\frac{dx}{\sqrt{1-x^2}} \end{aligned}}$$

$$= -\int e^{\cos^{-1} x} \times \left(-\frac{dx}{\sqrt{1-x^2}}\right)$$
$$= -\int e^u \, du$$
$$= -e^u + c$$
$$= -e^{\cos^{-1} x} + c$$

Question 11

$$\int_1^{e^{\frac{\pi}{2}}} \frac{\cos(\ln x)}{x} dx$$
$$= \int_1^{e^{\frac{\pi}{2}}} \frac{1}{x} \cos(\ln x) \, dx$$
$$= \Big[\sin(\ln x)\Big]_1^{e^{\frac{\pi}{2}}}$$
$$= \sin\left(\frac{\pi}{2}\right) - \sin 0$$
$$= 1 - 0$$
$$= 1$$

Alt:

$$\int_1^{e^{\frac{\pi}{2}}} \frac{\cos(\ln x)}{x} dx$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{dx}{x} \end{aligned}}$$

$$= \int_1^{e^{\frac{\pi}{2}}} \cos(\ln x) \times \frac{dx}{x}$$
$$= \int_0^{\frac{\pi}{2}} \cos u \, du$$
$$= \Big[\sin u\Big]_0^{\frac{\pi}{2}}$$
$$= \sin\left(\frac{\pi}{2}\right) - \sin 0$$
$$= 1 - 0$$
$$= 1$$

Question 12

$$\int (3x^2 + 2x)\sqrt{x^3 + x^2} \, dx$$
$$= \int (3x^2 + 2x)(x^3 + x^2)^{\frac{1}{2}} dx$$
$$= \frac{2(x^3 + x^2)^{\frac{3}{2}}}{3} + c$$
$$= \frac{2\sqrt{(x^3 + x^2)^3}}{3} + c$$

Alt:

$$\int (3x^2 + 2x) \; dx$$

$$\boxed{\begin{aligned} u^2 &= x^3 + x^2 \\ 2u \, du &= (3x^2 + 2x)dx \end{aligned}}$$

$$= \int \sqrt{x^3 + x^2} \times (3x^2 + 2x)dx$$
$$= \int u \times 2u \, du$$
$$= 2\int u^2 \, du$$
$$= 2 \times \frac{u^3}{3} + c$$
$$= \frac{2(x^3 + x^2)^{\frac{3}{2}}}{3} + c$$
$$= \frac{2\sqrt{(x^3 + x^2)^2}}{3} + c$$

Question 13

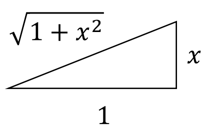
$$\int x^2 e^{x^3}\,dx$$
$$= \frac{1}{3}\int 3x^2 e^{x^3}\,dx$$
$$= \frac{1}{3}e^{x^3} + c$$

Alt:

$$\int x^2 e^{x^3}\,dx \qquad \boxed{\begin{gathered} u = x^3 \\ du = 3x^2 dx \end{gathered}}$$
$$= \frac{1}{3}\int e^{x^3} \times 3x^2 dx$$
$$= \frac{1}{3}\int e^u\,du$$
$$= \frac{1}{3}e^u + c$$
$$= \frac{1}{3}e^{x^3} + c$$

Question 14

$$\int \frac{\sin(\tan^{-1} x)}{1+x^2}\,dx$$
$$= \int \frac{1}{1+x^2} \times \sin(\tan^{-1} x)\,dx$$
$$= -\cos(\tan^{-1} x) + c$$
$$= -\frac{1}{\sqrt{1+x^2}} + c$$



Alt:

$$\int \frac{\sin(\tan^{-1} x)}{1+x^2}\,dx \qquad \boxed{\begin{gathered} u = \tan^{-1} x \\ du = \frac{dx}{1+x^2} \end{gathered}}$$
$$= \int \sin(\tan^{-1} x) \times \frac{dx}{1+x^2}$$
$$= \int \sin u\,du$$
$$= -\cos u + c$$
$$= -\cos(\tan^{-1} x) + c$$
$$= -\frac{1}{\sqrt{1+x^2}} + c$$

Alt II:

$$\int \frac{\sin(\tan^{-1} x)}{1+x^2}\,dx$$
$$= \int \frac{\left(x/\sqrt{1+x^2}\right)}{1+x^2}\,dx$$
$$= \int x\left(1+x^2\right)^{-\frac{3}{2}}\,dx$$
$$= \frac{1}{2}\int 2x\left(1+x^2\right)^{-\frac{3}{2}}\,dx$$
$$= \frac{1}{2} \times (-2) \times \left(1+x^2\right)^{-\frac{1}{2}} + c$$
$$= -\frac{1}{\sqrt{1+x^2}} + c$$

Question 15

$$\int (e^{t^2}+16)te^{t^2}\,dt$$
$$= \frac{1}{2}\int 2te^{t^2}\left(e^{t^2}+16\right)dt$$
$$= \frac{1}{2} \times \frac{\left(e^{t^2}+16\right)^2}{2} + c$$
$$= \frac{1}{4}\left(e^{t^2}+16\right)^2 + c$$

Alt:

$$\int (e^{t^2}+16)te^{t^2}\,dt \qquad \boxed{\begin{gathered} u = e^{t^2}+16 \\ du = 2te^{t^2}dt \end{gathered}}$$
$$= \frac{1}{2}\int (e^{t^2}+16) \times 2te^{t^2}dt$$
$$= \frac{1}{2}\int u\,du$$
$$= \frac{1}{2} \times \frac{u^2}{2} + c$$
$$= \frac{1}{4}\left(e^{t^2}+16\right)^2 + c$$

Question 16

$$\int \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x}\,dx$$
$$= -\int \frac{-\operatorname{cosec} x \cot x}{1+(\operatorname{cosec} x)^2}\,dx$$
$$= -\tan^{-1}(\operatorname{cosec} x) + c$$

Alt:

$$\int \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x}\,dx \qquad \boxed{\begin{gathered} u = \operatorname{cosec} x \\ du = -\cot x \operatorname{cosec} x\,dx \end{gathered}}$$
$$= -\int \frac{1}{1+\operatorname{cosec}^2 x} \times (-\cot x \operatorname{cosec} x\,dx)$$
$$= -\int \frac{1}{1+u^2}\,du$$
$$= -\tan^{-1} u + c$$
$$= -\tan^{-1}(\operatorname{cosec} x) + c$$

Question 17

$$\int e^x e^{e^x}\,dx$$
$$= e^{e^x} + c$$

Alt:

$$\int e^x e^{e^x}\,dx \qquad \boxed{\begin{gathered} u = e^x \\ du = e^x dx \end{gathered}}$$
$$= \int e^{e^x} \times e^x dx$$
$$= \int e^u\,du$$
$$= e^u + c$$
$$= e^{e^x} + c$$

Question 18

$$\int \frac{\ln^3 x}{x}\,dx$$
$$= \int \frac{1}{x}(\ln x)^3\,dx$$
$$= \frac{\ln^4 x}{4} + c$$

Alt:

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$$\int \frac{\ln^3 x}{x}\,dx$$
$$= \int (\ln x)^3 \times \frac{1}{x}\,dx$$
$$= \int u^3\,du$$
$$= \frac{u^4}{4} + c$$
$$= \frac{\ln^4 x}{4} + c$$

Question 19

$$\int \sqrt{\frac{2+x}{2-x}}\,dx$$
$$= \int \sqrt{\frac{2+x}{2-x}} \times \sqrt{\frac{2+x}{2+x}}\,dx$$
$$= \int \frac{2+x}{\sqrt{4-x^2}}\,dx$$
$$= \int \left(\frac{2}{\sqrt{4-x^2}} + \frac{1}{2} \times (-2x)\left(4-x^2\right)^{-\frac{1}{2}}\right)dx$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2} \times 2\left(4-x^2\right)^{\frac{1}{2}} + c$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \sqrt{4-x^2} + c$$

Question 20

$$\int \frac{\sin(\tan\theta)}{\cos^2\theta}\,d\theta$$
$$= \int \sec^2\theta \sin(\tan\theta)\,d\theta$$
$$= -\cos(\tan\theta) + c$$

Question 21

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$$
$$= \int \frac{\cos^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \frac{1-\sin^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \cos x\left((\sin x)^{-\frac{1}{2}} - (\sin x)^{\frac{3}{2}}\right)dx$$
$$= 2(\sin x)^{\frac{1}{2}} - \frac{2}{5}(\sin x)^{\frac{5}{2}} + c$$
$$= 2\sqrt{\sin x} - \frac{2\sqrt{\sin^5 x}}{5} + c$$

Alt:

$$\boxed{\begin{aligned} u^2 &= \sin x \\ 2u\,du &= \cos x\,dx \end{aligned}}$$

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$$
$$= \int \frac{\cos^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \frac{1-\sin^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \frac{1-u^4}{u} \times 2u\,du$$
$$= 2\int (1-u^4)\,du$$
$$= 2u - \frac{2u^5}{5} + c$$
$$= 2\sqrt{\sin x} - \frac{2\sqrt{\sin^5 x}}{5} + c$$

Question 22

$$\int \frac{x\sin\left(\sqrt{2x^2-1}\right)}{\sqrt{2x^2-1}}\,dx$$
$$= \frac{1}{2}\int \left(\frac{1}{2}\left(2x^2-1\right)^{-\frac{1}{2}} \times 4x\right)\sin\left(\sqrt{2x^2-1}\right)dx$$
$$= -\frac{1}{2}\cos\sqrt{2x^2-1} + c$$

Alt:

$$\boxed{\begin{aligned} u &= \left(2x^2-1\right)^{\frac{1}{2}} \\ du &= \frac{1}{2}\left(2x^2-1\right)^{-\frac{1}{2}}(4x)\,dx \\ &= \frac{2x}{\sqrt{2x^2-1}}\,dx \end{aligned}}$$

$$\int \frac{x\sin\left(\sqrt{2x^2-1}\right)}{\sqrt{2x^2-1}}\,dx$$
$$= \frac{1}{2}\int \sin\left(\sqrt{2x^2-1}\right) \times \frac{2x}{\sqrt{2x^2-1}}\,dx$$
$$= \frac{1}{2}\int \sin u\,du$$
$$= -\frac{1}{2}\cos u + c$$
$$= -\frac{1}{2}\cos\sqrt{2x^2-1} + c$$

Question 23

$$\int \frac{e^{\sqrt{\sin x}}}{\sec x\sqrt{\sin x}}\,dx$$
$$= 2\int \frac{\cos x}{2\sqrt{\sin x}}e^{\sqrt{\sin x}}\,dx$$
$$= 2e^{\sqrt{\sin x}} + c$$

Alt:

$$\int \frac{e^{\sqrt{\sin x}}}{\sec x\sqrt{\sin x}}\,dx \qquad \boxed{\begin{aligned} u^2 &= \sin x \\ 2u\,du &= \cos x\,dx \end{aligned}}$$
$$= \int \frac{e^{\sqrt{\sin x}}}{\sqrt{\sin x}} \times \cos x\,dx$$
$$= \int \frac{e^u}{u} \times 2u\,du$$
$$= 2\int e^u\,du$$
$$= 2e^u + c$$
$$= 2e^{\sqrt{\sin x}} + c$$

Question 24

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$$
$$= \int \frac{1}{x^3\left(\frac{1}{x^2}+1\right)^{\frac{3}{2}}}\,dx$$
$$= \int \left(x^{-2}+1\right)^{-\frac{3}{2}}\left(x^{-3}\right)dx$$
$$= \left(x^{-2}+1\right)^{-\frac{1}{2}} + c$$
$$= \frac{1}{\sqrt{\frac{1}{x^2}+1}} + c$$
$$= \frac{x}{\sqrt{1+x^2}} + c$$

Question 25

$$\int_1^{e^{\frac{\pi}{2}}} \frac{\sin(\ln|x|)}{x}\,dx$$
$$= \int_1^{e^{\frac{\pi}{2}}} \frac{1}{x}\sin(\ln|x|)\,dx$$
$$= -\Big[\cos(\ln|x|)\Big]_1^{e^{\frac{\pi}{2}}}$$
$$= -\left(\cos\frac{\pi}{2} - \cos 0\right)$$
$$= -(0-1)$$
$$= 1$$

Alt:

$$\int_1^{e^{\frac{\pi}{2}}} \frac{\sin(\ln|x|)}{x}\,dx \qquad \boxed{\begin{aligned} u &= \ln|x| \\ du &= \frac{dx}{x} \end{aligned}}$$
$$= \int_1^{e^{\frac{\pi}{2}}} \sin(\ln|x|) \times \frac{dx}{x}$$
$$= \int_0^{\frac{\pi}{2}} \sin u\,du$$
$$= -\Big[\cos u\Big]_0^{\frac{\pi}{2}}$$
$$= -\left(\cos\frac{\pi}{2} - \cos 0\right)$$
$$= -(0-1)$$
$$= 1$$

Question 26

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{2e^{\frac{2}{x^2}}}{x^3}\,dx$$
$$= -\frac{1}{2}\int_{\sqrt{2}}^{\sqrt{3}} \left(-4x^{-3}\right)e^{2x^{-2}}\,dx$$
$$= -\frac{1}{2}\left[e^{\frac{2}{x^2}}\right]_{\sqrt{2}}^{\sqrt{3}}$$
$$= -\frac{1}{2}\left(e^{\frac{2}{3}} - e\right)$$
$$= \frac{e - e^{\frac{2}{3}}}{2}$$

Alt:

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{2e^{\frac{2}{x^2}}}{x^3}\,dx \qquad \boxed{\begin{aligned} u &= e^{\frac{2}{x^2}} \\ &= e^{2x^{-2}} \\ \frac{du}{dx} &= -4x^{-3}e^{\frac{2}{x^2}} \\ -\frac{du}{4} &= \frac{e^{\frac{2}{x^2}}}{x^3}\,dx \end{aligned}}$$
$$= 2\int_e^{e^{\frac{2}{3}}} -\frac{du}{4}$$
$$= -\frac{1}{2}\int_e^{e^{\frac{2}{3}}} du$$
$$= -\frac{1}{2}\Big[u\Big]_e^{e^{\frac{2}{3}}}$$
$$= -\frac{1}{2}\left(e^{\frac{2}{3}} - e\right)$$
$$= \frac{e - e^{\frac{2}{3}}}{2}$$

Question 27

$\frac{d}{dx}\left(e^{\sqrt{2x^2-1}}\right)=\frac{1}{2}\left(2x^2-1\right)^{-\frac{1}{2}}(4x)e^{\sqrt{2x^2-1}}$

$=\frac{2xe^{\sqrt{2x^2-1}}}{\sqrt{2x^2-1}}$

$\int\frac{xe^{\sqrt{2x^2-1}}\sin\left(e^{\sqrt{2x^2-1}}\right)}{\sqrt{2x^2-1}}dx$

$=\frac{1}{2}\int\frac{2xe^{\sqrt{2x^2-1}}}{\sqrt{2x^2-1}}\sin\left(e^{\sqrt{2x^2-1}}\right)dx$

$=-\frac{1}{2}\cos\left(e^{\sqrt{2x^2-1}}\right)+c$

Question 28a

$\int\sin^3 2x\cos 2x\,dx$

$=\frac{1}{2}\int(2\cos 2x)(\sin 2x)^3\,dx$

$=\frac{1}{2}\times\frac{(\sin 2x)^4}{4}+c$

$=\frac{\sin^4 2x}{8}+c$

Alt:

$\int\sin^3 2x\cos 2x\,dx\quad\boxed{\begin{array}{c}u=\sin 2x\\du=2\cos 2x\,dx\end{array}}$

$=\frac{1}{2}\int\sin^3 2x\times 2\cos 2x\,dx$

$=\frac{1}{2}\int u^3\,du$

$=\frac{1}{2}\times\frac{u^4}{4}+c$

$=\frac{1}{8}\sin^4 2x+c$

Question 28b

$\int\sin^3 2x\cos 2x\,dx$

$=\int\sin 2x\left(1-\cos^2 2x\right)\cos 2x\,dx$

$=\int\sin 2x\cos 2x\,dx-\int\sin 2x\cos^3 2x\,dx$

$=\frac{1}{2}\int\sin 4x\,dx+\frac{1}{2}\int\left(-2\sin 2x\cos^3 2x\right)dx$

$=-\frac{1}{2}\times\frac{\cos 4x}{4}+\frac{1}{2}\times\frac{\cos^4 2x}{4}+c$

$=\frac{\cos^4 2x}{8}-\frac{\cos 4x}{8}+c$

Question 28c

$\int\sin^3 2x\cos 2x\,dx$

$=\int\sin 2x\times\sin^2 2x\times\cos 2x\,dx$

$=\int\sin 2x\times(1-\cos^2 2x)\times\cos 2x\,dx$

$=\int\left(\sin 2x\cos 2x-\sin 2x\cos^3 2x\right)dx$

$=\frac{1}{2}\int(2\cos 2x\sin 2x)\,dx$

$+\frac{1}{2}\int\left(-2\sin 2x\cos^3 2x\right)dx$

$=\frac{1}{2}\times\frac{\sin^2 2x}{2}+\frac{1}{2}\times\frac{\cos^4 2x}{4}+c$

$=\frac{\cos^4 2x}{8}+\frac{\sin^2 2x}{4}+c$

Question 28d

$\int\sin^3 2x\cos 2x\,dx$

$=\int\sin^2 2x\sin 2x\cos 2x\,dx$

$=\int\left(1-\cos^2 2x\right)\sin 2x\cos 2x\,dx$

$=\int\left(\sin 2x\cos 2x-\sin 2x\cos^3 2x\right)dx$

$=\frac{1}{2}\int\left(2\sin 2x\cos 2x-2\sin 2x\cos^3 2x\right)dx$

$=-\frac{1}{2}\times\frac{\cos^2 2x}{2}+\frac{1}{2}\times\frac{\cos^4 2x}{4}+c$

$=\frac{\cos^4 2x}{8}-\frac{\cos^2 2x}{4}+c$

Alt:

$\int\sin^3 2x\cos 2x\,dx\quad\boxed{\begin{array}{c}u=\cos 2x\\du=-2\sin 2x\,dx\end{array}}$

$=-\frac{1}{2}\int\sin^2 2x\cos 2x\times(-2\sin 2x\,dx)$

$=-\frac{1}{2}\int(1-\cos^2 2x)\cos 2x\times(-2\sin 2x\,dx)$

$=-\frac{1}{2}\int\left(1-u^2\right)u\,du$

$=-\frac{1}{2}\int\left(u-u^3\right)du$

$=-\frac{1}{2}\times\frac{u^2}{2}+\frac{1}{2}\times\frac{u^4}{4}+c$

$=\frac{u^4}{8}-\frac{u^2}{4}+c$

$=\frac{\cos^4 2x}{8}-\frac{\cos^2 2x}{4}+c$

Question 28e

$$\int \sin^3 2x \cos 2x \, dx$$
$$= \int \sin^2 2x \times \sin 2x \cos 2x \, dx$$
$$= \frac{1}{2}\int \sin^2 2x \sin 4x \, dx$$
$$= \frac{1}{2}\int \frac{1}{2}(1 - \cos 4x) \sin 4x \, dx$$
$$= \frac{1}{4}\int (\sin 4x - \sin 4x \cos 4x) \, dx$$
$$= \frac{1}{4}\int \sin 4x \, dx - \frac{1}{8}\int \sin 8x \, dx$$
$$= -\frac{1}{4} \times \frac{\cos 4x}{4} + \frac{1}{8} \times \frac{\cos 8x}{8} + c$$
$$= -\frac{\cos 4x}{16} + \frac{\cos 8x}{64} + c$$

Question 28f

$$\int \sin^3 2x \cos 2x \, dx$$
$$= \int \sin^2 2x \sin 2x \cos 2x \, dx$$
$$= \frac{1}{2}\int \sin^2 2x \sin 4x \, dx$$
$$= \frac{1}{2}\int \frac{1}{2}(1 - \cos 4x) \sin 4x \, dx$$
$$= \frac{1}{4}\int (\sin 4x - \sin 4x \cos 4x) \, dx$$
$$= \frac{1}{4}\int \sin 4x \, dx - \frac{1}{4}\int 4 \cos 4x \sin 4x \, dx$$
$$= -\frac{1}{4} \times \frac{\cos 4x}{4} - \frac{1}{4} \times \frac{\sin^2 4x}{4} + c$$
$$= -\frac{\cos 4x}{16} - \frac{\sin^2 4x}{16} + c$$

Question 1

$$\int \frac{x^3}{\sqrt{1+x^2}}dx \qquad \boxed{\begin{aligned} u^2 &= 1+x^2 \\ 2u\,du &= 2x\,dx \end{aligned}}$$
$$= \frac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}} \times 2x\,dx$$
$$= \frac{1}{2}\int \frac{u^2-1}{u} \times 2u\,du$$
$$= \int (u^2-1)\,du$$
$$= \frac{u^3}{3} - u + c$$
$$= \frac{\sqrt{(1+x^2)^3}}{3} - \sqrt{1+x^2} + c$$

Question 2

$$\int x^3\sqrt{x^2+1}\,dx \qquad \boxed{\begin{aligned} u^2 &= x^2+1 \\ 2u\,du &= 2x\,dx \end{aligned}}$$
$$= \frac{1}{2}\int x^2\sqrt{x^2+1} \times 2x\,dx$$
$$= \frac{1}{2}\int (u^2-1) \times u \times 2u\,du$$
$$= \int (u^4-u^2)\,du$$
$$= \frac{u^5}{5} - \frac{u^3}{3} + c$$
$$= \frac{\sqrt{(x^2+1)^5}}{5} - \frac{\sqrt{(x^2+1)^3}}{3} + c$$

Question 3

$$\int x^5\sqrt{x^2+1}\,dx \qquad \boxed{\begin{aligned} u^2 &= x^2+1 \\ 2u\,du &= 2x\,dx \end{aligned}}$$
$$= \frac{1}{2}\int x^4\sqrt{x^2+1} \times 2x\,dx$$
$$= \frac{1}{2}\int (u^2-1)^2 \times u \times 2u\,du$$
$$= \int (u^4-2u^2+1)u^2\,du$$
$$= \int (u^6-2u^4+u^2)\,du$$
$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + c$$
$$= \frac{\sqrt{(x^2+1)^7}}{7} - \frac{2\sqrt{(x^2+1)^5}}{5} + \frac{\sqrt{(x^2+1)^3}}{3} + c$$

Question 4

$$\int_0^1 x^3\sqrt{1-x^2}\,dx \qquad \boxed{\begin{aligned} u^2 &= 1-x^2 \\ 2u\,du &= -2x\,dx \end{aligned}}$$
$$= -\frac{1}{2}\int_0^1 x^2\sqrt{1-x^2} \times (-2x)dx$$
$$= -\frac{1}{2}\int_1^0 (1-u^2)u \times 2u\,du$$
$$= \int_0^1 (u^2-u^4)\,du$$
$$= \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1$$
$$= \left(\left(\frac{1}{3}-\frac{1}{5}\right) - (0-0)\right)$$
$$= \frac{2}{15}$$

Question 5

$$\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\,dx \qquad \boxed{\begin{aligned} u^2 &= 1-x \\ 2u\,du &= -dx \end{aligned}}$$
$$= \int_1^{\frac{1}{2}} \frac{1-u^2}{u}(-2udu)$$
$$= 2\int_{\frac{1}{2}}^1 (1-u^2)\,du$$
$$= 2\left[u - \frac{u^3}{3}\right]_{\frac{1}{2}}^1$$
$$= 2\left(\left(1-\frac{1}{3}\right) - \left(\frac{1}{2}-\frac{1}{24}\right)\right)$$
$$= \frac{5}{12}$$

Question 6

$$\int_0^3 x\sqrt{x+1}\,dx \qquad \boxed{\begin{aligned} u^2 &= x+1 \\ 2u\,du &= dx \end{aligned}}$$
$$= \int_1^2 (u^2-1)u \times 2udu$$
$$= 2\int_1^2 (u^4-u^2)du$$
$$= 2\left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^2$$
$$= 2\left(\left(\frac{2^5}{5}-\frac{2^3}{3}\right) - \left(\frac{1^5}{5}-\frac{1^3}{3}\right)\right)$$
$$= \frac{116}{15}$$

Question 7

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx \qquad \boxed{\begin{aligned} u^2 &= e^x+1 \\ 2u\,du &= e^x dx \end{aligned}}$$

$$= \int \frac{e^x}{\sqrt{e^x+1}} \times e^x dx$$

$$= \int \frac{u^2-1}{u} \times 2u\,du$$

$$= 2\int \left(u^2-1\right) du$$

$$= \frac{2u^3}{3} - 2u + c$$

$$= \frac{2\sqrt{(e^x+1)^3}}{3} - 2\sqrt{e^x+1} + c$$

Question 8

$$\int \frac{1}{2+\sqrt{x}}\,dx \qquad \boxed{\begin{aligned} u^2 &= x \\ 2u\,du &= dx \end{aligned}}$$

$$= \int \frac{1}{2+u} \times 2u\,du$$

$$= 2\int \frac{u}{u+2}\,du$$

$$= 2\int \frac{u+2-2}{u+2}\,du$$

$$= 2\int \left(1 - \frac{2}{u+2}\right) du$$

$$= 2u - 4\ln\left|u+2\right| + c$$

$$= 2\sqrt{x} - 4\ln\left|2+\sqrt{x}\right| + c$$

Question 9

$$\int x^5\sqrt{2-x^3}\,dx \qquad \boxed{\begin{aligned} u^2 &= 2-x^3 \\ 2u\,du &= -3x^2 dx \end{aligned}}$$

$$= -\frac{1}{3}\int x^3\sqrt{2-x^3} \times \left(-3x^2\right) dx$$

$$= -\frac{1}{3}\int \left(2-u^2\right) \times u \times 2u\,du$$

$$= -\frac{2}{3}\int \left(2-u^2\right)u^2\,du$$

$$= -\frac{2}{3}\int \left(2u^2-u^4\right) du$$

$$= -\frac{2}{3}\left(\frac{2u^3}{3} - \frac{u^5}{5}\right) + c$$

$$= -\frac{4\sqrt{(2-x^3)^3}}{9} + \frac{2\sqrt{(2-x^3)^5}}{15} + c$$

Question 10

$$\int \frac{2x^3}{(x^2-1)^2}\,dx \qquad \boxed{\begin{aligned} u &= x^2-1 \\ du &= 2x\,dx \end{aligned}}$$

$$= \int \frac{x^2}{(x^2-1)^2} \times 2x\,dx$$

$$= \int \frac{u+1}{u^2}\,du$$

$$= \int \left(\frac{1}{u} + u^{-2}\right) du$$

$$= \ln\left|u\right| - \frac{1}{u} + c$$

$$= \ln\left|x^2-1\right| - \frac{1}{x^2-1} + c$$

Question 11

$$\int \frac{x^3}{(x^2+1)^2}\,dx \qquad \boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{x^2}{(x^2+1)^2} \times 2x\,dx$$

$$= \frac{1}{2}\int \frac{u-1}{u^2}\,du$$

$$= \frac{1}{2}\int \left(\frac{1}{u} - u^{-2}\right) du$$

$$= \frac{1}{2}\ln\left|u\right| + \frac{1}{2u} + c$$

$$= \frac{1}{2}\ln\left|x^2+1\right| + \frac{1}{2(x^2+1)} + c$$

Question 12

$$\int \frac{e^{2x}}{e^x+1}\,dx \qquad \boxed{\begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned}}$$

$$= \int \frac{e^x}{e^x+1} \times e^x dx$$

$$= \int \frac{u}{u+1} \times du$$

$$= \int \left(\frac{u+1-1}{u+1}\right) du$$

$$= \int \left(1 - \frac{1}{u+1}\right) du$$

$$= u - \ln\left|u+1\right| + c$$

$$= e^x - \ln\left|e^x+1\right| + c$$

Question 13

$$\int_1^2 \frac{dx}{x\sqrt{2x-1}} \qquad \boxed{\begin{aligned} u^2 &= 2x-1 \\ 2u\,du &= 2dx \end{aligned}}$$

$$= \frac{1}{2}\int_1^2 \frac{1}{x\sqrt{2x-1}} \times 2dx$$

$$= \frac{1}{2}\int_1^{\sqrt{3}} \frac{1}{\frac{u^2+1}{2} \times u} \times 2u\,du$$

$$= 2\int_1^{\sqrt{3}} \frac{1}{u^2+1}\,du$$

$$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{6}$$

Question 14

$$\int \frac{x}{\sqrt{x+5}}\,dx \qquad \boxed{\begin{aligned} u^2 &= x+5 \\ 2u\,du &= dx \end{aligned}}$$

$$= \int \frac{u^2-5}{u} \times 2u\,du$$

$$= 2\int (u^2-5)\,du$$

$$= \frac{2u^3}{3} - 10u + c$$

$$= \frac{2\sqrt{(x+5)^3}}{3} - 10\sqrt{x+5} + c$$

$$= \frac{2(x+5)\sqrt{x+5}}{3} - 10\sqrt{x+5} + c$$

$$= \frac{2}{3}\sqrt{x+5}(x+5-15) + c$$

$$= \frac{2}{3}(x-10)\sqrt{x+5} + c$$

Question 15

$$d = \int_0^{0.5} t\sqrt{\frac{1+2t^2}{1-2t^2}}\,dx \qquad \boxed{\begin{aligned} u &= t^2 \\ du &= 2t\,dt \end{aligned}}$$

$$= \frac{1}{2}\int_0^{0.5} \sqrt{\frac{1+2t^2}{1-2t^2}} \times 2t\,dt$$

$$= \frac{1}{2}\int_0^{0.25} \sqrt{\frac{1+2u}{1-2u}}\,du$$

$$= \frac{1}{2}\int_0^{0.25} \sqrt{\frac{1+2u}{1-2u}} \times \sqrt{\frac{1+2u}{1+2u}}\,du$$

$$= \frac{1}{2}\int_0^{0.25} \frac{1+2u}{\sqrt{1-4u^2}}\,du$$

$$= \frac{1}{2}\int_0^{0.25} \frac{du}{\sqrt{1-4u^2}} + \int_0^{0.25} \frac{u}{\sqrt{1-4u^2}}\,du$$

$$= \frac{1}{4}\int_0^{0.25} \frac{2}{\sqrt{1-(2u)^2}}\,du$$

$$-\frac{1}{8}\int_0^{0.25} (-8u)\left(1-4u^2\right)^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\left[\sin^{-1}(2u)\right]_0^{0.25} - \frac{1}{8}\left[2\left(1-4u^2\right)^{\frac{1}{2}}\right]_0^{0.25}$$

$$= \frac{1}{4}\left(\frac{\pi}{6} - 0\right) - \frac{1}{8}\left(2\sqrt{\frac{3}{4}} - 2\right)$$

$$= \frac{\pi}{24} - \frac{\sqrt{3}}{8} + \frac{1}{4} \text{ metres}$$

$= 164$ mm (nearest millimetre)

Question 16

$$\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\,dx \qquad \boxed{\begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{(1+x^2)^{\frac{3}{2}}} \times 2x\,dx$$

$$= \frac{1}{2}\int_1^4 \frac{u-1}{u^{\frac{3}{2}}}\,du$$

$$= \frac{1}{2}\int_1^4 \left(u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\right)du$$

$$= \frac{1}{2}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]_1^4$$

$$= \frac{1}{2}\big((4+1)-(2+2)\big)$$

$$= \frac{1}{2}$$

Question 17a

$$\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx \qquad \boxed{\begin{aligned} u^2 &= x \\ 2u\,du &= dx \end{aligned}}$$

$$= \int \frac{u}{1-u} \times 2u\;du$$

$$= 2\int \frac{u^2}{1-u}\,du$$

$$= 2\int \frac{-u(1-u)-(1-u)+1}{1-u}\,du$$

$$= 2\int \left(-u-1+\frac{1}{1-u}\right)du$$

$$= -u^2 - 2u - 2\ln|1-u| + c$$

$$= -x - 2\sqrt{x} - 2\ln\left|1-\sqrt{x}\right| + c$$

Question 17b

$$\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx \qquad \boxed{\begin{aligned} u &= 1-x^{\frac{1}{2}} \\ du &= -\frac{1}{2}x^{-\frac{1}{2}} \\ du &= -\frac{du}{2\sqrt{x}} \end{aligned}}$$

$$= \int \frac{1-u}{u} \times \big(-2(1-u)\big)\,du$$

$$= -2\int \frac{(1-u)^2}{u}\,du$$

$$= -2\int \left(\frac{1}{u} - 2 + u\right)du$$

$$= -2\ln|u| + 4u - u^2 + c$$

$$= -2\ln\left|1-\sqrt{x}\right| + 4\left(1-\sqrt{x}\right) - \left(1-\sqrt{x}\right)^2 + c$$

$$= -2\ln\left|1-\sqrt{x}\right| + 4 - 4\sqrt{x} - 1 + 2\sqrt{x} - x + c$$

$$= -x - 2\sqrt{x} - 2\ln\left|1-\sqrt{x}\right| + c$$

Question 18

$$\int \frac{dx}{\sqrt{e^{2x}-1}} \qquad \boxed{\begin{aligned} u &= \frac{1}{e^x} \Leftrightarrow e^x = \frac{1}{u} \\ &= e^{-x} \\ du &= -e^{-x}dx \\ dx &= -e^x du = -\frac{1}{u}du \end{aligned}}$$

$$= \int \frac{-\frac{1}{u}}{\sqrt{\left(\frac{1}{u}\right)^2 - 1}}\,du$$

$$= -\int \frac{du}{u\sqrt{\left(\frac{1}{u}\right)^2 - 1}}$$

$$= -\int \frac{du}{\sqrt{1-u^2}}$$

$$= \cos^{-1} u + c$$

$$= \cos^{-1}\left(\frac{1}{e^x}\right) + c$$

Question 19

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx \qquad \boxed{\begin{aligned} u &= -x \\ du &= -dx \\ dx &= -du \end{aligned}}$$

$$= \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\sin^2(-u)}{1+2^{-u}}(-du)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(u)}{1+2^{-u}} \times \frac{2^u}{2^u}\,du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^u \sin^2(u)}{2^u+1}\,du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^x \sin^2 x}{1+2^x}\,dx \quad \text{change of variable}$$

$$\therefore 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^x \sin^2 x}{1+2^x}\,dx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+2^x)\sin^2 x}{1+2^x}\,dx$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\left(\frac{\pi}{2}-0\right) - \left(-\frac{\pi}{2}-0\right)\right)$$

$$= \frac{\pi}{4}$$

Question 20

$$\int \sqrt{\frac{x}{x+1}}\,dx$$
$$= \int \sqrt{\frac{x}{x+1}} \times \sqrt{\frac{x}{x}}\,dx$$
$$= \int \frac{x}{\sqrt{x^2+x}}\,dx$$
$$= \int \frac{x}{\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}}}\,dx$$
$$= \int \frac{x}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$
$$= \int \frac{x+\frac{1}{2}-\frac{1}{2}}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$
$$= \int \frac{x+\frac{1}{2}}{\sqrt{x^2+x}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$
$$= \frac{1}{2}\int (2x+1)\left(x^2+x\right)^{-\frac{1}{2}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$
$$= \frac{1}{2} \times 2\left(x^2+x\right)^{\frac{1}{2}} - \frac{1}{2}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$$
$$= \sqrt{x^2+x} - \frac{1}{2}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$$
$$= \sqrt{x^2+x} - \frac{1}{2}\ln\left|\left(\sqrt{\frac{x}{2}}+\sqrt{\frac{x+1}{2}}\right)^2\right| + c$$
$$= \sqrt{x^2+x} - \ln\left|\sqrt{\frac{x}{2}}+\sqrt{\frac{x+1}{2}}\right| + c$$
$$= \sqrt{x^2+x} - \ln\left(\sqrt{x}+\sqrt{x+1}\right) + \ln\sqrt{2} + c$$
$$= \sqrt{x^2+x} - \ln(\sqrt{x}+\sqrt{x+1}) \ + c$$

Question 1

$$\int \frac{x^2-2x-2}{x+1}\,dx$$
$$= \int \frac{x(x+1)-3(x+1)+1}{x+1}\,dx$$
$$= \int \left(x-3+\frac{1}{x+1}\right)dx$$
$$= \frac{x^2}{2}-3x+\ln\left|x+1\right|+c$$

Question 2

$$\int_1^3 \frac{2x-3}{x+1}\,dx$$
$$= \int_1^3 \frac{2(x+1)-5}{x+1}\,dx$$
$$= \int_1^3 \left(2-\frac{5}{x+1}\right)dx$$
$$= \left[2x-5\ln\left|x+1\right|\right]_1^3$$
$$= (6-5\ln 4)-(2-5\ln 2)$$
$$= 6-5\ln 4-2+5\ln 2$$
$$= 4-10\ln 2+5\ln 2$$
$$= 4-5\ln 2$$

Question 3

$$\int \frac{2x^2+19}{x^2+9}\,dx$$
$$= \int \frac{2(x^2+9)+1}{x^2+9}\,dx$$
$$= \int \left(2+\frac{1}{x^2+3^2}\right)dx$$
$$= 2x+\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$$

Question 4

$$\int \frac{3x+2}{x^2+6x+10}\,dx$$
$$= \int \frac{\frac{3}{2}(2x+6)-7}{x^2+6x+9+1}\,dx$$
$$= \frac{3}{2}\int \frac{2x+6}{x^2+6x+10}\,dx-7\int \frac{1}{(x+3)^2+1^2}\,dx$$
$$= \frac{3}{2}\ln\left|x^2+6x+10\right|-7\tan^{-1}\left(x+3\right)+c$$

Question 5

$$\int \frac{2}{x(x-1)}\,dx$$
$$= 2\int \left(\frac{1}{x-1}-\frac{1}{x}\right)dx$$
$$= 2\ln\left|x-1\right|-2\ln\left|x\right|+c$$

Question 6

$$\int \frac{2}{x(x^2+4)}\,dx$$
$$= \frac{1}{2}\int \left(\frac{1}{x}-\frac{x}{x^2+4}\right)dx$$
$$= \frac{1}{2}\int \left(\frac{1}{x}-\frac{1}{2}\times\frac{2x}{x^2+4}\right)dx$$
$$= \frac{1}{2}\ln\left|x\right|-\frac{1}{4}\ln\left|x^2+4\right|+c$$

Question 7

$$\int_{-2}^{2} \frac{2x}{x^2+4x+20}\,dx$$
$$= \int_{-2}^{2} \frac{2x+4-4}{x^2+4x+4+16}\,dx$$
$$= \int_{-2}^{2} \frac{2x+4}{x^2+4x+20}\,dx-4\int_{-2}^{2}\frac{dx}{(x+2)^2+4^2}$$
$$= \left[\ln\left|x^2+4x+20\right|\right]_{-2}^{2}$$
$$-4\times\frac{1}{4}\left[\tan^{-1}\left(\frac{x+2}{4}\right)\right]_{-2}^{2}$$
$$= \left(\ln(4+8+20)-\ln(4-8+20)\right)$$
$$-\left(\tan^{-1}(1)-\tan^{-1}0\right)$$
$$= \ln 32-\ln 16-\frac{\pi}{4}+0$$
$$= \ln 2-\frac{\pi}{4}$$

Question 8

$$\int \frac{e^{2x}+e^x}{e^{2x}+1}\,dx$$
$$= \int \left(\frac{1}{2}\times\frac{2e^{2x}}{e^{2x}+1}+\frac{e^x}{(e^x)^2+1}\right)dx$$
$$= \frac{1}{2}\ln\left|e^{2x}+1\right|+\tan^{-1}(e^x)+c$$

Question 9

$$\int \frac{x^4+1}{x^2+2}\,dx$$
$$= \int \frac{x^2\left(x^2+2\right)-2\left(x^2+2\right)+5}{x^2+2}\,dx$$
$$= \int \left(x^2-2+\frac{5}{x^2+2}\right)dx$$
$$= x^3-2x+\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$$

Question 10

$$\int_0^2 \frac{x^3}{x^2+2}\,dx$$
$$= \int_0^2 \frac{x\left(x^2+2\right)-2x}{x^2+2}\,dx$$
$$= \int_0^2 \left(x - \frac{2x}{x^2+2}\right)dx$$
$$= \left[\frac{x^2}{2} - \ln\left|x^2+2\right|\ \right]_0^2$$
$$= (2 - \ln 6) - (0 - \ln 2)$$
$$= 2 - \ln 6 + \ln 2$$
$$= 2 - \ln 3$$

Question 11

$$\int \frac{x^6+3x^2-1}{x^3+1}\,dx$$
$$= \int \frac{x^3\left(x^3+1\right)-\left(x^3+1\right)+3x^2}{x^3+1}\,dx$$
$$= \int \left(x^3 - 1 + \frac{3x^2}{x^3+1}\right)dx$$
$$= \frac{x^4}{4} - x + \ln\left|x^3+1\right| + c$$

Question 12

$$\int \frac{x^2-4x+2}{(x-2)^3}\,dx$$
$$= \int \frac{(x-2)^2-2}{(x-2)^3}\,dx$$
$$= \int \left(\frac{1}{x-2} - 2(x-2)^{-3}\right)du$$
$$= \ln\left|x-2\right| - 2\times\left(-\frac{1}{2}\right)(x+1)^{-2} + c$$
$$= \ln\left|x-2\right| + \frac{1}{(x-2)^2} + c$$

Question 13

$$\int_0^1 \frac{e^{2x}-e^x}{e^x+1}\,dx$$
$$= \int_0^1 \frac{\left(e^x(e^x+1)-2e^x\right)}{e^x+1}\,dx$$
$$= \int_0^1 \left(e^x - \frac{2e^x}{e^x+1}\right)dx$$
$$= \left[e^x - 2\ln\left|e^x+1\right|\ \right]_0^1$$
$$= \left(e - 2\ln(e+1)\right) - \left(1 - 2\ln 2\right)$$
$$= e - 2\ln(e+1) - 2 + 2\ln 2$$
$$= \ln(e^e) - \ln\left((e+1)^2\right) - \ln\left(e^2\right) + \ln 4$$
$$= \ln\left(\frac{4e^e}{[e(e+1)]^2}\right)$$

Question 14

$$V = \pi\int_0^{\ln 2} \left(\frac{e^{2x}}{\sqrt{e^{2x}+1}}\right)^2 dx$$
$$= \pi\int_0^{\ln 2} \frac{e^{4x}}{e^{2x}+1}\,dx$$
$$= \pi\int_0^{\ln 2} \frac{e^{2x}\left(e^{2x}+1\right)-\left(e^{2x}+1\right)+1}{e^{2x}+1}\,dx$$
$$= \pi\int_0^{\ln 2} \left(e^{2x} - 1 + \frac{1}{e^{2x}+1}\right)dx$$
$$= \pi\int_0^{\ln 2} \left(e^{2x} - 1 + \frac{e^{-2x}}{1+e^{-2x}}\right)dx$$
$$= \pi\left[\frac{1}{2}e^{2x} - x - \frac{1}{2}\ln(1+e^{-2x})\right]_0^{\ln 2}$$
$$= \pi\left(\left(\frac{1}{2}e^{2\ln 2} - 1 - \frac{1}{2}\ln\left(1+e^{-2\ln 2}\right)\right)\right.$$
$$\left.- \left(\frac{1}{2} - 0 - \frac{1}{2}\ln 2\right)\right)$$
$= 2.309076\ldots$

$= 2.31$ units^3 (2 dp)

Question 15

$$\int \frac{\cos x}{\sin^2 x - 3\sin x + 2}\,dx$$
$$= \int \frac{\cos x}{(\sin x - 2)(\sin x - 1)}\,dx$$
$$= \int \left(\frac{\cos x}{\sin x - 2} - \frac{\cos x}{\sin x - 1}\right)dx$$
$$= \ln\left|\sin x - 2\right| - \ln\left|\sin x - 1\right| + c$$
$$= \ln\left|\frac{\sin x - 2}{\sin x - 1}\right| + c$$

Question 16

$$\int \frac{8x}{1+e^2x}\,dx$$
$$= \int \frac{\frac{8}{e^2}\left(1+e^2x\right) - \frac{8}{e^2}}{1+e^2x}\,dx$$
$$= \int \left(\frac{8}{e^2} - \frac{8}{e^4}\times\frac{e^2}{1+e^2x}\right)dx$$
$$= \frac{8x}{e^2} - \frac{8\ln\left|1+e^2x\right|}{e^4} + c$$

Question 17

$$\int \frac{\sec x \tan x}{\sec x + \sec^2 x}\,dx \qquad \boxed{\begin{aligned} u &= \sec x \\ du &= \sec x \tan x\,dx \end{aligned}}$$
$$= \int \frac{1}{\sec x + \sec^2 x} \times \sec x \tan x\,dx$$
$$= \int \frac{1}{u + u^2}\,du$$
$$= \int \frac{1}{u(1+u)}\,du$$
$$= \int \left(\frac{1}{u} - \frac{1}{1+u}\right) du$$
$$= \ln|u| - \ln|1+u| + c$$
$$= \ln\left|\frac{u}{1+u}\right| + c$$
$$= \ln\left|\frac{\sec x}{1+\sec x}\right| + c$$
$$= \ln\left|\frac{1}{\cos x + 1}\right| + c$$
$$= -\ln\left|\cos x + 1\right| + c$$

Question 18

$$\int \frac{\tan^2 x + \tan x + \sec^2 x - 2}{\tan x - 1}\,dx$$
$$= \int \frac{\tan x(\tan x - 1) + 2(\tan x - 1) + \sec^2 x}{\tan x - 1}\,dx$$
$$= \int \left(\tan x + 2 + \frac{\sec^2 x}{\tan x - 1}\right) dx$$
$$= -\ln|\cos x| + 2x + \ln|\tan x - 1| + c$$
$$= \ln\left|\frac{\tan x - 1}{\cos x}\right| + 2x + c$$

Question 19a

$$I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx \qquad \boxed{\begin{aligned} u &= 4 - x \\ du &= -dx \end{aligned}}$$
$$= -\int_3^1 \frac{\cos^2\left(\frac{\pi}{8}(4-u)\right)}{(4-u)u}\,du$$
$$= \int_1^3 \frac{\cos^2\left(\frac{\pi}{2} - \frac{\pi}{8}u\right)}{u(4-u)}\,du$$
$$= \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

Question 19b

$$2I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx + \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$$
$$= \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx + \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$$
$$= \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right) + \sin^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$$
$$= \int_1^3 \frac{1}{x(4-x)}\,dx$$
$$= \frac{1}{4}\int_1^3 \left(\frac{1}{x} + \frac{1}{4-x}\right) dx$$
$$\therefore I = \frac{1}{8}\Big[\ln x - \ln|4-x|\Big]_1^3$$
$$= \frac{1}{8}(\ln 3 - \ln 1 - \ln 1 + \ln 3)$$
$$= \frac{\ln 3}{4}$$

Question 20

$$\int \frac{\sin x}{\sin x + \cos x + e^x}\,dx$$
$$= -\int \frac{\cos x - \sin x + e^x - \cos x - e^x}{\sin x + \cos x + e^x}\,dx$$
$$= -\int \frac{\cos x - \sin x + e^x}{\sin x + \cos x + e^x}\,dx + \int \frac{\sin x + \cos x + e^x - \sin x}{\sin x + \cos x + e^x}\,dx$$
$$= -\ln|\sin x + \cos x + e^x| + \int \left(1 - \frac{\sin x}{\sin x + \cos x + e^x}\right) dx$$
$$= -\ln|\sin x + \cos x + e^x| + x - \int \frac{\sin x}{\sin x + \cos x + e^x}\,dx$$
$$\therefore 2\int \frac{\sin x}{\sin x + \cos x + e^x}\,dx = -\ln|\sin x + \cos x + e^x| + x + c$$
$$\int \frac{\sin x}{\sin x + \cos x + e^x}\,dx = \frac{x}{2} - \frac{1}{2}\ln|\sin x + \cos x + e^x| + c$$

Question 1a

$$\frac{2x+1}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2}$$

$$\frac{2x+1}{(x+1)(x+2)} = \frac{a(x+2)+b(x+1)}{(x+1)(x+2)}$$

$$2x+1 = (a+b)x + 2a + b$$

$$\therefore 2 = a+b \quad (1)$$
$$1 = 2a+b \quad (2)$$

$$(2)-(1) \quad -1 = a \quad \rightarrow \quad a = -1$$
$$\text{sub in } (1) \quad 2 = -1 + b \quad \rightarrow \quad b = 3$$

$$\therefore \frac{2x+1}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{3}{x+2}$$

Question 1b

$$\int \frac{2x+1}{(x+1)(x+2)}dx$$
$$= \int\left(-\frac{1}{x+1} + \frac{3}{x+2}\right)dx$$
$$= -\ln\left|x+1\right| + 3\ln\left|x+2\right| + c$$

Question 2a

$$\frac{2x+3}{x^2(x+1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1}$$
$$2x+3 = ax(x+1) + b(x+1) + cx^2$$

Let $x = 0 \;\therefore 2(0)+3 = a(0) + b(0+1) + c(0)$

$\rightarrow \; 3 = b \;\rightarrow\; b = 3$

Let $x = -1 \;\therefore 2(-1)+3 = a(0) + b(0) + c(1)$

$\rightarrow \; 1 = c \;\rightarrow c = 1$

equating coefficients of x^2:

$0 = a + c \;\rightarrow\; 0 = a + 1 \;\rightarrow\; a = -1$

$$\therefore \frac{2x+3}{x^2(x+1)} = -\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}$$

Question 2b

$$\int_2^3 \frac{2x+3}{x^2(x+1)}dx$$
$$= \int_2^3\left(-\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}\right)dx$$
$$= \left[-\ln\left|x\right| - \frac{3}{x} + \ln\left|x+1\right|\right]_2^3$$
$$= \left(-\ln 3 - 1 + \ln 4\right) - \left(-\ln 2 - \frac{3}{2} - \ln 3\right)$$
$$= \ln 4 + \ln 2 - 2\ln 3 + \frac{1}{2}$$
$$= \ln\left(\frac{8}{9}\right) + \frac{1}{2}$$

Question 3

$$a = \frac{2(-1)+1}{-1+2} = -1$$
$$b = \frac{2(-2)+1}{(-2)+1} = 3$$
$$\therefore \frac{2x+1}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{3}{x+2}$$

Question 4

$$b = \frac{2(0)+3}{0+1} = 3$$
$$c = \frac{2(-1)+3}{(-1)^2} = 1$$

equating coefficients of x^2:

$0 = a + c \;\rightarrow\; 0 = a + 1 \;\rightarrow\; a = -1$

$$\therefore \frac{2x+3}{x^2(x+1)} = -\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}$$

Question 5a

$$a = \frac{1-(-1)}{(-1)+2} = 2$$
$$b = \frac{1-(-2)}{(-2)+1} = -3$$

Question 5b

$$\text{b} \int \frac{1-x}{(x+1)(x+2)}dx$$
$$= \int\left(\frac{2}{x+1} - \frac{3}{x+2}\right)dx$$
$$= 2\ln\left|x+1\right| - 3\ln\left|x+2\right| + c$$

Question 6

$$\frac{x+1}{x(x-1)^2} \equiv \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$$
$$a = \frac{(0)+1}{\left((0)-1\right)^2} = 1$$
$$c = \frac{(1)+1}{1} = 2$$

coefficients of x^2: $a + b = 0 \;\rightarrow b = -1$

$$\therefore \frac{x+1}{x(x-1)^2} \equiv \frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

Question 7

$$\frac{x+2}{x^2+7x+12} = \frac{x+2}{(x+3)(x+4)} = \frac{a}{x+3} + \frac{b}{x+4}$$
$$a = \frac{(-3)+2}{(-3)+4} = -1$$
$$b = \frac{(-4)+2}{(-4)+3} = 2$$
$$\int_0^1 \frac{x+2}{x^2+7x+12}dx$$
$$= \int_0^1\left(-\frac{1}{x+3} + \frac{2}{x+4}\right)dx$$
$$= \left[-\ln\left|x+3\right| + 2\ln\left|x+4\right|\right]_0^1$$
$$= (-\ln 4 + 2\ln 5) - (-\ln 3 + 2\ln 4)$$
$$= -\ln 4 + \ln 25 + \ln 3 - \ln 16$$
$$= \ln\frac{75}{64}$$

Question 8

$a = \dfrac{5(-2) - 5}{(-2)^2 + 1} = -3$

Let $x = i \quad bi + c = \dfrac{5i - 5}{i + 2} \times \dfrac{i - 2}{i - 2}$

$= \dfrac{-5 - 10i - 5i + 10}{i^2 - 2^2}$

$bi + c = -1 + 3i$

$\therefore c = -1, b = 3$

$\therefore \dfrac{5x - 5}{(x + 2)(x^2 + 1)} = -\dfrac{3}{x + 2} + \dfrac{3x - 1}{x^2 + 1}$

$\therefore \displaystyle\int_3^4 \frac{5x - 5}{(x + 2)(x^2 + 1)}\,dx$

$= \displaystyle\int_3^4 \left(-\frac{3}{x + 2} + \frac{3x - 1}{x^2 + 1}\right) dx$

$= \displaystyle\int_3^4 \left(-\frac{3}{x + 2} + \frac{3}{2} \times \frac{2x}{x^2 + 1} - \frac{1}{1 + x^2}\right) dx$

$= \left[-3\ln|x + 2| + \dfrac{3}{2}\ln|x^2 + 1| - \tan^{-1} x\right]_3^4$

$= \left(-3\ln 6 + \dfrac{3}{2}\ln 17 - \tan^{-1} 4\right)$

$\qquad - \left(-3\ln 5 + \dfrac{3}{2}\ln 10 - \tan^{-1} 3\right)$

$= \dfrac{3}{2}\ln\dfrac{17}{10} + 3\ln\dfrac{5}{6} + \tan^{-1} 3 - \tan^{-1} 4$

Question 9

$\displaystyle\int_{\frac{1}{2}}^2 \frac{2}{x^3 + x^2 + x + 1}\,dx$

$= \displaystyle\int_{\frac{1}{2}}^2 \left(\frac{1}{x + 1} - \frac{x}{x^2 + 1} + \frac{1}{x^2 + 1}\right) dx$

$= \left[\ln\left|x + 1\right| - \dfrac{1}{2}\ln\left|x^2 + 1\right| + \tan^{-1} x\right]_{\frac{1}{2}}^2$

$= \left(\ln 3 - \dfrac{1}{2}\ln 5 + \tan^{-1} 2\right)$

$\qquad - \left(\ln\dfrac{3}{2} - \dfrac{1}{2}\ln\dfrac{5}{4} + \tan^{-1}\left(\dfrac{1}{2}\right)\right)$

$= \ln 3 - \ln\dfrac{3}{2} - \dfrac{1}{2}\ln 5 + \dfrac{1}{2}\ln\dfrac{5}{4}$

$\qquad + \tan^{-1} 2 - \tan^{-1}\dfrac{1}{2}$

$= \ln 2 - \dfrac{1}{2}\ln 4 + \tan^{-1} 2 - \tan^{-1}\dfrac{1}{2}$

$= \tan^{-1} 2 - \tan^{-1}\dfrac{1}{2}$

Question 10

$\displaystyle\int_0^1 \frac{8(1 - x)}{(2 - x^2)(2 - 2x + x^2)}\,dx$

$= \displaystyle\int_0^1 \left(\frac{4 - 2x}{2 - 2x + x^2} - \frac{2x}{2 - x^2}\right) dx$

$= \displaystyle\int_0^1 \left(-\frac{2x - 2}{x^2 - 2x + 2} + \frac{2}{(x - 1)^2 + 1^2} - \frac{2x}{2 - x^2}\right) dx$

$= \left[-\ln\left|x^2 - 2x + 2\right| + 2\tan^{-1}(x - 1) + \ln\left|2 - x^2\right|\right]_0^1$

$= (0 + 0 + 0) - \left(-\ln 2 + 2\left(-\dfrac{\pi}{4}\right) + \ln 2\right)$

$= \dfrac{\pi}{2}$

Question 11

$\dfrac{x - 6}{x^2 + 3x - 4} = \dfrac{x - 6}{(x + 4)(x - 1)} = \dfrac{a}{x + 4} + \dfrac{b}{x - 1}$

$a = \dfrac{(-4) - 6}{(-4) - 1} = 2$

$b = \dfrac{(1) - 6}{(1) + 4} = -1$

$\therefore \displaystyle\int_2^5 \frac{x - 6}{x^2 + 3x - 4}\,dx$

$= \displaystyle\int_2^5 \left(\frac{2}{x + 4} - \frac{1}{x - 1}\right) dx$

$= \left[2\ln\left|x + 4\right| - \ln\left|x - 1\right|\right]_2^5$

$= (2\ln 9 - \ln 4) - (2\ln 6 - 0)$

$= \ln 81 - \ln 4 - \ln 36$

$= \ln\dfrac{81}{144}$

$= \ln\dfrac{9}{16}$

Question 12a

$b = \dfrac{-5(0) + 2}{(0) - 1} = -2$

$c = \dfrac{-5(1) + 2}{(1)^2} = -3$

coefficients of x^2: $a + c = 0 \ \rightarrow a = 3$

Question 12b

$\displaystyle\int_2^3 \frac{-5x + 2}{x^2(x - 1)}\,dx$

$= \displaystyle\int_2^3 \left(\frac{3}{x} - \frac{2}{x^2} - \frac{3}{x - 1}\right) dx$

$= \left[3\ln|x| + \dfrac{2}{x} - 3\ln|x - 1|\right]_2^3$

$= \left(3\ln 3 + \dfrac{2}{3} - 3\ln 2\right) - (3\ln 2 + 1 + 0)$

$= 3\ln\dfrac{3}{4} - \dfrac{1}{3}$

Question 13a

$$\frac{16x-43}{(x-3)^2(x+2)}=\frac{a}{(x-3)^2}+\frac{b}{x-3}+\frac{c}{x+2}$$
$$a=\frac{16(3)-43}{(3)+2}=1$$
$$c=\frac{16(-2)-43}{\left((-2)-3\right)^2}=-3$$
equating coefficients of x^2: $b+c=0 \;\to\; b=3$

Question 13b

$$\int\frac{16x-43}{(x-3)^2(x+2)}\,dx$$
$$=\int\left(\frac{1}{(x-3)^2}+\frac{3}{x-3}-\frac{3}{x+2}\right)dx$$
$$=-\frac{1}{x-3}+3\ln\left|x-3\right|-3\ln\left|x+2\right|+c$$
$$=3\ln\left|\frac{x-3}{x+2}\right|-\frac{1}{x-3}+c$$

Question 14

$$a=\frac{2(-1)+3}{(-1)^2+4}=\frac{1}{5}$$
equating coefficients of x^2:
$$a+b=0 \;\to\; b=-\frac{1}{5}$$
equating constants:
$$4a+c=3 \;\to\; \frac{4}{5}+c=3 \;\to c=\frac{11}{5}$$
$$\int_0^2\frac{2x+3}{(x+1)(x^2+4)}\,dx$$
$$=\frac{1}{5}\int_0^2\left(\frac{1}{x+1}-\frac{x-11}{x^2+4}\right)dx$$
$$=\frac{1}{5}\int_0^2\left(\frac{1}{x+1}-\frac{1}{2}\times\frac{2x}{x^2+4}+\frac{11}{x^2+4}\right)dx$$
$$=\frac{1}{5}\left[\ln|x+1|-\frac{1}{2}\ln\left|x^2+4\right|+\frac{11}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$
$$=\frac{1}{5}\left(\left(\ln 3-\frac{1}{2}\ln 8+\frac{11}{2}\times\left(\frac{\pi}{4}\right)\right)-\left(0-\frac{1}{2}\ln 4-0\right)\right)$$
$$=\frac{1}{5}\left(\ln 3+\frac{1}{2}\ln\frac{4}{8}+\frac{11\pi}{8}\right)$$
$$=\frac{1}{5}\left(\ln\frac{3}{\sqrt{2}}+\frac{11\pi}{8}\right)$$
$$=\frac{1}{5}\left(\ln\frac{3\sqrt{2}}{2}+\frac{11\pi}{8}\right)$$

Question 15

$$\frac{3x^2+8}{x(x^2+4)}=\frac{a}{x}+\frac{bx+c}{x^2+4}$$
$$a=\frac{3(0)+8}{(0)^2+4}=2$$
$$b(2i)+c=\frac{3(2i)^2+8}{2i}$$
$$c+2bi=-\frac{4}{2i}\times\frac{i}{i}$$
$$=2i$$
$$\therefore b=1, c=0$$

$$\int\frac{3x^2+8}{x(x^2+4)}\,dx=\int\left(\frac{2}{x}+\frac{x}{x^2+4}\right)dx$$
$$=2\ln\left|x\right|+\frac{1}{2}\ln\left|x^2+4\right|+c$$

Question 16

$$\int\frac{2x^3-x^2-8x-2}{x(x-2)}\,dx$$
$$=\int\frac{2x\left(x^2-2x\right)+3\left(x^2-2x\right)-2x-2}{x(x-2)}\,dx$$
$$=\int\left(2x+3-\frac{2x+2}{x(x-2)}\right)dx \qquad (*)$$

Let $\dfrac{2x+2}{x(x-2)}=\dfrac{a}{x}+\dfrac{b}{x-2}$
$$a=\frac{2(0)+2}{(0)-2}=-1$$
$$b=\frac{2(2)+2}{(2)}=3$$
sub in $(*)$
$$\int\frac{2x^3-x^2-8x-2}{x(x-2)}\,dx$$
$$=\int\left(2x+3+\frac{1}{x}-\frac{3}{x-2}\right)dx$$
$$=x^2+3x+\ln\left|x\right|-3\ln\left|x-2\right|+c$$

Question 17

$$\frac{e^x+1}{e^{2x}-5e^x+6}=\frac{e^x+1}{(e^x-3)(e^x-2)}=\frac{a}{e^x-3}+\frac{b}{e^x-2}$$

$$a=\frac{e^{\ln 3}+1}{e^{\ln 3}-2}=\frac{3+1}{3-2}=4$$

$$b=\frac{e^{\ln 2}+1}{e^{\ln 2}-3}=\frac{2+1}{2-3}=-3$$

$$\int\frac{e^x+1}{e^{2x}-5e^x+6}\,dx$$

$$=\int\left(\frac{4}{e^x-3}-\frac{3}{e^x-2}\right)dx$$

$$=\int\left(\frac{4e^{-x}}{1-3e^{-x}}-\frac{3e^{-x}}{1-2e^{-x}}\right)dx$$

$$=\frac{4}{3}\ln\left|1-3e^{-x}\right|-\frac{3}{2}\ln\left|1-2e^{-x}\right|+c$$

Question 18

$$\int_0^1\frac{3x^3+x^2+12x-12}{(x^2+4)(x^2+2x+4)}\,dx$$

$$=\int_0^1\left(\frac{x-3}{x^2+2x+4}+\frac{2x}{x^2+4}\right)dx$$

$$=\int_0^1\left(\frac{\frac{1}{2}(2x+2)-4}{x^2+2x+4}+\frac{2x}{x^2+4}\right)dx$$

$$=\int_0^1\left(\frac{1}{2}\times\frac{2x+2}{x^2+2x+4}-\frac{4}{(x+1)^2+\left(\sqrt{3}\right)^2}+\frac{2x}{x^2+4}\right)dx$$

$$=\left[\frac{1}{2}\ln\left|x^2+2x+4\right|-\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+\ln\left|x^2+4\right|\right]_0^1$$

$$=\left(\frac{1}{2}\ln 7-\frac{4}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}+\ln 5\right)$$

$$-\left(\frac{1}{2}\ln 4-\frac{4}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}+\ln 4\right)$$

$$=\ln\sqrt{7}-\frac{4}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}+\ln 5-\ln 2-\ln 4+\frac{4}{\sqrt{3}}\left(\frac{\pi}{6}\right)$$

$$=\ln\frac{5\sqrt{7}}{8}+\frac{4}{\sqrt{3}}\left(\frac{\pi}{6}-\tan^{-1}\frac{2}{\sqrt{3}}\right)$$

Question 19

Let $\dfrac{3!}{x(x+1)(x+2)(x+3)}=\dfrac{a}{x}+\dfrac{b}{x+1}+\dfrac{c}{x+2}+\dfrac{d}{x+3}$

$$a=\frac{3!}{(0+1)(0+2)(0+3)}=1$$

$$b=\frac{3!}{(-1)(-1+2)(-1+3)}=-3$$

$$c=\frac{3!}{(-2)(-2+1)(-2+3)}=3$$

$$d=\frac{3!}{(-3)(-3+1)(-3+2)}=-1$$

$$\therefore\frac{3!}{x(x+1)(x+2)(x+3)}=\frac{1}{x}-\frac{3}{x+1}+\frac{3}{x+2}-\frac{1}{x+3}$$

Question 20a

RHS $= \dfrac{(A+2t)\left(t^2-2t+2\right)+(B-2t)\left(t^2+2t+2\right)}{(t^4+4)}$

equating coefficients of t $\quad -2A+4t+2B-4t=0 \quad \therefore A=B$

equating the constant terms $\;2A+2B=16 \quad \therefore A=B=4$

Question 20b

$$\begin{aligned}
x &= \int_0^t \frac{16}{t^4+4}\,dt \\
&= \int_0^t \left(\frac{4+2t}{t^2+2t+2}+\frac{4-2t}{t^2-2t+2}\right)dt \\
&= \int_0^t \left(\frac{2t+2}{t^2+2t+2}+\frac{2}{(t+1)^2+1}-\frac{2t-2}{t^2-2t+2}+\frac{2}{(t-1)^2+1}\right)dt \\
&= \left[\ln\left(t^2+2t+2\right)+2\tan^{-1}(t+1)-\ln\left(t^2-2t+2\right)+2\tan^{-1}(t-1)\right]_0^t \\
&= \left(\ln\left(t^2+2t+2\right)+2\tan^{-1}(t+1)-\ln\left(t^2-2t+2\right)+2\tan^{-1}(t-1)\right) \\
&\qquad -(\ln 2+2\tan^{-1}1-\ln 2+2\tan^{-1}1) \\
&= \ln\left(\frac{t^2+2t+2}{t^2-2t+2}\right)+2\tan^{-1}(t+1)+2\tan^{-1}(t-1)
\end{aligned}$$

Question 20c

$$x = \ln\left(\frac{1+\frac{2}{t}+\frac{2}{t^2}}{1-\frac{2}{t}+\frac{2}{t^2}}\right)+2\tan^{-1}(t+1)+2\tan^{-1}(t-1)$$

As $t\to\infty, x\to \ln(1)+2\tan^{-1}(\infty)+2\tan^{-1}(\infty)$

$\to \ln(1)+2\left(\frac{\pi}{2}\right)+2\left(\frac{\pi}{2}\right)$

$=2\pi$ metres

Since $\dfrac{16}{t^4+4}>0$ for all t the particle is always moving to the right, so the particle approaches but never reaches a displacement of $x=2\pi$ metres.

Question 1

$$\int_0^a f(a-x)dx \qquad \boxed{\begin{aligned} u &= a-x \\ du &= -dx \\ dx &= -du \end{aligned}}$$

$$= \int_a^0 f(u) \times (-du)$$

$$= \int_0^a f(u)du$$

$$= \int_0^a f(x)dx$$

$$\int_0^{2\pi} x\cos x\, dx$$

$$= \int_0^{2\pi} (2\pi - x)\cos(2\pi - x)\, dx$$

$$= 2\pi \int_0^{2\pi} \cos(2\pi - x)\, dx - \int_0^{2\pi} x\cos(2\pi - x)\, dx$$

$$= 2\pi \int_0^{2\pi} \cos x\, dx - \int_0^{2\pi} x\cos x\, dx$$

$$\therefore 2\int_0^{2\pi} x\cos x\, dx = 2\pi \int_0^{2\pi} \cos x\, dx$$

$$\int_0^{2\pi} x\cos x\, dx = \pi\Big[\sin x\Big]_0^{2\pi}$$

$$= \pi(0-0)$$

$$= 0$$

Question 2

$$\int_0^2 x\sqrt{2-x}\, dx$$

$$= \int_0^2 (2-x)\sqrt{x}\, dx$$

$$= \int_0^2 \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$$

$$= \frac{4}{3} \times 2\sqrt{2} - \frac{2}{5} \times 4\sqrt{2}$$

$$= \frac{16\sqrt{2}}{15}$$

Question 3

$$\int_{-2}^2 (x + x^3 + x^5)(1 + x^2 + x^4)dx = 0$$

[since an odd function × an even function is an odd function].

Question 4

$$\int_{-\pi}^0 \sin\left(x + \frac{\pi}{2}\right)\cos\left(x + \frac{\pi}{2}\right) dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos x\, dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\, dx \quad \text{where } f(x) \text{ is odd}$$

$$= 0$$

Question 5

$x^3 > x^2$ for $x > 1$, so $1 \le a < b$.

Question 6

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

$$\therefore 2\int_0^a f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(a-x)\, dx$$

$$\int_0^a f(x)\, dx = \frac{1}{2}\int_0^a \big(f(x) + f(a-x)\big)\, dx$$

$$= \frac{1}{2}\int_0^a f(a)\, dx$$

$$= \frac{f(a)}{2}\Big[x\Big]_0^a$$

$$= \frac{f(a)}{2}(a-0)$$

$$= \frac{a}{2}f(a)$$

Question 7

A: $e^{x^2} > 0$ and $\cos x \ge 0$ in the domain. True.
B: $x^3\cos x$ is odd since x^3 is odd and $\cos x$ is even. False
C: $\sin^2 x - \cos^2 x = -\cos 2x$ which is negative in the domain. False
D; $\sin^{-1}(x^3)$ is the odd function of an odd function so is odd. False.
ANSWER A

Question 8

$$\int_{-a}^a f(x)\, dx$$

$$= \int_{-a}^0 f(x)\, dx + \int_0^a f(x)\, dx$$

$$= \int_0^a f(-x)\, dx + \int_0^a f(x)\, dx$$

$$= \int_0^a \big(f(x) + f(-x)\big)\, dx$$

ANSWER C

Question 10

$$\int_a^{3a} f(x-a)\,dx \qquad \boxed{\begin{aligned} u &= 2a - x \\ du &= -dx \\ dx &= -du \end{aligned}}$$

$$= \int_a^{-a} f(a-u)\,du$$

$$= \int_{-a}^{a} f(a-x)\,dx$$

Question 11

for $0 \le x \le \frac{\pi}{6}$ $\sin^2 x < \sin x < 1 - \sin x < 1 - \sin^2 x$

$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x\,dx$ is the smallest integral

ANSWER B

Question 12

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx \qquad (1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx \qquad (2)$$

From (1) and (2):

$$2\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{1}{2}\Big[x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

Question 13

$$\int_0^2 \left(1 + \sin\left(\pi(1-x)^3\right)\right)dx$$

$$= \int_0^2 \left(1 - \sin\left(\pi(x-1)^3\right)\right)dx \qquad \text{sine is odd}$$

$$= \int_{-1}^{1} \left(1 - \sin \pi x^3\right)dx \qquad \text{translating 1 unit left}$$

$$= \int_{-1}^{1} 1\,dx - \int_{-1}^{1} \sin \pi x^3\,dx$$

$$= \Big[x\Big]_{-1}^{1} - 0 \qquad \text{since sine and } x^3 \text{ are odd}$$

$$= \left(1 - (-1)\right)$$

$$= 2$$

Question 14

$x - a \ge 0$ for all $x \in [0,2]$

$\therefore a \le 0 \qquad (1)$

$1 - \sqrt{x-a} \ne 0$

$\sqrt{x-a} \ne 1$

$x - a \ne 1$ for all $x \in [0,2]$

$\therefore a < -1, a > 1 \qquad (2)$

The integral is defined only if $a < -1$, as both (1) and (2) are satisfied.

Question 15

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2\left(x - \frac{\pi}{4}\right)}$$

$$= \int_0^{\frac{\pi}{2}} \sec^2\left(x - \frac{\pi}{4}\right)dx$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\,dx$$

$$= \Big[\tan x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 1 - (-1)$$

$$= 2$$

Question 16

$$xy - x = 1$$

$$x(y-1) = 1$$

$$x = \frac{1}{y-1}$$

$$\int_3^5 x\,dy = \int_3^5 \frac{dy}{y-1}$$

$$= \Big[\ln(y-1)\Big]_3^5$$

$$= \ln 4 - \ln 2$$

$$= \ln 2$$

Question 17

Since $\tan^{-1} x$ is odd and $1 + \sin^2 x$ is even then $\frac{\tan^{-1} x}{1 + \sin^2 x}$ is odd.

$\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd

$$\therefore \int_{-1}^{1} \frac{\tan^{-1} x}{1 + \sin^2 x}\,dx = 0$$

Question 18

$$\int_0^1 (x^2 - a)^2 \, dx$$
$$= \int_0^1 (x^4 - 2ax^2 + a^2) \, dx$$
$$= \left[\frac{x^5}{5} - \frac{2a}{3}x^3 + a^2 x\right]_0^1$$
$$= \left(\frac{1}{5} - \frac{2a}{3} + a^2\right) - (0)$$
$$= a^2 - \frac{2a}{3} + \frac{1}{5}$$

Minimum when $a = -\frac{-\frac{2}{3}}{2(1)} = \frac{1}{3}$

$$\min = \left(\frac{1}{3}\right)^2 - \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{5} = \frac{4}{45}$$

Question 19

If $f(a+x) = f(a-x)$ then $f(x)$ has line symmetry on $x = a$ (1).

Question 19a

$$\int_0^{2a} f(x) \, dx$$
$$= \int_0^a f(x) \, dx + \int_a^{2a} f(x) \, dx \text{ splitting the integral}$$
$$= \int_0^a f(x) \, dx + \int_0^a f(x) \, dx \quad \text{from (1)}$$
$$= 2\int_0^a f(x) \, dx$$

Question 19b

$$\int_0^a f(2x) \, dx$$
$$= \frac{1}{2}\int_0^{2a} f(x) \, dx \text{ using a horizontal dilation}$$

of $f(x)$ by a factor of 2

$$= \int_0^a f(x) \, dx \text{ from part (a)}$$

Question 19c

$$\int_0^a f(x+a) \, dx$$
$$= \int_0^a f(a-x) \, dx \text{ since } f(x+a) = f(a-x)$$
$$= \int_0^a f(x) \, dx \text{ reflecting about } x = \frac{a}{2}$$

Question 19d

$$\int_0^a f(x-a) \, dx$$
$$= \int_{-a}^0 f(x) \, dx \text{ translating } a \text{ units to the left}$$
$$= \int_{2a}^{3a} f(x) \, dx \text{ since } f(x) \text{ is symmetrical on } x = a$$

Question 20

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x + \cos^2 x - 2\sin x \cos x} dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\sin x - \cos x)^2} dx \qquad (1)$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right)}{1 + \left(\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)\right)^2} dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + (\cos x - \sin x)^2} dx \qquad (2)$$

$(1) + (2)$:

$$2\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1 + (\sin x - \cos x)^2} dx$$
$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} dx = \frac{1}{2}\left[\tan^{-1}(\sin x - \cos x)\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$
$$= \frac{\pi}{4}$$

Question 21

$$\int_0^a [f(a-x) + f(a+x)] dx$$
$$= \int_0^a f(a-x) \, dx + \int_0^a f(x+a) \, dx$$
$$= \int_0^a f(x) \, dx + \int_a^{2a} f(x) \, dx$$
$$= \int_0^{2a} f(x) \, dx$$

ANSWER B

Question 22

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$
$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx$$
$$= \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$
$$2\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$
$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = -\frac{\pi}{2}\left[\tan^{-1}(\cos x)\right]_0^{\pi}$$
$$= -\frac{\pi}{2}(\tan^{-1}(-1) - \tan^{-1}(1))$$
$$= -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$
$$= \frac{\pi^2}{4}$$

Question 23a

$$\int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}\,dx$$

$$\boxed{\begin{aligned} u &= \frac{1}{x} = x^{-1} \\ \frac{du}{dx} &= -x^{-2} = -\frac{1}{x^2} \\ dx &= -\frac{du}{u^2} \end{aligned}}$$

$$= \int_{a}^{\frac{1}{a}} \frac{f\left(\frac{1}{u}\right)}{\left(\frac{1}{u}\right)\left(f\left(\frac{1}{u}\right)+f(u)\right)}\left(-\frac{du}{u^2}\right)$$

$$= \int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{u}\right)}{u\left(f(u)+f\left(\frac{1}{u}\right)\right)}\,du$$

$$= \int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x}\right)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}\,dx$$

Question 23b

$$I = \int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x+\sin\frac{1}{x}\right)}\,dx$$

$$= \int_{\frac{1}{2}}^{2} \frac{\sin\frac{1}{x}}{x\left(\sin x+\sin\frac{1}{x}\right)}\,dx$$

$$\therefore 2I = \int_{\frac{1}{2}}^{2} \frac{\sin x+\sin\frac{1}{x}}{x\left(\sin x+\sin\frac{1}{x}\right)}\,dx$$

$$I = \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{dx}{x}$$

$$= \frac{1}{2}\Big[\ln x\Big]_{\frac{1}{2}}^{2}$$

$$= \frac{1}{2}\left(\ln 2 - \ln\left(\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\big(\ln 2 - (-\ln 2)\big)$$

$$= \ln 2$$

Question 24

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \sin^2 x}{1+e^x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{e^x \sin^2 x}{1+e^x} + \frac{e^{-x}\sin^2(-x)}{1+e^{-x}}\right)dx \quad \text{given result}$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{e^x \sin^2 x}{1+e^x} + \frac{e^{-x}\sin^2 x}{1+e^{-x}} \times \frac{e^x}{e^x}\right)dx \quad \sin^2 x \text{ even}$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{e^x \sin^2 x}{1+e^x} + \frac{\sin^2 x}{1+e^x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{(1+e^x)\sin^2 x}{1+e^x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{2}-0\right)-(0-0)\right)$$

$$= \frac{\pi}{4}$$

Question 25a

$$I = \int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,dx$$

$$= \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}\,dx$$

$$2I = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}\,dx$$

$$= \Big[x\Big]_a^b$$

$$= b-a$$

$$\therefore I = \frac{b-a}{2}$$

Question 25b

$$\int_2^5 \frac{x^2}{\sqrt{29-x^2}+x}\,dx$$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \\ dx &= \frac{du}{2\sqrt{u}} \end{aligned}}$$

$$= \int_4^{25} \frac{u}{\sqrt{29-u}+\sqrt{u}} \times \frac{du}{2\sqrt{u}}$$

$$= \frac{1}{2}\int_4^{25} \frac{\sqrt{u}}{\sqrt{4+25-u}+\sqrt{u}}\,du$$

$= \frac{1}{2}\int_4^{25} \frac{\sqrt{x}}{\sqrt{4+25-x}+\sqrt{x}}\,du$ change of variable

$= \frac{1}{2}\int_a^{b} \frac{f(x)}{f(a+b-x)+f(x)}\,dx$ where $f(x) = \sqrt{x}, a = 4, b = 25$

$= \frac{1}{2} \times \frac{25-4}{2}$ from (a)

$$= \frac{21}{4}$$

Question 26

$$\int_3^7 \frac{\ln(x+2)}{\ln(24+10x-x^2)}\,dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln(x+2)}{\ln(24+10x-x^2)} + \frac{\ln\big((10-x)+2\big)}{\ln(24+10(10-x)-(10-x)^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln(x+2)}{\ln(24+10x-x^2)} + \frac{\ln(12-x)}{\ln(24+100-10x-100+20x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln(x+2)}{\ln(24+10x-x^2)} + \frac{\ln(12-x)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln(x+2)+\ln(12-x)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln\big((x+2)(12-x)\big)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln\big(12x-x^2+24-2x\big)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 \left(\frac{\ln\big(24+10x-x^2\big)}{\ln(24+10x-x^2)}\right)dx$$

$$= \frac{1}{2}\int_3^7 dx$$

$$= \frac{1}{2}\Big[x\Big]_3^7$$

$$= \frac{1}{2}(7-3)$$

$$= 2$$

Question 1

$\sin 4x \cos 3x$

$= \frac{1}{2}\left[\sin(4x+3x)+\sin(4x-3x)\right]$

$= \frac{1}{2}(\sin 7x + \sin x)$

Question 2

$\cos 75° \cos 15°$

$= \frac{1}{2}\left(\cos(75°-15°)+\cos(75°+15°)\right)$

$= \frac{1}{2}\left(\cos 60° + \cos 90°\right)$

$= \frac{1}{2}\left(\frac{1}{2}+0\right)$

$= \frac{1}{4}$

Question 3

$\frac{\sin(A+B)+\sin(A-B)}{\cos(A-B)+\cos(A+B)}$

$= \frac{\sin A \cos B}{\cos A \cos B}$

$= \tan A$

Question 4

$\cos 130° + \cos 10°$

$= 2\cos\left(\frac{130+10}{2}\right)\cos\left(\frac{130-10}{2}\right)$

$= 2\cos 70° \cos 60°$

$= 2\cos 70° \times \frac{1}{2}$

$= \cos 70°$

Question 5

$\cos\frac{\pi}{12} - \cos\frac{5\pi}{12}$

$= \cos\left(\frac{3\pi}{12}-\frac{2\pi}{12}\right) - \cos\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right)$

$= \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$

$= 2\sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$= 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{1}{\sqrt{2}}$

$= \cos\frac{\pi}{4}$

Question 6

$\cos(A-B) - \cos(A+B)$

$= \cos A \cos B + \sin A \sin B$

$- (\cos A \cos B - \sin A \sin B)$

$= 2\sin A \sin B$

$\therefore \sin A \sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$

Question 7a

$\int \cos 3x \cos 2x \, dx$

$= \frac{1}{2}\int \left(\cos(3x-2x)+\cos(3x+2x)\right) dx$

$= \frac{1}{2}\int (\cos x + \cos 5x)\, dx$

$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + c$

Question 7b

$\int \sin 8x \cos 5x \, dx$

$= \frac{1}{2}\int (\sin(8x+5x)+\sin(8x-5x))\, dx$

$= \frac{1}{2}\int (\sin 13x + \sin 3x)\, dx$

$= -\frac{1}{26}\cos 13x - \frac{1}{6}\cos 3x + c$

Question 8a

$\int_0^{\frac{\pi}{4}} \sin 4x \sin 3x \, dx$ (note the changed order)

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos(4x-3x)-\cos(4x+3x))\, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos x - \cos 7x)\, dx$

$= \frac{1}{2}\left[\sin x - \frac{1}{7}\sin 7x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left(\left(\frac{1}{\sqrt{2}} - \frac{1}{7}\left(-\frac{1}{\sqrt{2}}\right)\right) - (0-0)\right)$

$= \frac{1}{2} \times \frac{8}{7} \times \frac{1}{\sqrt{2}}$

$= \frac{2\sqrt{2}}{7}$

Question 8b

$\int_{\frac{\pi}{2}}^{\pi} \cos 5x \sin 3x \, dx$

$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (\sin(5x+3x)-\sin(5x-3x))\, dx$

$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (\sin 8x - \sin 2x)\, dx$

$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\pi}$

$= \frac{1}{2}\left(\left(-\frac{1}{8}\cos 8\pi + \frac{1}{2}\cos 2\pi\right)\right.$

$\left. - \left(-\frac{1}{8}\cos 4\pi + \frac{1}{2}\cos \pi\right)\right)$

$= \frac{1}{2}\left(\left(-\frac{1}{8}+\frac{1}{2}\right) - \left(-\frac{1}{8}-\frac{1}{2}\right)\right)$

$= \frac{1}{2}$

Question 9a

$$\sin 4x - \sin 2x = 0$$
$$2\sin 2x\cos 2x - \sin 2x = 0$$
$$\sin 2x\,(2\cos 2x - 1) = 0$$
$$\sin 2x = 0 \qquad \cos 2x = \frac{1}{2}$$
$$2x = 0, \pi, 2\pi \qquad 2x = \frac{\pi}{3}, \frac{5\pi}{3}$$
$$x = 0, \frac{\pi}{2}, \pi \qquad x = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$$

Question 9b

$$\sin 4x - \sin 2x = 0$$
$$\sin(3x + x) - \sin(3x - x) = 0$$
$$2\cos 3x\sin x = 0$$
$$\cos 3x = 0 \qquad \sin x = 0$$
$$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \qquad x = 0, \pi$$
$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$
$$\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$$

Question 10a

$$\int \sin x \sin 2x\,dx$$
$$= \int 2\sin^2 x\cos x\,dx$$
$$= 2\int (\sin x)^2 \times \frac{d}{dx}(\sin x)\,dx$$
$$= \frac{2\sin^3 x}{3} + c$$

Question 10b

$$\int \sin x \sin 2x\,dx$$
$$= \int \frac{1}{2}\left[\cos(2x - x) - \cos(2x + x)\right]dx$$
$$= \frac{1}{2}\int \left[\cos x - \cos 3x\right]dx$$
$$= \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + c$$
$$= \frac{1}{2}\sin x - \frac{1}{6}(\sin 2x\cos x + \cos 2x\sin x) + c$$
$$= \frac{1}{2}\sin x - \frac{1}{6}\left(2\sin x\cos^2 x + \sin x - 2\sin^3 x\right) + c$$
$$= \frac{1}{2}\sin x - \frac{1}{3}\sin x\left(1 - \sin^2 x\right) - \frac{1}{6}\sin x + \frac{1}{3}\sin^3 x + c$$
$$= \frac{1}{2}\sin x - \frac{1}{3}\sin x + \frac{1}{3}\sin^3 x - \frac{1}{6}\sin x + \frac{1}{3}\sin^3 x + c$$
$$= \frac{2\sin^3 x}{3} + c$$

Question 10c

The double angle results are much quicker and less complex, so are more efficient.

Question 11

$$\text{LHS} = \frac{1}{2}\left[\cos(6x - 3x) + \cos(6x + 3x)\right]$$
$$-\frac{1}{2}\left[\cos(7x - 4x) - \cos(7x + 4x)\right]$$
$$= \frac{1}{2}\left[\cos 3x + \cos 9x - \cos 3x + \cos 11x\right]$$
$$= \frac{1}{2}\left[\cos 9x + \cos 11x\right]$$
$$= \frac{1}{2}\left[\cos(10x - x) + \cos(10x + x)\right]$$
$$= \frac{1}{2}\left[2\cos 10x\cos x\right]$$
$$= \cos 10x\cos x$$

$\therefore A = 10$ and $B = 1$ (or vice versa)

Question 12

$$\cos x\sin y = \frac{1}{2}\left[\sin(x + y) - \sin(x - y)\right]$$
$$= \frac{1}{2}(N - M)$$
$$= \frac{N - M}{2}$$

Question 13

$$\frac{\cos(A - B) + \cos(A + B)}{\cos(A - B) - \cos(A + B)}$$
$$= \frac{2\cos A\cos B}{2\sin A\sin B}$$
$$= \cot A\cot B$$

Question 14

$$\sin 3x + \sin x$$
$$= 2\sin\left(\frac{3x + x}{2}\right)\cos\left(\frac{3x - x}{2}\right)$$
$$= 2\sin 2x\cos x$$
$$= 2(2\sin x\cos x)\cos x$$
$$= 4\sin x\cos^2 x$$
$$= 4\sin x\left(1 - \sin^2 x\right)$$
$$= 4\sin x - 4\sin^3 x$$

Question 15

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)}{2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)}$$
$$= \frac{\sin\left(\frac{A + B}{2}\right)}{\cos\left(\frac{A + B}{2}\right)}$$
$$= \tan\left(\frac{A + B}{2}\right)$$

Question 16

$$\sin 3x + \cos 2x - \sin x = 0$$
$$\sin(2x + x) - \sin(2x - x) + \cos 2x = 0$$
$$2\cos 2x \sin x + \cos 2x = 0$$
$$\cos 2x\,(2\sin x + 1) = 0$$
$$\cos 2x = 0 \quad \sin x = -\frac{1}{2}$$
$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \qquad x = \frac{7\pi}{6}, \frac{11\pi}{6}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$$

Question 17

$$\int_0^{\pi} \sin(mx)\cos(nx)\,dx$$
$$= \frac{1}{2}\int_0^{\pi} (\sin(mx + nx) + \sin(mx - nx))\,dx$$
$$= \frac{1}{2}\int_0^{\pi} (\sin(m + n)x + \sin(m - n)x)\,dx$$
$$= \frac{1}{2}\left[-\frac{\cos(m + n)x}{m + n} - \frac{\cos(m - n)x}{m - n}\right]_0^{\pi}$$
$$= \frac{1}{2}\left(\left(-\frac{\cos(m + n)\pi}{m + n} - \frac{\cos(m - n)\pi}{m - n}\right)\right.$$
$$\left.-\left(-\frac{\cos 0}{m + n} - \frac{\cos 0}{m - n}\right)\right)$$
$$= \frac{1}{2}\left(\left(-\frac{1}{m + n} - \frac{1}{m - n}\right)\right.$$
$$\left.-\left(-\frac{1}{m + n} - \frac{1}{m - n}\right)\right) \text{ since } \cos 2k\pi = 1$$
$$= 0$$

Question 18a

$\sin 2\theta + \sin 4\theta - \sin 6\theta$
$= \sin(3\theta + \theta) + \sin(3\theta - \theta) - \sin 2(3\theta)$
$= 2\sin 3\theta\cos\theta - 2\sin 3\theta\cos 3\theta$
$= 2\sin 3\theta\,(\cos\theta - \cos 3\theta)$
$= 2\sin 3\theta\,(\cos(2\theta - \theta) - \cos(2\theta + \theta))$
$= 2\sin 3\theta\,(2\sin 2\theta\sin\theta)$
$= 4\sin\theta\sin 2\theta\sin 3\theta$

Question 18b

$$\sin 6\theta = \sin 2\theta + \sin 4\theta$$
$$\sin 2\theta + \sin 4\theta - \sin 6\theta = 0$$
$$4\sin\theta\sin 2\theta\sin 3\theta = 0$$
$$\sin\theta = 0 \quad \sin 2\theta = 0 \qquad \sin 3\theta = 0$$
$$\theta = 0, \pi \qquad 2\theta = 0, \pi, 2\pi \quad 3\theta = 0, \pi, 2\pi, 3\pi$$
$$\theta = 0, \frac{\pi}{2}, \pi \qquad \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$$
$$\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$$

Question 19

$$\sin 2x + \sin 3x + \cos 2x + \cos 3x = 0$$
$$\sin\left(\frac{5x}{2} + \frac{x}{2}\right) + \sin\left(\frac{5x}{2} - \frac{x}{2}\right) + \cos\left(\frac{5x}{2} + \frac{x}{2}\right) + \cos\left(\frac{5x}{2} - \frac{x}{2}\right) = 0$$
$$2\sin\frac{5x}{2}\cos\frac{x}{2} + 2\cos\frac{5x}{2}\cos\frac{x}{2} = 0$$
$$2\cos\frac{x}{2}\left(\sin\frac{5x}{2} + \cos\frac{5x}{2}\right) = 0$$
$$\cos\frac{x}{2} = 0 \qquad \sin\frac{5x}{2} + \cos\frac{5x}{2} = 0$$
$$\frac{x}{2} = \frac{\pi}{2} \qquad \sin\frac{5x}{2} = -\cos\frac{5x}{2}$$
$$x = \pi \qquad \tan\frac{5x}{2} = -1$$
$$\frac{5x}{2} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}$$
$$x = \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{11\pi}{10}, \frac{3\pi}{2}, \frac{19\pi}{10}$$

$$\therefore x = \frac{3\pi}{10}, \frac{7\pi}{10}, \pi, \frac{11\pi}{10}, \frac{3\pi}{2}, \frac{19\pi}{10}$$

Question 20a

$$\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x}$$
$$= \frac{\sin(3x+2x) + \sin(3x-2x) + \sin 3x}{\cos(3x+2x) + \cos(3x-2x) + \cos 3x}$$
$$= \frac{2\sin 3x \cos 2x + \sin 3x}{2\cos 3x \cos 2x + \cos 3x}$$
$$= \frac{\sin 3x\,(2\cos 2x + 1)}{\cos 3x\,(2\cos 2x + 1)}$$
$$= \frac{\sin 3x}{\cos 3x} \quad \text{since } \cos 2x \neq -\frac{1}{2}$$
$$= \tan 3x \quad \text{since} \cos 3x \neq 0$$

Question 20b

$$\sin x + \sin 3x + \sin 5x = \cos x + \cos 3x + \cos 5x$$
$$\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = 1$$
$$\tan 3x = 1$$
$$3x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$$
$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$$

Question 21

$$\cos\left(x+\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{6}\right) = \frac{1}{2}\left[\sin\left(x+\frac{\pi}{6}+x-\frac{\pi}{6}\right) - \sin\left(x+\frac{\pi}{6}-x+\frac{\pi}{6}\right)\right]$$
$$= \frac{1}{2}\left[\sin 2x - \sin\frac{\pi}{3}\right]$$
$$= \frac{1}{2}\left[\sin 2x - \frac{\sqrt{3}}{2}\right]$$
$$\therefore \frac{1}{2}\left[\sin 2x - \frac{\sqrt{3}}{2}\right] = -\frac{\sqrt{3}}{4}$$
$$\sin 2x - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$
$$\sin 2x = 0$$
$$2x = 0, \pi, 2\pi, 3\pi, 4\pi$$
$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

Question 22

a $$\text{LHS} = \frac{1}{2}\left[\cos\left[(a+b)-(a-b)\right] - \cos\left[(a+b)+(a-b)\right]\right]$$
$$= \frac{1}{2}\left[\cos 2b - \cos 2a\right]$$
$$= \frac{1}{2}\left[\left(1-2\sin^2 b\right) - \left(1-2\sin^2 a\right)\right]$$
$$= \frac{1}{2}\left[2\sin^2 a - 2\sin^2 b\right]$$
$$= \sin^2 a - \sin^2 b$$

b $$\sin(4x+2x)\sin(4x-2x) = \sin 6x \quad \text{from (a)}$$
$$\sin 6x \sin 2x = \sin 6x$$
$$\sin 6x\,(\sin 2x - 1) = 0$$
$$\sin 6x = 0 \quad \text{or} \quad \sin 2x - 1 = 0$$
$$6x = 0, \pi, 2\pi, 3\pi \qquad \sin 2x = 1$$
$$x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2} \qquad 2x = \frac{\pi}{2}$$
$$x = \frac{\pi}{4}$$
$$\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$$

Question 23

$$4\cos 2x \sin x + \cos 3x - 2\sin x + 1 = 2(\sin 3x - \sin x) + \cos 3x - 2\sin x + 1$$
$$= 2\sin 3x + \cos 3x + 1$$
$$R = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 0.464$$
$$\therefore 4\cos 2x \sin x + \cos 3x - 2\sin x + 1 = \sqrt{5}\sin\left(3x + 0.464\right) + 1$$

Question 24

a LHS $= \frac{1}{2}(\sin A + \sin B) + \frac{1}{2}(\sin A - \sin B)$

$= \frac{1}{2}(\sin A + \sin B + \sin A - \sin B)$

$= \frac{1}{2} \times 2\sin A$

$= \sin A$

b LHS $= \sin\left(\frac{2x + \frac{\pi}{3}}{2}\right)\cos\left(\frac{2x - \frac{\pi}{3}}{2}\right) + \cos\left(\frac{2x + \frac{\pi}{3}}{2}\right)\sin\left(\frac{2x - \frac{\pi}{3}}{2}\right)$

$= \sin 2x$

$\therefore \sin 2x = -\frac{\sqrt{3}}{2}$

$2x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

$x = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$

Question 25

$\int x\cos 4x^2 \cos 2x^2 \, dx$

$= \int x \times \frac{1}{2}(\cos 2x^2 + \cos 6x^2)\, dx$

$= \frac{1}{2}\int x\cos 2x^2 \, dx + \frac{1}{2}\int x\cos 6x^2 \, dx$

$= \frac{1}{8}\int 4x\cos 2x^2 \, dx + \frac{1}{24}\int 12x\cos 6x^2 \, dx$

$= \frac{\sin 2x^2}{8} + \frac{\sin 6x^2}{24} + c$

Question 1a

$$\int \sin x\cos^3 x\ dx$$

$$= -\int (-\sin x)(\cos x)^3\,dx$$

$$= -\frac{\cos^4 x}{4} + c$$

Question 1b

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 x\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\cos^2 x\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,(1-\sin^2 x)\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\cos x - \cos x\,(\sin x)^2\right)dx$$

$$= \left[\sin x - \frac{\sin^3 x}{3}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(1-\frac{1}{3}\right) - \left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right)$$

$$= \frac{2}{3} - \frac{5}{6\sqrt{2}}$$

$$= \frac{8-5\sqrt{2}}{12}$$

Question 2a

$$\int \sec^4 x\tan^2 x\,dx$$

$$= \int \sec^2 x\sec^2 x\tan^2 x\,dx$$

$$= \int \sec^2 x\,(1+\tan^2 x)\tan^2 x\,dx$$

$$= \int \sec^2 x\,(\tan^2 x + \tan^4 x)\,dx$$

$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + c$$

Question 2b

$$\int \sec^4 x\,dx$$

$$= \int \sec^2 x\sec^2 x\,dx$$

$$= \int \sec^2 x\,(\tan^2 x + 1)\,dx$$

$$= \int \sec^2 x\,(\tan x)^2 dx + \int \sec^2 x\,dx$$

$$= \frac{\tan^3 x}{3} + \tan x + c$$

Question 3

$$\int \operatorname{cosec} x\,dx$$

$$= \int \operatorname{cosec} x \times \frac{\cot x + \operatorname{cosec} x}{\operatorname{cosec} x + \cot x}dx$$

$$= \int \frac{\operatorname{cosec} x\cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x}dx$$

$$= -\int \frac{-\operatorname{cosec} x\cot x - \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x}dx$$

$$= -\ln|\cot x + \operatorname{cosec} x| + c$$

Question 4

$$\int \tan x\sec^3 x\,dx$$

$$= \int \tan x\sec x\,(\sec x)^2\,dx$$

$$= \frac{\sec^3 x}{3} + c$$

Question 5

$$\int \cos^4 x\,dx$$

$$= \int \left(\cos^2 x\right)^2$$

$$= \int \left(\frac{1}{2}(1+\cos 2x)\right)^2\,dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right)dx$$

$$= \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}\left(\frac{1}{4}\sin 4x + x\right)\right) + c$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + \frac{x}{2} + c$$

$$= \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + \frac{3x}{8} + c$$

Question 6

$$\int \tan^3 x\,dx$$

$$= \int \tan^2 x\tan x\,dx$$

$$= \int \left(\sec^2 x - 1\right)\tan x\,dx$$

$$= \int (\sec^2 x\tan x - \tan x)\,dx$$

$$= \frac{\tan^2 x}{2} + \ln|\cos x| + c$$

Question 7

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot^3 x \operatorname{cosec}^2 x\, dx$

$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (-\operatorname{cosec}^2 x)(\cot x)^3\, dx$

$= -\left[\frac{\cot^4 x}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= -\frac{1}{4}\left(\left(\frac{1}{\sqrt{3}}\right)^4 - \left(\sqrt{3}\right)^4\right)$

$= -\frac{1}{4}\left(\frac{1}{9} - 9\right)$

$= \frac{20}{9}$

Question 8

$\int \operatorname{cosec}^4 x\, dx$

$= \int \operatorname{cosec}^2 x \times \operatorname{cosec}^2 x\, dx$

$= \int \operatorname{cosec}^2 x\,(1 + \cot^2 x)\, dx$

$= \int (\operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cot^2 x)\, dx$

$= -\cot x - \frac{\cot^3 x}{3} + c$

Question 9

$\int \sqrt{\cos x} \sin^3 x\, dx$

$= \int \sin x\,(\cos x)^{\frac{1}{2}}\left(1 - \cos^2 x\right) dx$

$= -\int (-\sin x)\left((\cos x)^{\frac{1}{2}} - (\cos x)^{\frac{5}{2}}\right) dx$

$= -\frac{2}{3}(\cos x)^{\frac{3}{2}} + \frac{2}{7}(\cos x)^{\frac{7}{2}} + c$

$= \frac{2\sqrt{\cos^7 x}}{7} - \frac{2\sqrt{\cos^3 x}}{3} + c$

Question 10

$\int \sin^2 x \cos^4 x\ \ dx$

$= \int \frac{1}{2}(1 - \cos 2x)\left(\frac{1}{2}(1 + \cos 2x)\right)^2\ dx$

$= \frac{1}{8}\int (1 - \cos 2x)\left((1 + \cos 2x)\right)^2\ dx$

$= \frac{1}{8}\int (1 - \cos 2x)\left(1 + 2\cos 2x + \cos^2 2x\right) dx$

$= \frac{1}{8}\int (1 + 2\cos 2x + \cos^2 2x - \cos 2x - 2\cos^2 2x - \cos^3 2x) dx$

$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx$

$= \frac{1}{8}\int \left(1 + \cos 2x - \frac{1}{2}(1 + \cos 4x) - \cos 2x\left(1 - \sin^2 2x\right)\right) dx$

$= \frac{1}{8}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x + \cos 2x \sin^2 2x\right) dx$

$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + c$

Question 11

$\int \frac{\cos 2x}{\cos x}\, dx$

$= \int \frac{2\cos^2 x - 1}{\cos x}\, dx$

$= \int (2\cos x - \sec x)\, dx$

$= 2\sin x - \ln|\tan x + \sec x| + c$

Question 12

$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan x}{\cos^2 x}\, dx$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$

$= \left[\frac{\tan^2 x}{2}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \frac{1}{2}\left(1^2 - (-1)^2\right)$

$= 0$

Alternative solution:

$\tan x$ is odd and $\cos^2 x$ is even, so $\frac{\tan x}{\cos^2 x}$ is odd

$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan x}{\cos^2 x}\, dx = 0$

Question 13

$\int (\cos^4 x - \sin^4 x)\, dx$

$= \int (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)\, dx$

$= \int (\cos^2 x - \sin^2 x)\, dx$

$= \int \cos 2x\, dx$

$= \frac{1}{2}\sin 2x + c$

Question 14

$\int (\tan^4 x - \sec^4 x)\, dx$

$= \int (\tan^2 x - \sec^2 x)(\tan^2 x + \sec^2 x)\, dx$

$= \int (-1)\left(2\sec^2 x - 1\right) dx$

$= -2\tan x + x + c$

Question 15

$$\int \sec^6 x\,dx$$
$$= \int \sec^2 x \sec^4 x\,dx$$
$$= \int \sec^2 x \left(\tan^2 x + 1\right)^2 dx$$
$$= \int \sec^2 x \left(\tan^4 x + 2\tan^2 x + 1\right) dx$$
$$= \frac{\tan^5 x}{5} + \frac{2\tan^3 x}{3} + \tan x + c$$

Question 16

$$V = \pi \int_0^{\frac{\pi}{4}} \left(\sec^2 x \tan^3 x\right)^2 dx$$
$$= \pi \int_0^{\frac{\pi}{4}} \sec^4 x \tan^6 x\,dx$$
$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \sec^2 x \tan^6 x\,dx$$
$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \left(\tan^2 x + 1\right) \tan^6 x\,dx$$
$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \left(\tan^8 x + \tan^6 x\right) dx$$
$$= \pi \left[\frac{\tan^9 x}{9} + \frac{\tan^7 x}{7}\right]_0^{\frac{\pi}{4}}$$
$$= \pi \left(\left(\frac{1}{9} + \frac{1}{7}\right) - (0 + 0)\right)$$
$$= \frac{16\pi}{63} \text{ units}^3$$

Question 17

$$\int \frac{(\cos x + \sin x)^3}{\sin 2x + 1} dx$$
$$= \int \frac{(\cos x + \sin x)^3}{2\sin x \cos x + \cos^2 x + \sin^2 x} dx$$
$$= \int \frac{(\cos x + \sin x)^3}{(\cos x + \sin x)^2} dx$$
$$= \int (\cos x + \sin x)\,dx$$
$$= \sin x - \cos x + c$$

Question 18

$$\int \frac{\sqrt{1 + \sin x}}{\sec x} dx$$
$$= \int \cos x\,(1 + \sin x)^{\frac{1}{2}} dx$$
$$= \frac{2}{3}(1 + \sin x)^{\frac{3}{2}} + c$$
$$= \frac{2}{3}\sqrt{(1 + \sin x)^3} + c$$

Question 19

$$\int \cot^2 x\,dx$$
$$= \int \left(\operatorname{cosec}^2 x - 1\right) dx$$
$$= -\cot x - x + c$$

Question 20

$$\int \frac{\sin^5 x}{\cos^4 x} dx$$
$$= \int \sin x \frac{\sin^4 x}{\cos^4 x} dx$$
$$= \int \sin x \times \frac{\left(1 - \cos^2 x\right)^2}{\cos^4 x} dx$$
$$= \int \sin x \times \frac{1 - 2\cos x + \cos^2 x}{\cos^4 x} dx$$
$$= \int \sin x \times \left((\cos x)^{-4} - 2(\cos x)^{-3}\right.$$
$$\left. + (\cos x)^{-2}\right) dx$$
$$= \frac{(\cos x)^{-3}}{-3} - \frac{2(\cos x)^{-2}}{-2} + \frac{(\cos x)^{-1}}{-1} + c$$
$$= -\frac{\sec^3 x}{3} + \sec^2 x + \sec x + c$$

Question 21

$$\int \cot^3 x\,dx$$
$$= \int \cot^2 x \cot x\,dx$$
$$= \int \left(\operatorname{cosec}^2 x - 1\right) \cot x\,dx$$
$$= \int \left(\cot x \operatorname{cosec}^2 x - \cot x\right) dx$$
$$= \int \left(-(-\cot x)\operatorname{cosec}^2 x - \frac{\cos x}{\sin x}\right) dx$$
$$= -\frac{\operatorname{cosec}^3 x}{3} - \ln|\sin x| + c$$

Question 22

$$\int \cot^3 x \operatorname{cosec}^3 x\,dx$$
$$= \int \cot x \operatorname{cosec} x \cot^2 x \operatorname{cosec}^2 x\,dx$$
$$= \int \cot x \operatorname{cosec} x \left(\operatorname{cosec}^2 x - 1\right) \operatorname{cosec}^2 x\,dx$$
$$= -\int -\cot x \operatorname{cosec} x \left(\operatorname{cosec}^4 x - \operatorname{cosec}^2 x\right) dx$$
$$= -\frac{\operatorname{cosec}^5 x}{5} + \frac{\operatorname{cosec}^3 x}{3} + c$$

Question 23

$$\int \sqrt{1 + \sin x}\,dx$$
$$= \int \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + \sin 2\left(\frac{x}{2}\right)}\,dx$$
$$= \int \sqrt{\cos^2 \frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2} + \sin^2 \frac{x}{2}}\,dx$$
$$= \int \sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\,dx$$
$$= \int \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) dx$$
$$= 2\sin\frac{x}{2} - 2\cos\frac{x}{2} + c$$

Question 1

$$\cos A = \cos\left[2\left(\frac{A}{2}\right)\right]$$
$$= \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$$
using the double angle result
$$= \frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2}}$$
since $\cos^2\theta + \sin^2\theta = 1$
$$= \frac{1 - \tan^2\frac{A}{2}}{1 + \tan^2\frac{A}{2}}$$
dividing the numerator and denominator by $\cos^2\frac{A}{2}$
$$= \frac{1 - t^2}{1 + t^2}$$

Question 2a

$$\frac{1 + \cos A}{\sin A}$$
$$= \frac{1 + \frac{1 - t^2}{1 + t^2}}{\frac{2t}{1 + t^2}}$$
$$= \frac{1 + t^2 + 1 - t^2}{2t}$$
$$= \frac{2}{2t}$$
$$= \frac{1}{t}$$
$$= \cot\frac{A}{2}$$

Question 2b

$$\cot 15° = \cot\left(\frac{30°}{2}\right)$$
$$= \frac{1 + \cos 30°}{\sin 30°}$$
from (a)
$$= \frac{1 + \frac{\sqrt{3}}{2}}{\frac{1}{2}}$$
$$= 2(1) + 2\left(\frac{\sqrt{3}}{2}\right)$$
$$= 2 + \sqrt{3}$$

Question 3

$$2\tan\frac{x}{2} = \sqrt{3}\left(1 - \tan^2\frac{x}{2}\right)$$
$$\frac{2\tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}} = \sqrt{3}$$
$$\tan x = \sqrt{3}$$
$$x = \frac{\pi}{3}, \frac{4\pi}{3}$$

Question 4a

$\tan\theta\sin\theta$
$$= \frac{2t}{1 - t^2} \times \frac{2t}{1 + t^2}$$
$$= \frac{4t^2}{1 - t^4}$$

Question 4b

$\cos 2\theta$
$$= 2\cos^2\theta - 1$$
$$= \frac{2\left(1 - t^2\right)^2}{(1 + t^2)^2} - 1$$
$$= \frac{2\left(1 - 2t^2 + t^4\right) - \left(1 + 2t^2 + t^4\right)}{(1 + t^2)^2}$$
$$= \frac{1 - 6t^2 + t^4}{(1 + t^2)^2}$$

Question 5a

$$\frac{2}{1 - \cos x}$$
$$= \frac{2}{1 - \frac{1 - t^2}{1 + t^2}}$$
$$= \frac{2 + 2t^2}{1 + t^2 - 1 + t^2}$$
$$= \frac{2\left(1 + t^2\right)}{2t^2}$$
$$= \frac{1 + t^2}{t^2}$$
$$= \frac{1 + \tan^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)}$$
$$= \frac{\sec^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)}$$
$$= \frac{1}{\cos^2\left(\frac{x}{2}\right)} \times \frac{\cos^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$$
$$= \frac{1}{\sin^2\left(\frac{x}{2}\right)}$$
$$= \operatorname{cosec}^2\left(\frac{x}{2}\right)$$

Question 5b

$$\frac{1 + \cos x + \sin x}{1 - \cos x + \sin x}$$
$$= \frac{1 + \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2}}{1 - \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2}}$$
$$= \frac{1 + t^2 + 1 - t^2 + 2t}{1 + t^2 - 1 + t^2 + 2t}$$
$$= \frac{2 + 2t}{2t + 2t^2}$$
$$= \frac{2(1 + t)}{2t(1 + t)}$$
$$= \frac{1}{t}$$
$$= \frac{1}{\tan\left(\frac{x}{2}\right)}$$
$$= \cot\left(\frac{x}{2}\right)$$

Question 6a

let $t = \tan\frac{\pi}{8}$

$$\tan\frac{\pi}{4} = \frac{2t}{1-t^2}$$
$$1 = \frac{2t}{1-t^2}$$
$$1 - t^2 = 2t$$
$$t^2 + 2t - 1 = 0$$
$$t = \frac{-2 + \sqrt{(2)^2 - 4(1)(-1)}}{2 \times 1}$$

only use the positive since $\tan\frac{\pi}{8} > 0$

$$= \frac{-2+\sqrt{8}}{2}$$
$$= \frac{-2+2\sqrt{2}}{2}$$
$$= -1 + \sqrt{2}$$
$$= \sqrt{2} - 1$$

Question 6b

let $t = \tan\frac{\pi}{8}$

$$\sin\frac{\pi}{4} = \frac{2t}{1+t^2}$$
$$\frac{1}{\sqrt{2}} = \frac{2t}{1+t^2}$$
$$1 + t^2 = 2\sqrt{2}t$$
$$t^2 - 2\sqrt{2}t + 1 = 0$$
$$t = \frac{2\sqrt{2} - \sqrt{\left(-2\sqrt{2}\right)^2 - 4(1)(1)}}{2 \times 1}$$

only use the negative since $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

giving $t = \tan\frac{3\pi}{8}$ which is larger

$$= \frac{2\sqrt{2}-\sqrt{4}}{2}$$
$$= \frac{2\sqrt{2}-2}{2}$$
$$= \sqrt{2} - 1$$

Question 7

let $t = \tan\theta$

$$\frac{1+\cos 2\theta}{1-\cos 2\theta} = \frac{\left(1 + \frac{1-t^2}{1+t^2}\right)}{\left(1 - \frac{1-t^2}{1+t^2}\right)}$$
$$= \frac{1+t^2+1-t^2}{1+t^2-1+t^2}$$
$$= \frac{2}{2t^2}$$
$$= \frac{1}{t^2}$$
$$= \cot^2\theta$$

Question 8a

$$\sin x - 2\cos x = R(\sin x\cos\alpha - \cos x\sin\alpha)$$
$$R\cos\alpha = 1 \quad (1) \quad (\cos\alpha > 0)$$
$$-R\sin\alpha = -2 \quad (2) \quad (\sin\alpha > 0)$$

$\therefore \alpha$ is in the 1st quadrant

$$(1)^2 + (2)^2\text{:}\quad R^2\left(\cos^2\alpha + \sin^2\alpha\right) = 1^2 + 2^2$$
$$R^2 = 5$$
$$R = \sqrt{5}$$

$$(2) \div (1)\text{:}\quad \frac{-R\sin\alpha}{R\cos\alpha} = \frac{-2}{1}$$
$$\tan\alpha = 2$$
$$\alpha = 1.107149$$
$$\therefore \sin x - 2\cos x = \sqrt{5}\sin(x - 1.107149)$$
$$\therefore \sqrt{5}\sin(x - 1.107149) = 2$$
$$\sin(x - 1.107149) = \frac{2}{\sqrt{5}}$$
$$x - 1.107149 = 1.107149, \pi$$
$$x = 2.214297, \pi$$
$$= 2.21 \text{ (2 dp)}, \pi$$

Question 8b

$$\frac{2t}{1+t^2} - 2\frac{1-t^2}{1+t^2} = 2$$
$$2t - 2 + 2t^2 = 2 + 2t^2$$
$$2t - 2 = 2$$
$$t = 2$$
$$\tan\frac{A}{2} = 2$$
$$\frac{A}{2} = 1.107149$$
$$A = 2.214297$$

Check $x = \pi$:

$\sin\pi - 2\cos\pi = 0 - 2(-1) = 2$ ✓

$\therefore x = 2.214297, \pi$

Question 9

$$\cos x - 2\sin x = 1$$
$$\frac{1-t^2}{1+t^2} - 2\left(\frac{2t}{1+t^2}\right) = 1$$
$$1 - t^2 - 4t = 1 + t^2$$
$$2t^2 + 4t = 0$$
$$2t(t+2) = 0$$
$$t = 0 \qquad t + 2 = 0$$
$$\tan\frac{x}{2} = 0 \qquad \tan\frac{x}{2} = -2$$
$$\frac{x}{2} = 0 \qquad \frac{x}{2} = -\tan^{-1}(2)$$
$$x = 0 \qquad x = -2\tan^{-1}(2)$$
$$= -2.214297$$

Check $\theta = -\pi$:

$\cos(-\pi) - 2\sin(-\pi) = -1 - 2(0) = -1 \neq 1$ ✗

Check $\theta = \pi$:

$\cos(\pi) - 2\sin(\pi) = -1 - 2(0) = -1 \neq 1$ ✗

$\therefore x = 0, -2.21$ (2 dp)

Question 10

$$\sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$$

$$\begin{aligned}
&\sqrt{3}\sin x - \cos x \\
&= 2\left(\sin x \times \frac{\sqrt{3}}{2} - \cos x \times \frac{1}{2}\right) \\
&= 2\left(\sin x \cos\frac{\pi}{6} - \cos x \sin\frac{\pi}{6}\right) \\
&= 2\sin\left(x - \frac{\pi}{6}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore 2\sin\left(x - \frac{\pi}{6}\right) &= 1 \\
\sin\left(x - \frac{\pi}{6}\right) &= \frac{1}{2} \\
x - \frac{\pi}{6} &= \frac{\pi}{6}, \frac{5\pi}{6} \\
x &= \frac{\pi}{3}, \pi
\end{aligned}$$

The only solutions in the domain $[0, 2\pi]$ are $x = \frac{\pi}{3}$ and $x = \pi$.

Alternative solution using *t* formulas:

$$\begin{aligned}
\sqrt{3}\sin x - \cos x &= 1 \\
\sqrt{3}\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} &= 1 \\
2\sqrt{3}t - 1 + t^2 &= 1 + t^2 \\
2\sqrt{3}t &= 2 \\
t &= \frac{1}{\sqrt{3}} \\
\tan\frac{x}{2} &= \frac{1}{\sqrt{3}} \\
\frac{x}{2} &= \frac{\pi}{6} \\
x &= \frac{\pi}{3}
\end{aligned}$$

Check $x = \pi$:

$\sqrt{3}\sin\pi - \cos\pi = 0 - (-1) = 1$ ✓

The only solutions in the domain $[0, 2\pi]$ are $x = \frac{\pi}{3}$ and $x = \pi$.

Question 11

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\begin{aligned}
&\sin x + \cos x \\
&= \sqrt{2}\left(\sin x \times \frac{1}{\sqrt{2}} + \cos x \times \frac{1}{\sqrt{2}}\right) \\
&= \sqrt{2}\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right) \\
&= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) &= \frac{1}{2} \\
\sin\left(x + \frac{\pi}{4}\right) &= \frac{1}{2\sqrt{2}} \\
x + \frac{\pi}{4} &= \pi - \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right), 2\pi \\
x + 0.785398 &= 2.780226, 6.644552 \\
x &= 1.994827\ldots, 5.859154 \\
x &= 2.00, 5.86 \text{ (2 dp)}
\end{aligned}$$

The only solutions in the domain $[0, 2\pi]$ are $x = 2.00$ and $x = 5.86$, correct to 2 decimal places.

Question 11 continued

Alternative solution using *t* formulas:

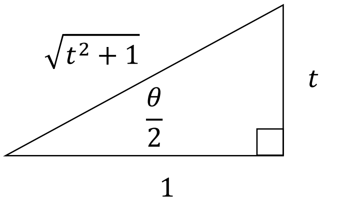
$$\begin{aligned}
\sin x + \cos x &= \frac{1}{2} \\
\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} &= \frac{1}{2} \\
4t + 2 - 2t^2 &= 1 + t^2 \\
3t^2 - 4t - 1 &= 0 \\
t &= \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-1)}}{2 \times 3} \\
\tan\frac{x}{2} &= \frac{4 \pm \sqrt{28}}{6} \\
\frac{x}{2} &= \tan^{-1}\left(\frac{4 + \sqrt{28}}{6}\right), \pi \\
x &= 2\tan^{-1}\left(\frac{4 + \sqrt{28}}{6}\right), 2\pi \\
&= 1.994827\ldots, 5.859154\ldots
\end{aligned}$$

Check $x = \pi$:

$\sin\pi + \cos\pi = 0 + (-1) = -1 \neq 0.5$ ✗

The only solutions in the domain $[0, 2\pi]$ are $x = 2.00$ and $x = 5.86$, correct to 2 decimal places.

Question 12



$$\begin{aligned}
\frac{2t}{1+t^2} - \frac{t}{\sqrt{1+t^2}} &= 0 \\
\frac{2t}{1+t^2} &= \frac{t}{\sqrt{1+t^2}} \\
2t &= t\sqrt{1+t^2} \\
4t^2 &= t^2 + t^4 \\
t^4 - 3t^2 &= 0 \\
t^2\left(t^2 - 3\right) &= 0 \\
t = 0 \quad t &= \pm\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
\tan\frac{\theta}{2} &= 0 & \tan\frac{\theta}{2} &= \pm\sqrt{3} \\
\frac{\theta}{2} &= 0 & \frac{\theta}{2} &= \pm\frac{\pi}{3} \\
\theta &= 0 & \theta &= \pm\frac{2\pi}{3}
\end{aligned}$$

Check $\theta = -\pi$:

$\sin(-\pi) - \sin\left(-\frac{\pi}{2}\right) = 0 - (-1) = 1 \neq 0$ ✗

Check $\theta = \pi$:

$\sin(\pi) - \sin\left(\frac{\pi}{2}\right) = 0 - (1) = -1 \neq 0$ ✗

The solutions are $\theta = -\frac{2\pi}{3}, 0, \frac{2\pi}{3}$.

Question 13

$\sec x + \tan x$

$= \dfrac{1+t^2}{1-t^2} + \dfrac{2t}{1-t^2}$

$= \dfrac{(t^2+2t+1)}{1-t^2}$

$= \dfrac{(t+1)^2}{(1+t)(1-t)}$

$= \dfrac{1+t}{1-t}$ since $t \neq -1$

$= \dfrac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}$

$= \tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$

$\therefore a = \dfrac{1}{2}, b = \dfrac{\pi}{4}.$

Question 14

$3\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) = 3$

$3\left(\dfrac{1-t^2}{1+t^2}\right) - \dfrac{2t}{1+t^2} = 3$ where $t = \tan\left(\frac{x}{4}\right)$

$3 - 3t^2 - 2t = 3 + 3t^2$

$6t^2 + 2t = 0$

$2t(3t+1) = 0$

$t = 0$ or $3t + 1 = 0$

$\tan\left(\frac{x}{4}\right) = 0 \quad \tan\left(\frac{x}{4}\right) = -\dfrac{1}{3}$

$\dfrac{x}{4} = 0$ no solution for $0 \leq \dfrac{x}{4} \leq \dfrac{\pi}{2}$

$x = 0$

Check $\theta = \pi$:

$3\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = 3(0) - (1) = -1 \neq 3$ ✗

$\therefore x = 0$ is the only solution.

Question 15a

$m\cos x - \sin x = 1$

$m\left(\dfrac{1-t^2}{1+t^2}\right) - \dfrac{2t}{1+t^2} = 1$

$m - mt^2 - 2t = 1 + t^2$

$m - mt^2 = t^2 + 2t + 1$

$m\left(1-t^2\right) = (t+1)^2$

$m(1-t)(1+t) = (t+1)^2$

$m(1-t) = t+1$ since $t + 1 \neq 0$ for 0

$m - mt = 1 + t$

$t + mt = m - 1$

$t(m+1) = m - 1$

$t = \dfrac{m-1}{m+1}$

Question 15b

$\tan\dfrac{x}{2} = \dfrac{\tan(\tan^{-1} m) - \tan\frac{\pi}{4}}{1 + \tan(\tan^{-1} m) \times \tan\frac{\pi}{4}}$

$= \tan\left(\tan^{-1} m - \dfrac{\pi}{4}\right)$

$\dfrac{x}{2} = \tan^{-1} m - \dfrac{\pi}{4}$

$x = 2\tan^{-1} m - \dfrac{\pi}{2}$

Question 16

$2\sin x - \cos x = 0$

$2\left(\dfrac{2t}{1+t^2}\right) - \dfrac{1-t^2}{1+t^2} = 0$

$4t - 1 + t^2 = 0$

$t^2 + 4t - 1 = 0$

$\therefore \tan\dfrac{\alpha}{2} \times \tan\dfrac{\beta}{2} = \dfrac{c}{a}$

$= \dfrac{-1}{1}$

$\therefore \tan\dfrac{\alpha}{2} = -\dfrac{1}{\tan\frac{\beta}{2}}$

Question 17a

$\sqrt{\dfrac{1-\cos x}{1+\sin x}}$

$= \sqrt{\dfrac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2}}}$

$= \sqrt{\dfrac{1+t^2-1+t^2}{1+t^2+2t}}$

$= \sqrt{\dfrac{2t^2}{(t+1)^2}}$

$= \sqrt{\left(\dfrac{\sqrt{2}t}{t+1}\right)^2}$

$= \left|\dfrac{\sqrt{2}t}{t+1}\right|$

Question 17b

$\sqrt{\dfrac{1-\cos x}{1+\sin x}} = \dfrac{1}{\sqrt{2}}$

$\left|\dfrac{\sqrt{2}t}{t+1}\right| = \dfrac{1}{\sqrt{2}}$

$\left|\dfrac{t}{t+1}\right| = \dfrac{1}{2}$

$\dfrac{t}{t+1} = \pm\dfrac{1}{2}$

$\dfrac{t+1-1}{t+1} = \pm\dfrac{1}{2}$

$1 - \dfrac{1}{t+1} = \pm\dfrac{1}{2}$

$\dfrac{1}{t+1} = 1 \pm \dfrac{1}{2}$

$= \dfrac{1}{2}, \dfrac{3}{2}$

$t + 1 = \dfrac{2}{3}, 2$

$t = -\dfrac{1}{3}, 1$

$\tan\dfrac{x}{2} = -\dfrac{1}{3} \qquad \tan\dfrac{x}{2} = 1$

$\dfrac{x}{2} = \pi - \tan^{-1}\left(\dfrac{1}{3}\right) \qquad \dfrac{x}{2} = \dfrac{\pi}{4}$

$x = 2\pi - 2\tan^{-1}\left(\dfrac{1}{3}\right) \qquad x = \dfrac{\pi}{2}$

$= 5.639684$

$\therefore x = \dfrac{\pi}{2}, 5.64$ (2 dp)

Question 1

$$\int \frac{dx}{1+\sin x}$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int \frac{1}{1+t^2+2t}\,dt$$
$$= 2\int \frac{1}{(t+1)^2}\,dt$$
$$= 2\int (t+1)^{-2}\,dt$$
$$= -\frac{2}{t+1} + c$$
$$= -\frac{2}{\tan\frac{x}{2}+1} + c$$

Question 2

$$\int \frac{5}{12+13\cos x}\,dx$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int \frac{5}{12+13\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$
$$= 10\int \frac{1}{12+12t^2+13-13t^2}\,dt$$
$$= 10\int \frac{1}{25-t^2}\,dt$$
$$= \int \left(\frac{1}{5-t}+\frac{1}{5+t}\right)dt$$
$$= -\ln|5-t| + \ln|5+t| + c$$
$$= \ln\left|\frac{5+\tan\frac{x}{2}}{5-\tan\frac{x}{2}}\right| + c$$

Question 3

$$\int x^3\sqrt{1-x^2}\,dx$$

$$\boxed{\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\,d\theta \end{aligned}}$$

$$= \int \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$$
$$= \int \sin^3\theta\sqrt{\cos^2\theta}\cos\theta\,d\theta$$
$$= \int \sin^3\theta\cos^2\theta\,d\theta$$
$$= \int \sin\theta\sin^2\theta\cos^2\theta\,d\theta$$
$$= \int \sin\theta\left(1-\cos^2\theta\right)\cos^2\theta\,d\theta$$
$$= -\int (-\sin\theta)(\cos^2\theta-\cos^4\theta)\,d\theta$$
$$= \frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3} + c$$

$$\boxed{\begin{aligned} \cos\theta &= \sqrt{1-\sin^2\theta} \\ &= \sqrt{1-x^2} \end{aligned}}$$

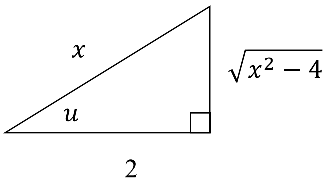
$$= \frac{\sqrt{(1-x^2)^5}}{5} - \frac{\sqrt{(1-x^2)^3}}{3} + c$$

Question 4

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx$$

$$\boxed{\begin{aligned} x &= 2\sec u \\ dx &= 2\sec u\tan u\,du \end{aligned}}$$

$$= \int \frac{1}{2\sec u\sqrt{4\sec^2 u-4}} \times 2\sec u\tan u\,du$$
$$= \int \frac{\tan u}{\sqrt{4\tan^2 u}}\,du$$
$$= \int \frac{\tan u}{2\tan u}\,du$$
$$= \frac{1}{2}\int du$$
$$= \frac{u}{2} + c$$
$$= \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2-4}}{2} + c$$



Alternatively:

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx$$

$$\boxed{\begin{aligned} u^2 &= x^2-4 \\ 2u\,du &= 2x\,dx \\ dx &= \frac{u\,du}{x} \end{aligned}}$$

$$= \int \frac{1}{xu} \times \left(u\frac{du}{x}\right)$$
$$= \int \frac{1}{x^2}\,du$$
$$= \int \frac{1}{u^2+4}\,du$$
$$= \frac{1}{2}\tan^{-1}\frac{u}{2} + c$$
$$= \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2-4}}{2} + c$$

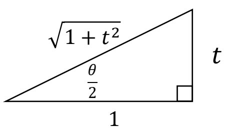
Hint: Integrands in the form $\frac{1}{x\sqrt{x^2-c}}$ can be solved using a u^2 substitution.

Question 5a

$$t = \tan\frac{\theta}{2}$$
$$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$$
$$= \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$$
$$= \frac{1}{2}(1+t^2)$$

Question 5b

$$\tan\frac{\theta}{2} = \frac{t}{1}$$
$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$$
$$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$$



$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$= 2 \times \frac{t}{\sqrt{1+t^2}} \times \frac{1}{\sqrt{1+t^2}}$$
$$= \frac{2t}{1+t^2}$$

Question 5c

$$\int \operatorname{cosec}\theta\ d\theta$$

$$= \int \frac{1+t^2}{2t} \times \frac{2dt}{1+t^2} \text{ from (a)}$$

$$= \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$= \ln\left|\tan\frac{\theta}{2}\right| + c$$

Question 6

$$\int \frac{1}{1-\cos x}\,dx$$

$$= \int \frac{1}{1-\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{1+t^2-1+t^2}\,dt$$

$$= 2\int \frac{dt}{2t^2}$$

$$= \int t^{-2}\,dt$$

$$= -\frac{1}{t} + c$$

$$= -\cot\frac{x}{2} + c$$

Question 7

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$$

$$\boxed{\begin{aligned} t &= \tan\frac{\theta}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int_0^1 \frac{1}{2-\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{1}{2+2t^2-1+t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{1+3t^2}\,dt$$

$$= \frac{2}{\sqrt{3}}\int_0^1 \frac{\sqrt{3}}{1+3t^2}\,dt$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}t\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}\right)$$

$$= \frac{2\sqrt{3}\pi}{9}$$

Question 8

$$\int \sqrt{x+4}\,dx$$

$$\boxed{\begin{aligned} x &= 4\tan^2 u \\ dx &= 8\tan u \sec^2 u\,du \end{aligned}}$$

$$= \int \sqrt{4\tan^2 u + 4} \times 8\tan u \sec^2 u\,du$$

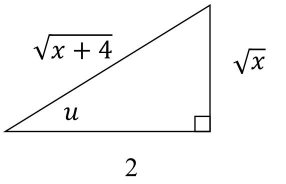
$$= 8\int \tan u \sec^2 u \sqrt{4\sec^2 u}\,du$$

$$= 16\int \tan u \sec^3 u\,du$$

$$= 16\int \tan u \sec u\,(\sec u)^2\,du$$

$$= \frac{16\sec^3 u}{3} + c$$

$$= \frac{2\sqrt{(x+4)^3}}{3} + c$$



Question 9

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) + 5} \times \frac{2dt}{1+t^2}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{6t-4+4t^2+5+5t^2}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{9t^2+6t+1}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{(3t+1)^2}\,dt$$

$$= 2\left[\frac{(3t+1)^{-1}}{-1\times 3}\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= -\frac{2}{3}\left[\frac{1}{3t+1}\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{1}{\sqrt{3}+1}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{\sqrt{3}-1}{2}\right)$$

$$= -\frac{2}{3}\frac{\left(3-2\sqrt{3}\right)}{4}$$

$$= \frac{2\sqrt{3}-3}{6}$$

Question 10

$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\, dx$ $\boxed{\begin{aligned} x &= \sin^2\theta \\ dx &= 2\sin\theta\cos\theta\, d\theta \end{aligned}}$

$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \times 2\sin\theta\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta$

$= \int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\, d\theta$

$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \left(\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0-0)\right)$

$= \frac{\pi}{4} - \frac{1}{2}$

Question 11

$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta}$

$= \int_1^{\sqrt{3}} \frac{1+t^2}{2t} \times \frac{2dt}{1+t^2}$

$= \int_1^{\sqrt{3}} \frac{dt}{t}$

$= \left[\ln t\right]_1^{\sqrt{3}}$

$= \ln\sqrt{3} - \ln 1$

$= \ln 3^{\frac{1}{2}}$

$= \frac{1}{2}\ln 3$

Question 12

$\int \operatorname{cosec} x\, dx$ $\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$

$= \int \frac{1+t^2}{2t} \times \frac{2dt}{1+t^2}$

$= \int \frac{dt}{t}$

$= \ln|t| + c$

$= \ln\left|\tan\frac{x}{2}\right| + c$

Question 13

$\int \frac{1}{\sin x + \cos x}\, dx$ $\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$

$= \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$

$= 2\int \frac{dt}{2t+1-t^2}$

$= 2\int \frac{dt}{2-(t-1)^2}$

$= 2\int \frac{dt}{(\sqrt{2}+t-1)(\sqrt{2}-t+1)}$

$= \frac{1}{\sqrt{2}}\int \left(\frac{1}{t-1+\sqrt{2}} + \frac{1}{1+\sqrt{2}-t}\right) dt$

$= \frac{1}{\sqrt{2}}\left(\ln\left|t-1+\sqrt{2}\right| - \ln\left|1+\sqrt{2}-t\right|\right) + c$

$= \frac{1}{\sqrt{2}}\ln\left|\frac{t-1+\sqrt{2}}{1+\sqrt{2}-t}\right| + c$

$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{1+\sqrt{2}-\tan\frac{x}{2}}\right| + c$

Question 14a

$\operatorname{cosec} x + \cot x$

$= \frac{1+t^2}{2t} + \frac{1-t^2}{2t}$

$= \frac{2}{2t}$

$= \frac{1}{t}$

$= \cot\frac{x}{2}$

Question 14b

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{cosec} x + \cot x)\, dx$

$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\frac{x}{2}\, dx$

$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\, dx$

$= 2\left[\ln\left(\sin\frac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= 2\left(\ln\left(\sin\frac{\pi}{4}\right) - \ln\left(\sin\frac{\pi}{6}\right)\right)$

$= 2\left(\ln\left(\frac{1}{\sqrt{2}}\right) - \ln\left(\frac{1}{2}\right)\right)$

$= 2\left(-\ln\left(\sqrt{2}\right) + \ln(2)\right)$

$= 2\left(-\frac{1}{2}\ln 2 + \ln(2)\right)$

$= 2\left(\frac{1}{2}\ln 2\right)$

$= \ln 2$

Question 15

$$\int \frac{\sec^2 x}{1-\tan^2 x}\,dx$$
$$= \int \frac{1+\tan^2 x}{1-\tan^2 x}\,dx$$
$$= \int \sec 2x\,dx$$
$$= \frac{1}{2}\ln\left|\tan 2x + \sec 2x\right| + c$$

Question 16

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx \qquad \boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta \end{aligned}}$$
$$= \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} \times 2\cos\theta\,d\theta$$
$$= \int \frac{4\sin^2\theta}{2\cos\theta} \times 2\cos\theta\,d\theta$$
$$= 4\int \sin^2\theta\,d\theta$$
$$= 4\int \frac{1}{2}(1-\cos 2\theta)\,d\theta$$
$$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + c$$
$$= 2\theta - 2\sin\theta\cos\theta + c$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + c$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + c$$

Question 17

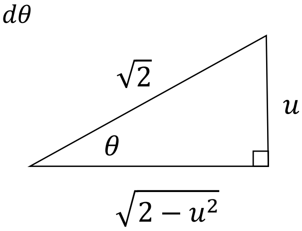
$$\int \frac{3+\cos x}{2-\cos x}\,dx \qquad \boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$
$$= \int \frac{3+\dfrac{1-t^2}{1+t^2}}{2-\dfrac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int \frac{3+3t^2+1-t^2}{2+2t^2-1+t^2} \times \frac{dt}{1+t^2}$$
$$= 2\int \frac{2t^2+4}{(3t^2+1)(1+t^2)}\,dt$$
$$= 2\int \left(\frac{5}{3t^2+1} - \frac{1}{1+t^2}\right)dt$$
$$= \frac{10}{\sqrt{3}}\int \frac{\sqrt{3}}{3t^2+1}\,dt - 2\int \frac{1}{1+t^2}\,dt$$
$$= \frac{10}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}t\right) - 2\tan^{-1} t + c$$
$$= \frac{10}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{x}{2}\right) - x + c$$

Question 18

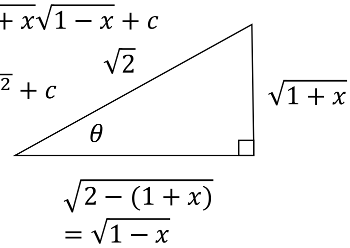
$$\int \sqrt{2x-x^2}\,dx$$
$$= \int \sqrt{-(x^2-2x+1-1)}\,dx$$
$$= \int \sqrt{1-(x-1)^2}\,dx \qquad \boxed{\begin{aligned} x-1 &= \sin\theta \\ dx &= \cos\theta\,d\theta \end{aligned}}$$
$$= \int \sqrt{1-\sin^2\theta} \times \cos\theta\,d\theta$$
$$= \int \cos^2\theta\,d\theta$$
$$= \frac{1}{2}\int (1+\cos 2\theta)d\theta$$
$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + c$$
$$= \frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4} + c$$
$$= \frac{\sin^{-1}(x-1)}{2} + \frac{(x-1)\sqrt{1-(x-1)^2}}{2} + c$$
$$= \frac{\sin^{-1}(x-1)}{2} + \frac{(x-1)\sqrt{2x-x^2}}{2} + c$$

Question 19

$$\int \sqrt{\frac{1-x}{1+x}}\,dx \qquad \boxed{\begin{aligned} u^2 &= 1+x \ \rightarrow x = u^2-1 \\ 2u\,du &= dx \end{aligned}}$$
$$= \int \sqrt{\frac{1-(u^2-1)}{u^2}}\,2u\,du$$
$$= 2\int \sqrt{\frac{2-u^2}{u^2}}\,u\,du$$
$$= 2\int \sqrt{2-u^2}\,du \qquad \boxed{\begin{aligned} u &= \sqrt{2}\sin\theta \\ du &= \sqrt{2}\cos\theta\,d\theta \end{aligned}}$$
$$= 2\int \sqrt{2-2\sin^2\theta} \times \sqrt{2}\cos\theta\,d\theta$$
$$= 2\sqrt{2}\int \sqrt{2\cos^2\theta} \times \cos\theta\,d\theta$$
$$= 4\int \cos^2\theta\,d\theta$$

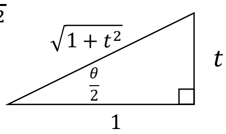


$$= 2\int (1+\cos 2\theta)\,d\theta$$
$$= 2\theta + \sin 2\theta + c$$
$$= 2\theta + 2\sin\theta\cos\theta + c$$
$$= 2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + 2 \times \frac{u}{\sqrt{2}} \times \frac{\sqrt{2-u^2}}{\sqrt{2}} + c$$
$$= 2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + u\sqrt{2-u^2} + c$$
$$= 2\sin^{-1}\left(\frac{\sqrt{1+x}}{\sqrt{2}}\right) + \sqrt{1+x}\sqrt{1-x} + c$$
$$= 2\sin^{-1}\left(\frac{\sqrt{1+x}}{\sqrt{2}}\right) + \sqrt{1+x}\sqrt{1-x} + c$$
$$= 2\sin^{-1}\left(\frac{\sqrt{1+x}}{\sqrt{2}}\right) + \sqrt{1-x^2} + c$$



Question 20

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(1+\sin x)^3}$$
$$= \int_0^{\frac{\pi}{2}} \frac{dx}{\left(1+\sin\left(\frac{\pi}{2}-x\right)\right)^3}$$
$$= \int_0^{\frac{\pi}{2}} \frac{dx}{(1+\cos x)^3}$$
$$= \int_0^{\frac{\pi}{2}} \frac{dx}{\left(2\cos^2\left(\frac{x}{2}\right)\right)^3}$$
$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} \sec^6\left(\frac{x}{2}\right) dx \qquad \boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2}\end{aligned}}$$
$$= \frac{1}{8}\int_0^1 \left(\sqrt{1+t^2}\right)^6 \times \frac{2dt}{1+t^2}$$



$$= \frac{1}{4}\int_0^1 \left(1+t^2\right)^2 dt$$
$$= \frac{1}{4}\int_0^1 (1+2t^2+t^4)\, dt$$
$$= \frac{1}{4}\left[t+\frac{2t^3}{3}+\frac{t^5}{5}\right]_0^1$$
$$= \frac{1}{4}\left(\left(1+\frac{2}{3}+\frac{1}{5}\right)-(0+0+0)\right)$$
$$= \frac{7}{15}$$

Question 21a

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\frac{1}{2}\sin x} \qquad \boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2}\end{aligned}}$$
$$= \int_0^1 \frac{1}{1+\frac{t}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int_0^1 \frac{dt}{1+t^2+t}$$
$$= 2\int_0^1 \frac{dt}{t^2+t+\frac{1}{4}+\frac{3}{4}}$$
$$= 2\int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \frac{4}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$$
$$= \frac{4}{\sqrt{3}}\left(\tan^{-1}\sqrt{3}-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$
$$= \frac{4}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$
$$= \frac{2\pi}{3\sqrt{3}}$$

Question 21b

$$\int_0^a f(2a-x)\, dx \qquad \boxed{\begin{aligned} u &= 2a-x \\ du &= -dx \\ dx &= -du\end{aligned}}$$
$$= \int_{2a}^a f(u)\,(-du)$$
$$= \int_a^{2a} f(u)\, du$$
$$= \int_a^{2a} f(x)\, dx$$

$$\int_0^{2a} f(x)\, dx$$
$$= \int_0^a f(x)\, dx + \int_a^{2a} f(x)\, dx$$
$$= \int_0^a f(x)\, dx + \int_0^a f(2a-x)\, dx$$
$$= \int_0^a [f(x)+f(2a-x)]\, dx$$

Question 21c

$$\int_0^{\pi} \frac{x\, dx}{1+\frac{1}{2}\sin x}$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{x}{1+\frac{1}{2}\sin x}+\frac{\pi-x}{1+\frac{1}{2}\sin(\pi-x)}\right) dx$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{x}{1+\frac{1}{2}\sin x}+\frac{\pi-x}{1+\frac{1}{2}\sin x}\right) dx$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{1+\frac{1}{2}\sin x}\right) dx$$
$$= \pi\int_0^{\frac{\pi}{2}} \frac{dx}{1+\frac{1}{2}\sin x}$$
$$= \pi\left(\frac{2\pi}{3\sqrt{3}}\right) \text{ from } (i)$$
$$= \frac{2\pi^2}{3\sqrt{3}}$$

Question 1

$$\int x^2 \ln x\,dx \qquad \boxed{\begin{array}{ll} u = \ln x & \dfrac{dv}{dx} = x^2 \\ \dfrac{du}{dx} = \dfrac{1}{x} & v = \dfrac{x^3}{3} \end{array}}$$

$$= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \times \frac{1}{x}\,dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$$

$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + c$$

Question 2

$$\int_0^{\frac{\pi}{2}} e^x \cos x\,dx \qquad \boxed{\begin{array}{ll} u = \cos x & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = -\sin x & v = e^x \end{array}}$$

$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x\,dx \qquad \boxed{\begin{array}{ll} u = \sin x & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = \cos x & v = e^x \end{array}}$$

$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \left(\left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x\,dx\right)$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \cos x\,dx = \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \left[e^x \sin x\right]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} e^x \sin x\,dx = \frac{1}{2}\left(\left(0-1\right) + \left(e^{\frac{\pi}{2}} - 0\right)\right)$$

$$= \frac{e^{\frac{\pi}{2}} - 1}{2}$$

Alternative using tabular IBP

S	D	I
$+$	$\cos x$	e^x
$-$	$-\sin x$	e^x
$+$	$-\cos x$	e^x

$$\int_0^{\frac{\pi}{2}} e^x \cos x\,dx = \left[e^x \cos x + e^x \sin x\right]_0^{\frac{\pi}{2}}$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \cos x\,dx = e^{\frac{\pi}{2}}(0+1) - 1(1+0)$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \sin x\,dx = \frac{e^{\frac{\pi}{2}} - 1}{2}$$

Question 3

$$\int x^3 \sin x^2\,dx \qquad \boxed{\begin{array}{ll} u = \dfrac{x^2}{2} & \dfrac{dv}{dx} = 2x \sin x^2 \\ \dfrac{du}{dx} = x & v = -\cos x^2 \end{array}}$$

$$= -\frac{x^2 \cos x^2}{2} + \int x \cos x^2\,dx$$

$$= -\frac{x^2 \cos x^2}{2} + \frac{1}{2}\int 2x\ \cos x^2\,dx$$

$$= -\frac{x^2 \cos x^2}{2} + \frac{\sin x^2}{2} + c$$

Question 4

$$\int \ln x\,dx \qquad \boxed{\begin{array}{ll} u = \ln x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{x} & v = x \end{array}}$$

$$= x \ln x - \int x \times \frac{1}{x}\,dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c$$

Question 5

$$\int x e^{2x}\,dx \qquad \boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = e^{2x} \\ \dfrac{du}{dx} = 1 & v = \dfrac{1}{2}e^{2x} \end{array}}$$

$$= \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$$

Question 6

$$\int_0^{\pi} x \cos x\,dx \qquad \boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = 1 & v = \sin x \end{array}}$$

$$= \left[x \sin x\right]_0^{\pi} - \int_0^{\pi} \sin x\,dx$$

$$= 0 - 0 + \left[\cos x\right]_0^{\pi}$$

$$= -1 - 1$$

$$= -2$$

Question 7

$$\int_0^2 te^{-t}\,dt \qquad \boxed{\begin{array}{ll} u = t & \dfrac{dv}{dt} = e^{-t} \\ \dfrac{du}{dt} = 1 & v = -e^{-t} \end{array}}$$

$$= -\left[te^{-t}\right]_0^2 + \int_0^2 e^{-t}\,dt$$

$$= -\frac{2}{e^2} - \left[e^{-t}\right]_0^2$$

$$= -\frac{2}{e^2} - \frac{1}{e^2} + 1$$

$$= 1 - \frac{3}{e^2}$$

Question 8

$$\int_0^1 \tan^{-1} x\, dx$$

$$\boxed{\begin{array}{ll} u = \tan^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x^2} & v = x \end{array}}$$

$$= \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$$
$$= \frac{\pi}{4} - 0 - \frac{1}{2}\left[\ln\left|1+x^2\right|\right]_0^1$$
$$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0)$$
$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$

Question 9

$$\int \ln(1+x)\, dx$$

$$\boxed{\begin{array}{ll} u = \ln(1+x) & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x} & v = x \end{array}}$$

$$= x\ln(1+x) - \int \frac{x}{1+x}\, dx$$
$$= x\ln(1+x) - \int \frac{1+x-1}{1+x}\, dx$$
$$= x\ln(1+x) - \int \left(1 - \frac{1}{1+x}\right) dx$$
$$= x\ln(1+x) - x + \ln|1+x| + c$$
$$= (x+1)\ln(1+x) - x + c$$

Question 10

$$\int x2^x\, dx$$

$$\boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = 2^x \\ \dfrac{du}{dx} = 1 & v = \dfrac{2^x}{\ln 2} \end{array}}$$

$$= \frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\, dx$$
$$= \frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\left(\frac{2^x}{\ln 2}\right) + c$$
$$= \frac{x2^x}{\ln 2} - \frac{2^x}{\ln^2 2} + c$$

Question 11

$$\int \sin^{-1} x\, dx$$

$$\boxed{\begin{array}{ll} u = \sin^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}} & v = x \end{array}}$$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$
$$= x\sin^{-1} x + \frac{1}{2}\int (-2x)\left(1-x^2\right)^{-\frac{1}{2}} dx$$
$$= x\sin^{-1} x + \frac{1}{2} \times 2\left(1-x^2\right)^{\frac{1}{2}} + c$$
$$= x\sin^{-1} x + \sqrt{1-x^2} + c$$

Question 12

$$\int_4^{e+3} \ln|x-3|\, dx$$

$$\boxed{\begin{array}{ll} u = \ln|x-3| & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{x-3} & v = x \end{array}}$$

$$= \left[x\ln|x-3|\right]_4^{e+3} - \int_4^{e+3} \frac{x}{x-3}\, dx$$
$$= \left((e+3) - 0\right) - \int_4^{e+3} \frac{x-3+3}{x-3}\, dx$$
$$= e + 3 - \int_4^{e+3} \left(1 + \frac{3}{x-3}\right) dx$$
$$= e + 3 - \left[x + 3\ln|x-3|\right]_4^{e+3}$$
$$= e + 3 - \left(\left((e+3)+3\right) - (4+0)\right)$$
$$= 1$$

Question 13

$$\int \tan^{-1} x\, dx$$

$$\boxed{\begin{array}{ll} u = \tan^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x^2} & v = x \end{array}}$$

$$= x\tan^{-1} x - \int \frac{x}{1+x^2}\, dx$$
$$= x\tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right| + c$$

Question 14

$$\int \ln|\cos x| \operatorname{cosec}^2 x\, dx$$

$$\boxed{\begin{array}{ll} u = \ln|\cos x| & \dfrac{dv}{dx} = \operatorname{cosec}^2 x \\ \dfrac{du}{dx} = -\tan x & v = -\cot x \end{array}}$$

$$= -\ln|\cos x| \cot x - \int \tan x \cot x\, dx$$
$$= -\ln|\cos x| \cot x - \int dx$$
$$= -\ln|\cos x| \cot x - x + c$$

Question 15

$$I = \int 2^x \sin x \, dx \qquad \boxed{\begin{array}{ll} u = \sin x & \dfrac{dv}{dx} = 2^x \\ \dfrac{du}{dx} = \cos x & v = \dfrac{2^x}{\ln 2} \end{array}}$$

$$= \frac{2^x \sin x}{\ln 2} - \frac{1}{\ln 2}\int 2^x \cos x \, dx$$

$$\boxed{\begin{array}{ll} u = \cos x & \dfrac{dv}{dx} = 2^x \\ \dfrac{du}{dx} = -\sin x & v = \dfrac{2^x}{\ln 2} \end{array}}$$

$$= \frac{2^x \sin x}{\ln 2} - \frac{1}{\ln 2}\left[\frac{2^x \cos x}{\ln 2} + \frac{1}{\ln 2}\int 2^x \sin x \, dx\right]$$

$$= \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2} - \frac{1}{\ln^2 2}\int 2^x \sin x \, dx$$

$$\therefore \left(1 + \frac{1}{\ln^2 2}\right) I = \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2} + c$$

$$I = \frac{\ln^2 2}{\ln^2 2 + 1}\left(\frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2}\right) + c$$

$$= \frac{2^x(\ln 2 \sin x - \cos x)}{\ln^2 2 + 1} + c$$

Alternative using tabular IBP

S	**D**	**I**
$+$	$\sin x$	2^x
$-$	$\cos x$	$\dfrac{2^x}{\ln 2}$
$+$	$-\sin x$	$\dfrac{2^x}{(\ln 2)^2}$

$$\int 2^x \sin x \, dx = \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2}$$

$$\therefore \left(1 + \frac{1}{\ln^2 2}\right) I = \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2} + c$$

$$I = \frac{\ln^2 2}{\ln^2 2 + 1}\left(\frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{\ln^2 2}\right) + c$$

$$= \frac{2^x(\ln 2 \sin x - \cos x)}{\ln^2 2 + 1} + c$$

Question 16

a $\displaystyle\int x^2\sqrt{x-1}\, dx \qquad \boxed{\begin{array}{ll} u = x^2 & \dfrac{dv}{dx} = (x-1)^{\frac{1}{2}} \\ \dfrac{du}{dx} = 2x & v = \dfrac{2}{3}(x-1)^{\frac{3}{2}} \end{array}}$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\int x(x-1)^{\frac{3}{2}}\, dx$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\left[\frac{2x(x-1)^{\frac{5}{2}}}{5} - \frac{2}{5}\int (x-1)^{\frac{5}{2}}\, dx\right]$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{8x(x-1)^{\frac{5}{2}}}{15} + \frac{16}{105}(x-1)^{\frac{7}{2}} + c$$

$$= \frac{2x^2\sqrt{(x-1)^3}}{3} - \frac{8x\sqrt{(x-1)^5}}{15} + \frac{16\sqrt{(x-1)^7}}{105} + c$$

b $\displaystyle\int x^2\sqrt{x-1}\, dx \qquad \boxed{\begin{array}{r} u^2 = x-1 \\ 2u\, du = dx \end{array}}$

$$= \int x^2 u \times 2u\, du$$

$$= 2\int \left(u^2+1\right)^2 u^2\, du$$

$$= 2\int \left(u^6 + 2u^4 + u^2\right) du$$

$$= \frac{2u^7}{7} + \frac{4u^5}{5} + \frac{2u^3}{3} + c$$

$$= \frac{2\sqrt{(x-1)^7}}{7} + \frac{4\sqrt{(x-1)^5}}{5} + \frac{2\sqrt{(x-1)^3}}{3} + c$$

Question 17

a $\displaystyle\int \frac{\ln x - 2}{(\ln x - 1)^2}\, dx$

$$\boxed{\begin{array}{ll} u = x(\ln x - 2) & \dfrac{dv}{dx} = \dfrac{1}{x}(\ln x - 1)^{-2} \\ \dfrac{du}{dx} = x\left(\dfrac{1}{x}\right) + (\ln x - 2)(1) & v = -(\ln x - 1)^{-1} \\ \quad = \ln x - 1 & \quad = -\dfrac{1}{\ln x - 1} \end{array}}$$

$= -\dfrac{x(\ln x - 2)}{\ln x - 1} + \displaystyle\int \frac{\ln x - 1}{\ln x - 1}\, dx$

$= -\dfrac{x(\ln x - 2)}{\ln x - 1} + \displaystyle\int dx$

$= \dfrac{2x - x\ln x}{\ln x - 1} + x + c$

$= \dfrac{2x - x\ln x}{\ln x - 1} + \dfrac{x\ln x - x}{\ln x - 1} + c$

$= \dfrac{x}{\ln x - 1} + c$

b $\therefore \displaystyle\int \frac{\ln x - 2}{(\ln x - 1)^2}\, dx = \int \frac{\ln x - 1 - 1}{(\ln x - 1)^2} dx$

$= \displaystyle\int \frac{(\ln x - 1)(1) - (x)\left(\frac{1}{x}\right)}{(\ln x - 1)^2} dx$

$= \displaystyle\int \frac{d}{dx}\left(\frac{x}{\ln x - 1}\right) dx$

$= \dfrac{x}{\ln x - 1} + c$

Question 18

$\displaystyle\int x^2 \ln x\, dx$

$$\boxed{\begin{array}{ll} u = x^2 & \dfrac{dv}{dx} = \ln x \\ \dfrac{du}{dx} = 2x & v = x\ln x - x \end{array}}$$

$= x^2(x\ln x - x) - 2\displaystyle\int x(x\ln x - x)\, dx$

$= x^3 \ln x - x^3 - 2\displaystyle\int \left(x^2 \ln x - x^2\right) dx$

$= x^3 \ln x - x^3 - 2\displaystyle\int x^2 \ln x\, dx + 2\int x^2 dx$

$\therefore 3\displaystyle\int x^2 \ln x\, dx = x^3 \ln x - x^3 + \frac{2x^3}{3} + c$

$\displaystyle\int x^2 \ln x\, dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + c$

Question 19

$I = \displaystyle\int x \sin x \cos x\, dx$

$$\boxed{\begin{array}{ll} u = x\sin x & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = \sin x + x\cos x & v = \sin x \end{array}}$$

$= x\sin^2 x - \displaystyle\int \left(\sin^2 x + x\sin x\cos x\right) dx$

$= x\sin^2 x - \displaystyle\int \sin^2 x\, dx - \int x\sin x\cos x\, dx$

$\therefore 2I = x\sin^2 x - \dfrac{1}{2}\displaystyle\int (1 - \cos 2x)\, dx$

$I = \dfrac{x\sin^2 x}{2} - \dfrac{1}{4}\left(x - \dfrac{1}{2}\sin 2x\right) + c$

$= \dfrac{x\sin^2 x}{2} - \dfrac{x}{4} + \dfrac{\sin 2x}{8} + c$

Question 20

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

$$\boxed{\begin{array}{ll} u = xe^x & \dfrac{dv}{dx} = \dfrac{1}{(1+x)^2} \\ \dfrac{du}{dx} = xe^x + e^x & v = -\dfrac{1}{1+x} \\ \quad = e^x(x+1) & \end{array}}$$

$$= -\frac{xe^x}{1+x} + \int e^x\,dx$$

$$= -\frac{xe^x}{1+x} + e^x + c$$

$$= \frac{-xe^x + e^x + xe^x}{1+x} + c$$

$$= \frac{e^x}{1+x} + c$$

Alternatively:

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

$$= \int \frac{(1+x)e^x - e^x}{(1+x)^2}\,dx$$

$$= \int \frac{d}{dx}\left(\frac{e^x}{1+x}\right)dx$$

$$= \frac{e^x}{1+x} + c$$

Question 1

$I_n = \int x^n e^{2x}\,dx$

$$\boxed{\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = e^{2x} \\ \dfrac{du}{dx} = nx^{n-1} & v = \dfrac{1}{2}e^{2x} \end{array}}$$

$= \dfrac{x^n e^{2x}}{2} - \dfrac{n}{2}\int x^{n-1}e^{2x}\,dx$

$= \dfrac{x^n e^{2x}}{2} - \dfrac{n}{2}I_{n-1}$

Question 2

$I_n = \int \sin^n x\,dx$

$$\boxed{\begin{array}{ll} u = \sin^{n-1} x & \dfrac{dv}{dx} = \sin x \\ \dfrac{du}{dx} = (n-1)\sin^{n-2} x\cos x & v = -\cos x \end{array}}$$

$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos x\cos x\,dx$

$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)dx$

$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$

$\therefore I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n$

$nI_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$

$I_n = -\dfrac{\sin^{n-1} x\cos x}{n} + \dfrac{n-1}{n}I_{n-2}$

Question 3

$I_n + I_{n-2}$

$= \int (\cot^n x + \cot^{n-2} x)\,dx$

$= \int \cot^{n-2} x\,(\cot^2 x + 1)\,dx$

$= \int \operatorname{cosec}^2 x\cot^{n-2} x\,\,dx$

$= -\dfrac{1}{n-1}\cot^{n-1} x$

$\therefore I_n = -\dfrac{1}{n-1}\cot^{n-1} x - I_{n-2}$

Question 4

$I_0 = \int x^0 e^{2x}\,dx$

$= \int e^{2x}\,dx$

$= \dfrac{1}{2}e^{2x} + c$

$\int x^2 e^{2x}\,dx = I_2$

$= \dfrac{x^2 e^{2x}}{2} - \dfrac{2}{2}I_{2-1}$

$= \dfrac{x^2 e^{2x}}{2} - \left(\dfrac{xe^{2x}}{2} - \dfrac{1}{2}I_0\right)$

$= \dfrac{x^2 e^{2x}}{2} - \dfrac{xe^{2x}}{2} + \dfrac{1}{4}e^{2x} + c$

Question 5

$$\int_0^1 x^3\left(x^2-1\right)^5 dx$$
$$= I_3$$
$$= \frac{3-1}{3+11} I_1$$
$$= \frac{1}{7}\int_0^1 x\left(x^2-1\right)^5 dx$$
$$= \frac{1}{14}\int_0^1 2x\left(x^2-1\right)^5 dx$$
$$= \frac{1}{14}\left[\frac{\left(x^2-1\right)^6}{6}\right]_0^1$$
$$= \frac{1}{14}\left(0-\frac{1}{6}\right)$$
$$= -\frac{1}{84}$$

Question 6a

$$I_n + I_{n-1}$$
$$= \int_0^{\frac{\pi}{4}} \tan^{2n}\theta\, d\theta + \int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta\,(1+\tan^2\theta)\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \sec^2\theta\,(\tan\theta)^{2n-2}\, d\theta$$
$$= \left[\frac{\tan^{2n-1}\theta}{2n-1}\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2n-1} - 0$$
$$\therefore I_n = \frac{1}{2n-1} - I_{n-1}$$

Question 6b

$$I_3 = \frac{1}{2(3)-1} - I_2$$
$$= \frac{1}{5} - \left(\frac{1}{2(2)-1} - I_1\right)$$
$$= \frac{1}{5} - \frac{1}{3} + I_1$$
$$= -\frac{2}{15} + \int_0^{\frac{\pi}{4}} \tan^2\theta\, d\theta$$
$$= -\frac{2}{15} + \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - 1\right) d\theta$$
$$= -\frac{2}{15} + \left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$$
$$= -\frac{2}{15} + \left(\left(1-\frac{\pi}{4}\right) - (0-0)\right)$$
$$= \frac{13}{5} - \frac{\pi}{4}$$

Question 7

$$I_n = \int_1^{e^2} (\ln x)^n \, dx \qquad \boxed{\begin{array}{ll} u = (\ln x)^n & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = n(\ln x)^{n-1} \times \dfrac{1}{x} & v = x \end{array}}$$

$$= \Big[x(\ln x)^n\Big]_1^{e^2} - n\int_1^{e^2} (\ln x)^{n-1} \, dx$$

$$\therefore I_n = e^2 \times (2)^n - 0 - nI_{n-1}$$
$$I_n = e^2 2^n - nI_{n-1}$$

Question 8a

$$I_n = \int_0^x \sec^n t \, dt \qquad \boxed{\begin{array}{ll} u = \sec^{n-2} t & \dfrac{dv}{dt} = \sec^2 t \\ \dfrac{du}{dt} = (n-2)\sec^{n-3} t \sec t \tan t & v = \tan t \\ \quad = (n-2)\sec^{n-2} t \tan t & \end{array}}$$

$$= \Big[\sec^{n-2} t \tan t\Big]_0^x - (n-2)\int \sec^{n-2} t \tan^2 t \, dt$$
$$= \sec^{n-2} x \tan x - 0 - (n-2)\int \sec^{n-2} t\left(\sec^2 t - 1\right) dt$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n t \, dt + (n-2)\int \sec^{n-2} t \, dt$$
$$\therefore I_n = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$
$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$
$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Question 8b

$$\int_0^{\frac{\pi}{3}} \sec^4 t \, dt$$
$$= I_4$$
$$= \frac{\sec^2 \frac{\pi}{3} \tan \frac{\pi}{3}}{3} + \frac{2}{3} I_2$$
$$= 2^2 \times \frac{\sqrt{3}}{3} + \frac{2}{3}\left(\frac{1 \times \tan \frac{\pi}{3}}{1} + 0\right)$$
$$= \frac{4\sqrt{3}}{3} + \frac{2}{3}\left(\sqrt{3}\right)$$
$$= 2\sqrt{3}$$

Question 9

$$I_n = \int x \ln^n x \, dx \qquad \boxed{\begin{array}{ll} u = (\ln x)^n & \dfrac{dv}{dx} = x \\ \dfrac{du}{dx} = \dfrac{n(\ln x)^{n-1}}{x} & v = \dfrac{x^2}{2} \end{array}}$$

$$= \frac{x^2(\ln x)^n}{2} - \frac{n}{2}\int x(\ln x)^{n-1} \, dx$$
$$\therefore I_n = \frac{x^2(\ln x)^n}{2} - \frac{n}{2} I_{n-1}$$
$$I_0 = \int x \, dx = \frac{x^2}{2} + c$$
$$I_2 = \frac{x^2(\ln x)^2}{2} - \frac{2}{2} I_1$$
$$= \frac{x^2(\ln x)^2}{2} - \left(\frac{x^2 \ln x}{2} - \frac{1}{2} I_0\right)$$
$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{1}{2}\left(\frac{x^2}{2} + c\right)$$
$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + c$$

Question 10

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$$

$$\boxed{\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = nx^{n-1} & v = \sin x \end{array}}$$

$$= \Big[x^n \sin x\Big]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$\boxed{\begin{array}{ll} u = x^{n-1} & \dfrac{dv}{dx} = \sin x \\ \dfrac{du}{dx} = (n-1)x^{n-2} & v = -\cos x \end{array}}$$

$$= \left(\frac{\pi}{2}\right)^n + n\left\{\Big[x^{n-1} \cos x\Big]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx\right\}$$

$$= \left(\frac{\pi}{2}\right)^n + n(0-0) - n(n-1)I_{n-2}$$

$$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

Alternative solution using tabular IBP

$$I_n = \Big[x^n \sin x + nx^{n-1}\cos x\Big]_0^{\frac{\pi}{2}} - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$$

$$= \left(\left(\left(\frac{\pi}{2}\right)^n (1) + 0\right) - (0+0)\right) - n(n-1)I_{n-2}$$

$$= \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

S	**D**	**I**
$+$	x^n	$\cos x$
$-$	nx^{n-1}	$\sin x$
$+$	$n(n-1)x^{n-2}$	$-\cos x$

Question 11

$$I_n = \int x^n \sqrt{2x+1} \, dx$$

$$\boxed{\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = (2x+1)^{\frac{1}{2}} \\ \dfrac{du}{dx} = nx^{n-1} & v = \dfrac{1}{3}(2x+1)^{\frac{3}{2}} \end{array}}$$

$$= \frac{x^n(2x+1)^{\frac{3}{2}}}{3} - \frac{n}{3}\int x^{n-1}(2x+1)^{\frac{3}{2}} \, dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}\int (2x+1)x^{n-1}(2x+1)^{\frac{1}{2}} \, dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}\int 2x \times x^{n-1}(2x+1)^{\frac{1}{2}} \, dx - \frac{n}{3}\int x^{n-1}(2x+1)^{\frac{1}{2}} \, dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{2n}{3}\int x^n\sqrt{2x+1} \, dx - \frac{n}{3}\int x^{n-1}\sqrt{2x+1} \, dx$$

$$\therefore I_n = \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{2n}{3}I_n - \frac{n}{3}I_{n-1}$$

$$\frac{2n+3}{3}I_n = \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}I_{n-1}$$

$$\therefore I_n = \frac{x^n\sqrt{(2x+1)^3}}{2n+3} - \frac{n}{2n+3}I_{n-1}$$

Question 12

$$I_{n-1} = \int x^{-(n-1)} e^{2x}\,dx$$

$$\boxed{\begin{array}{ll} u = x^{-(n-1)} & \dfrac{dv}{dx} = e^{2x} \\ \dfrac{du}{dx} = -(n-1)x^{-n} & v = \dfrac{1}{2}e^{2x} \end{array}}$$

$$= \frac{e^{2x}}{2x^{n-1}} + \frac{n-1}{2}\int x^{-n}e^{2x}\,dx$$

$$I_{n-1} = \frac{e^{2x}}{2x^{n-1}} + \frac{n-1}{2}I_n$$

$$\frac{n-1}{2}I_n = -\frac{e^{2x}}{2x^{n-1}} + I_{n-1}$$

$$I_n = -\frac{e^{2x}}{(n-1)x^{n-1}} + \frac{2}{n-1}I_{n-1}$$

Question 13a

$$I_{m,n} = \int x^m (\ln x)^n\,dx$$

$$\boxed{\begin{array}{ll} u = (\ln x)^n & \dfrac{dv}{dx} = x^m \\ \dfrac{du}{dx} = n(\ln x)^{n-1} \times \dfrac{1}{x} & v = \dfrac{x^{m+1}}{m+1} \end{array}}$$

$$= \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\,dx$$

$$\therefore I_{m,n} = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}I_{m,n-1}$$

Question 13b

$$\int x^2 (\ln x)^2$$

$$= I_{2,2}$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2}{3}I_{2,1}$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2}{3}\left(\frac{x^3 \ln x}{3} - \frac{1}{3}I_{2,0}\right)$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2x^3 \ln x}{9} + \frac{2}{9}\int x^2\,dx$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2x^3 \ln x}{9} + \frac{2}{9}\left(\frac{x^3}{3}\right) + c$$

$$= \frac{x^3(\ln x)^2}{3} - \frac{2x^3 \ln x}{9} + \frac{2x^3}{27} + c$$

Question 14

$$I_n = \int \frac{\sin 3x}{x^n}\,dx$$

$$\boxed{\begin{array}{ll} u = \sin 3x & \dfrac{dv}{dx} = x^{-n} \\ \dfrac{du}{dx} = 3\cos 3x & v = \dfrac{x^{-n+1}}{-n+1} \\ & \quad = -\dfrac{1}{(n-1)x^{n-1}} \end{array}}$$

$$= -\frac{\sin 3x}{(n-1)x^{n-1}} + \frac{3}{n-1}\int \frac{\cos 3x}{x^{n-1}}\,dx$$

$$= -\frac{\sin 3x}{(n-1)x^{n-1}} + \frac{3}{n-1}J_{n-1}$$

Question 15

$$I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx$$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^{n-1} & \frac{dv}{dx} = 2x\left(x^2+1\right)^{-2} \\ \frac{du}{dx} = \frac{1}{2}(n-1)x^{n-2} & v = -\left(x^2+1\right)^{-1} \end{array}}$$

$$= -\frac{1}{2}\left[\frac{x^{n-1}}{x^2+1}\right]_0^1 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{x^2+1}\,dx$$

$$= -\frac{1}{4} - 0 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}\left(x^2+1\right)}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}\int_0^1 \frac{x^n}{(x^2+1)^2}\,dx + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}I_n + \frac{n-1}{2}I_{n-2}$$

$$\therefore \frac{3-n}{2}I_n = -\frac{1}{4} + \frac{n-1}{2}I_{n-2}$$

$$I_n = \frac{1}{2(n-3)} - \frac{n-1}{n-3}I_{n-2}$$

Question 16

$$I_m = \int_0^1 x^m\left(x^2-1\right)^5\,dx$$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^{m-1} & \frac{dv}{dx} = 2x\left(x^2-1\right)^5 \\ \frac{du}{dx} = \frac{m-1}{2}x^{m-2} & v = \frac{1}{6}\left(x^2-1\right)^6 \end{array}}$$

$$= \frac{1}{12}\left[x^{m-1}\left(x^2-1\right)^6\right]_0^1 - \frac{(m-1)}{12}\int_0^1 x^{m-2}\left(x^2-1\right)^6\,dx$$

$$= 0 - \frac{m-1}{12}\int_0^1 x^{m-2}\left(x^2-1\right)\left(x^2-1\right)^5\,dx$$

$$= -\frac{m-1}{12}\int_0^1 x^m\left(x^2-1\right)^5\,dx + \frac{m-1}{12}\int_0^1 x^{m-2}\left(x^2-1\right)^5\,dx$$

$$\therefore \frac{m+11}{12}I_m = \frac{m-1}{12}I_{m-2}$$

$$\therefore I_m = \frac{m-1}{m+11}I_{m-2}$$

Question 17

$$I_n = \int_{-3}^0 x^n\sqrt{x+3}\,dx$$

$$\boxed{\begin{array}{ll} u = x^n & \frac{dv}{dx} = (x+3)^{\frac{1}{2}} \\ \frac{du}{dx} = nx^{n-1} & v = \frac{2}{3}(x+3)^{\frac{3}{2}} \end{array}}$$

$$= \frac{2}{3}\left[x^n(x+3)^{\frac{3}{2}}\right]_{-3}^0 - \frac{2n}{3}\int_{-3}^0 x^{n-1}(x+3)^{\frac{3}{2}}\,dx$$

$$= 0 - \frac{2n}{3}\int_{-3}^0 x^{n-1}(x+3)(x+3)^{\frac{1}{2}}\,dx$$

$$= -\frac{2n}{3}\int_{-3}^0 x^n\sqrt{x+3}\,dx - 2n\int_{-3}^0 x^{n-1}\sqrt{x+3}\,dx$$

$$\therefore I_n = -\frac{2n}{3}I_n - 2nI_{n-1}$$

$$\frac{2n+3}{3}I_n = -2nI_{n-1}$$

$$I_n = -\frac{6n}{2n+3}I_{n-1}$$

Question 18

$$I_n = \int \frac{dx}{\sin^n x}$$

$$\boxed{\begin{array}{ll} u = \operatorname{cosec}^{n-2} x & \dfrac{dv}{dx} = \operatorname{cosec}^2 x \\ \dfrac{du}{dx} = (n-2)\operatorname{cosec}^{n-3} x\,(-\operatorname{cosec} x \cot x) & v = -\cot x \end{array}}$$

$$\begin{aligned}
&= \int \operatorname{cosec}^n x\,dx \\
&= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\,dx \\
&= -\frac{\cos x}{\sin x} \times \frac{1}{\sin^{n-2} x} - (n-2)\int \operatorname{cosec}^{n-2} x\left(\operatorname{cosec}^2 x - 1\right)dx \\
&= -\frac{\cos x}{\sin^{n-1} x} - (n-2)\int \operatorname{cosec}^n x\,dx + (n-2)\int \operatorname{cosec}^{n-2} x\,dx
\end{aligned}$$

$$\therefore I_n = -\frac{\cos x}{\sin^{n-1} x} - (n-2)I_n + (n-2)I_{n-2}$$

$$(n-1)I_n = -\frac{\cos x}{\sin^{n-1} x} + (n-2)I_{n-2}$$

$$\therefore I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}I_{n-2}$$

Question 19a

$$I_{2n+1} = \int_0^1 x^{2n+1}e^{x^2}\,dx$$

$$\boxed{\begin{array}{ll} u = \dfrac{1}{2}x^{2n} & \dfrac{dv}{dx} = 2xe^{x^2} \\ \dfrac{du}{dx} = nx^{2n-1} & v = e^{x^2} \end{array}}$$

$$\begin{aligned}
&= \frac{1}{2}\left[x^{2n}e^{x^2}\right]_0^1 - n\int_0^1 x^{2n-1}e^{x^2}\,dx \\
&= \frac{1}{2}(e-0) - nI_{2n-1} \\
&= \frac{e}{2} - nI_{2n-1}
\end{aligned}$$

Question 19b

$$\begin{aligned}
I_5 &= \frac{e}{2} - 2I_3 \\
&= \frac{e}{2} - 2\left(\frac{e}{2} - I_1\right) \\
&= -\frac{e}{2} + 2\int_0^1 xe^{x^2}\,dx \\
&= -\frac{e}{2} + \left[e^{x^2}\right]_0^1 \\
&= -\frac{e}{2} + e - 1 \\
&= \frac{e}{2} - 1
\end{aligned}$$

Question 20

$$I_n = \int_0^1 \sqrt{x}(1-x)^n\,dx \qquad \boxed{\begin{array}{ll} u = (1-x)^n & \dfrac{dv}{dx} = x^{\frac{1}{2}} \\ \dfrac{du}{dx} = -n(1-x)^{n-1} & v = \dfrac{2}{3}x^{\frac{3}{2}} \end{array}}$$

$$= \frac{2}{3}\left[x^{\frac{3}{2}}(1-x)^n\right]_0^1 + \frac{2n}{3}\int_0^1 x^{\frac{3}{2}}(1-x)^{n-1}\,dx$$

$$= \frac{2}{3}(0-0) + \frac{2n}{3}\int_0^1 x\sqrt{x}(1-x)^{n-1}\,dx$$

$$= -\frac{2n}{3}\int_0^1 (1-x-1)\sqrt{x}(1-x)^{n-1}\,dx$$

$$= -\frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^n\,dx + \frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^{n-1}\,dx$$

$$= -\frac{2n}{3}I_n + \frac{2n}{3}I_{n-1}$$

$$\frac{2n+3}{3}I_n = \frac{2n}{3}I_{n-1}$$

$$I_n = \frac{2n}{2n+3}I_{n-1}$$

$$I_3 = \frac{2(3)}{2(3)+3}I_2$$

$$= \frac{2}{3}\left(\frac{2(2)}{2(2)+3}\right)I_1$$

$$= \frac{8}{21}\left(\frac{2(1)}{2(1)+3}\right)I_0$$

$$= \frac{16}{105}\int_0^1 x^{\frac{1}{2}}\,dx$$

$$= \frac{16}{105}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1$$

$$= \frac{32}{315}$$

Question 21a

$$I_n = \int_0^1 \left(1-x^2\right)^n\,dx \qquad \boxed{\begin{array}{ll} u = \left(1-x^2\right)^n & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = n\left(1-x^2\right)^{n-1}(-2x) & v = x \end{array}}$$

$$= \left[x\left(1-x^2\right)^n\right]_0^1 + 2n\int_0^1 x^2\left(1-x^2\right)^{n-1}\,dx$$

$$= 0 - 0 + 2nJ_{n-1}$$

$$= 2nJ_{n-1}$$

Question 21b

$$\therefore I_n = -2n\int_0^1 \left(1-x^2-1\right)\left(1-x^2\right)^{n-1}\,dx$$

$$= -2n\int_0^1 \left(1-x^2\right)^n\,dx + 2n\int_0^1 \left(1-x^2\right)^{n-1}\,dx$$

$$\mathrm{I}_n = -2nI_n + 2nI_{n-1}$$

$$(2n+1)I_n = 2nI_{n-1}$$

$$I_n = \frac{2n}{2n+1}I_{n-1}$$

Question 21c

$$\begin{aligned}
I_n - I_{n+1} &= \int_0^1 \left(1-x^2\right)^n dx - \int_0^1 \left(1-x^2\right)^{n+1} dx \\
&= \int_0^1 \left[\left(1-x^2\right)^n - \left(1-x^2\right)^{n+1}\right] dx \\
&= \int_0^1 \left[\left(1-x^2\right)^n \left(1-1+x^2\right)\right] dx \\
&= \int_0^1 x^2\left(1-x^2\right)^n dx \\
&= J_n
\end{aligned}$$

$\therefore J_n = I_n - 2(n+1)J_n$ from (a)

$(2n+3)J_n = I_n$

$$J_n = \frac{1}{2n+3} I_n$$

Question 21d

$$\begin{aligned}
J_n &= \frac{1}{2n+3}\left(2nJ_{n-1}\right) \\
&= \frac{2n}{2n+3} J_{n-1}
\end{aligned}$$

Question 22a

$$\int \frac{dx}{5-4\cos x} \qquad \boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2}\end{aligned}}$$

$$\begin{aligned}
&= \int \frac{1}{5 - \frac{4(1-t^2)}{1+t^2}} \times \frac{2dt}{1+t^2} \\
&= 2\int \frac{dt}{5+5t^2-4+4t^2} \\
&= 2\int \frac{dt}{9t^2+1} \\
&= \frac{2}{3}\int \frac{3}{(3t)^2+1} dt \\
&= \frac{2}{3}\tan^{-1} 3t + c \\
&= \frac{2}{3}\tan^{-1}\left(3\tan\frac{x}{2}\right) + c
\end{aligned}$$

Question 22b

$$\begin{aligned}
&\int_0^\pi \frac{\cos x}{5-4\cos x} dx \\
&= \int_0^\pi \frac{-\frac{1}{4}(5-4\cos x) + \frac{5}{4}}{5-4\cos x} dx \\
&= -\frac{1}{4}\int_0^\pi dx + \frac{5}{4}\int_0^\pi \frac{dx}{5-4\cos x} \\
&= -\frac{1}{4}\Big[x\Big]_0^\pi + \frac{5}{4}\left(\frac{\pi}{3}\right) \\
&= -\frac{\pi}{4} + \frac{5\pi}{12} \\
&= \frac{\pi}{6}
\end{aligned}$$

Question 22c

$$u_{n+1}+u_{n-1}-\frac{5}{2}u_n$$
$$=\int_0^{\pi}\frac{\cos\big((n+1)x\big)+\cos((n-1)x)-\frac{5}{2}\cos nx}{5-4\cos x}\,dx$$
$$=\int_0^{\pi}\frac{\cos nx\cos x-\sin nx\sin x+\cos nx\cos x+\sin nx\sin x-\frac{5}{2}\cos nx}{5-4\cos x}\,dx$$
$$=\int_0^{\pi}\frac{2\cos nx\cos x-\frac{5}{2}\cos nx}{5-4\cos x}\,dx$$
$$=\frac{1}{2}\int_0^{\pi}\frac{\cos nx\,(4\cos x-5)}{5-4\cos x}\,dx$$
$$=-\frac{1}{2}\int_0^{\pi}\cos nx\,dx$$
$$=-\frac{1}{2}\left[\frac{1}{n}\sin nx\right]_0^{\pi}$$
$$=-\frac{1}{2n}(0-0)$$
$$=0$$

Question 23a

$$I_1=\int_0^{\frac{1}{2}}\frac{dx}{1+4x^2}$$
$$=\frac{1}{2}\int_0^{\frac{1}{2}}\frac{2}{1+(2x)^2}\,dx$$
$$=\frac{1}{2}\left[\tan^{-1}2x\right]_0^{\frac{1}{2}}$$
$$=\frac{1}{2}\left(\frac{\pi}{4}-0\right)$$
$$=\frac{\pi}{8}$$

Question 23b

$$I_n=\int_0^{\frac{1}{2}}\frac{dx}{(1+4x^2)^n}$$
$$=\int_0^{\frac{1}{2}}\left(1+4x^2\right)^{-n}dx$$

$$\boxed{\begin{array}{ll} u=\left(1+4x^2\right)^{-n} & \dfrac{dv}{dx}=1 \\ \dfrac{du}{dx}=-n\left(1+4x^2\right)^{-(n+1)}(8x) & v=x \end{array}}$$

$$=\left[\frac{x}{(1+4x^2)^n}\right]_0^{\frac{1}{2}}+2n\int_0^{\frac{1}{2}}\frac{4x^2}{(1+4x^2)^{n+1}}\,dx$$
$$=\frac{1}{2^{n+1}}-0+2n\int_0^{\frac{1}{2}}\frac{1+4x^2-1}{(1+4x^2)^{n+1}}\,dx$$
$$=\frac{1}{2^{n+1}}+2n\int_0^{\frac{1}{2}}\frac{1}{(1+4x^2)^n}\,dx-2n\int_0^{\frac{1}{2}}\frac{1}{(1+4x^2)^{n+1}}\,dx$$
$$I_n=\frac{1}{2^{n+1}}+2nI_n-2nI_{n+1}$$
$$(1-2n)I_n=-2nI_{n+1}+\frac{1}{2^{n+1}}$$
$$I_n=\frac{2nI_{n+1}}{2n-1}+\frac{1}{2^{n+1}(1-2n)}$$

Question 23c

$$\begin{aligned}
I_n &= \frac{2nI_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)} \\
I_n - \frac{1}{2^{n+1}(1-2n)} &= \frac{2nI_{n+1}}{2n-1} \\
I_{n+1} &= \frac{2n-1}{2n} I_n + \frac{1}{n2^{n+2}} \\
I_3 &= \frac{2(2)-1}{2(2)} I_2 + \frac{1}{(2)2^{(2)+2}} \\
&= \frac{3}{4} I_2 + \frac{1}{32} \\
&= \frac{3}{4}\left(\frac{2(1)-1}{2(1)} I_1 + \frac{1}{(1)2^{(1)+2}}\right) + \frac{1}{32} \\
&= \frac{3}{4}\left(\frac{1}{2} \times \frac{\pi}{8} + \frac{1}{8}\right) + \frac{1}{32} \\
&= \frac{3}{4}\left(\frac{\pi+2}{16}\right) + \frac{1}{32} \\
&= \frac{3\pi+8}{64}
\end{aligned}$$

Question 24a

$$\begin{aligned}
&\left(1-t^2\right)^{\frac{n-3}{2}} - \left(1-t^2\right)^{\frac{n-1}{2}} \\
&= \left(1-t^2\right)^{\frac{n-3}{2}}\left(1-\left(1-t^2\right)\right) \\
&= t^2\left(1-t^2\right)^{\frac{n-3}{2}}
\end{aligned}$$

Question 24b

$$I_n = \int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}} dt$$

$$\boxed{\begin{array}{ll} u = \left(1-t^2\right)^{\frac{n-1}{2}} & \dfrac{dv}{dt} = 1 \\ \dfrac{du}{dt} = \dfrac{n-1}{2}\left(1-t^2\right)^{\frac{n-3}{2}}(-2t) & v = t \end{array}}$$

$$\begin{aligned}
&= \left[t\left(1-t^2\right)^{\frac{n-1}{2}}\right]_0^1 + (n-1)\int_0^1 t^2\left(1-t^2\right)^{\frac{(n-3)}{2}} dt \\
&= 0 + (n-1)\int_0^1 \left(\left(1-t^2\right)^{\frac{n-3}{2}} - \left(1-t^2\right)^{\frac{n-1}{2}}\right) dt \\
&= (n-1)\int_0^1 \left(1-t^2\right)^{\frac{n-3}{2}} dt - (n-1)\int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}} dt
\end{aligned}$$

$$\begin{aligned}
I_n &= (n-1)I_{n-2} - (n-1)I_n \\
nI_n &= (n-1)I_{n-2}
\end{aligned}$$

Question 24c

Let $P(n)$ represent the proposition.

$P(1)$ is true since

$$J_1 = I_1 I_0$$
$$= \int_0^1 dt \times \int_0^1 \left(1-t^2\right)^{-\frac{1}{2}} dt$$
$$= \Big[t\Big]_0^1 \times \Big[\sin^{-1} t\Big]_0^1$$
$$= (1-0) \times \left(\frac{\pi}{2} - 0\right)$$
$$= \frac{\pi}{2}$$

If $P(k)$ is true for arbitrary $k \geq 1$ then

$J_k = kI_k I_{k-1} = \frac{\pi}{2}$

RTP: $P(k+1)$ $J_{k+1} = (k+1)I_{k+1}I_k = \frac{\pi}{2}$

$(k+1)I_{k+1}I_k$

$= kI_{k-1}I_k$ from (ii)

$= \frac{\pi}{2}$ from $P(k)$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

Question 24d

$0 \leq 1 - t^2 \leq 1$ for $0 \leq t \leq 1$

$$\therefore\ 0 \leq \left(1-t^2\right)^{\frac{n-1}{2}} \leq \left(1-t^2\right)^{\frac{n-2}{2}}$$
$$\therefore 0 < \int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}} dt < \int_0^1 \left(1-t^2\right)^{\frac{n-2}{2}} dt$$
$$0 < I_n < I_{n-1}$$

Question 24e

$$J_n = nI_n I_{n-1} = \frac{\pi}{2}$$
$$I_n I_{n-1} = \frac{\pi}{2n}$$
$$I_n^2 < \frac{\pi}{2n} \text{ from (iv)}$$
$$I_n < \sqrt{\frac{\pi}{2n}}$$
$$J_{n+1} = (n+1)I_{n+1}I_n = \frac{\pi}{2}$$
$$I_{n+1}I_n = \frac{\pi}{2(n+1)}$$
$$I_n^2 > \frac{\pi}{2n} \text{ from (iv)}$$
$$\therefore \sqrt{\frac{\pi}{2(n+1)}} < I_n^2$$
$$\therefore \sqrt{\frac{\pi}{2(n+1)}} < I_n < \sqrt{\frac{\pi}{2n}}$$

Question 25a

$$\begin{aligned}(2k)!! &= (2k)(2k-2)(2k-4)\ldots 2\\ &= 2(k)2(k-1)2(k-2)\ldots 2(1)\\ &= 2^k k!\end{aligned}$$

Question 25b

$(2k+1)!!\,2^k k!$
$= [(2k+1)(2k-1)(2k-3)\ldots 1][2(k)2(k-1)2(k-2)\ldots 2(1)]$
$= [(2k+1)(2k-1)(2k-3)\ldots 1][(2k)(2k-2)(2k-4)\ldots 2]$
$= (2k+1)(2k)(2k-1)(2k-2)(2k-3)(2k-4)\ldots 2\cdot 1$
$= (2k+1)!$

$$\therefore (2k+1)!! = \frac{(2k+1)!}{2^k k!}$$

Question 25c

$$\begin{aligned}I_{2n} &= \int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx\\
&= -\Big[\cos x \sin^{2n-1} x\Big]_0^{\frac{\pi}{2}} + (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n-2} x \cos^2 x\,dx\\
&= -(0-0) + (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n-2} x\left(1-\sin^2 x\right)dx\\
&= (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n-2} x\,dx - (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx\\
I_{2n} &= (2n-1)I_{2n-2} - (2n-1)I_{2n}\\
2nI_{2n} &= (2n-1)I_{2n-2}\\
I_{2n} &= \frac{2n-1}{2n} I_{2n-2}\\
&= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2} I_{2n-4}\\
&= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\frac{2n-5}{2n-4} I_{2n-6}\\
&= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\frac{2n-5}{2n-4}\times\ldots\times\frac{1}{2} I_0\\
&= \frac{(2n-1)!!}{(2n)!!}\int_0^{\frac{\pi}{2}} dx\\
&= \frac{\frac{(2n)!}{2^n n!}}{2^n n!}\times\Big[x\Big]_0^{\frac{\pi}{2}}\\
&= \frac{(2n)!}{2^{2n}(n!)^2}\left(\frac{\pi}{2}-0\right)\\
&= \frac{\pi(2n)!}{2^{2n+1}(n!)^2}\end{aligned}$$

Question 26

$$I_{2n} = \int_{-1}^{1} \left(1-x^2\right)^n dx$$

$$\boxed{\begin{array}{ll} u = \left(1-x^2\right)^n & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = n\left(1-x^2\right)^{n-1}(-2x) & v = x \end{array}}$$

$$\begin{aligned}
&= \Big[x\left(1-x^2\right)^n\Big]_{-1}^{1} + 2n\int_{-1}^{1} x^2\left(1-x^2\right)^{n-1} dx \\
&= 0 - 2n\int_{-1}^{1}\left(1-x^2-1\right)\left(1-x^2\right)^{n-1} dx \\
&= -2n\int_{-1}^{1}\left(1-x^2\right)^n dx + 2n\int_{-1}^{1}\left(1-x^2\right)^{n-1} dx
\end{aligned}$$

$$\begin{aligned}
I_{2n} &= -2nI_{2n} + 2nI_{2n-2} \\
(2n+1)I_{2n} &= 2nI_{2n-2} \\
I_{2n} &= \frac{2n}{2n+1} I_{2n-2} \\
&= \frac{2n}{2n+1}\times\frac{2n-2}{2n-1} I_{2n-4} \\
&= \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-3} I_{2n-6} \\
&= \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-3}\times\ldots\times\frac{2}{3} I_0 \\
&= \frac{2n}{2n+1}\times\frac{2n}{2n}\times\frac{2n-2}{2n-1}\times\frac{2n-2}{2n-2}\times\frac{2n-4}{2n-3}\times\frac{2n-4}{2n-4}\times\ldots\times\frac{2}{3}\times\frac{2}{2}\times\int_{-1}^{1} dx \\
&= \frac{\left(2(n)2(n-1)2(n-2)\ldots2(1)\right)^2}{(2n+1)!}\Big[x\Big]_{-1}^{1} \\
&= \frac{2^{2n}(n!)^2}{(2n+1)!}\left(1-(-1)\right) \\
&= \frac{2^{2n+1}(n!)^2}{(2n+1)!}
\end{aligned}$$

Question 1

$$\int \frac{dx}{x^2+6x+10}$$
$$= \int \frac{dx}{(x+3)^2+1^2}$$
$$= \tan^{-1}(x+3)+c$$

Question 2

$$\int \frac{1}{x^2+2x+2}\,dx$$
$$= \int \frac{1}{(x+1)^2+1^2}\,dx$$
$$= \tan^{-1}(x+1)+c$$

Question 3a

$$I_n = \int_1^e (\ln t)^n\,dt \qquad \boxed{\begin{array}{ll} u=(\ln t)^n & \dfrac{dv}{dt}=1 \\ \dfrac{du}{dt}=\dfrac{n(\ln t)^{n-1}}{t} & v=t \end{array}}$$
$$= \Big[t(\ln t)^n\Big]_1^e - n\int (\ln t)^{n-1}\,dt$$
$$= e - n\int (\ln t)^{n-1}\,dt$$
$$= e - nI_{n-1}$$

Question 3b

$$I_3 = e - 3I_2$$
$$= e - 3(e - 2I_1)$$
$$= -2e + 6(e - I_0)$$
$$= 4e - 6\int_1^e dt$$
$$= 4e - 6\Big[t\Big]_1^e$$
$$= 4e - 6(e-1)$$
$$= -2e + 6$$

Question 4

$$\int x\tan^{-1}x\,dx \qquad \boxed{\begin{array}{ll} u=\tan^{-1}x & \dfrac{dv}{dx}=x \\ \dfrac{du}{dx}=\dfrac{1}{1+x^2} & v=\dfrac{x^2}{2} \end{array}}$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,dx$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c$$

Question 5

$$\int \sin x\cos(\cos x)dx$$
$$= -\int -\sin x\cos(\cos x)dx$$
$$= -\sin(\cos x)+c$$

Question 6

$$\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{16-(3+x)^2}}\,dx$$
$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

ANSWER D

Question 7

$$\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx \qquad \boxed{\begin{array}{l} u=e^x+1 \\ du=e^x dx \end{array}}$$
$$= \int_2^{e+1} \frac{du}{u^2}$$
$$= \int_2^{e+1} u^{-2}\,du$$
$$= -\left[\frac{1}{u}\right]_2^{e+1}$$
$$= -\left[\frac{1}{u}\right]_2^{e+1}$$
$$= -\left(\frac{1}{e+1}-\frac{1}{2}\right)$$
$$= \frac{1}{2}-\frac{1}{e+1}$$

Question 8

$$\int \frac{1}{x\ln x}\,dx$$
$$= \int \frac{1/x}{\ln x}\,dx$$
$$= \ln\left|\ln x\right| + c$$

Question 9

$$\int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}\,dx$$
$$= -\int_0^1 \left(-\frac{1}{\sqrt{1-x^2}}\right)\times(\cos^{-1}x)^1\,dx$$
$$= -\frac{1}{2}\Big[(\cos^{-1}x)^2\Big]_0^1$$
$$= -\frac{1}{2}\left(0-\left(\frac{\pi}{2}\right)^2\right)$$
$$= \frac{\pi^2}{8}$$

ANSWER C

Question 10

$$\int_0^{\frac{\pi}{3}} \sec^3 x\tan x\,dx$$
$$= \int_0^{\frac{\pi}{3}} \sec x\tan x\,(\sec x)^2\,dx$$
$$= \left[\frac{\sec^3 x}{3}\right]_0^{\frac{\pi}{3}}$$
$$= \frac{8}{3}-\frac{1}{3}$$
$$= \frac{7}{3}$$

Question 11

The identity is true for all values of n since

$$\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$$

ANSWER D

Question 12

$$\int \frac{dx}{(x-1)(x+2)}$$
$$= \frac{1}{3}\int\left(\frac{1}{x-1} - \frac{1}{x+2}\right)dx$$
$$= \frac{1}{3}(\ln|x-1| - \ln|x+2|) + c$$
$$= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + c$$

Question 13

$$\int_0^1 xe^{-x^2}\,dx$$
$$= -\frac{1}{2}\int_0^1 (-2x)e^{-x^2}\,dx$$
$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$
$$= -\frac{1}{2}(e^{-1} - 1)$$
$$= \frac{1}{2} - \frac{1}{2e}$$

Question 14a

$$a = \frac{10}{(-1)^2 + 4} = 2$$

equating coefficients of x^2:

$$a + b = 0 \;\Rightarrow\; b = -2$$

equating coefficients of x:

$$b + c = 0 \;\Rightarrow\; c = 2$$

Question 14b

$$\int\left(\frac{2}{x+1} + \frac{-2x+2}{x^2+4}\right)dx$$
$$= \int\left(\frac{2}{x+1} - \frac{2x}{x^2+4} + \frac{2}{x^2+4}\right)dx$$
$$= 2\ln|x+1| - \ln\left|x^2+4\right| + \tan^{-1}\frac{x}{2} + c$$

Question 15

$$\int \frac{t^2-1}{t^3}\,dt$$
$$= \int\left(\frac{1}{t} - t^{-3}\right)dt$$
$$= \ln|t| - \frac{t^{-2}}{-2} + c$$
$$= \ln|t| + \frac{1}{2t^2} + c$$

Question 16

$$\int \frac{dx}{\sqrt{6-x-x^2}}$$
$$= \int \frac{dx}{\sqrt{-\left(x^2+x+\frac{1}{4}-\frac{25}{4}\right)}}$$
$$= \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(\frac{1}{2}+x\right)^2}}$$
$$= \sin^{-1}\left(\frac{x+\frac{1}{2}}{5/2}\right) + c$$
$$= \sin^{-1}\left(\frac{2x+1}{5}\right) + c$$

Question 17

$$\int x^2 \log_e(3x)\,dx$$

$$\boxed{\begin{array}{ll} u = \log_e 3x & \dfrac{dv}{dx} = x^2 \\ \dfrac{du}{dx} = \dfrac{1}{x} & v = \dfrac{x^3}{3} \end{array}}$$

$$= \frac{x^3}{3}\log_e 3x - \frac{1}{3}\int x^2\,dx$$
$$= \frac{x^3}{3}\log_e 3x - \frac{x^3}{9} + c$$

Question 18

$$\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$
$$= \int \cos\theta(\sin\theta)^{-4}\,d\theta$$
$$= \frac{(\sin\theta)^{-3}}{-3} + c$$
$$= -\frac{1}{3\sin^3\theta} + c$$

Question 19

$$\int \frac{dx}{\sqrt{2-4x-2x^2}}$$
$$= \int \frac{dx}{\sqrt{2(1-2x-x^2)}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{2-(1+x)^2}}$$
$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

Question 20a

$$\frac{\cos 2x - \cos 4x}{\sin 4x + \sin 2x}$$
$$= \frac{2\sin 3x \sin x}{2\sin 3x \cos x}$$
$$= \frac{\sin x}{\cos x} \quad \text{since } \sin 3x \neq 0$$
$$= \tan x$$

Question 20b

$$\sin 3x + \sin x$$
$$= 2\sin 2x \cos x$$
$$= 2(2\sin x\cos x)\cos x$$
$$= 4\sin x\cos^2 x$$

Question 20c

$2\cos 2x\cos x$
$= 2\left(2\cos^2 x - 1\right)\cos x$
$= 4\cos^3 x - 2\cos x$

Question 20d

$2\tan x\cos 3x$
$= \dfrac{2\sin x\cos 3x}{\cos x}$
$= \dfrac{\sin 4x - \sin 2x}{\cos x}$
$= \sec x\,(\sin 4x - \sin 2x)$

Question 21

$\int x\cos x\,dx$ $\quad \boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = 1 & v = \sin x \end{array}}$

$= x\sin x - \int \sin x\,dx$
$= x\sin x + \cos x + c$

Question 22

$\int \dfrac{x+1}{x-2}\,dx$
$= \int \dfrac{x-2+3}{x-2}\,dx$
$= \int \left(1 + \dfrac{3}{x-2}\right)dx$
$= x + 3\ln|x-2| + c$

Question 23

$\int \dfrac{x}{\sqrt{16-x^2}}\,dx$
$= -\dfrac{1}{2}\int (-2x)\left(16-x^2\right)^{-\frac{1}{2}}\,dx$
$= -\dfrac{1}{2}\times 2\left(16-x^2\right)^{\frac{1}{2}} + c$
$= -\sqrt{16-x^2} + c$

ANSWER B

Question 24

$\int_0^{\frac{\pi}{2}} \dfrac{dx}{1+\sin x}$ $\quad \boxed{\begin{array}{l} t = \tan\dfrac{x}{2} \\ dx = \dfrac{2dt}{1+t^2} \end{array}}$

$= \int_0^1 \dfrac{1}{1+\dfrac{2t}{1+t^2}} \times \dfrac{2dt}{1+t^2}$
$= 2\int_0^1 \dfrac{1}{1+t^2+2t}\,dt$
$= 2\int_0^1 (t+1)^{-2}\,dt$
$= -2\left[\dfrac{1}{t+1}\right]_0^1$
$= -2\left(\dfrac{1}{2} - 1\right)$
$= 1$

Question 25

$\int \dfrac{e^x dx}{\sqrt{1-e^{2x}}}$ $\quad \boxed{\begin{array}{l} u = e^x \\ du = e^x dx \end{array}}$

$= \int \dfrac{e^x}{\sqrt{1-u^2}} \times \dfrac{du}{e^x}$
$= \int \dfrac{1}{\sqrt{1-u^2}}\,du$
$= \sin^{-1} u + c$
$= \sin^{-1} e^x + c$

Question 26

$\int \dfrac{4x^3 - 2x^2 + 1}{2x-1}\,dx$
$= \int \dfrac{2x^2(2x-1)+1}{2x-1}\,dx$
$= \int \left(2x^2 + \dfrac{1}{2x-1}\right)dx$
$= \dfrac{2}{3}x^3 + \dfrac{1}{2}\ln|2x-1| + c$

Question 27

$\int \dfrac{dx}{\sqrt{3-4x-4x^2}}$
$= \int \dfrac{dx}{\sqrt{-(4x^2+4x+1-4)}}$
$= \int \dfrac{dx}{\sqrt{2^2-(2x+1)^2}}$
$= \dfrac{1}{2}\int \dfrac{2\,dx}{\sqrt{2^2-(2x+1)^2}}$
$= \dfrac{1}{2}\sin^{-1}\left(\dfrac{2x+1}{2}\right) + c$

Question 28

$\cos^4 x\sin^5 x$ is the product of an even and an odd function which is odd.
$\int_{-a}^{a} f(x)\,dx = 0$ for odd functions
$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\sin^5 x\,dx = 0$

Question 29

$\int_0^1 \tan^{-1} x\,dx$ $\quad \boxed{\begin{array}{ll} u = \tan^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x^2} & v = x \end{array}}$

$= \left[x\tan^{-1} x\right]_0^1 - \int_0^1 \dfrac{x}{1+x^2}\,dx$
$= \dfrac{\pi}{4} - \dfrac{1}{2}\left[\ln\left|1+x^2\right|\right]_0^1$
$= \dfrac{\pi}{4} - \dfrac{1}{2}(\ln 2 - 0)$
$= \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$

Question 30

$$\int \frac{dx}{x^2-6x+13}$$

$$= \int \frac{dx}{(x-3)^2+2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c$$

Question 31

$\sin 130° + \sin 10°$

$= \sin(70° + 60°) + \sin(70° - 60°)$

$= 2\sin 70° \cos 60°$

$= 2\sin 70° \times \frac{1}{2}$

$= \sin 70°$

Alt:

$\sin 130° + \sin 10°$

$$= 2\sin\left(\frac{130+10}{2}\right)\cos\left(\frac{130-10}{2}\right)$$

$= 2\sin 70° \cos 60°$

$= 2\sin 70° \times \frac{1}{2}$

$= \sin 70°$

Question 32

$$f(x) = \sin 3x \cos x - \cos 2x \sin 2x$$

$$= \frac{1}{2}\left[\sin 4x + \sin 2x\right] - \frac{1}{2}\sin 4x$$

$$= \frac{1}{2}\sin 2x$$

The amplitude is $\frac{1}{2}$ and the period is $\frac{2\pi}{2} = \pi$.

Question 33

$$\frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{\left(1 - \frac{1-t^2}{1+t^2}\right)}{\frac{2t}{1+t^2}}$$

$$= \frac{1+t^2-1+t^2}{2t}$$

$$= \frac{2t^2}{2t}$$

$= t$

Answer B

Question 34

$$\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\,dx$$

$$= \int_1^4 \frac{x^3}{u^{\frac{3}{2}}} \times \frac{du}{2x}$$

$$= \frac{1}{2}\int_1^4 (u-1)u^{-\frac{3}{2}}\,du$$

$$= \frac{1}{2}\int_1^4 \left(u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\right)du$$

$$= \frac{1}{2}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]_1^4$$

$$= \frac{1}{2}\big((4+1)-(2+2)\big)$$

$$= \frac{1}{2}$$

Question 35

$$\int x\sqrt{x}\left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right)dx$$

$$= \int \left(x^2 + x^3\right)dx$$

$$= \frac{x^3}{3} + \frac{x^4}{4} + c$$

Question 36

$$\int_0^{\log_3 2} 3^{x\log_3 2}\,dx$$

$$= \int_0^{\log_3 2} 3^{\log_3 2^x}\,dx$$

$$= \int_0^{\log_3 2} 2^x\,dx$$

$$= \frac{1}{\ln 2}\left[2^x\right]_0^{\log_3 2}$$

$$= \frac{1}{\ln 2}\left(2^{\log_3 2} - 1\right)$$

Question 37

$$\frac{\sin B - \sin A}{\cos A - \cos B}$$

$$= -\frac{\sin A - \sin B}{\cos A - \cos B}$$

$$= -\frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$= \cot\left(\frac{A+B}{2}\right)$$

Question 38

$$\int_0^{\frac{\pi}{6}} \frac{d\theta}{9-8\cos^2\theta}$$

$$\boxed{\begin{aligned} t &= \tan\theta \\ \frac{dt}{d\theta} &= \sec^2\theta \\ &= 1+t^2 \\ d\theta &= \frac{dt}{1+t^2} \end{aligned}}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9-\frac{8}{\sec^2\theta}} \times \frac{dt}{1+t^2}$$
$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9-\frac{8}{1+t^2}} \times \frac{dt}{1+t^2}$$
$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9+9t^2-8}\, dt$$
$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9t^2}\, dt$$
$$= \frac{1}{3}\int_0^{\frac{1}{\sqrt{3}}} \frac{3}{1+(3t)^2}\, dt$$
$$= \frac{1}{3}\Big[\tan^{-1} 3t\Big]_0^{\frac{1}{\sqrt{3}}}$$
$$= \frac{1}{3}\left(\frac{\pi}{3}-0\right)$$
$$= \frac{\pi}{9}$$

Question 39

$$\int \frac{e^{2x}}{e^x+1}\, dx$$

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x dx \\ dx &= \frac{du}{u} \end{aligned}}$$

$$= \int \frac{u^2}{u+1} \times \frac{du}{u}$$
$$= \int \frac{u}{u+1}\, du$$
$$= \int \left(\frac{u+1}{u+1} - \frac{1}{u+1}\right) du$$
$$= \int \left(1 - \frac{1}{u+1}\right) du$$
$$= u - \ln(u+1) + c$$
$$= e^x - \ln|e^x+1| + c$$

Question 40a

$$a = \frac{8-2(-1)}{4+(-1)^2} = 2$$

equating coefficients of x^2: $a+b=0 \quad \Rightarrow \quad b = -2$

equating constants: $4a+c=8 \quad \Rightarrow \quad c=0$

$\therefore a=2, b=-2, c=0$

Question 40b

$$\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\, dx$$
$$= \int_0^4 \left(\frac{2}{1+x} - \frac{2x}{4+x^2}\right) dx$$
$$= \Big[2\ln(1+x) - \ln(4+x^2)\Big]_0^4$$
$$= (2\ln 5 - \ln 20) - (2\ln 1 - \ln 4)$$
$$= \ln 25 - \ln 20 + \ln 4$$
$$= \ln \frac{25\times 4}{20}$$
$$= \ln 5$$

Question 41

The 5 can be absorbed into the values of A and B, so is not needed in either denominator as a factor. The easiest way to get the required denominator of $(x+h)^2$ is to have a constant over $(x+h)^2$ and another constant over $x+h$.

ANSWER C

Question 42

$$\int_{-1}^1 \frac{1}{1+e^{-x}} \times \frac{e^x}{e^x}\, dx$$
$$= \int_{-1}^1 \frac{e^x}{e^x+1}\, dx$$
$$= \Big[\ln(e^x+1)\Big]_{-1}^1$$
$$= \ln(e+1) - \ln(e^{-1}+1)$$
$$= \ln(e+1) - \ln\left(\frac{1+e}{e}\right)$$
$$= \ln\left(\frac{e+1}{(1+e)/e}\right)$$
$$= \ln e$$
$$= 1$$

ANSWER B

Question 43

$$\begin{aligned} 2\cos\theta + 3\sin\theta &= -2 \\ 2\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right) &= -2 \\ 2-2t^2+6t &= -2-2t^2 \\ 6t &= -4 \\ t &= -\frac{2}{3} \\ \tan\frac{\theta}{2} &= -\frac{2}{3} \\ \frac{\theta}{2} &= \pi - \tan^{-1}\left(\frac{2}{3}\right) \\ \theta &= 2\left(\pi - \tan^{-1}\left(\frac{2}{3}\right)\right) \\ &= 5.107180 \end{aligned}$$

Check $\theta = \pi$:

$2\cos\pi + 3\sin\pi = -2 + 0 = -2$ ✓

$\therefore t = 5.11\ (2\ dp), \pi$

Question 44a

I_n

$= \int_0^1 \sqrt{x}(1$

$- x)^n\, dx$ $\boxed{\begin{array}{ll} u = (1-x)^n & \dfrac{dv}{dx} = x^{\frac{1}{2}} \\ \dfrac{du}{dx} = -n(1-x)^{n-1} & v = \dfrac{2}{3}x^{\frac{3}{2}} \end{array}}$

$= \frac{2}{3}\left[x^{\frac{3}{2}}(1-x)^n\right]_0^1 + \frac{2n}{3}\int_0^1 x^{\frac{3}{2}}(1-x)^{n-1}\, dx$

$= 0 + \frac{2n}{3}\int_0^1 \sqrt{x}\cdot x(1-x)^{n-1}\, dx$

$= -\frac{2n}{3}\int_0^1 \sqrt{x}(1-x-1)(1-x)^{n-1}\, dx$

$= -\frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^n\, dx$

$+ \frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^{n-1}\, dx$

$\therefore\ I_n = -\frac{2n}{3}I_n + \frac{2n}{3}I_{n-1}$

$\frac{2n+3}{3}I_n = \frac{2n}{3}I_{n-1}$

$I_n = \left(\frac{2n}{2n+3}\right)I_{n-1}$

Question 44b

$I_3 = \left(\frac{2(3)}{2(3)+3}\right)I_2$

$= \frac{2}{3}\left(\frac{2(2)}{2(2)+3}\right)I_1$

$= \frac{8}{21}\left(\frac{2(1)}{2(1)+3}\right)I_0$

$= \frac{16}{105}\int_0^1 \sqrt{x}\, dx$

$= \frac{16}{105}\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^1$

$= \frac{32}{315}$

Question 45

$\sin 3x + \sin x + \sin 2x$

$= \sin(2x+x) + \sin(2x-x) + \sin 2x$

$= 2\sin 2x \cos x + \sin 2x$

$= \sin 2x\,(2\cos x + 1)$

Question 46a

$c = \frac{16}{2^2+4} = 2$

equating coefficients of x^2:

$-a + c = 0 \quad \Rightarrow \quad a = 2$

equating constants:

$2b + 4c = 16 \quad \Rightarrow \quad b = 4$

$a = 2, b = 4, c = 2$

Question 46b

$\int \frac{16}{(x^2+4)(2-x)}\, dx$

$= \int \left(\frac{2x+4}{x^2+4} + \frac{2}{2-x}\right) dx$

$= \int \left(\frac{2x}{x^2+4} + \frac{4}{x^2+4} + \frac{2}{2-x}\right) dx$

$= \ln\left|x^2+4\right| + 2\tan^{-1}\left(\frac{x}{2}\right) - 2\ln|2-x| + c$

Question 47

$\int_0^1 \sin^{-1} x\, dx$ $\boxed{\begin{array}{ll} u = \sin^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}} & v = x \end{array}}$

$= \left[x\sin^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$

$= \frac{\pi}{2} - 0 + \frac{1}{2}\int_0^1 (-2x)\left(1-x^2\right)^{-\frac{1}{2}}\, dx$

$= \frac{\pi}{2} + \frac{1}{2}\times 2\left[\left(1-x^2\right)^{\frac{1}{2}}\right]_0^1$

$= \frac{\pi}{2} - 1$

Question 48

$\int_0^1 \frac{x}{(x+1)(2x+1)}\, dx$

$= \int_0^1 \left(\frac{1}{x+1} - \frac{1}{2x+1}\right) dx$

$= \left[\ln|x+1| - \frac{1}{2}\ln|2x+1|\right]_0^1$

$= \left(\ln 2 - \frac{1}{2}\ln 3\right) - \left(\ln 1 - \frac{1}{2}\ln 1\right)$

$= \ln\frac{2}{\sqrt{3}}$

Question 49

$\int_0^{\frac{\pi}{4}} x\sec^2 x\, dx$ $\boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = \sec^2 x \\ \dfrac{du}{dx} = 1 & v = \tan x \end{array}}$

$= \left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\, dx$

$= \frac{\pi}{4} - 0 + \left[\ln|\cos x|\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} + \ln\left|\frac{1}{\sqrt{2}}\right|$

$= \frac{\pi}{4} + \ln 1 - \ln\sqrt{2}$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

ANSWER A

Question 50

$$\int (\tan^3 2x + \tan 2x)\,dx$$
$$= \int \tan 2x\,(\tan^2 2x + 1)\,dx$$
$$= \frac{1}{2}\int 2\sec^2 2x\,(\tan 2x)^1\,dx$$
$$= \frac{\tan^2 2x}{2\times 2} + c$$
$$= \frac{1}{4}\tan^2 2x + c$$

ANSWER C

Question 51

$$\cos x - 3\sin x + 3 = 0$$
$$\frac{1-t^2}{1+t^2} - 3\left(\frac{2t}{1+t^2}\right) + 3 = 0$$
$$1 - t^2 - 6t + 3 + 3t^2 = 0$$
$$2t^2 - 6t + 4 = 0$$
$$t^2 - 3t + 2 = 0$$
$$(t-1)(t-2) = 0$$
$$t = 1, 2$$
$$\tan\frac{x}{2} = 1 \qquad \tan\frac{x}{2} = 2$$
$$\frac{x}{2} = \frac{\pi}{4} \qquad \frac{x}{2} = \tan^{-1}(2)$$
$$x = \frac{\pi}{2} \qquad x = 2.214297$$

Check $\theta = \pi$:

$\cos\pi - 3\sin\pi + 3 = -1 + 0 + 3 = 2 \neq 0$ ✗

$\therefore x = \frac{\pi}{2}$, 2.21 (2 dp)

Question 52

$$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin\theta}\,d\theta \qquad \boxed{\begin{aligned} t &= \tan\frac{\theta}{2} \\ d\theta &= \frac{2dt}{1+t^2}\end{aligned}}$$
$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1-\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+t^2-2t}\,dt$$
$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(t-1)^2}\,dt$$
$$= -2\left[\frac{1}{t-1}\right]_0^{\frac{1}{\sqrt{3}}}$$
$$= -2\left(\frac{1}{\frac{1}{\sqrt{3}}-1} + 1\right)$$
$$= -2\left(\frac{\sqrt{3}}{1-\sqrt{3}} + \frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$$
$$= -2\left(\frac{1}{1-\sqrt{3}}\right) \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$
$$= -\frac{2(1+\sqrt{3})}{1-3}$$
$$= \sqrt{3} + 1$$

Question 53

$$\int \sqrt{\frac{5-x}{5+x}}\,dx$$
$$= \int \sqrt{\frac{5-x}{5+x}} \times \sqrt{\frac{5-x}{5-x}}\,dx$$
$$= \int \frac{5-x}{\sqrt{25-x^2}}\,dx$$
$$= \int \frac{5}{\sqrt{25-x^2}}\,dx + \frac{1}{2}\int (-2x)(25-x^2)^{-\frac{1}{2}}\,dx$$
$$= 5\sin^{-1}\left(\frac{x}{5}\right) + \frac{1}{2}\times 2(25-x^2)^{\frac{1}{2}} + c$$
$$= 5\sin^{-1}\left(\frac{x}{5}\right) + \sqrt{25-x^2} + c$$

Question 54a

$$A = \frac{10}{1+(-3)^2} = 1$$

equating coefficients of x^2:

$A + B = 0 \quad \Rightarrow \quad B = -1$

equating constants:

$A + 3C = 10 \quad \Rightarrow \quad C = 3$

$A = 1, B = -1, C = 3$

Question 54b

$$\int \frac{10}{3+\tan\theta}\,d\theta \qquad \boxed{\begin{aligned} t &= \tan\theta \\ d\theta &= \frac{dt}{1+t^2}\end{aligned}}$$
$$= \int \frac{10}{3+t} \times \frac{dt}{1+t^2}$$
$$= \int \frac{10}{(3+t)(1+t^2)}\,dt$$
$$= \int \left(\frac{1}{3+t} - \frac{t-3}{1+t^2}\right)dt$$
$$= \int \left(\frac{1}{3+t} - \frac{t}{1+t^2} + \frac{3}{1+t^2}\right)dt$$
$$= \ln|3+t| - \frac{1}{2}\ln\left|1+t^2\right| + 3\tan^{-1}t + c$$
$$= \ln|3+\tan\theta| - \frac{1}{2}\ln\left|1+\tan^2\theta\right| + 3\tan^{-1}(\tan\theta) + c$$
$$= \ln|\tan\theta + 3| - \frac{1}{2}\ln\left|\sec^2\theta\right| + 3\theta + c$$

Question 55a

$$I_0 = \int_0^1 \frac{1}{\sqrt{1+x}}\,dx$$
$$= \int_0^1 (1+x)^{-\frac{1}{2}}\,dx$$
$$= \left[\frac{(1+x)^{\frac{1}{2}}}{1/2}\right]_0^1$$
$$= 2\left(\sqrt{2}-1\right)$$
$$= 2\sqrt{2} - 2$$

Question 55b

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx \qquad \boxed{\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = \dfrac{1}{\sqrt{1+x}} \\ \dfrac{du}{dx} = nx^{n-1} & v = 2(1+x)^{\frac{1}{2}} \end{array}}$$

$$= 2\left[x^n(1+x)^{\frac{1}{2}}\right]_0^1 - 2n\int_0^1 x^{n-1}(1+x)^{\frac{1}{2}}\,dx$$

$$= 2\sqrt{2} - 0 - 2n\int_0^1 x^{n-1}\sqrt{1+x}\,dx$$

$= 2\sqrt{2} - 2n(I_n + I_{n-1})$ given

$$\therefore (2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$$

$$I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$$

Question 55c

$$I_1 = \frac{2\sqrt{2} - 2(1)I_0}{2(1)+1}$$

$$= \frac{2\sqrt{2} - 2(2\sqrt{2} - 2)}{3}$$

$$= \frac{4 - 2\sqrt{2}}{3}$$

$$I_2 = \frac{2\sqrt{2} - 2(2)I_1}{2(2)+1}$$

$$= \frac{2\sqrt{2} - 4\left(\frac{4-2\sqrt{2}}{3}\right)}{5}$$

$$= \frac{6\sqrt{2} - 16 + 8\sqrt{2}}{15}$$

$$= \frac{14\sqrt{2} - 16}{15}$$

Question 56

$$\int_1^2 \frac{dx}{x(1+x^2)}$$

$$= \int_1^2 \left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$$

$$= \left[\ln|x| - \frac{1}{2}\ln|1+x^2|\right]_1^2$$

$$= \left(\ln 2 - \frac{1}{2}\ln 5\right) - \left(\ln 1 - \frac{1}{2}\ln 2\right)$$

$$= \frac{3}{2}\ln 2 - \frac{1}{2}\ln 5$$

Question 57

$$\int \frac{x}{\sqrt{1-x}}\,dx \qquad \boxed{\begin{array}{r} u^2 = 1 - x \\ 2u\,du = -dx \\ dx = -2u\,du \end{array}}$$

$$= \int \frac{(1-u^2)}{u})(-2u\,du)$$

$$= 2\int (u^2 - 1)\,du$$

$$= \frac{2u^3}{3} - 2u + C$$

$$= \frac{2\sqrt{(1-x)^3}}{3} - 2\sqrt{1-x} + c$$

Question 58

$$I = \int e^{-2x}\cos x\,dx \qquad \boxed{\begin{array}{ll} u = \cos x & \dfrac{dv}{dx} = e^{-2x} \\ \dfrac{du}{dx} = -\sin x & v = -\dfrac{1}{2}e^{-2x} \end{array}}$$

$$I = -\frac{1}{2}e^{-2x}\cos x - \frac{1}{2}\int e^{-2x}\sin x\,dx$$

$$\boxed{\begin{array}{ll} u = \sin x & \dfrac{dv}{dx} = e^{-2x} \\ \dfrac{du}{dx} = \cos x & v = -\dfrac{1}{2}e^{-2x} \end{array}}$$

$$= -\frac{1}{2}e^{-2x}\cos x$$

$$-\frac{1}{2}\left(-\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\int e^{-2x}\cos x\,dx\right)$$

$$= -\frac{1}{2}e^{-2x}\cos x + \frac{1}{4}e^{-2x}\sin x - \frac{1}{4}I + c$$

$$\frac{5}{4}I = -\frac{1}{2}e^{-2x}\cos x + \frac{1}{4}e^{-2x}\sin x + c$$

$$I = \frac{e^{-2x}(\sin x - 2\cos x)}{5} + c$$

Question 59

for $0 < x < 1 \quad x^n > x^{n+1}$

$\therefore \dfrac{1}{1+x^n} < \dfrac{1}{1+x^{n+1}}$ ∴ statement I is true

$\sin x$ and $\cos x$ are symmetrical about $\frac{\pi}{4}$,

so $\int_0^{\frac{\pi}{2}} \sin x\,dx = \int_0^{\frac{\pi}{2}} \cos x\,dx$ ∴ statement II is true

ANSWER D

Question 60

$$\int_0^1 \cos^{-1} x\,dx \qquad \boxed{\begin{array}{ll} u = \cos^{-1} x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = -\dfrac{1}{\sqrt{1-x^2}} & v = x \end{array}}$$

$$= \left[x\cos^{-1} x\right]_0^1 + \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= 0 - 0 - \frac{1}{2}\int_0^1 (-2x)(1-x^2)^{-\frac{1}{2}}\,dx$$

$$= -\frac{1}{2} \times 2\left[(1-x^2)^{\frac{1}{2}}\right]_0^1$$

$$= -(0 - 1)$$

$$= 1$$

Question 61a

$$I_1 = \int_0^1 \frac{x}{\sqrt{1-x}}\,dx \qquad \boxed{\begin{array}{r} u^2 = 1 - x \\ 2u\,du = -dx \\ dx = -2u\,du \end{array}}$$

$$= \int_1^0 \frac{1-u^2}{u} \times (-2u\,du)$$

$$= 2\int_0^1 (1-u^2)\,du$$

$$= 2\left[u - \frac{u^3}{3}\right]_0^1$$

$$= 2\left(1 - \frac{1}{3}\right)$$

$$= \frac{4}{3}$$

Question 61b

$I_{n-1} - I_n$

$= \int_0^1 \frac{x^{n-1}}{\sqrt{1-x}}\,dx - \int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx$

$= \int_0^1 \frac{x^{n-1}(1-x)}{\sqrt{1-x}}\,dx$

$= \int_0^1 x^{n-1}\sqrt{1-x}\,dx$

Question 61c

$I_{n-1} - I_n$

$= \int_0^1 x^{n-1}\sqrt{1-x}\,dx$

$$\boxed{\begin{array}{ll} u = \sqrt{1-x} & \frac{dv}{dx} = x^{n-1} \\ \frac{du}{dx} = -\frac{1}{2}(1-x)^{-\frac{1}{2}} & v = \frac{x^n}{n} \\ \quad = -\frac{1}{2\sqrt{1-x}} & \end{array}}$$

$= \frac{1}{n}\left[x^n\sqrt{1-x}\right]_0^1 + \frac{1}{2n}\int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx$

$I_{n-1} - I_n = \frac{1}{2n} I_n$

$2nI_{n-1} - 2nI_n = I_n$

$(2n+1)I_n = 2nI_{n-1}$

$I_n = \frac{2n}{2n+1} I_{n-1}$

Question 62

$\int \sec x\,dx$

$$\boxed{\begin{array}{l} t = \tan\frac{x}{2} \\ dx = \frac{2dt}{1+t^2} \end{array}}$$

$= \int \frac{1+t^2}{1-t^2}\left(\frac{2dt}{1+t^2}\right)$

$= 2\int \frac{1}{1-t^2}\,dt$

$= \int \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt$

$= \ln|1+t| - \ln|1-t| + c$

$= \ln\left|\frac{1+t}{1-t}\right| + c$

ANSWER B

Question 63

$\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx$

$$\boxed{\begin{array}{ll} u = x & \frac{dv}{dx} = \tan^2 x \\ & \quad = \sec^2 x - 1 \\ \frac{du}{dx} = 1 & v = \tan x - x \end{array}}$$

$= \left[x(\tan x - x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (\tan x - x)\,dx$

$= \left(\frac{\pi}{4}\left(1 - \frac{\pi}{4}\right)\right) - 0 - \left[-\ln|\cos x| - \frac{x^2}{2}\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4}\left(1 - \frac{\pi}{4}\right) - \left(\left(-\ln\frac{1}{\sqrt{2}} - \frac{\pi^2}{32}\right) - (0 - 0)\right)$

$= \frac{\pi}{4} - \frac{\pi^2}{16} + \frac{\pi^2}{32} - \frac{1}{2}\ln 2$

$= \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\ln 2$

Question 64

$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

$= 2\int \left(\frac{1}{2}x^{-\frac{1}{2}}\right)\cos\sqrt{x}\,dx$

$= 2\sin\sqrt{x} + c$

ANSWER B

Question 65a

$B = \frac{3(1)^2 + 3(1) - 2}{(1+1)^2} = 1$

equating coefficients of x^2:

$A + B = 3 \quad \Rightarrow \quad A = 2$

$A = 2, B = 1$

Question 65b

$\int \frac{3x^2 + 3x - 2}{(x-1)(x+1)^2}\,dx$

$= \int \left(\frac{2}{x+1} + \frac{1}{x-1} + \frac{1}{(x+1)^2}\right) dx$

$= 2\ln|x+1| + \ln|x-1| - \frac{1}{x+1} + c$

$= \ln\left|(x+1)^2(x-1)\right| - \frac{1}{x+1} + c$

Question 66

$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$$

$$\boxed{\begin{aligned} x &= 2\cos\theta \\ \frac{dx}{d\theta} &= -2\sin\theta \\ dx &= -2\sin\theta\, d\theta \end{aligned}}$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{1}{4\cos^2\theta\sqrt{4-4\cos^2\theta}}(-2\sin\theta\, d\theta)$$
$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos^2\theta \sin\theta} \times \sin\theta\, d\theta$$
$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2\theta\, d\theta$$
$$= \frac{1}{4}\Big[\tan\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \frac{1}{4}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{4}\left(\frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{2\sqrt{3}}$$
$$= \frac{\sqrt{3}}{6}$$

Question 67a

$$\frac{x^2 - x + 1}{x^2 + 2x + 1} = \frac{1(x^2+2x+1) - 3(x+1) + 3}{x^2+2x+1}$$
$$= 1 - \frac{3}{x+1} + \frac{3}{(x+1)^2}$$

$A = 1, B = -3, C = 3$

Question 67b

$$\int \frac{x^2 - x + 1}{(x+1)^2}\,dx$$
$$= \int \left(1 - \frac{3}{x+1} + \frac{3}{(x+1)^2}\right) dx$$
$$= x - 3\ln|x+1| - \frac{3}{x+1} + c$$

Question 68

$$\int_0^{\frac{\pi}{3}} x\sec^2 x\, dx$$

$$\boxed{\begin{aligned} u &= x & \frac{dv}{dx} &= \sec^2 x \\ \frac{du}{dx} &= 1 & v &= \tan x \end{aligned}}$$

$$= \Big[x\tan x\Big]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan x\, dx$$
$$= \frac{\pi\sqrt{3}}{3} + \Big[\ln|\cos x|\Big]_0^{\frac{\pi}{3}}$$
$$= \frac{\pi\sqrt{3}}{3} + \ln\frac{1}{2} - \ln 1$$
$$= \frac{\pi\sqrt{3}}{3} - \ln 2$$

Question 69a

$$I_n = \int_0^{\pi} x^n \sin x\, dx$$

$$\boxed{\begin{aligned} u &= x^n & \frac{dv}{dx} &= \sin x \\ \frac{du}{dx} &= nx^{n-1} & v &= -\cos x \end{aligned}}$$

$$= -\Big[x^n\cos x\Big]_0^{\pi} + n\int_0^{\pi} x^{n-1}\cos x\, dx$$
$$= -(-\pi^n - 0) + n\int_0^{\pi} x^{n-1}\cos x\, dx$$

$$\boxed{\begin{aligned} u &= x^{n-1} & \frac{dv}{dx} &= \cos x \\ \frac{du}{dx} &= (n-1)x^{n-2} & v &= \sin x \end{aligned}}$$

$$= \pi^n + n\left\{\Big[x^{n-1}\sin x\Big]_0^{\pi} - (n-1)\int x^{n-2}\sin x\, dx\right\}$$
$$I_n = \pi^n + n\{0 - (n-1)I_{n-2}$$
$$= \pi^n - n(n-1)I_{n-2}$$

Question 69b

$$\int_0^{\pi} x^4 \sin x\, dx$$
$$= I_4$$
$$= \pi^4 - 4(3)I_2$$
$$= \pi^4 - 12(\pi^2 - 2(1)I_0)$$
$$= \pi^4 - 12\pi^2 + 24\int_0^{\pi} \sin x\, dx$$
$$= \pi^4 - 12\pi^2 + 24\Big[-\cos x\Big]_0^{\pi}$$
$$= \pi^4 - 12\pi^2 + 24(1+1)$$
$$= \pi^4 - 12\pi^2 + 48$$

Question 70

$I - J$
$$= \int_0^{\ln 2} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$
$$= \Big[\ln|e^x + e^{-x}|\Big]_0^{\ln 2}$$
$$= \ln\left(2 + \frac{1}{2}\right) - \ln(1+1)$$
$$= \ln\left(\frac{5}{4}\right)$$

ANSWER D

Question 71

$f(t) + g(t)$
$$= 3\sin 4t + 3\sin\left(4t + \frac{\pi}{3}\right)$$
$$= 3\left(\sin 4t + \sin\left(4t + \frac{\pi}{3}\right)\right)$$
$$= 3\left(\sin\left(4t + \frac{\pi}{6} - \frac{\pi}{6}\right) + \sin\left(4t + \frac{\pi}{6} + \frac{\pi}{6}\right)\right)$$
$$= 3\left(2\sin\left(4t + \frac{\pi}{6}\right)\cos\frac{\pi}{6}\right)$$
$$= 6\sin\left(4t + \frac{\pi}{6}\right) \times \frac{\sqrt{3}}{2}$$
$$= 3\sqrt{3}\sin\left(4t + \frac{\pi}{6}\right)$$

Question 72a

i $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$$\boxed{\begin{array}{ll} u = \sin^{n-1} x & \dfrac{dv}{dx} = \sin x \\ \dfrac{du}{dx} = (n-1)\sin^{n-2} x \cos x & v = -\cos x \end{array}}$$

$= -\left[\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}}$

$+ (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\, dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,(1-\sin^2 x)\, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$\therefore I_n = (n-1)I_{n-2} - (n-1)I_n$

$nI_n = (n-1)I_{n-2}$

$I_n = \dfrac{n-1}{n} I_{n-2}$

Question 72b

$\int_0^{\frac{\pi}{2}} \sin^5 x\, dx$

$= I_5$

$= \dfrac{4}{5} I_3$

$= \dfrac{4}{5} \times \dfrac{2}{3} I_1$

$= \dfrac{8}{15}\int_0^{\frac{\pi}{2}} \sin x\, dx$

$= \dfrac{8}{15}\left[-\cos x\right]_0^{\frac{\pi}{2}}$

$= \dfrac{8}{15}(0+1)$

$= \dfrac{8}{15}$

Question 73

$\int \sin^3 x\, dx$

$= \int \sin^2 x \sin x\, dx$

$= \int \left(1-\cos^2 x\right) \sin x\, dx$

$= \int \sin x\, dx - \int \sin x \cos^2 x\, dx$

$= -\cos x + \dfrac{\cos^3 x}{3} + c$

ANSWER B

Question 74

$$\begin{aligned} \sin 4x + \sin 3x + \sin 2x &= 0 \\ \sin 4x + \sin 2x + \sin 3x &= 0 \\ \sin(3x+x) + \sin(3x-x) + \sin 3x &= 0 \\ 2\sin 3x \cos x + \sin 3x &= 0 \\ \sin 3x\,(2\cos x + 1) &= 0 \end{aligned}$$

$\sin 3x = 0 \quad 2\cos x + 1 = 0$

$3x = 0, \pi, 2\pi, 3\pi \quad \cos x = -\dfrac{1}{2}$

$x = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi \qquad x = \pi - \dfrac{\pi}{3}$

$= \dfrac{2\pi}{3}$

$\therefore x = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi$

Question 75

$\int x\sqrt{x-1}\, dx \qquad \boxed{\begin{array}{c} u^2 = x-1 \\ 2u\, du = dx \end{array}}$

$= \int \left(u^2+1\right)u \times 2u\, du$

$= 2\int \left(u^4 + u^2\right) du$

$= \dfrac{2u^5}{5} + \dfrac{2u^3}{3} + c$

$= \dfrac{2}{5}(x-1)^{\frac{5}{2}} + \dfrac{2}{3}(x-1)^{\frac{3}{2}} + c$

Question 76

$\int \dfrac{x}{\sqrt{2-x^2}}\, dx \qquad \boxed{\begin{array}{c} x = \sqrt{2}\sin\theta \\ dx = \sqrt{2}\cos\theta\, d\theta \end{array}}$

$= \int \dfrac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}} \times \sqrt{2}\cos\theta\, d\theta$

$= \int \dfrac{\sqrt{2}\sin\theta}{\sqrt{2}\times\sqrt{1-\sin^2\theta}} \times \sqrt{2}\cos\theta\, d\theta$

$= \sqrt{2}\int \dfrac{\sin\theta}{\cos\theta} \times \cos\theta\, d\theta$

$= \sqrt{2}\int \sin\theta\, d\theta$

$= -\sqrt{2}\cos\theta + c$

$= -\sqrt{2}\sqrt{1-\sin^2\theta} + c$

$= -\sqrt{2} \times \sqrt{1 - \dfrac{x^2}{2}} + c$

$= -\sqrt{2-x^2} + c$

Question 77

$\int \dfrac{\cos^3 x + \sin^3 x}{\cos x + \sin x}\, dx$

$= \int \dfrac{(\cos x + \sin x)\left(\cos^2 - \cos x \sin x + \sin^2 x\right)}{\cos x + \sin x}\, dx$

$= \int (1 - \cos x \sin x)\, dx$

$= \int \left(1 - \dfrac{1}{2}\sin 2x\right) dx$

$= x + \dfrac{1}{4}\cos 2x + c$

ANSWER A

Question 78

$$\int \frac{\sin 2x + \sin x}{\cos^2 x}\,dx$$
$$= \int \left(\frac{2\sin x\cos x}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right)dx$$
$$= \int \left(2\tan x + \sin x\,(\cos x)^{-2}\right)dx$$
$$= -2\ln|\cos x| + \frac{1}{\cos x} + c$$
$$= -2\ln|\cos x| + \sec x + c$$

Question 79

$$\int e^x \sin x\,dx \qquad \boxed{\begin{array}{ll} u = \sin x & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = \cos x & v = e^x \end{array}}$$
$$= e^x \sin x$$
$$-\int e^x \cos x\,dx \qquad \boxed{\begin{array}{ll} u = \cos x & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = -\sin x & v = e^x \end{array}}$$
$$= e^x \sin x - \left\{e^x \cos x + \int e^x \sin x\,dx\right\}$$
$$\therefore 2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$$
$$\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2}$$

Question 80

$$\int \sin 2x \sin 4x\,dx$$
$$= \frac{1}{2}\int \Big[\cos(2x - 4x) - \cos(2x + 4x)\Big]dx$$
$$= \frac{1}{2}\int \Big[\cos(-2x) - \cos 6x\Big]dx$$
$$= \frac{1}{2}\int \Big[\cos 2x - \cos 6x\Big]dx$$
$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{6}\sin 6x\right] + c$$
$$= \frac{\sin 2x}{4} - \frac{\sin 6x}{12} + c$$

Question 81

$$\int \frac{\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)}{\cos\left(x - \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)}\,dx$$
$$= \int \frac{2\sin x \cos\frac{\pi}{4}}{2\cos x\cos\frac{\pi}{4}}\,dx$$
$$= \int \frac{\sin x}{\cos x}\,dx$$
$$= -\ln|\cos x| + c$$

Question 82

$$\int \frac{x}{\sqrt{x+5}}\,dx \qquad \boxed{\begin{array}{r} u^2 = x + 5 \\ 2u\,du = dx \end{array}}$$
$$= \int \frac{u^2 - 5}{u} \times 2u\,du$$
$$= 2\int (u^2 - 5)\,du$$
$$= \frac{2u^3}{3} - 10u + C$$
$$= \frac{2\sqrt{(x+5)^3}}{3} - 10\sqrt{x+5} + c$$
$$= \frac{2(x+5)\sqrt{x+5}}{3} - 10\sqrt{x+5} + c$$
$$= \frac{2}{3}\sqrt{x+5}(x + 5 - 15) + c$$
$$= \frac{2}{3}(x - 10)\sqrt{x+5} + c$$

ANSWER D

Question 83

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1 + \cos x + \sin x}$$
$$= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{1 + \cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)}\,dx$$
$$= \int_o^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{1 + \sin x + \cos x}\,dx$$
$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x + \sin x} - \int_0^{\frac{\pi}{2}} \frac{x\,dx}{1 + \cos x + \sin x}$$
$$\therefore 2\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1 + \cos x + \sin x} = \frac{\pi}{2}\ln 2$$
$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1 + \cos x + \sin x} = \frac{\pi}{4}\ln 2$$

Question 84

$$\int \frac{dx}{x^3\sqrt{x^2 - 4}} \qquad \boxed{\begin{array}{r} x = 2\sec\theta \\ dx = 2\sec\theta\tan\theta\,d\theta \end{array}}$$
$$= \int \frac{1}{8\sec^3\theta\,\sqrt{4\sec^2\theta - 4}} \times 2\sec\theta\tan\theta\,d\theta$$
$$= \int \frac{2\sec\theta\tan\theta}{8\sec^3\theta\,2\tan\theta}\,d\theta$$
$$= \frac{1}{8}\int \cos^2\theta\,d\theta$$
$$= \frac{1}{8}\int \left(\frac{1}{2}(1 + \cos 2\theta)\right)d\theta$$
$$= \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + c$$
$$= \frac{1}{16}\sec^{-1}\frac{x}{2} + \frac{1}{16} \times \frac{\sqrt{x^2 - 4}}{x} \times \frac{2}{x} + c$$
$$= \frac{1}{16}\cos^{-1}\frac{2}{x} + \frac{\sqrt{x^2 - 4}}{8x^2} + c$$

Question 85a

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx \qquad \boxed{\begin{aligned} u &= \pi - x \\ du &= -dx \\ dx &= -du \end{aligned}}$$

$$= \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}(-du)$$

$$= \int_0^{\pi} \frac{(\pi-u)(\sin u)}{1+(-\cos u)^2}\,du$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$$

Question 85b

$$\therefore \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$= \int_0^{\pi} \left(\frac{\pi\sin x}{1+\cos^2 x} - \frac{x\sin x}{1+\cos^2 x}\right) du$$

$$2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx \quad \boxed{\begin{aligned} v &= \cos x \\ \frac{dv}{dx} &= -\sin x \\ dx &= -\frac{dv}{\sin x} \end{aligned}}$$

$$= \pi\int_1^{-1} \frac{\sin x}{1+v^2} \times \left(-\frac{dv}{\sin x}\right)$$

$$= \pi\int_{-1}^{1} \frac{1}{1+v^2}\,dv$$

$$= \pi\Big[\tan^{-1} v\Big]_{-1}^{1}$$

$$= \pi\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi^2}{2}$$

$$\therefore \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$$

Question 86

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}\,dx$$

$$\therefore 2\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$= \Big[\,x\,\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$$

ANSWER D

Question 87a

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$$

$$= \int_{\frac{\pi}{4}}^{0} \ln\left(1+\tan\left(\frac{\pi}{4} - u\right)\right)(-du)$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{\tan\frac{\pi}{4} - \tan u}{1+\tan\frac{\pi}{4}\tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1-\tan u}{1+\tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\tan u + 1 - \tan u}{1+\tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right) dx$$

Question 87b

$$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$$
$$= \int_0^{\frac{\pi}{4}} \left(\ln 2 - \ln(1+\tan x)\right)\,dx$$
$$2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,dx$$
$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{1}{2}\ln 2\left[\,x\,\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\ln 2 \times \frac{\pi}{4}$$
$$= \frac{\pi \ln 2}{8}$$

Question 88

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\,dx$$
$$= \int (e^x - e^{-x})(e^x + e^{-x})^{-2}\,dx$$
$$= -\frac{1}{e^x + e^{-x}} + \boldsymbol{c}$$

Question 89a

$$I_n - I_{n-1}$$
$$= \int_0^{\frac{\pi}{6}} (\sin^{2n}\theta \sec\theta - \sin^{2(n-1)}\sec\theta)\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \sin^{2n-2}\ \theta \sec\theta\,(\sin^2\theta - 1)\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \sin^{2n-2}\ \theta \sec\theta\,(-\cos^2\theta)\,d\theta$$
$$= -\int_0^{\frac{\pi}{6}} \cos\theta \sin^{2n-2}\theta\,d\theta$$
$$= -\left[\frac{\sin^{2n-1}\theta}{2n-1}\right]_0^{\frac{\pi}{6}}$$
$$= -\frac{\left(\frac{1}{2}\right)^{2n-1}}{2n-1} - 0$$
$$= -\frac{1}{2^{2n-1}(2n-1)}$$

Question 89b

$$\therefore \sum_{k=1}^{n}(I_k - I_{k-1}) = \sum_{k=1}^{n}\frac{-1}{2^{2k-1}(2k-1)}$$
$$(I_1 - I_0) + (I_2 - I_1) + (I_3 - I_2) + \ldots + (I_n - I_{n-1})$$
$$= -\sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$
$$I_n - I_0 = -\sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$
$$I_n = I_0 - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$
$$= \int_0^{\frac{\pi}{6}}\sec\theta\,d\theta - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$
$$= \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\frac{\pi}{6}} - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$
$$= \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \ln\left|1+0\right| - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$
$$= \ln\sqrt{3} - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$
$$= \frac{1}{2}\ln 3 - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$

Question 90a

$$I_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cot x\,dx$$
$$= \Big[\ln|\sin x|\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$
$$= \ln\frac{1}{\sqrt{2}} - \ln\frac{1}{2}$$
$$= \ln 1 - \ln\sqrt{2} - \ln 1 + \ln 2$$
$$= 0 - \frac{1}{2}\ln 2 - 0 + \ln 2$$
$$= \frac{1}{2}\ln 2$$

Question 90b

$$I_{n-2} + I_n$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(\cot^{n-2}x + \cot^n x)\,dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cot^{n-2}x(1+\cot^2 x)\,dx$$
$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(-\operatorname{cosec}^2 x)\cot^{n-2}x\,dx$$
$$= -\left[\frac{\cot^{n-1}x}{n-1}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$
$$= -\frac{1}{n-1}\left(1 - \left(\sqrt{3}\right)^{n-1}\right)$$
$$= \frac{\left(\sqrt{3}\right)^{n-1} - 1}{n-1}$$
$$= \frac{1}{n-1}\left(3^{\frac{1}{2}(n-1)} - 1\right)$$

Question 90c

$$I_5 = \frac{1}{5-1}\left(3^{\frac{1}{2}(5-1)} - 1\right) - I_3$$
$$= 2 - \left(\frac{1}{3-1}\left(3^{\frac{1}{2}(3-1)} - 1\right) - I_1\right)$$
$$= 1 + \frac{1}{2}\ln 2$$

Question 91

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} \qquad \boxed{\begin{aligned} t &= \tan\frac{\theta}{2} \\ d\theta &= \frac{2dt}{1+t^2}\end{aligned}}$$
$$= \int_0^1 \frac{1}{4\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) + 6} \times \frac{2dt}{1+t^2}$$
$$= 2\int_0^1 \frac{1}{8t - 2 + 2t^2 + 6 + 6t^2}\,dt$$
$$= 2\int_0^1 \frac{1}{8t^2 + 8t + 4}\,dt$$
$$= \int_0^1 \frac{1}{4t^2 + 4t + 2}\,dt$$
$$= \frac{1}{2}\int_0^1 \frac{2}{(2t+1)^2 + 1^2}\,dt$$
$$= \frac{1}{2}\Big[\tan^{-1}(2t+1)\Big]_0^1$$
$$= \frac{1}{2}(\tan^{-1}3 - \tan^{-1}1)$$
$$= \frac{1}{2}\tan^{-1}(\tan(\tan^{-1}3 - \tan^{-1}1))$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{\tan(\tan^{-1}3) - \tan(\tan^{-1}1)}{1 + \tan(\tan^{-1}3)\tan(\tan^{-1}1)}\right)$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{3-1}{1+(3)(1)}\right)$$
$$= \frac{1}{2}\tan^{-1}\frac{1}{2}$$

Question 92

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$$

$$\boxed{\begin{aligned} x &= \frac{1-u}{1+u} \\ \frac{dx}{du} &= \frac{(1+u)(-1) - (1-u)(1)}{(1+u)^2} \\ &= -\frac{2}{(1+u)^2} \\ dx &= -\frac{2du}{(1+u)^2}\end{aligned}}$$

$$= \int_1^0 \frac{\ln\left(1 + \frac{1-u}{1+u}\right)}{1 + \frac{(1-u)^2}{(1+u)^2}} \times \left(-\frac{2du}{(1+u)^2}\right)$$
$$= 2\int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{1 + 2u + u^2 + 1 - 2u + u^2}\,du$$
$$= \int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{u^2+1}\,du$$
$$= \int_0^1 \left(\frac{\ln 2}{u^2+1} - \frac{\ln(1+u)}{u^2+1}\right)du$$
$$\therefore I = \ln 2\int_0^1 \frac{1}{u^2+1}\,du - I$$
$$2I = \ln 2\Big[\tan^{-1}u\Big]_0^1$$
$$I = \frac{\ln 2}{2}\left(\frac{\pi}{4} - 0\right)$$
$$= \frac{\pi\ln 2}{8}$$

Question 93a

$$\frac{5-5x^2}{(1+2x)(1+x^2)} = \frac{a}{1+2x} + \frac{bx+c}{1+x^2}$$
$$a = \frac{5 - 5\left(-\frac{1}{2}\right)^2}{1 + \left(-\frac{1}{2}\right)^2} = 3$$

equating coefficients of x^2: $a + 2b = -5 \quad \Rightarrow \quad b = -4$

equating constants: $a + c = 5 \quad \Rightarrow \quad c = 2$

$$\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx$$
$$= \int_0^1 \left(\frac{3}{1+2x} - \frac{4x-2}{1+x^2}\right)dx$$
$$= \int_0^1 \left(\frac{3}{1+2x} - \frac{4x}{1+x^2} + \frac{2}{1+x^2}\right)dx$$
$$= \left[\frac{3}{2}\ln|1+2x| - 2\ln\left|1+x^2\right| + 2\tan^{-1}x\right]_0^1$$
$$= \left(\frac{3}{2}\ln 3 - 2\ln 2 + \frac{\pi}{2}\right) - \left(\frac{3}{2}\ln 1 - 2\ln 1 + 0\right)$$
$$= \frac{1}{2}(\ln 3^3 - \ln 2^4 + \pi)$$
$$= \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$$

Question 93b

$$\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1+2\sin 2x+\cos 2x}\,dx$$

$$\boxed{\begin{aligned} t &= \tan x \\ \frac{dt}{dx} &= \sec^2 x \\ dx &= \frac{dt}{\sec^2 x} \\ &= \frac{dt}{1+\tan^2 x} \\ &= \frac{dt}{1+t^2} \end{aligned}}$$

$$= \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+\frac{4t}{1+t^2}+\frac{1-t^2}{1+t^2}} \times \frac{dt}{1+t^2}$$

$$= \int_0^1 \frac{1-t^2}{1+t^2+4t+1-t^2} \times \frac{dt}{1+t^2}$$

$$= \int_0^1 \frac{1-t^2}{2(1+2t)(1+t^2)}\,dt$$

$$= \frac{1}{10}\int_0^1 \frac{5-5t^2}{(1+2t)(1+t^2)}\,dt$$

$$= \frac{1}{10} \times \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$$

$$= \frac{1}{20}\left(\pi + \ln\frac{27}{16}\right)$$

Question 94a

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$$

$$\boxed{\begin{array}{ll} u = x^n & \frac{dv}{dx} = \cos x \\ \frac{du}{dx} = nx^{n-1} & v = \sin x \end{array}}$$

$$= \left[x^n \sin x\right]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$$

$$= \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$$

$$\boxed{\begin{array}{ll} u = x^{n-1} & \frac{dv}{dx} = \sin x \\ \frac{du}{dx} = (n-1)x^{n-2} & v = -\cos x \end{array}}$$

$$= \left(\frac{\pi}{2}\right)^n - n\left\{\left[-x^{n-1}\cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x\,dx\right\}$$

$$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n\{0 + (n-1)I_{n-2}\}$$

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

Question 94b

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} x^4 \cos x\,dx \\ &= I_4 \\ &= \left(\frac{\pi}{2}\right)^4 - 4\times 3\times I_2 \\ &= \frac{\pi^4}{16} - 12\left(\left(\frac{\pi}{2}\right)^2 - 2\times 1\times I_0\right) \\ &= \frac{\pi^4}{16} - 3\pi^2 + 24\int_0^{\frac{\pi}{2}} \cos x\,dx \\ &= \frac{\pi^4}{16} - 3\pi^2 + 24\left[\sin x\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi^4}{16} - 3\pi^2 + 24 \end{aligned}$$

Question 95

I_n

$$= \int_0^1 \frac{dx}{(1+x^2)^n}$$

$$\boxed{\begin{array}{ll} u = \left(1+x^2\right)^{-n} & \frac{dv}{dx} = 1 \\ \frac{du}{dx} = -n\left(1+x^2\right)^{-n-1}(2x) & v = x \end{array}}$$

$$= \left[\frac{x}{(1+x^2)^n}\right]_0^1$$

$$+ 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} - 0 + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{1}{(1+x^2)^n}\,dx - 2n\int_0^1 \frac{1}{(1+x^2)^{n+1}}\,dx$$

$$\therefore I_n = \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$$

$$(1-2n)I_n = \frac{1}{2^n} - 2nI_{n+1}$$

$$I_n = \frac{1}{2^n(1-2n)} + \frac{2n}{2n-1}I_{n+1}$$

Question 96a

$I_n = \int_0^1 x^n \sqrt{1-x^3}\, dx$

$u = x^{n-2}$	$\frac{dv}{dx} = x^2(1-x^3)^{\frac{1}{2}}$
$\frac{du}{dx} = (n-2)x^{n-3}$	$v = -\frac{2}{9}(1-x^3)^{\frac{3}{2}}$

$= -\frac{2}{9}\left[x^{n-2}(1-x^3)^{\frac{3}{2}}\right]_0^1$

$+\frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)^{\frac{3}{2}}\, dx$

$= 0 + \frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)\sqrt{1-x^3}\, dx$

$= \frac{2}{9}(n$

$-2)\int_0^1 \left(x^{n-3}\sqrt{1-x^3} - x^n\sqrt{1-x^3}\right) dx$

$I_n = \frac{2n-4}{9} I_{n-3} - \frac{2n-4}{9} I_n$

$\frac{2n+5}{9} I_n = \frac{2n-4}{9} I_{n-3}$

$I_n = \frac{2n-4}{2n+5} I_{n-3}$

Question 96b

$I_8 = \frac{2(8)-4}{2(8)+5} I_5$

$= \frac{4}{7}\left(\frac{2(5)-4}{2(5)+5}\right) I_2$

$= \frac{8}{35}\int_0^1 x^2\sqrt{1-x^3}\, dx$

$= \frac{8}{35}\int_1^0 x^2 \times u \times \left(-\frac{2u\, du}{3x^2}\right)$

$= \frac{16}{105}\int_0^1 u^2\, du$

$= \frac{16}{105}\left[\frac{u^3}{3}\right]_0^1$

$= \frac{16}{315}$

Question 97a

$2\sin x \sin((2k+1)x)$

$= \cos((2k+1)x - x) - \cos((2k+1)x + x)$

$= \cos(2kx + x - x) - \cos(2kx + x + x)$

$= \cos(2kx) - \cos(2(k+1)x)$

Question 97b

Let $P(n)$ represent the proposition

$P(1)$ is true since LHS $= \sin x$; RHS $=$ $\frac{1-\cos(2x)}{2\sin x} = \frac{1-(1-2\sin^2 x)}{2\sin x} = \frac{2\sin^2 x}{2\sin x} = \sin x$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$\sin x + \sin 3x + \sin 5x + \ldots + \sin((2k-1)x)$

$= \frac{1-\cos(2kx)}{2\sin x}$

$RTP\ P(k+1)$:

$\sin x + \sin 3x + \sin 5x + \ldots + \sin((2k-1)x)$

$+ \sin((2k+1)x) = \frac{1-\cos(2(k+1)x)}{2\sin x}$

LHS

$= \frac{1-\cos(2kx)}{2\sin x} + \sin((2k+1)x)$ from $P(k)$

$= \frac{1-\cos(2kx) + 2\sin x\sin((2k+1)x)}{2\sin x}$

$= \frac{1-\cos(2kx) + \cos(2kx) - \cos(2(k+1)x)}{2\sin x}$

$= \frac{1-\cos(2(k+1)x)}{2\sin x}$

$= RHS$

$\therefore P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for $n \geq 1$ by induction.

Question 98a

$(1+t^2)^{n-1} + t^2(1+t^2)^{n-1}$

$= (1+t^2)^{n-1}(1+t^2)$

$= (1+t^2)^{n-1}$

Question 98b

$= \int_0^x (1+t^2)^n\, dt$

$u = (1+t^2)^n$	$\frac{dv}{dt} = 1$
$\frac{du}{dx} = n(1+t^2)^{(n-1)(2t)}$	$v = t$

$= \left[t(1+t^2)^n\right]_0^x - 2n\int t^2(1+t^2)^{n-1}\, dt$

$= x(1+x^2)^n$

$- 2n\int\left((1+t^2)^n - (1+t^2)^{n-1}\right) dt$

$\therefore\ \mathrm{I}_n = x(1+x^2)^n - 2nI_n + 2n\, I_{n-1}$

$(2n+1)I_n = x(1+x^2)^n + 2nI_{n-1}$

$I_n = \frac{x(1+x^2)^n}{2n+1} + \frac{2n}{2n+1} I_{n-1}$

Question 99a

$$I_n = \int_{e^{-1}}^{1} (1 + \log_e x)^n \, dx$$

$u = (1 + \ln x)^n$	$\dfrac{dv}{dx} = 1$
$\dfrac{du}{dx} = n(1 + \ln x)^{n-1} \times \dfrac{1}{x}$	$v = x$

$$= \left[x(1 + \ln x)^n\right]_{e^{-1}}^{1} - n\int_{e^{-1}}^{1} (1 + \ln x)^{n-1} \, dx$$

$$= 1 - 0 - nI_{n-1}$$

$$= 1 - nI_{n-1}$$

Question 99b

$$J_n = \int_{e^{-1}}^{1} (\log_e x)(1 + \log_e x)^n \, dx$$

$$= \int_{e^{-1}}^{1} (1 + \log_e x - 1)(1 + \log_e x)^n \, dx$$

$$= \int_{e^{-1}}^{1} (1 + \log_e x)^{n+1} \, dx$$

$$- \int_{e^{-1}}^{1} (1 + \log_e x)^n \, dx$$

$$= I_{n+1} - I_n$$

$$= 1 - (n+1)I_n - I_n$$

$$= 1 - (n+2)I_n$$

Question 99c

$$J_3 = 1 - (3+2)I_3$$

$$= 1 - 5(1 - 3I_2)$$

$$= -4 + 15(1 - 2I_1)$$

$$= 11 - 30(1 - I_0)$$

$$= -19 + 30\int_{e^{-1}}^{1} dx$$

$$= -19 + 30\left[\, x \,\right]_{e^{-1}}^{1}$$

$$= -19 + 30\left(1 - \frac{1}{e}\right)$$

$$= 11 - \frac{30}{e}$$

Question 100

$$\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} \, dx$$

$$= \int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} \times \frac{\sin 3x}{\sin 3x} \, dx$$

$$= \int \frac{\sin 3x\,(\cos 5x + \cos 4x)}{\sin 3x - 2\sin 3x \cos 3x} \, dx$$

$$= \int \frac{2\sin\frac{3x}{2}\cos\frac{3x}{2}\left(2\cos\frac{9x}{2}\cos\frac{x}{2}\right)}{\sin 3x - \sin 6x} \, dx$$

$$= \int \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}} \, dx$$

$$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2} \, dx$$

$$= -\int (\cos 2x + \cos x) \, dx$$

$$= -\frac{1}{2}\sin 2x - \sin x + c$$