1000 Extension 2 Revision Questions

Steve Howard

A revision resource for the NEW Mathematics Extension 2 (2017) NSW Syllabus

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About the Author

Steve is a Mathematics Teacher at Cowra High School, a medium sized rural high school in Central West NSW, where he has taught for over 25 years. He has also taught gifted and talented students online through xsel and Aurora College.

He has a particular love and passion for Mathematics Extension 2, writing this book and the supporting digital textbook that will be released during 2020. For access to the free resources as they are created visit howardmathematics.com.

In his spare time he writes commercial online courses for teachers covering the new Mathematics Extension 1 and 2 syllabuses through TTA. They will be extended to Advanced during 2020. The online courses for teachers cover this material in much greater depth and include hundreds of recorded examples. For the courses currently offered see http://tta.edu.au/contributors/56438/steve-howard

For the 2020 HSC you can also access Steve's lessons covering the whole Extension 2 course for free. Go to https://students.tta.edu.au/bundles/hsc-extension-2-student-lessons-6-course-set. This course will be updated from beta form and available commercially after this year's HSC.

Steve loves finding more efficient techniques for solving mathematical questions, by trawling through other teachers' solutions or making up his own approaches when there must be a better way. Many of the approaches you see here are unique, with emphases on both understanding the work, plus little tricks to help students succeed in exams.

Steve did 4 Unit Mathematics as a student way back in 1988, gaining a mark of 198/200 and training as an actuary. Working in an office in the city wasn't for him, so he went back to uni to retrain as a teacher then headed to the country, where he lives in an owner built mud house, with chickens, goats and rescued native birds (which you will sometimes hear in his recordings)!!

Contents

Introduction

Welcome! I hope all students and teachers find this book useful and enjoyable in your journey in Extension 2 Maths. This book and the matching digital textbook are being released chapter by chapter as they are created during 2019 - 2020, and will be continually changed and updated over time - the two digital books work best together. For free downloads of the latest versions please visit howardmathematics.com.

The first 500 questions in Section I are arranged topic by topic matching the chapters from the textbook to help students study and revise as the course progresses and before assessment tasks. The second 500 questions are from mixed topics to help students prepare for their Trials and the HSC.

If you find any mistakes, or have any ideas that would make either the textbook or the revision questions better, please contact me via email below.

Cheers
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There are also others who have helped, before I started this list, so thanks to them too!

BASIC

1 If m and n are odd, $m \geq n$, prove:
a $m+n$ is even b $m-n$ is even c mn is odd
d m^2 is odd e m^3 is odd f m^3+n^3 is even

2 Prove that $\sqrt{3}+\sqrt{7}<2\sqrt{5}$

3 Prove that there are no integers a and b such that $4a+8b=34$

4 Two sides of a triangle are 3 cm and 8 cm. What are the possible values for the third side?

5 Prove $a^2+\frac{1}{a^2}\geq 2$

6 Prove that if the product of any two integers is even, then both of them cannot be odd.

7 Prove that if n is an even integer that n^2-4n+3 is odd.

8 True or False: A triangle could have side lengths 4 cm, 5 cm and 10 cm.

9 Prove $x^2+y^2\geq 2xy$

10 Prove that the sum of two consecutive even positive powers of 2 is always a multiple of 20.

11 Prove that if $3a^2-4a+5$ is even, then a is odd.

12 Give a counterexample to prove each of the following statements is false.
i For real values of x, $\cos(90-|x|)^\circ=\sin x^\circ$.
ii $|3x+2|\leq 11 \Rightarrow |x|\leq 3$

13 Prove for real x, that $|x^2-x|+|x-1|\geq|(x+1)(x-1)|$

14 The product of any two even positive numbers is divisible by 4.

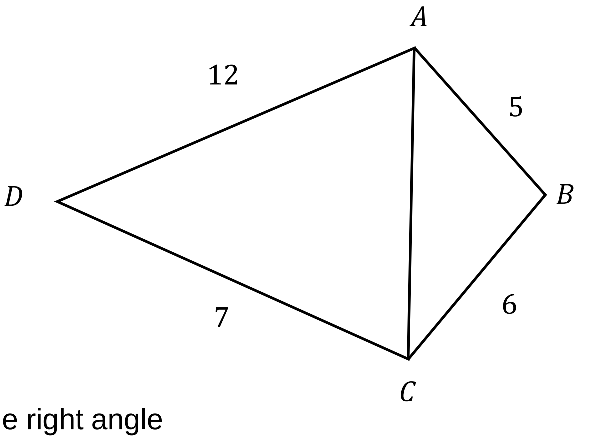
15 Prove by contradiction that if a^2 is even then a is even.

16 Prove that the sum of any two consecutive numbers equals the difference of their squares.

17 Prove $x^2 + x + 1$ is odd for all integral x

18 Prove for integral x, x^2 is divisible by 4 if and only if x is even

19 In the diagram, $AB = 5, BC = 6, CD = 7$ and $AD = 12$. Find the possible lengths of AC.



20 Prove that a triangle can have no more than one right angle

21 Prove that if $a^2 - 6a$ is odd for integral a, then a is odd

22 Given $a^k - b^k = (a-b)\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 \ldots + b^{k-1}\right)$ prove that $\frac{4^k}{3}$ always has a remainder of 1.

23 Prove by contradiction that there are infinite number of even integers.

24 Prove by contradiction that there are no integers m, n which satisfy $3n + 21m = 137$

25 Prove the sum of three consecutive numbers is divisible by 3

26 Prove $\sqrt{1+x} \leq 1 + \frac{x}{2}$ for $x \geq 1$

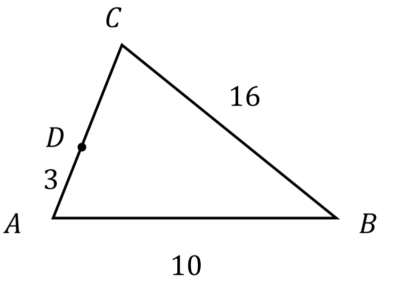
27 Prove that there is no positive integer n that satisfies $3n < n^2 < 4n$

28 Given $a + b = 1$, for $a, b > 0$, prove that $a^2 + b = b^2 + a$

29 Prove that the product of two rational numbers is rational.

30 If $a = b + c$ for $a, b, c > 0$, prove $a^2 > b^2 + c^2$

31 In ΔABC, $AB = 10, BC = 16$ and $AD = 3$.
What are the possible values for DC?



32 Prove that $a^2 - 4b - 2 = 0$ has no integral solutions

33 Prove $xy + yz + zx \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$

34 For m, n positive integers, $m > n$, show that $2m(m^2 + 3n^2)$ can be written as the sum of two cubes.

35 Prove that there are no rational solutions to $x^3 + 3x + 3 = 0$ using contradiction

36 Two sides of an isosceles triangle are 4 cm and 8 cm. Why is the third side 8 cm and not 4 cm?

37 Prove that if $a^2 + b^2 = c^2$ for a, b, c integral, then a or b is even.

38 Prove that for integral a, b that $a^3 - b^3$ is even if and only if $a - b$ is even

39 Prove $2(x^2 + y^2) \geq (x + y)^2$

40 Prove every odd number can be written as the difference of two squares

41 Given a, b and c are integers, if bc is not divisible by a then c is not divisible by a.

42 Prove that $\sqrt{5}$ is irrational by contradiction

43 Prove that $\cos\theta + \sin\theta \leq \sqrt{2}$

44 Prove that the product of any five consecutive integers is divisible by $5!$

45 Prove that a number is divisible by 8 if and only if the last three digits form a number divisible by 8

46 Prove that there are infinitely many prime numbers

47 Prove that $|a-b| \geq \big||a|-|b|\big|$

48 Show that $x^4+y^4+z^2+1 \geq 2x(xy^2-x+z+1)$

49 If $a,b,c>0$ prove that $\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a} \geq ab+bc+ca$.
You may assume that $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

50 If $a,b,c>0$ then prove $a^5+b^5+c^5 \geq a^3bc+b^3ca+c^3ab$

51 Prove by contradiction that the square root of the irrational number m is also irrational.

52 Given a,b,c are positive real numbers with $a>b$ and $c^2>ab$, prove by contradiction that

$$\frac{a+c}{\sqrt{a^2+c^2}}-\frac{b+c}{\sqrt{b^2+c^2}}>0$$

53 The perimeter of a triangle with integral sides a,b,c is 10 cm. What is the maximum length of the longest side c?

54 Let $a,b,c>0$. Prove that

$$a^3+b^3+c^3+ab^2+bc^2+ca^2 \geq 2(a^2b+b^2c+c^2a)$$

55 If $a,b,c>0$ and $(1+a)(1+b)(1+c)=8$, then $abc \leq 1$

56 Without using induction, prove that a^3+2a is divisible by 3 when a is a positive integer.

57 Prove that there are no integers which satisfy $a^2-8b=7$

58 The sides of an isosceles triangle with integer side lengths are $2x+1, 7-x$ and $3x+3$. Find x.

59 Let $a,b>0$ and $ab \geq 1$, prove that

$$\frac{1}{a^2+1}+\frac{1}{b^2+1} \geq \frac{2}{ab+1}$$

60 Assuming $\frac{a}{b}+\frac{b}{a} \geq 2$ prove $\left(\frac{a}{1-a}\right)\left(\frac{b}{1-b}\right)\left(\frac{c}{1-c}\right) \geq 8$ if $a+b+c=2,\ \ 0<a,b,c<1$

1 Let $m = 2k + 1, n = 2j + 1$ for integral k, j

a

$m + n = 2k + 1 + 2j + 1$
$= 2(k + j + 1)$
$= 2p$ for integral p since k, j are integral

$\therefore$ If m and n are odd then $m + n$ is even □

b

$m - n = 2k + 1 - 2j - 1$
$= 2(k - j)$
$= 2p$ for integral p since k, j are integral

$\therefore$ If m and n are odd then $m - n$ is even □

c

$mn = (2k + 1)(2j + 1)$
$= 4kj + 2k + 2j + 1$
$= 2(2kj + j + k) + 1$
$= 2p + 1$ for integral p since k, j are integral

$\therefore$ If m and n are odd then mn is even □

d

$m^2 = (2k + 1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
$= 2p + 1$ where p is integral since k is integral

$\therefore$ If m is odd then m^2 is odd □

e

$m^3 = (2k + 1)^3$
$= 8k^3 + 12k^2 + 6k + 1$
$= 2(4k^3 + 6k^2 + 3k) + 1$
$= 2p + 1$ where p is integral since k is integral

$\therefore$ If m is odd then m^3 is odd □

f

$m^3 + n^3 = (2k + 1)^3 + (2j + 1)^3$
$= 8k^3 + 12k^2 + 6k + 1 + 8j^3 + 12j^2 + 6j + 1$
$= 2(4k^3 + 6k^2 + 3k + 4j^3 + 6j^2 + 3j + 1)$
$= 2p$ where p is integral since k,j are integral

$\therefore$ If m and n are odd then $m^3 + n^3$ is even □

2 Suppose by contradiction that $\sqrt{3} + \sqrt{7} \geq 2\sqrt{5}$

$\therefore \left(\sqrt{3} + \sqrt{7}\right)^2 \geq \left(2\sqrt{5}\right)^2$
$3 + 2\sqrt{21} + 7 \geq 20$
$2\sqrt{21} \geq 10$
$\sqrt{21} \geq 5$
$21 \geq 25$ #

Which is a contradiction, so $\sqrt{3} + \sqrt{7} < 2\sqrt{5}$ □

3 $4(a+2b)=34$ #
There are no integral solutions, as the LHS would then be a multiple of 4 while the RHS is not □

4 Let the third side be x
If x is one of the small sides, then $x+3>8 \rightarrow x>5$
If x is the largest side, then $3+8>x \rightarrow x<11$
$\therefore 5<x<11$ □

5

$$\left(a-\frac{1}{a}\right)^2 \geq 0$$
$$a^2-2+\frac{1}{a^2} \geq 0$$
$$a^2+\frac{1}{a^2} \geq 2 \quad □$$

6 Suppose by contradiction that both integers are odd and their product is even (*).
Let the numbers be $2m+1$ and $2n+1$ where m, n are integral
$P=(2m+1)\cdot(2n+1)$
$=4mn+2m+2n+1$
$=2(2mn+m+n)+1$
$=2p+1$ for integral p since m, n integral #
Which contradicts $(*)$ since the product cannot be odd and even.
$\therefore$ if the product of any two integers is even, then both of them cannot be odd □

7 Let $n=2k$ for integral k
$n^2-4n+3=(2k)^2-4(2k)+3$
$=4k^2-8k+3$
$=2(2k^2-4k+1)+1$
$=2p+1$ for integral p since k is integral
$\therefore$ if n is an even integer then n^2-4n+3 is odd □

8 False, as $4+5<10$ □

9

$$(x-y)^2 \geq 0$$
$$x^2-2xy+y^2 \geq 0$$
$$x^2+y^2 \geq 2xy$$

10 Let k be a positive even integer.
2^k+2^{k+2}
$=2^k(1+2^2)$
$=5\cdot 2^k$
$=5\cdot 2^2\cdot 2^{k-2}$
$=20\cdot 2^{k-2}$
$=20p$ for integral p since $k \geq 2$ □

11 Suppose a is even
Let $a = 2n$
$\therefore 3a^2 - 4a + 5 = 3(2n)^2 - 4(2n) - 5$
$= 2(6n^2 - 4n - 3) + 1$
$= 2p + 1$ for integral p since n is integral
$\therefore$ if a is even then $3a^2 - 4a + 5$ is odd
$\therefore$ if $3a^2 - 4a + 5$ is even then a is odd by contrapositive □

12 **i** Let $x = -30 \therefore \cos(90 - |-30|)° = \cos 60° = \sin 30° \neq \sin(-30°)$

ii Let $x = -4 \quad |3(-4) + 2| = |-10| \leq 11 \quad |-4| > 3$

13 $|x^2 - x| + |x - 1| \geq |(x^2 - x) + (x - 1)|$
$\geq |x^2 - 1|$
$\geq |(x + 1)(x - 1)|$ □

14 Let the numbers be $2m$ and $2n$ where m, n are integral

$P = 2m \cdot 2n$
$= 4mn$ □

15 Suppose a^2 is even and a is odd (*)
Let $a = 2k + 1$ for integral k.
$\therefore a^2 = (2k + 1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
$= 2p + 1$ for integral p since k is integral #
Which contradicts (*) since a^2 cannot be even and odd, $\therefore$ if a^2 is even then a is even □

16 Let the consecutive numbers be k and $k + 1$ for integral k
$(k + 1)^2 - k^2$
$= k^2 + 2k + 1 - k^2$
$= 2k + 1$
$= k + (k + 1)$ □

17 Let $x = 2k + j$ for integral k and $j = 0,1$
$x^2 + x + 1 = (2k + j)^2 + (2k + j) + 1$
$= 4k^2 + 2kj + j^2 + 2k + j + 1$
$= 2(2k^2 + kj + k) + (j^2 + j) + 1$
$= 2(2k^2 + kj + k) + j(j + 1) + 1$
$= 2p + 2q + 1$ for integral p, q since j, k integral and the product of two consecutive numbers is even
$= 2(p + q) + 1$
$= 2m + 1$ for integral m since p, q are integral
$\therefore x^2 + x + 1$ is odd for all integral x

18 Prove that if x^2 is divisible by 4 then x is even by contrapositive

Suppose x is odd

Let $x = 2k + 1$ for integral k

$\therefore x^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 4(k^2 + k) + 1$

$= 4p + 1$ for integral p since k is integral

$\therefore$ if x is odd then x^2 is not divisible by 4

$\therefore$ if x^2 is divisible by 4 then x is even by contrapositive.

Conversely, if x is even let $x = 2j$ for integral j

$x^2 = (2j)^2$

$= 4j^2$

$\therefore x^2$ is divisible by 4.

$\therefore x^2$ is divisible by 4 if and only if x is even □

19 In ΔABC

$AC + 5 > 6 \ \rightarrow AC > 1$

$5 + 6 > AC \ \rightarrow AC < 11$

$\therefore 1 < AC < 11$

In ΔADC

$AC + 7 > 12 \ \rightarrow AC > 5$

$7 + 12 > AC \ \rightarrow AC < 19$

$\therefore 5 < AC < 19$

AC must satisfy the triangle inequality in both triangles, so $5 < AC < 11$ □

20 Suppose a triangle had two right angles

The angle sum of a triangle is $180°$, so the third angle would be zero # This is a contradiction as no angle in a triangle can be $0°$.

$\therefore$ a triangle can have no more than one right angle □

21 Suppose a is even

Let $a = 2n$ for integral n

$$\begin{aligned} a^2 - 6a &= (2n)^2 - 6(2n) \\ &= 4n^2 - 12n \\ &= 2(2n^2 - 6n) \\ &= 2p \text{ for integral } p \text{ since } n \text{ is integral} \end{aligned}$$

∴ if a is even then $a^2 - 6a$ is even

∴ if $a^2 - 6a$ is odd then a is odd by contrapositive □

22 4^k

$= 4^k - 1^k + 1$

$= (4-1)\left(4^{k-1} + 4^{k-2} \times 1 + 4^{k-3} \times 1^2 \ldots + 1^{k-1}\right) + 1$

$= 3\left(4^{k-1} + 4^{k-2} \times 1 + 4^{k-3} \times 1^2 \ldots + 1^{k-1}\right) + 1$

$= 3p + 1 \quad$ for integral p since k is integral

∴ $\frac{4^k}{3}$ always has remainder 1 □

23 Suppose there is a finite number of even integers (*)
Let the largest even integer be k
$k + 2$ is also even, so we have another even integer which contradicts (*)
∴ there are infinite number of even integers □

24 Suppose m, n are integers which do satisfy $3n + 21m = 137$

∴ $3(n + 7m) = 137$

$3p = 3 \times 45 + 2 \quad$ for integral p since n, m are integral $\quad$ #

Which is a contradiction since the LHS is a multiple of 3 but the RHS is not, hence there are no integers m, n which satisfy $3n + 21m = 137$

25 Let the integers be $k, k+1, k+2$ for integral k

$$\begin{aligned} k + k + 1 + k + 2 &= 3k + 3 \\ &= 3(k+1) \\ &= 3p \text{ for integral } p \text{ since } k \text{ is integral} \end{aligned}$$

∴ the sum of three consecutive numbers is divisible by 3 □

26

$$\left(\sqrt{1+x} - 1\right)^2 \geq 0$$

$$1 + x - 2\sqrt{1+x} + 1 \geq 0$$

$$x + 2 \geq 2\sqrt{1+x}$$

$$\sqrt{1+x} \leq 1 + \frac{x}{2}$$

27 Suppose there is a positive integer satisfying $3n < n^2 < 4n$ (*)

$\therefore 3 < n < 4 \quad \#$

Which contradicts (*) since there is no positive integer satisfying $3 < n < 4$, hence there is no positive integer n that satisfies $3n < n^2 < 4n$ □

28

$$\begin{aligned} \text{LHS} - \text{RHS} &= a^2 - b^2 + b - a \\ &= (a+b)(a-b) - (a-b) \\ &= (a-b)(a+b-1) \\ &= (a-b)(0) \quad \text{since } a+b=1 \\ &= 0 \end{aligned}$$

$\therefore a^2 - b^2 + b - a = 0$

$\therefore a^2 + b = b^2 + a$ □

Alternatively

$$\begin{aligned} \text{LHS} &= a^2 + b \\ &= (1-b)^2 + b \qquad \text{since } a+b=1 \\ &= 1 - 2b + b^2 + b \\ &= b^2 + 1 - b \\ &= b^2 + a \\ &= \text{RHS} \end{aligned}$$ □

29 Let the numbers be $\frac{p}{q}$ and $\frac{m}{n}$, where p, q, m, n are integral

$$\begin{aligned} \frac{p}{q} \cdot \frac{m}{n} &= \frac{pm}{qn} \\ &= \frac{a}{b} \quad \text{for integral } a, b \text{ since } p, q, m, n \text{ are integral} \end{aligned}$$

$\therefore$ the product of two rational numbers is rational □

30

$$\begin{aligned} a &= b + c \\ a^2 &= b^2 + 2bc + c^2 \\ a^2 &> b^2 + c^2 \qquad \text{since } 2bc > 0 \end{aligned}$$ □

31 If AC is one of the short sides then

$$\begin{aligned} AC + 10 &> 16 \\ DC + 3 + 10 &> 16 \\ DC &> 3 \end{aligned}$$

If AC is the largest side then

$$\begin{aligned} 10 + 16 &> AC \\ 26 &> 3 + DC \\ DC &< 23 \end{aligned}$$

$\therefore 3 < DC < 23$ □

32 Suppose $a^2 - 4b - 2 = 0$ has integral solutions

$\therefore 4b = a^2 - 2$

The LHS is even, so the RHS must be even

$\therefore a^2$ must be even

$\therefore a$ must be even

Let $a = 2k$ for integral k

$\therefore (2k)^2 - 2 = 4b$

$4k^2 - 2 = 4b$

Now the RHS is a multiple of 4 but the LHS isn't, so there are no integral solutions □

33 $\dfrac{xy + yz}{2} \geq \sqrt{xy^2z} \qquad (\text{AM} - \text{GM})$

$\dfrac{xy + yx}{2} \geq y\sqrt{xz} \qquad (1)$

Similarly

$\dfrac{yz + zx}{2} \geq z\sqrt{xy} \qquad (2)$

$\dfrac{xy + zx}{2} \geq x\sqrt{yz} \qquad (3)$

$(1) + (2) + (3)$:

$xy + yz + zx \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$ □

34 $2m(m^2 + 3n^2)$

$= 2m^3 + 6mn^2$

$= m^3 + 3m^2n + 3mn^2 + 1 + m^3 - 3m^2n + 3mn^2 - 1$

$= (m + n)^3 + (m - n)^3$ □

35 Suppose $x = \frac{p}{q}$ for integral p, q is a solution and p and q have no common factor except 1

$\therefore \left(\dfrac{p}{q}\right)^3 + 3\left(\dfrac{p}{q}\right) + 3 = 0$

$\dfrac{p^3}{q^3} + \dfrac{3p}{q} + 3 = 0$

$\dfrac{p^3 + 3pq^2 + 3q^3}{q^3} = 0$

$\therefore p^3 + 3pq^2 + 3q^3 = 0 \qquad (q \neq 0)$

Now the RHS is even

If p or q is odd then the LHS is odd, so no solution.

If p and q are both even then they have a common factor of 2 which contradicts (*)

$\therefore$ there are no rational solutions to $x^3 + 3x + 3 = 0$ □

36 If the sides are $4, 8$ and 8 then the sides satisfy the triangle inequality in any order: $4 + 8 > 8,\ \ 4 + 8 > 8,\ \ 8 + 8 > 4$

If the sides are $4, 4$ and 8 then the sides do not satisfy the triangle inequality in any order: $4 + 4 \not> 8,\ \ 4 + 4 \not> 8,\ \ 4 + 8 > 4$

The third side of the isosceles triangle must be 4 cm.

37 Suppose a, b are odd solutions to $a^2 + b^2 = c^2$ (*)

Let $a = 2k + 1, b = 2j + 1$ for integral k, j

$$a^2 + b^2 = (2k + 1)^2 + (2j + 1)^2$$
$$= 4k^2 + 4k + 1 + 4j^2 + 4j + 1$$
$$= 4\left(k^2 + k + j^2 + j + \frac{1}{2}\right)$$

Now $a^2 + b^2$ is the sum of two odd numbers, so must be even, so c^2 must be even and so c is even.

The square of an even number is divisible by 4, but the expression above is not (since $\frac{1}{2}$ is not integral), so there is no solution which contradicts (*).

$\therefore$ if $a^2 + b^2 = c^2$ for a, b, c integral, then a or b is even □

38 $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

If $a - b$ is even then the RHS is even, as an even number times an odd or even number gives an even product, hence $a^3 - b^3$ is even

$\therefore a^3 - b^3$ is even if $a - b$ is even

Suppose $a - b$ is odd, thus a and b have opposite parity

$\therefore a^2 + ab + b^2$ must be odd, as two of the terms would be even and one odd

$\therefore a^3 - b^3$ is odd as $a - b$ and $a^2 + a + b^2$ are both odd

$\therefore$ if $a - b$ is odd then $a^3 - b^3$ is odd

$\therefore$ if $a^3 - b^3$ is even then $a - b$ is even by contrapositive

$\therefore a^3 - b^3$ is even if and only if $a - b$ is even □

39 $\text{LHS} - \text{RHS} = 2x^2 + 2y^2 - x^2 - 2xy - y^2$
$= x^2 - 2xy + y^2$
$= (x - y)^2$
≥ 0
$\therefore 2(x^2 + y^2) \geq (x + y)^2$

40 Let the odd number be $2k + 1$
$2k + 1 = k^2 + 2k + 1 - k^2$
$= (k + 1)^2 - k^2$
$\therefore$ every odd number can be written as the difference of two squares □

41 Suppose that c is divisible by a, so let $c = ka$ for integral k.

$\therefore bc = b(ka)$

$= kb \times a$

$= p \times a$ for integral p since k, b are integral

$\therefore$ if c is divisible by a then bc is divisible by a

$\therefore$ if bc is not divisible by a then c is not divisible by a by contrapositive □

42 Suppose $\sqrt{5} = \frac{p}{q}$ for integral p, q is a solution and p and q have no common factor except 1 (*)
$\therefore 5 = \dfrac{p^2}{q^2}$
$5q^2 = p^2$
Now the LHS is a multiple of 5
$\therefore p^2$ is a multiple of 5
$\therefore p$ is a multiple of 5

Let $p = 5m$ for integral m
$\therefore 5q^2 = 25m^2$
$q^2 = 5m^2$

Now $5m^2$ is a multiple of 5
$\therefore q^2$ is a multiple of 5
$\therefore q$ is a multiple of 5 #

Now p and q have a common factor of 5 which contradicts (*), hence $\sqrt{5}$ is irrational □

43 Suppose that $\cos\theta + \sin\theta > \sqrt{2}$

$\therefore (\cos\theta + \sin\theta)^2 > 2$

$\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta > 2$

$\sin^2\theta + \cos^2\theta + \sin 2\theta > 2$

$1 + \sin 2\theta > 2$

$\sin 2\theta > 1$ #

Which is a contradiction since $-1 \leq \sin 2\theta \leq 1$,

hence $\cos\theta + \sin\theta \leq \sqrt{2}$ □

44 The five consecutive numbers must have:

- one multiple of 5
- one or two multiples of 4
- one or two multiples of 3
- one or two multiples of 2 (separate from the multiples of 4)

Let the numbers be $5a, 4b, 3c, 2d, e$

$P = 5a \times 4b \times 3c \times 2d \times e$

$\quad = 5! \times abcde$

$= 5!\, p$ for integral p since a, b, c, d, e are integral

∴ the product of any five consecutive integers is divisible by $5!$

45 Let the number be $1000a + 100b + 10c + d$, where a, b, c, d are integral and $0 \leq b, c, d \leq 9$

Let $1000a + 100b + 10c + d = 8m$ for integral m

$\therefore 1000a + 100b + 10c + d = 8m$

$8(125a) + 100b + 10c + d = 8m$

$100b + 10c + d = 8(m - 125a)$

$= 8p$ for integral p since m, a are integral

∴ if the number is divisible by 8 then the number formed by the last three digits is divisible by 8

Conversely, if the last three digits are divisible by 8, let $100b + 10c + d = 8n$ for integral n

$\therefore 1000a + 100b + 10c + d = 8(125a) + 8n$

$= 8(125a + n)$

$= 8p$ for integral p since a, n are integral

∴ if the last three digits form a number divisible by 8 then the number is divisible by 8

∴ a number is divisible by 8 if and only if the last three digits form a number divisible by 8

□

46 Suppose that there is a finite list of prime numbers, $p_1, p_2, p_3 \ldots p_n$ (*)
Consider $Q = p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_n + 1$
Dividing Q by any prime number on our list there is a remainder of 1, so Q isn't divisible by any prime in the finite list.
But every number has at least one prime factor, so there must be a prime number not on our list which contradicts (*)
Hence there are infinitely many prime numbers □

47 If $a \geq b$

$|a-b|+|b|>|a-b+b|$ triangle inequality

$|a-b|+|b|>|a|$

$\therefore |a-b|>|a|-|b|$

If $a<b$

$|b-a|+|a|>|b-a+a|$ triangle inequality

$|b-a|+|a|>|b|$

$\therefore |a-b|>|b|-|a|$ since $|a-b|=|b-a|$

$\therefore \ |a-b| \geq \big||a|-|b|\big|$ □

48

$$\begin{aligned}\text{LHS}-\text{RHS} &= x^4+y^4+z^2+1-2x(xy^2-x+z+1)\\ &= x^4+y^4+z^2+1-2x^2y^2+2x^2-2xz-2x\\ &= (x^4-2x^2y^2+y^4)+(z^2-2xz+x^2)+(x^2-2x+1)\\ &= (x^2-y^2)^2+(z-x)^2+(x-1)^2\\ &\geq 0+0+0\\ &\geq 0\end{aligned}$$

$\therefore x^4+y^4+z^2+1 \geq 2x(xy^2-x+z+1)$ □

49

$$\begin{aligned}\text{LHS}-\text{RHS} &= \frac{a^3}{b}-ca+\frac{b^3}{c}-ab+\frac{c^3}{a}-bc\\ &= \frac{a^3-abc}{b}+\frac{b^3-abc}{c}+\frac{c^3-abc}{a}\\ &\geq \frac{a^3+b^3+c^3-3abc}{n} \qquad \text{for } n \geq a,b,c\\ &\geq \frac{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}{n}\\ &\geq \frac{(a+b+c)(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2)}{2n}\\ &\geq \frac{(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]}{2n}\\ &\geq 0 \ \text{ since } a,b,c,n>0\end{aligned}$$

$\therefore \dfrac{a^3}{b}+\dfrac{b^3}{c}+\dfrac{c^3}{a} \geq ab+bc+ca$

* Improved solution with thanks to Declan Zammit and Matt Dunstan

50 $\dfrac{a^5+a^5+a^5+b^5+c^5}{5} \geq \sqrt[5]{a^{15}b^5c^5}$

$\therefore \dfrac{3a^5+b^5+c^5}{5} \geq a^3bc$

Similarly

$\dfrac{3b^5+a^5+c^5}{5} \geq b^3ac$

$\dfrac{3c^5+a^5+b^5}{5} \geq c^3ab$

Summing the above gives

$a^5+b^5+c^5 \geq a^3bc+b^3ca+c^3ab$ □

51 Suppose that $\sqrt{m}$ is rational.

$\therefore \sqrt{m} = \frac{p}{q}$ where p, q are integers with no common factor except 1

$mq^2 = p^2$ #

Now the RHS is an integer but the LHS is not since m is irrational which is a contradiction, hence the square root of the irrational number m is also irrational.

52 Suppose $\frac{a+c}{\sqrt{a^2+c^2}} - \frac{b+c}{\sqrt{b^2+c^2}} \le 0$

$$\therefore \frac{a+c}{\sqrt{a^2+c^2}} \le \frac{b+c}{\sqrt{b^2+c^2}}$$
$$(a+c)\sqrt{b^2+c^2} \le (b+c)\sqrt{a^2+c^2}$$
$$(a+c)^2(b^2+c^2) \le (b+c)^2(a^2+c^2) \quad \text{since } a,b,c > 0$$
$$(a^2+2ac+c^2)(b^2+c^2) \le (b^2+2bc+c^2)(a^2+c^2)$$
$$2ab^2c + 2ac^3 \le 2a^2bc + 2bc^3$$
$$ab^2 + ac^2 \le a^2b + bc^2$$
$$ab^2 - a^2b + ac^2 - bc^2 \le 0$$
$$ab(b-a) + c^2(a-b) \le 0$$
$$c^2(a-b) - ab(a-b) \le 0$$
$$(c^2 - ab)(a-b) \le 0 \quad \#$$

This is a contradiction since $c^2 > ab$ and $a > b$, so $(c^2 - ab)(a-b) > 0$

Hence $\frac{a+c}{\sqrt{a^2+c^2}} - \frac{b+c}{\sqrt{b^2+c^2}} > 0$ □

53 $a + b + c = 10$

$\therefore a + b = 10 - c \quad (1)$

$a + b > c \quad (2)$

$\therefore 10 - c > c$

$2c < 10$

$c < 5$

The longest integral side length is 4 cm □

54
$$\begin{aligned}\text{LHS}-\text{RHS} &= a^3 + b^3 + c^3 + ab^2 + bc^2 + ca^2 - 2(a^2b + b^2c + c^2a)\\ &= (a^3 + ab^2 - 2a^2b) + (b^3 + bc^2 - 2b^2c) + (c^3 + ca^2 - 2c^2a)\\ &= a(a^2 - 2ab + b^2) + b(b^2 - 2bc + c^2) + c(c^2 - 2ca + a^2)\\ &= a(a-b)^2 + b(b-c)^2 + c(c-a)^2\\ &\ge 0 + 0 + 0\\ &\ge 0\end{aligned}$$

$\therefore a^3 + b^3 + c^3 + ab^2 + bc^2 + ca^2 \ge 2(a^2b + b^2c + c^2a)$ □

55 $1 + a \ge 2\sqrt{a} \qquad (\text{AM} - \text{GM})$

Similarly

$1 + b \ge 2\sqrt{b}$

$1 + c \ge 2\sqrt{c}$

$\therefore (1+a)(1+b)(1+c) \ge 8\sqrt{abc}$

$8 \ge 8\sqrt{abc}$

$abc \le 1$

56 $a^3 + 2a$
$= a(a^2 + 2)$
$= a(a^2 + 3a + 2 - 3a)$
$= a\big((a+1)(a+2) - 3a\big)$
$= a(a+1)(a+2) - 3a^2$
$= 6k - 3a^2$ where k is integral since the product of 3 consecutive numbers is divisible by 6
$= 3(2k - a^2)$ □

57 7 is odd
$\therefore a^2 - 8b$ must be odd
$\therefore a^2$ must be odd
$\therefore a$ must be odd
Let $a = 2k + 1$ for integral k
$\therefore (2k+1)^2 - 8b = 7$
$4k^2 + 4k + 1 - 8b = 7$
$4k^2 + 4k - 8b = 6$ #
This is a contradiction since the LHS is a multiple of 4 but the RHS isn't, hence there are no integers which satisfy $a^2 - 8b = 7$ □

58
$$2x + 1 + 7 - x > 3x + 3$$
$$x + 8 > 3x + 3$$
$$5 > 2x$$
$$x < 2\frac{1}{2} \qquad (1)$$

$$2x + 1 + 3x + 3 > 7 - x$$
$$5x + 4 > 7 - x$$
$$6x > 11$$
$$x > 1\frac{5}{6} \qquad (2)$$
From (1) and (2) $x = 2$ since all side lengths are integers.
Check side lengths: $2(2) + 1 = 5, 7 - (2) = 5, 3(2) + 3 = 9$ □

59
$$\text{LHS} - \text{RHS} = \frac{1}{a^2+1} + \frac{1}{b^2+1} - \frac{2}{ab+1}$$
$$= \frac{(b^2+1)(ab+1) + (a^2+1)(ab+1) - 2(a^2+1)(b^2+1)}{(a^2+1)(b^2+1)(ab+1)}$$
$$= \frac{ab^3 + b^2 + ab + 1 + a^3b + a^2 + ab + 1 - 2a^2b^2 - 2a^2 - 2b^2 - 2}{(a^2+1)(b^2+1)(ab+1)}$$
$$= \frac{(a^3b - 2a^2b^2 + ab^3) - (a^2 - 2ab + b^2)}{(a^2+1)(b^2+1)(ab+1)}$$
$$= \frac{ab(a^2 - 2ab + b^2) - (a-b)^2}{(a^2+1)(b^2+1)(ab+1)}$$
$$= \frac{ab(a-b)^2 - (a-b)^2}{(a^2+1)(b^2+1)(ab+1)}$$
$$= \frac{(ab-1)(a-b)^2}{(a^2+1)(b^2+1)(ab+1)}$$
$$\geq 0 \quad \text{since } ab \geq 1 \text{ and all other terms positive}$$
$$\therefore \frac{1}{a^2+1} + \frac{1}{b^2+1} \geq \frac{2}{ab+1} \qquad \square$$

60 Let $x = 1 - a, y = 1 - b, z = 1 - c$

$x + y + z = 3 - (a + b + c) = 3 - 2 = 1$

$$\text{LHS} = \left(\frac{y+z}{x}\right)\left(\frac{z+x}{y}\right)\left(\frac{x+y}{z}\right)$$

$$= \frac{(x+y)(x+z)(y+z)}{xyz}$$

$$= \left(\frac{x+y}{\sqrt{xy}}\right)\left(\frac{x+z}{\sqrt{xz}}\right)\left(\frac{y+z}{\sqrt{yz}}\right)$$

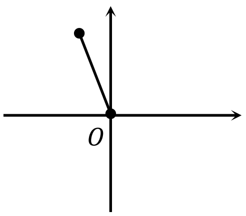
$$= \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\right)\left(\sqrt{\frac{x}{z}} + \sqrt{\frac{z}{x}}\right)\left(\sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}}\right)$$

$$\geq 2 \times 2 \times 2$$

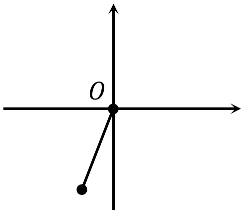
$$\geq 8$$

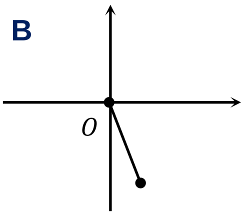
BASIC

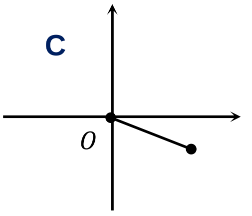
1 A particular complex number z is represented by the point on the following Argand diagram.

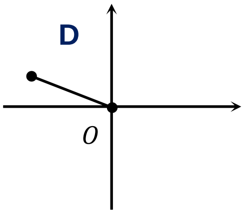


All axes below have the same scale as those in the diagram above. The complex number $i\bar{z}$ is best represented by

A


B


C


D


2 On an Argand diagram, sketch the locus of the points z such that $|z-1| = |z+i|$

3 What is $-\sqrt{3}+i$ expressed in modulus-argument form?

A $\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ **B** $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

C $\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ **D** $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

4 Let $z = 3 - i$. What is the value of $\overline{iz}$?

A $-1-3i$ **B** $-1+3i$ **C** 1−3i **D** $1+3i$

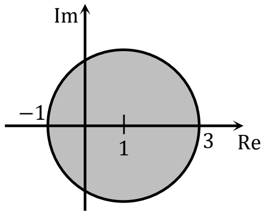
5 Given $z = 1+2i$ and $w = -2+i$, find:

A $|z|$ **B** zw **C** $\frac{5}{iw}$

6 Which of the following inequalities is represented by the Argand diagram?

A $|z-1| \le 2$ **B** $|z-i| \le 2$

C $|z+1| \le 2$ **D** $|z+i| \le 2$



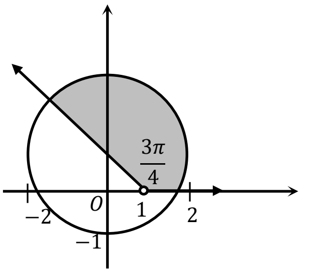
7 $\frac{2-i}{-2-i} = ?$

A $-\frac{3}{5}+\frac{4}{5}i$ **B** -1 **C** $-1+\frac{4}{3}\mathbf{i}$ **D** $-\frac{5}{3}$

8 Given that $z = 6i - 8$, find the square roots of z in the form $a+ib$.

9 Given $e^{i\theta} = \cos\theta + i\sin\theta$, prove $e^{i\pi} + 1 = 0$

10 Consider the Argand diagram at right.



Which inequality could define the shaded area?

A $|z-i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$

B $|z+i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$

C $|z-i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{\pi}{4}$

D $|z+i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{\pi}{4}$

11 Given that $z = \sqrt{2} - \sqrt{2}i$ and $w = -\sqrt{2}$, find, in the form $x + iy$:

i wz^2 **ii** $\arg z$ **iii** $\frac{z}{z+w}$ **iv** $|z|$ **v** z^{10}

12 Find the values of real numbers a and b such that $(a + ib)^2 = 5 - 12i$

13 Draw Argand diagrams to represent the following regions:

a $1 \leq |z + 4 - 3i| \leq 3$ **b** $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}$

14 Realising the denominator of $\frac{12-6i}{4+3i}$ gives:

A $1.2 + 2.4i$ **B** $\frac{30}{7} + \frac{60}{7}i$ **C** $\frac{30}{7} - \frac{60}{7}i$ **D** $1.2 - 2.4i$

15 Factorise $x^4 - 3x^2 - 10$ over:

i the rational field **ii** the real field **iii** the complex field

16 Find the square roots of $1 + \mathrm{i}\sqrt{3}$

17 **i** Express $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ in the form $\mathrm{r}(\cos\theta + i\sin\theta)$.

ii Hence or otherwise find z^{15} in the form $x + iy$.

18 Let $z = 3 - i$ and $w = 2 + i$. Find the following in the form $x + iy$.

i $\overline{zw}$ **ii** $\left|\frac{z}{w}\right|$

19 Simplify $-2i\left(e^{-\frac{\pi}{2}i}\right)$

20 The locus of a point P on the complex plane is defined by $|z - (1 + 2i)| = 3$.

i Sketch the locus of P.

ii Find the maximum value of $|z|$.

21 If $z = 1 + \sqrt{3}i$, then $z^4 =$

A $8 + 8\sqrt{3}i$ **B** $8 - 8\sqrt{3}i$ **C** $-8 + 8\sqrt{3}i$ **D** $-8 - 8\sqrt{3}i$

22 Convert $1 + \sqrt{3}i$ into exponential form.

23 Let $z = 3 + 4i$ and $w = 1 - 2i$. Find in the form $x + iy$,

i $\mathrm{Re}(z) - I(w)$ **ii** $\dfrac{z}{iw}$ **iii** $\sqrt{z}$

24 Sketch, on the same Argand diagram, the locus specified by,

i $|z - 9| = |z + 1|$

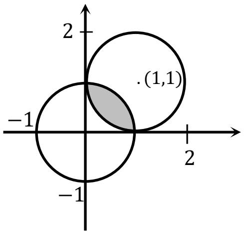
ii $\arg(z - 2i) = \dfrac{\pi}{4}$

iii Hence write down all values of z which simultaneously satisfy $|z - 9| = |z + 1|$ and $\arg(z - 2i) = \frac{\pi}{4}$

25 What is the value of $\arg \bar{z}$ given the complex number $z = 1 - i\sqrt{3}$?

A $-\dfrac{\pi}{3}$ **B** $\dfrac{\pi}{3}$ **C** $\dfrac{-2\pi}{3}$ **D** $\dfrac{2\pi}{3}$

26 Consider the Argand diagram at right.



Which inequality could define the shaded area?

A $|z| \le 1$ and $|z - (1 - i)| \ge 1$

B $|z| \le 1$ and $|z - (1 + i)| \ge 1$

C $|z| \le 1$ and $|z - (1 - i)| \le 1$

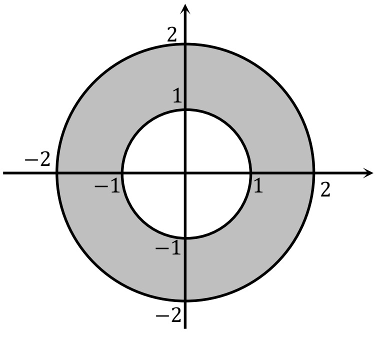
D $|z| \le 1$ and $|z - (1 + i)| \le 1$

27 If $A = 3 + 4i$ and $B = 5 - 13i$ write the following in the form $x + iy$

i AB **ii** $\dfrac{A}{B}$ **iii** $\sqrt{-A}$

28 On the Argand diagram, shade the region where both $|z - 1 - i| \le 2$ and $0 \le \arg z < \frac{\pi}{4}$

29 Consider the Argand diagram at right.



Which inequality best describes the shaded area?

(A) $0 \le |z| \le 2$ (B) $1 \le |z| \le 2$

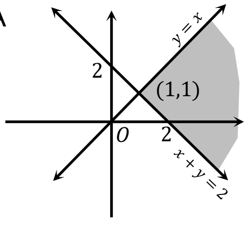
(C) $0 \le |z - 1| \le 2$ (D) $1 \le |z - 1| \le 2$

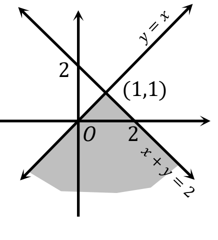
30 Factorise $x^4 + x^2 - 12$ completely over the field of:

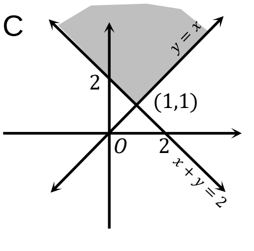
i Rational numbers. **ii** Real numbers. **iii** Complex numbers.

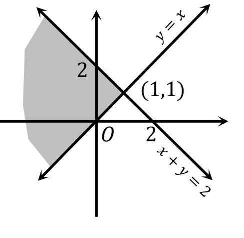
31 The complex numbers z_1, z_2, z_3 and z_4 are represented in the complex plane by the points A, B, C and D respectively. If $z_1 + z_3 = z_2 + z_4$ prove $ABCD$ is a parallelogram.

32 The complex number z satisfies the inequations $|z + 2i| \ge |z + 2|$ and $\mathrm{Im}(z) + \mathrm{Re}(z) \ge 2$ Which of these shows the shaded region in the Argand diagram that satisfies these inequations?

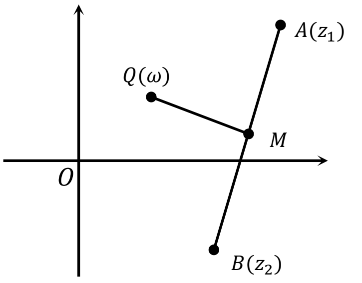
A


B


C


D


33 The complex numbers z_1 and z_2 respectively. M is the midpoint of the interval AB and QM is drawn perpendicular to AB. $QM = AM = BM$. If Q corresponds to the complex number ω, then $\omega =$?



A $i\left(\frac{z_1 - z_2}{2}\right)$ **B** $i\left(\frac{z_1 + z_2}{2}\right)$

C $\frac{z_1 + z_2}{2} + i\left(\frac{z_1 + z_2}{2}\right)$ **D** $\frac{z_1 + z_2}{2} + i\left(\frac{z_1 - z_2}{2}\right)$

MEDIUM

34 If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ is equal to

A $128\,\omega$

B -128ω

C $128\omega^2$

D $-128\omega^2$

35 If $z = x + iy$, the locus of points that lie on a circle of radius 2 centred at the origin on the Argand diagram can be represented by the equation

A $z\bar{z} = 2$

B $(z + \bar{z})^2 - (z - \bar{z})^2 = 16$

C $\text{Re}(z^2) + \text{Im}(z^2) = 4$

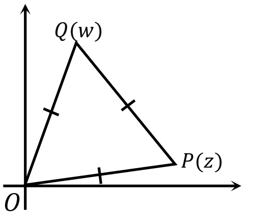
D $\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2 = 16$

36 Let $z = 1 + i\sqrt{3}$

i Write z in modulus-argument form. **ii** Hence evaluate $z^5 + 16z$.

37 Given that $z = x + iy$, find the value of x and the value of y such that $z + 3i\bar{z} = -1 + 13i$

38 In the Argand diagram, OPQ is an equilateral triangle. P represents the complex number z and Q represents the complex number w. Show that $w^3 + z^3 = 0$



39 Find $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^9$

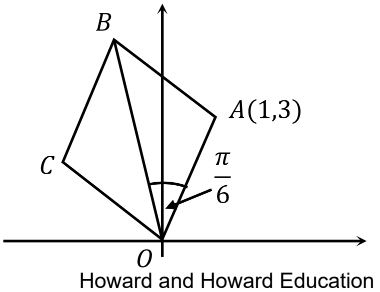
(A) -1 (B) 1 (C) 0 (D) 2

40 **i** Show that $(1 - 3i)^2 = -8 - 6i$

ii Hence solve the equation $2z^2 - 8z + (12 + 3i) = 0$

41 $OABC$ is a rhombus in the Argand diagram, where O is the origin and point B is in the second quadrant. The point A is $(1,3)$ and the angle AOB is $\frac{\pi}{6}$.

Find the complex numbers represented by the points B and C in the form $a + ib$.



42 The roots of $z^3 - 1 = 0$ are $1, \Omega$ and $\overline{\Omega}$, where Ω is one of the complex roots
i Explain why $1 + \Omega + \Omega^2 = 0$
ii Show that $\overline{\Omega} = \Omega^2$

43 Solve $1 + i = e^{a+bi}$.

44 Given $z = \frac{\sqrt{3}+i}{1+i}$
i Find the modulus and argument of z.
ii Find the smallest positive integer such that z^n is real.

45 The complex number z moves such that $\text{Im}\left[\frac{1}{\bar{z}-i}\right] = 2$. Show that the locus of z is a circle.

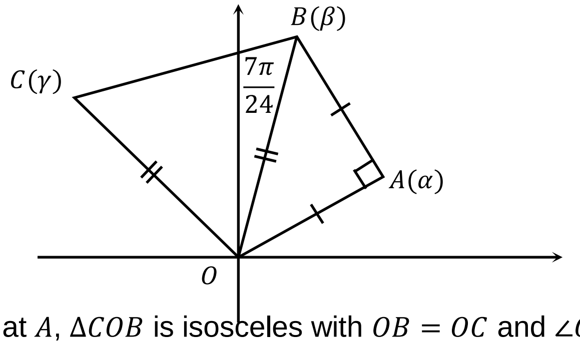
46 Sketch the region in the complex number plane where the inequalities $|z + 1 - i| < 2$ and $-\pi \leq \arg(z + 1 - i) \leq \frac{3\pi}{4}$ hold simultaneously.

47 Find the three different values of z for which $z^3 = \frac{1+i}{\sqrt{2}}$

48 Find the two complex numbers that satisfy $z\bar{z} = 37$ and $\frac{z}{\bar{z}} = \frac{35}{37} + \frac{12i}{37}$

49 If $w = \left(-1 + i\sqrt{3}\right)^{2012}$, find $\arg w$.

50 Points A, B and C represent the complex numbers α, β and γ in the Argand diagram respectively.



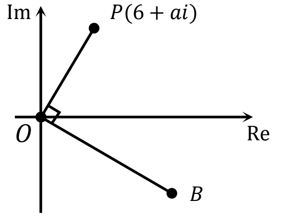
ΔOAB is right isosceles at A, ΔCOB is isosceles with $OB = OC$ and $\angle OBC = \frac{7\pi}{24}$.
i Find $\angle AOC$.
ii Explain why $\gamma = \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\alpha$
iii Hence find the value of $2\alpha^2 + \gamma^2 + \alpha\gamma\sqrt{2}$.

51 Find the modulus and argument of $e^{(\ln 2)(1+i)}$

52 What is the locus in the Argand diagram of the point z such that $z\bar{z} - 2(z + \bar{z}) = 5$

53 Find the value of z^{10} in Cartesian form, given that $z = \sqrt{2} - \sqrt{2}i$

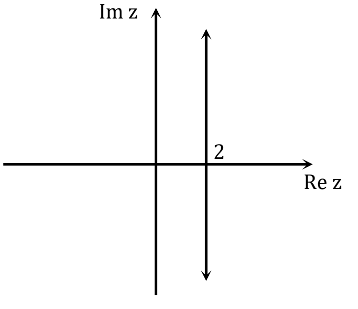
54 In the following Argand diagram, P represents the point $6 + ai$, and O is the origin.



Find the complex number represented by the point B, given $\angle POB = 90°$ and $2|OB| = 3|OP|$.

55 Which if the following is NOT a valid algebraic description of the line shown at right?

A $\text{Re}(z) = 2$ **B** $|z| = |z - 4|$
C $\arg(z - 4) + \arg z = \pi$ **D** $z + \bar{z} = 4$

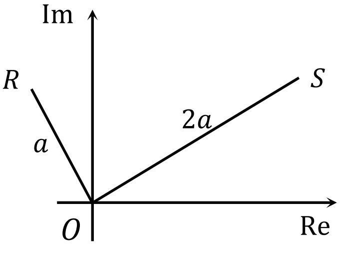


56 **i** Write $2 + 2\sqrt{3}i$ in modulus-argument form.
ii Hence express $\left(2 + 2\sqrt{3}i\right)^3$ in the form $x + iy$
iii Find all unique solutions to the equation $z^4 = 2 + 2\sqrt{3}i$, giving answers in modulus-argument form.

57 Given z is a complex number, sketch on a number plane the locus of a point P representing z such that $\arg z = \arg[z - (1 + i)]$.

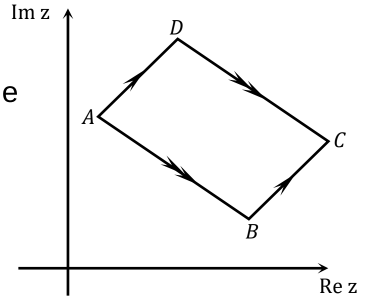
58 In the Argand diagram below the points R and S represent the complex numbers w and z, respectively where $\angle SOR = 90°$. The distance OS is 2a units, and distance OR is a units. Which of the following is correct?

A $w = 2iz$ **B** $w = i\bar{w}$
C $w = -\frac{iz}{2}$ **D** $w = -\frac{z}{2i}$



59 Use a binomial expansion and de Moivre's theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

60 The diagram above shows parallelogram $ABCD$ drawn in the first quadrant of the complex plane. The points A, B and C represent the complex numbers z_1, z_2 and z_3 respectively. The vector DB represents the complex number:



A $z_1 + z_3 - 2z_2$ **B** $z_2 - z_1 - z_3$
C $2z_2 - z_1 - z_3$ **D** $2z_2 - z_1 + z_3$

61 The complex number ω is a root of the equation $z^3 + 1 = 0$. Which of the following is FALSE?

A $\bar{\omega}$ is also a root **B** $\omega^2 + 1 - \omega = 0$ **C** $\frac{1}{\omega}$ is also a root **D** $(\omega - 1)^3 = -1$

62 **i** Find the two square roots of $16 - 30i$.
ii Hence solve $z^2 - 2z - (15 - 30i) = 0$

63 Show that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

64 **i** On an Argand diagram, draw and shade the region R given by $|z - 2 - 2i| \leq 2$
ii P is a point in R, representing the complex number z. What is the maximum value of $|z|$?
iii The tangent to the curve at P cuts the x-axis at the point T. By using the nature of ΔOPT, or otherwise, find the exact area of ΔOPT

65 On an Argand diagram, the points A and B represent the complex numbers $z_1 = -2i$ and $z_2 = 1 - \sqrt{3}i$. Which of the following statements is true?

A $\arg(z_1 + z_2) = -\dfrac{5\pi}{12}$ **B** $|z_1 - z_2| = 2 + \sqrt{3}$

C $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{\pi}{6}$ **D** $\arg(z_1 z_2) = -\dfrac{\pi}{3}$

66 It is given that $z = \cos\theta + i\sin\theta$, where $0 < \arg z < \frac{\pi}{2}$.
i Show that $z + 1 = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ and express $z - 1$ in polar form.
ii Hence show that $\text{Re}\left(\frac{z-1}{z+1}\right) = 0$

67 Which of the following complex numbers equals $\left(\sqrt{3} + i\right)^4$?

A $-2 + \dfrac{2}{\sqrt{3}}i$ **B** $-8 + \dfrac{8}{\sqrt{3}}i$ **C** $-2 + 2\sqrt{3}i$ **D** $-8 + 8\sqrt{3}i$

68 On the Argand diagram, let O be the origin and A be the point representing the complex number $\alpha = \frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)$. Point B represents the complex number β, where $\beta = \alpha \times \text{cis}\left(\frac{\pi}{3}\right)$
i Express β in polar form.
ii Hence find the area of ΔOAB

69 The complex number Z moves such that $\mathrm{Im}\left(\frac{1}{\bar{Z}-i}\right) = 1$. Show that the locus of Z is a circle and find its centre and radius.

70 Show that the complex number $z = \frac{1-t^2+2it}{1+t^2}$ lies on a unit circle centre the origin for all values of t.

71 Find the complex number $z = a + bi$, where a and b are real, such that $2\bar{z} - iz = 1 + 4i$

72 Show that $z = \cos\frac{\pi}{9} + i\sin\frac{\pi}{9}$ is a root of the equation $z^6 - z^3 + 1 = 0$

73 In an Argand diagram, $ABCD$ is a quadrilateral such that the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ represent the complex numbers a, b, c, d respectively. P, Q, R and S are the midpoints of AB, BC, CD and DA respectively. M and N are the midpoints of PR and QS respectively.

i Show that the vectors $\overrightarrow{OM}$ and $\overrightarrow{ON}$ both represent the complex number $\frac{1}{4}(a + b + c + d)$

ii Hence explain what type of quadrilateral $PQRS$ is.

74 Find the fourth roots of $2 + 2\sqrt{3}i$.

75 The complex number $z = x + iy$ satisfies the relation $(z - \bar{z})^2 + 18(z + \bar{z}) = 36$. Show that the locus of z on the Argand plane is a parabola.

76 If $z = a(\cos\theta + i\sin\theta)$ when a and θ are real, show that $\frac{z}{z^2+a^2}$ is equivalent to $\frac{1}{2a\cos\theta}$

77 Suppose $\omega^3 = 1, \omega \neq 1$ and k is a positive integer.
What are the two values of $1 + \omega^k + \omega^{2k}$?

A 3, 0 **B** 3,1 **C** 1,0 **D** None of the above

78 Given that a and b are real numbers and $\frac{a}{1+i} + \frac{b}{1+2i} = 1$ find the values of a and b.

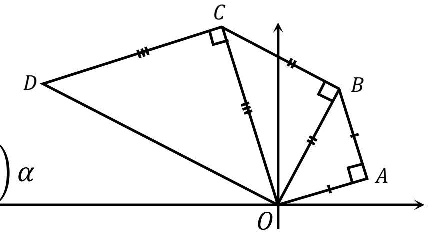
79 The complex numbers z_1, z_2, z_3 and z_4 are represented in the complex plane by the points A, B, C and D respectively. z represents a complex number such that $z^5 = 1$, where $z \neq 1$.

i Deduce that $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$

ii By substituting $x = z + \frac{1}{z}$ reduce the equation in (a) to a quadratic in x.

iii Hence deduce that $\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -\frac{1}{4}$

80 The points A and D in a complex plane represent the complex numbers α and β respectively. The triangles AOB, ABC and OCD are right angled isosceles triangles as shown.



i Show that B represents the complex number $\left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)\alpha$

ii Hence show that $\beta = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right) \times i\alpha$

iii Show that $64\alpha^4 + \beta^4 = 0$

81 Sketch the following on separate Argand diagrams.

i $z^2 - (\bar{z})^2 = 16i$

ii $\arg\left(\dfrac{z-i}{z-2}\right) = \dfrac{\pi}{2}$

82 Suppose z is any non-zero complex number.

i Explain why $\frac{z}{\bar{z}}$ has modulus 1 and argument twice the argument of z.

ii Find all complex numbers z so that $\frac{z}{\bar{z}} = i$. Give your answers in the form $a + ib$, where a and b are real.

CHALLENGING

83 **i** Show that $e^{n\theta i} + e^{-n\theta i} = 2\cos n\theta$

ii Show that $\left(e^{i\theta} + e^{-i\theta}\right)^3 = \left(e^{3\theta i} + e^{-3\theta i}\right) + 3\left(e^{i\theta} + e^{-i\theta}\right)$

iii Hence prove that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

84 Let $w = \frac{3+4i}{5}$ and $z = \frac{5+12i}{13}$, so that $|w| = |z| = 1$.

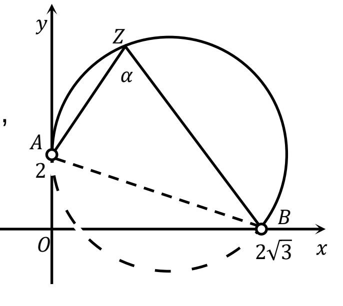
i Find wz in the form $x + iy$.

ii Hence, or otherwise, find two distinct ways of writing 65^2 as the sum of $a^2 + b^2$, where a and b are integers and $0 < a < b$.

85 Given $|z| < \frac{1}{2}$, show that $|(1+i)z^3 + iz| < \frac{3}{4}$.

86 The locus of the complex number Z, moving in the complex plane such that $\arg(Z - 2\sqrt{3}) - \arg(Z - 2i) = \frac{\pi}{3}$, is part of circle.

The angle between the lines from $2i$ to Z and then from $2\sqrt{3}$ to Z is α, as shown in the diagram below.



i Show that $\alpha = \frac{\pi}{3}$

ii Find the centre and radius of the circle.

87 **i** Solve $z^5 + 1 = 0$ by de Moivre's theorem, leaving your solutions in modulus-argument form.

ii Prove that the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^5 + 1 = 0$.

iii Show that if $z^4 - z^3 + z^2 - z + 1 = 0$ where $z = \text{cis}\,\theta$ then $4\cos^2\theta - 2\cos\theta - 1 = 0$.

Hint:

$$z^4 - z^3 + z^2 - z + 1 = 0 \Rightarrow z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0$$

iv Hence find the exact value of $\sec\frac{3\pi}{5}$.

88 **i** Use de Moivre's theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$.
ii Hence show that $x^4 - 10x^2 + 5 = 0$ has roots $\pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$.
iii Deduce that $\tan\frac{\pi}{5}.\tan\frac{2\pi}{5}.\tan\frac{3\pi}{5}.\tan\frac{4\pi}{5} = 5$

89 **i** Find the roots of the equation $z^5 - 1 = 0$, leaving answers in polar form.
ii Hence find the exact value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$.

90 Let $z = \cos\theta + i\sin\theta$.
i Show that $\sin n\theta = \frac{z^n - z^{-n}}{2i}$ and $\cos n\theta = \frac{z^n + z^{-n}}{2}$.
ii Hence, or otherwise, prove the identity $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta + 2$

91 The roots of the equation $z^5 + 1 = 0$ are $-1, \omega_1, \omega_2, \omega_3, \omega_4$ in cyclic order, anti-clockwise around the Argand diagram.
i Show that $\omega_1 = \overline{\omega_4}$
ii Find the values of a, b and c so that $(z+1)(z^4 + az^3 + bz^2 + cz + 1) = z^5 + 1$ and hence show that if ω is a root of $z^5 + 1 = 0$, not equal to -1, then $\omega^4 + \omega^2 + 1 = \omega^3 + \omega$
iii Show that ${\omega_1}^3 = \omega_3$.
(For the rest of this question you may assume : ${\omega_2}^3 = \omega_1$, ${\omega_4}^3 = \omega_2$ and ${\omega_3}^3 = \omega_4$)
iv Deduce that ${\omega_1}^3 + {\omega_2}^3 + {\omega_3}^3 + {\omega_4}^3 = 1$.
v By using the sum of the roots of $z^5 + 1 = 0$ in pairs, or otherwise, prove that

$$\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = -\frac{1}{2}$$

92 The fixed complex number α is such that $0 < \arg\alpha < \frac{\pi}{2}$. In an Argand diagram α is represented by the point A while $i\alpha$ is represented by the point B. z is a variable complex number which is represented by the point P.
i Draw a diagram showing A, B and the locus of P if $|z - \alpha| = |z - i\alpha|$
ii Draw a diagram showing A, B and the locus of P if $\arg(z - \alpha) = \arg(i\alpha)$
iii Find, in terms of α the complex number represented by the point of intersection of the two loci in (i) and (ii)

93 The number c is real and non-zero. It is also known that $(1 + ic)^5$ is real.
i Use binomial theorem to expand $(1 + ic)^5$.
ii Show that $c^4 - 10c^2 + 5 = 0$
iii Hence show that $c = \sqrt{5 - 2\sqrt{5}}, -\sqrt{5 - 2\sqrt{5}}, \sqrt{5 + 2\sqrt{5}}, -\sqrt{5 + 2\sqrt{5}}$
iv Let $1 + ic = r\text{cis}\,\theta$. Use de Moivre's theorem to show that the smallest positive value of θ is $\frac{\pi}{5}$.
v Hence evaluate $\tan\left(\frac{\pi}{5}\right)$.

94 **i** Given that $z = \cos\theta + i\sin\theta$ prove that $z^n + \frac{1}{z^n} = 2\cos n\theta$
ii Express $x^5 - 1$ as the product of three factors each containing real coefficients.
iii Prove that $\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = \frac{5}{4}$

95 The complex number v has modulus 1 and argument $\frac{\pi}{6}$, and the complex number w has modulus 2 and argument $-\frac{2\pi}{3}$.
i Express wv and iv in the modulus-argument form where each argument is between $-\pi$ and π.
ii Show that v is a solution of the equation $Z^4 = iZ$. Hence, or otherwise, state the other two non-zero roots of this equation. You may leave your answers in modulus-argument form.
iii Mark on an Argand diagram, the points P, Q, R and S representing v, w, wv and iv respectively.
iv Hence, or otherwise, show that PS is parallel to RQ.
v Hence, or otherwise, find a real number u such that $iv - v = u(w - wv)$

96 **i** Show that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ where $\cos\theta \neq 0$ and n is a positive integer.
ii Hence show that if z is a purely imaginary number, the roots of $(1+z)^4 + (1-z)^4 = 0$ are
$z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$.

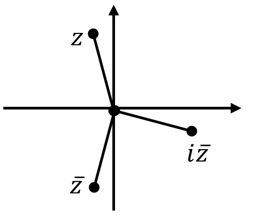
97 Let $w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$.
i Show that w^n is a root of $z^9 - 1 = 0$, n an integer.
ii Show that $w + w^8 = 2\cos\frac{2\pi}{9}$
iii Show that $(w^3 + w^6)(w^2 + w^7) = w + w^8 + w^4 + w^5$
iv Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$. You may assume that $\cos\frac{2\pi}{3} = -\frac{1}{2}$

98 Given that w is a root of $z^3 + iz^2 + ikz + 2i = 0$, where k is real, and $(1-i)w$ is real, find the possible value of k.

99 Let $z = \cos\theta + i\sin\theta$
i Show that $z^n + z^{-n} = 2\cos n\theta$ and find a similar expression for $z^n - z^{-n}$.
ii Hence prove that $2^5\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

100 **i** Show that $\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} = i\cot\frac{\theta}{2}$
ii Hence solve $\left(\frac{z-1}{z+1}\right)^8 = -1$

1 $\bar{z}$ is a reflection of z in the real axis, $\bar{z} \times i$ is an anticlockwise rotation of 90°
ANSWER C



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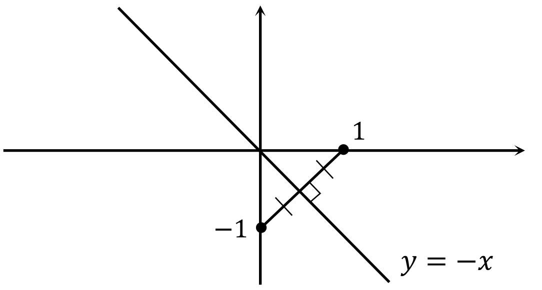


3

$$\left|-\sqrt{3}+i\right| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2$$

$$\arg(-\sqrt{3}+1) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$$

$$-\sqrt{3}+i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

ANSWER D

4

$$iz = i(3-i)$$
$$= 3i - i^2$$
$$= 1 + 3i$$
$$\therefore \overline{iz} = 1 - 3i$$

ANSWER C

5

i $|z| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$

ii $zw = (1+2i)(-2+i) = -2 + i - 4i - 2 = -4 - 3i$

iii $\dfrac{5}{iw} = \dfrac{5}{i(-2+i)} \times \dfrac{-1+2i}{-1+2i} = \dfrac{-5+10i}{(-1)^2+(2)^2} = -1+2i$

6 The region is on and in the circle centred at $(1,0)$ with radius 2. Its equation is

$$|z - (1+0i)| \le 2$$
$$\therefore |z-1| \le 2$$

ANSWER A

7

$$\frac{2-i}{-2-i} \times \frac{-2+i}{-2+i} = \frac{-4+2i+2i+1}{4+1}$$
$$= -\frac{3}{5} + \frac{4}{5}i$$

ANSWER A

8 Let $(a+ib)^2 = -8+6i$

$\therefore a^2 - b^2 = -8 \qquad 2ab = 6$

by inspection $a = 1, b = 3$ or $a = -1, b = -3$

the roots are $\pm(1+3i)$

Alternatively:

$$a = \pm\sqrt{\frac{|z| + \text{Re}(z)}{2}} = \pm\sqrt{\frac{\sqrt{(-8)^2+(6)^2} - 8}{2}}$$
$$= \pm 1$$
$$b = \text{sgn}(6) \pm \sqrt{\frac{|z| - \text{Re}(z)}{2}}$$
$$= (+) \pm \sqrt{\frac{\sqrt{(-8)^2+(6)^2} + 8}{2}} = \pm 3$$

the roots are $\pm(1+3i)$

9 Let $\theta = \pi$

$$e^{i\pi} = \cos\pi + i\sin\pi$$
$$e^{i\pi} = -1 + i(0)$$
$$\therefore e^{i\pi} + 1 = 0$$

10

The shaded area, when measured from $(1,0)$, has an argument between 0 and $\frac{3\pi}{4}$ inclusive (since the x and y intercepts are both 1 the gradient of the left hand ray is -1, so an angle of $\frac{3\pi}{4}$. So $0 \le \arg(z-1) \le \frac{3\pi}{4}$.
The shaded area is on or inside the circle of radius 2 centred at $(0,1)$, so $|z-i| \le 2$
ANSWER A

11

$$-\sqrt{2}\left(\sqrt{2} - \sqrt{2}i\right)^2 = -\sqrt{2}(2 - 4i - 2) = 4\sqrt{2}i$$

$$\arg z = -\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = -\frac{\pi}{4}$$

$$\frac{z}{z+w} = \frac{\sqrt{2}-\sqrt{2}i}{\sqrt{2}-\sqrt{2}i-\sqrt{2}} = \frac{\sqrt{2}-\sqrt{2}i}{-\sqrt{2}i} \times \frac{\sqrt{2}i}{\sqrt{2}i} = \frac{2+2i}{2} = 1+i$$

$$|z| = \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = 2$$

$$z^{10} = \left(2\text{ cis}\left(-\frac{\pi}{4}\right)\right)^{10} = 2^{10}\text{cis}\left(-\frac{5\pi}{2}\right) = 1024\text{ cis}\left(-\frac{\pi}{2}\right) = -1024i$$

12 Let $(a+ib)^2 = 5-12i$
$\therefore a^2-b^2=5 \qquad 2ab=-12$
by inspection $a=3, b=-2$ or $a=-3, b=2$
the roots are $\pm(3-2i)$
Alternatively:

$$a=\pm\sqrt{\frac{|z|+\text{Re}(z)}{2}}$$

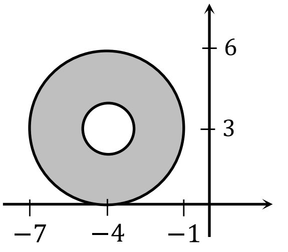
$$=\pm\sqrt{\frac{\sqrt{(5)^2+(-12)^2}+5}{2}}=\pm 3$$

$$b=\text{sgn}(-12i)\pm\sqrt{\frac{|z|+\text{Re}(z)}{2}}$$

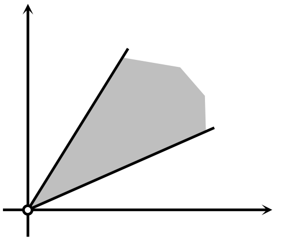
$$=(-)\pm\sqrt{\frac{\sqrt{(5)^2+(-12)^2}-5}{2}}=\mp 2$$

the roots are $\pm(3-2i)$

13 The region between the circles of radius 1 and 3 centred at $(-4,3)$.



The sector between the rays from the origin at arguments of $\frac{\pi}{6}$ and $\frac{\pi}{3}$.



14 $\dfrac{12-6i}{4+3i}\times\dfrac{4-3i}{4-3i}=\dfrac{48-36i-24i-18}{4^2+3^2}$
$=\dfrac{30}{25}-\dfrac{60}{25}i=1.2-2.4i$
ANSWER D

15 **i** $(x^2-5)(x^2+2)$
ii $(x+\sqrt{5})(x-\sqrt{5})(x^2+2)$
iii $(x+\sqrt{5})(x-\sqrt{5})(x+\sqrt{2}i)(x-\sqrt{2}i)$

16 $(a+ib)^2=1+i\sqrt{3}$ by inspection
$a^2-b^2=1 \quad 2ab=\sqrt{3}$ too hard - use another method:
$1+i\sqrt{3}=2\text{cis}\dfrac{\pi}{3}$
The square roots of $1+i\sqrt{3}$ are $\pm\sqrt{2}\text{cis}\frac{\pi}{6}$

$$=\pm\sqrt{2}\left(\frac{\sqrt{3}}{2}+\sqrt{2}i\right)=\pm\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i\right)$$

Alternatively:

$$a=\pm\sqrt{\frac{|z|+\text{Re}(z)}{2}}=\pm\sqrt{\frac{\sqrt{(1)^2+(\sqrt{3})^2}+1}{2}}$$

$$=\pm\sqrt{\frac{3}{2}}\times\sqrt{\frac{2}{2}}=\pm\frac{\sqrt{6}}{2}$$

$$b=\text{sgn}(\sqrt{3})\pm\sqrt{\frac{|z|-\text{Re}(z)}{2}}$$

$$=(+)\pm\sqrt{\frac{\sqrt{(1)^2+(\sqrt{3})^2}-1}{2}}=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$$

the square roots are $\pm\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i\right)$
Alternatively: let $(c+id)^2=2+2\sqrt{3}i$
$c^2-d^2=2 \qquad 2cd=2\sqrt{3}$
$c=\sqrt{3}, d=1$ or $c=-\sqrt{3}, d=-1$
The square roots of $2+2\sqrt{3}i$ are $\pm(\sqrt{3}+i)$
so the square roots of $1+\sqrt{3}i$ are

$$\pm\left(\frac{\sqrt{3}}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=\pm\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i\right)$$

17 **i** $r=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}=1$

$$\theta=\tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)=\frac{\pi}{6}$$

$$\frac{\sqrt{3}}{2}+\frac{i}{2}=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$$

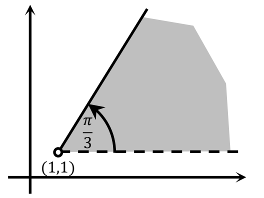
ii $z^{15}=\cos\dfrac{15\pi}{6}+i\sin\dfrac{15\pi}{6}=\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}$
$=i$

18 **i** $\overline{zw}=\bar{z}\times\bar{w}=(3+i)(2-i)$
$=6-3i+2i+1=7-i$

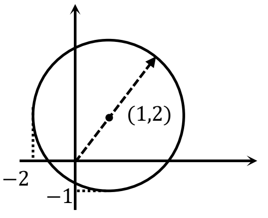
ii $\left|\dfrac{z}{w}\right|=\dfrac{|z|}{|w|}=\dfrac{\sqrt{(3)^2+(-1)^2}}{\sqrt{(2)^2+(1)^2}}=\dfrac{\sqrt{10}}{\sqrt{5}}=\sqrt{2}$

19 $-2i\left(e^{-\frac{\pi}{2}i}\right)=-2i(-i)=2i^2=-2$

ii The sector between the rays from $(1,1)$ at arguments of 0 and $\frac{\pi}{3}$



20 **i**



ii The maximum modulus occurs on the opposite side of the circle, on a line through the centre, and its distance si the distance from the centre plus the radius

$|z|_{\text{max}}=\sqrt{(1)^2+(2)^2}+3=3+\sqrt{5}$

21 $1+\sqrt{3}i=2\,cis\left(\frac{\pi}{3}\right)$

$$\left(1+\sqrt{3}i\right)^4=2^4\,cis\left(\frac{4\pi}{3}\right)$$
$$=16\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$
$$=16\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$
$$=-8-8\sqrt{3}i$$

ANSWER D

22 $\left|1+\sqrt{3}i\right|=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$

$\arg(1+\sqrt{3}i)=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\frac{\pi}{3}$

$\therefore 1+\sqrt{3}i=2e^{\frac{\pi}{3}i}$

23 **i** $\text{Re}(z)-\text{Im}(w)=3-(-2)=5$

ii $\frac{z}{iw}=\frac{3+4i}{i(1-2i)}=\frac{3+4i}{2+i}\times\frac{2-i}{2-i}$

$=\frac{6-3i+8i+4}{2^2+1^1}=2+i$

iii $(a+ib)^2=3+4i$

$a^2-b^2=3 \quad 2ab=4$

$a=2,b=1$ or $a=-2,b=-1$ by inspection

The square roots of $3+4i$ are $\pm(2+i)$

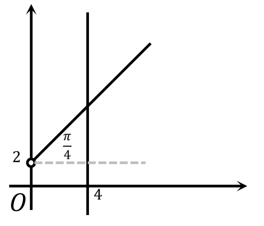
Alternative:

$$a=\pm\sqrt{\frac{|z|+\text{Re}(z)}{2}}=\pm\sqrt{\frac{\sqrt{3^2+4^2}+3}{2}}=\pm 2$$

$$b=\text{sgn}(4)\pm\sqrt{\frac{|z|-\text{Re}(z)}{2}}$$

$$=\pm\sqrt{\frac{\sqrt{3^2+4^2}-3}{2}}=\pm 1$$

24 $|z-9|=|z+1|$ is the perpendicular bisector of the points $(9,0)$ and $(-1,0)$, so $x=4$. $\arg(z-2)=\frac{\pi}{4}$ is the ray form $(0,2)$ with an argument of $\frac{\pi}{4}$.



Substituting $x=4$ into $y=x+2$, the point $(4,6)$ is the point of intersection, so $4+6i$ is the only value of z that satisfies both equations.

25 $\arg\bar{z}=\arg\left(1+i\sqrt{3}\right)=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$

ANSWER B

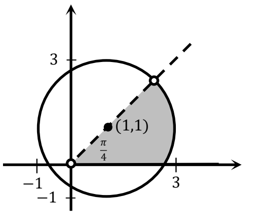
26 The shaded are is on and inside a circle centre the origin, radius 1, so $|z|\le 1$ and also on or inside a circle centred at $1+i$ with radius 1, so $|z-(1+i)|\le 1$

ANSWER D

27 **i** $AB = (3+4i)(5-13i)$
$= 15 - 39i + 20i + 52 = 67 - 19i$
ii $\frac{A}{B} = \frac{3+4i}{5-13i} \times \frac{5+13i}{5+13i}$
$= \frac{15+39i+20i-52}{5^2+13^2} = -\frac{37}{194} + \frac{59}{194}i$
iii $(a+ib)^2 = -3-4i$
$a^2 - b^2 = -3 \quad 2ab = -4$
by inspection $a = 1, b = -2$ or $a = -2, b = 1$
the square roots of A are $\pm(1-2i)$
Alternatively:

$$a = \pm\sqrt{\frac{|z| + \mathrm{Re}(z)}{2}} = \pm\sqrt{\frac{\sqrt{(-3)^2+(-4)^2} - 3}{2}}$$
$= \pm 1$
$$b = \mathrm{sgn}(-4) \pm \sqrt{\frac{|z| - \mathrm{Re}(z)}{2}}$$
$$= (-) \pm \sqrt{\frac{\sqrt{(-3)^2+(-4)^2} + 3}{2}} = \pm 2$$

28



29 The distance from the origin is between 1 and 2, so $1 \le |z| \le 2$
ANSWER B

30
i $(x^2+4)(x^2-3)$
ii $(x^2+4)(x-\sqrt{3})(x+\sqrt{3})$
iii $(x-2i)(x+2i)(x-\sqrt{3})(x+\sqrt{3})$

31 Let M be the midpoint of AC, $\therefore \overrightarrow{OM} = \frac{z_1+z_3}{2}$
Let N be the midpoint of BD, $\therefore \overrightarrow{ON} = \frac{z_2+z_4}{2}$
If $z_1 + z_3 = z_2 + z_4 \quad 2\overrightarrow{OM} = 2\overrightarrow{ON} \quad \therefore \overrightarrow{ON} = \overrightarrow{ON}$
∴ the diagonals bisect so $ABCD$ is a parallelogram.

32 $|z+2i| \ge |z+2|$ means that the region includes points that are closer to $(-2,0)$ than $(0,-2)$, so above the line $y = x$ (the perpendicular bisector), so C or D possible.
$Im\,(z) + Re\,(z) \ge 2$ is equivalent to $x + y \ge 2$, so on or above the line $x + y = 2$, so C
ANSWER C

33 M is the midpoint of AB, so $\frac{z_1+z_2}{2}$. The vector MQ is half the vector AB rotated anti-clockwise 90 degrees, so $\frac{z_1-z_2}{2}i$.
So $\omega = \frac{z_1+z_2}{2} + \frac{z_1-z_2}{2}i$
ANSWER D

34 If ω is an imaginary cube root of unity, then
$1 + \omega + \omega^2 = 0$
$(1+\omega-\omega^2)^7 = (1+\omega+\omega^2-2\omega^2)^7$
$= (-2\omega^2)^7 = -128\omega^{14} = -128\omega^{12}\omega^2$
$= -128\omega^2$
ANSWER D

35 $(z+\bar{z})^2 - (z-\bar{z})^2 = 16$
$(2x)^2 - (2iy)^2 = 16$
$4x^2 + 4y^2 = 16$
$x^2 + y^2 = 4$
ANSWER B

36
i $|1+i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$
$\arg(1+i\sqrt{3}) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$
$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$
ii $z^5 + 16z$
$$= 2^5\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) + 32\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= 32\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) + 32\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$= 64\cos\frac{\pi}{3}$
$= 32$

37 $z + 3i\bar{z} = -1 + 13i$
$x + iy + 3i(x - iy) = -1 + 13i$
$x + iy + 3ix + 3y = -1 + 13i$
$x + 3y = -1 \quad (1) \qquad 3x + y = 13 \quad (2)$
$3 \times (1) - (2) \qquad 8y = -16 \quad \Rightarrow \quad y = -2$
sub in (1) $\quad x + 3(-2) = -1 \quad \Rightarrow \quad x = 5$

38 $\angle POQ = \frac{\pi}{3}$ (angle in an equilateral triangle)
$\therefore w = z\,\mathrm{cis}\frac{\pi}{3}$
$\therefore w^3 + z^3 = \left(z\,\mathrm{cis}\frac{\pi}{3}\right)^3 + z^3 = z^3\mathrm{cis}\frac{3\pi}{3} + z^3$
$= z^3(-1) + z^3 = 0$

39 $\left(\mathrm{cis}\frac{\pi}{3}\right)^9 = \mathrm{cis}\,3\pi = \mathrm{cis}\,\pi = -1$
ANSWER A

40 **i** $(1-3i)^2 = 1-6i-9 = -8-6i$

ii $z = \dfrac{8 \pm \sqrt{64-4(2)(12+3i)}}{2(2)}$

$= \dfrac{8 \pm \sqrt{-32-24i}}{4}$

$= \dfrac{4 \pm \sqrt{-8-6i}}{2} = \dfrac{4 \pm (1-3i)}{2}$

$= \dfrac{5-3i}{2}, \dfrac{3+3i}{2}$

41

$\overrightarrow{OC} = \overrightarrow{OA} \times \text{cis}\left(2\left(\dfrac{\pi}{6}\right)\right)$

$= (1+3i)\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

$= (1+3i)\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right) = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2} + i\dfrac{3}{2} - \dfrac{3\sqrt{3}}{2}$

$= \dfrac{1-3\sqrt{3}}{2} + i\dfrac{\sqrt{3}+3}{2}$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC} = 1+3i + \dfrac{1-3\sqrt{3}}{2} + i\dfrac{\sqrt{3}+3}{2}$

$= \dfrac{3-3\sqrt{3}}{2} + i\dfrac{\sqrt{3}+9}{2}$

42

i $\Omega^3 - 1 = 0 \ \therefore (\Omega-1)(\Omega^2+\Omega+1) = 0$
$\Omega \neq 1 \ \therefore \ \Omega^2+\Omega+1 = 0$
$\therefore \Omega = \dfrac{-1 \pm \sqrt{1^2-4(1)(1)}}{2(1)} = \dfrac{-1 \pm \sqrt{3}i}{2}$

ii $\Omega^2 = \dfrac{1-2\sqrt{3}i-3}{4} = \dfrac{-1-\sqrt{3}i}{2} = \overline{\Omega}$

43 $1+i = e^{a+bi}$
$\sqrt{2}e^{\frac{\pi}{4}i} = e^a \cdot e^{bi}$
$\therefore e^a = \sqrt{2} \ \rightarrow \ a = \ln\sqrt{2} = \dfrac{1}{2}\ln 2$
$b = \dfrac{\pi}{4}$

44

i $\left|\dfrac{\sqrt{3}+i}{1+i}\right| = \dfrac{|\sqrt{3}+i|}{|1+i|} = \dfrac{\sqrt{\left(\sqrt{3}\right)^2+(1)^2}}{\sqrt{(1)^2+(1)^2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

$\arg\left(\dfrac{\sqrt{3}+i}{1+i}\right) = \arg(\sqrt{3}+i) - \arg(1+i)$

$= \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) - \tan^{-1}\left(\dfrac{1}{1}\right)$

$= \dfrac{\pi}{6} - \dfrac{\pi}{4}$

$= -\dfrac{\pi}{12}$

$\therefore \dfrac{\sqrt{3}+i}{1+i} = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{12}\right) + i\sin\left(-\dfrac{\pi}{12}\right)\right)$

$\left(\dfrac{\sqrt{3}+i}{1+i}\right)^n = \left(\sqrt{2}\left(\cos\left(-\dfrac{\pi}{12}\right) + i\sin\left(-\dfrac{\pi}{12}\right)\right)\right)^n$

$= 2^{\frac{n}{2}}\left(\cos\left(-\dfrac{n\pi}{12}\right) + i\sin\left(-\dfrac{n\pi}{12}\right)\right)$

Let $\sin\left(-\frac{n\pi}{12}\right) = 0$

$\therefore -\dfrac{n\pi}{12} = k\pi \quad k \text{ integral}$

$n = -12k$

$\therefore n = 12$ is the smallest positive such that z^n is purely real.

45 $\dfrac{1}{\bar{z}-i}=\dfrac{1}{x-iy-i}$

$=\dfrac{1}{x-i(y+1)}\times\dfrac{x+i(y+1)}{x+i(y+1)}$

$=\dfrac{x+i(y+1)}{x^2+(y+1)^2}$

$\therefore \text{Im}\left(\dfrac{1}{\bar{z}-i}\right)=\dfrac{y+1}{x^2+(y+1)^2}=2$

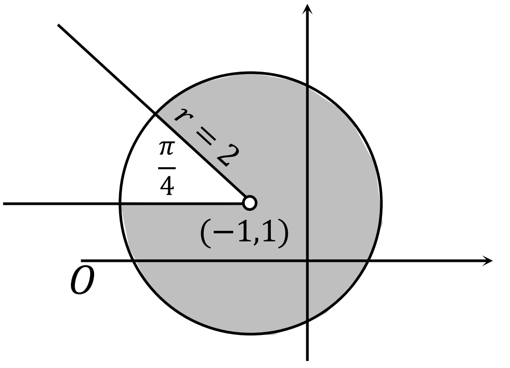
$\therefore y+1=2x^2+2y^2+4y+2$

$x^2+y^2+\dfrac{3}{2}y=-\dfrac{1}{2}$

$x^2+\left(y+\dfrac{3}{4}\right)^2=-\dfrac{1}{2}+\dfrac{9}{16}=\dfrac{1}{16}$

which is a circle

46



47

$\dfrac{1+i}{\sqrt{2}}=\dfrac{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)}{\sqrt{2}}=\text{cis}\left(\dfrac{\pi}{4}\right)$

$\therefore (r\text{ cis }\theta)^3=\text{cis}\dfrac{\pi}{4}$

$\therefore r^3=1 \qquad 3\theta=2k\pi+\dfrac{\pi}{4} \qquad k=-1,0,1$

$\theta=-\dfrac{7\pi}{12},\dfrac{\pi}{12},\dfrac{9\pi}{12}$

$z=\text{cis}\left(-\dfrac{7\pi}{12}\right),\text{cis}\left(\dfrac{\pi}{12}\right),\text{cis}\left(\dfrac{3\pi}{4}\right)$

48 $z\bar{z}=37$

$\therefore (x+iy)(x-iy)=37$

$\therefore x^2+y^2=37 \quad (1)$

$\dfrac{z}{\bar{z}}=\dfrac{35}{37}+\dfrac{12i}{37}$

$\therefore \dfrac{z}{\bar{z}}\times\dfrac{z}{z}=\dfrac{z^2}{37}=\dfrac{x^2-y^2+2xyi}{37}$

$\therefore \quad x^2-y^2=35 \quad (2) \qquad xy=6 \qquad (3)$

by inspection $x=6,y=1$ or $x=-6,y=-1$

$\therefore z=\pm(6+i)$

49

$\arg\left(-1+i\sqrt{3}\right)=\pi-\tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right)=\dfrac{2\pi}{3}$

$\arg w=2012\arg\left(-1+i\sqrt{3}\right)=\dfrac{4024\pi}{3}=-\dfrac{2\pi}{3}$

50 **i** $\angle AOB=\frac{\pi}{4}$ (angle in a right angled isosceles triangle)

$\angle COB=\pi-2\left(\frac{7\pi}{24}\right)=\frac{5\pi}{12}$ (apex angle isosceles triangle)

$\angle AOC=\dfrac{\pi}{4}+\dfrac{5\pi}{12}=\dfrac{2\pi}{3}$

ii $\gamma=\left|\dfrac{OC}{OB}\times\dfrac{OB}{OA}\times OA\right|\text{cis}\left(\arg\alpha+\dfrac{\pi}{4}+\dfrac{5\pi}{12}\right)$

$=\left|1\times\dfrac{\sqrt{2}|\alpha|}{|\alpha|}\times|\alpha|\right|\times\text{cis}(\arg(\alpha))\text{cis}\left(\dfrac{2\pi}{3}\right)$

$=\sqrt{2}\text{cis}\left(\dfrac{2\pi}{3}\right)|\alpha|\text{cis}(\arg\alpha)$

$=\sqrt{2}\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)\alpha$

iii $2\alpha^2+\gamma^2+\alpha\gamma\sqrt{2}$

$=2\alpha^2+\left(\sqrt{2}\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)\alpha\right)^2+\alpha\left(\sqrt{2}\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)\alpha\right)\sqrt{2}$

$=2\alpha^2\left(1+\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}+\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$

$=2\alpha^2\left(1-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)$

$=0$

51 $e^{(\ln 2)(1+i)} = e^{\ln 2 + i \ln 2}$

$= e^{\ln 2} \cdot e^{i \ln 2}$

$= 2e^{(\ln 2)i}$

The modulus is 2 and the argument is $\ln 2$.

52 $z\bar{z} - 2(z + \bar{z}) = 5$

$x^2 + y^2 - 4x = 5$

$(x-2)^2 + y^2 = 9$

A circle centred at (2,0) with radius 3.

53 $\sqrt{2} - \sqrt{2}i = 2 \text{ cis}\left(-\frac{\pi}{4}\right)$

$\left(\sqrt{2} - \sqrt{2}i\right)^{10} = \left(2 \text{ cis}\left(-\frac{\pi}{4}\right)\right)^{10}$

$= 2^{10}\text{cis}\left(-\frac{5\pi}{2}\right)$

$= 1024 \text{ cis}\left(-\frac{\pi}{2}\right) = -1024i$

54 $\overrightarrow{OB} = \frac{3}{2} \times i^3 \times \overrightarrow{OP} = -\frac{3}{2}i(6 + ai) = -9i + \frac{3a}{2}$

$\therefore B$ represents $\frac{3a}{2} - 9i$

55 The vertical line is of the form $Re(z) = 2$

A is correct

B is $|z - (0 + 0i)| = |z - (4 + 0i)|$, the perpendicular bisector of (0,0) and (4,0), which is correct.

C would be an arc with ends (0,0) and (4,0). This is incorrect.

D $z + \bar{z} = (x + iy) + (x - iy) = 2x \Rightarrow \therefore 2x = 4 \equiv Re(z) = 2$ thus correct

ANSWER C

56

i $\left|2 + 2\sqrt{3}i\right| = \sqrt{(2)^2 + \left(2\sqrt{3}\right)^2} = 4$

$\arg\left(2 + 2\sqrt{3}i\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\therefore\ 2 + 2\sqrt{3}i = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

ii $\left(2 + 2\sqrt{3}i\right)^3 = 4^3\left(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right) = -64$

$z^4 = 2 + 2\sqrt{3}i$

$\therefore r^4\text{cis } 4\theta = 4 \text{ cis}\left(\frac{\pi}{3} + 2k\pi\right) \quad k = -2, -1, 0, 1$

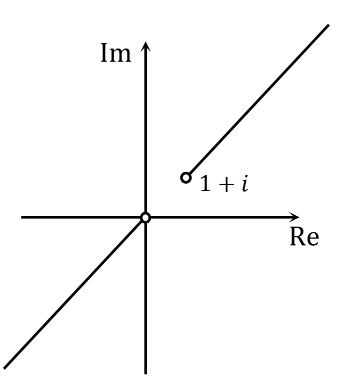
$\therefore r^4 = 4 \quad 4\theta = -\frac{11\pi}{3}, -\frac{5\pi}{3}, \frac{\pi}{3}, \frac{7\pi}{3}$

$r = \sqrt{2} \quad \theta = -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

$z = \sqrt{2}\text{cis}\left(-\frac{11\pi}{12}\right), \sqrt{2}\text{cis}\left(-\frac{5\pi}{12}\right), \sqrt{2}\text{cis}\left(\frac{\pi}{12}\right), \sqrt{2}\text{cis}\left(\frac{7\pi}{12}\right)$

57

The vectors $z - 0$ and $z - (1 + i)$ have the same argument, which occurs on the line through (0,0) and (1,1) outside the two points.



58 $\overrightarrow{OR} = \frac{1}{2}i\overrightarrow{OS} \Rightarrow \quad w = \frac{i}{2}z$

Since this is not one of the options, multiply the RHS top and bottom by i

$w = \frac{i^2 z}{2i} = -\frac{z}{2i}$

ANSWER D

59 Let $c = \cos\theta, s = \sin\theta$.

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$, and also

$= c^5 + 5c^4si + 10c^3s^2i^2 + 10c^2s^3i^3 + 5cs^4i^4 + s^5i^5$

$= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i$

$\therefore \cos 5\theta$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta(1 - \cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

60

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \overrightarrow{OA} + \overrightarrow{BC} = z_1 + (z_3 - z_2)$

$\overrightarrow{DB} = z_2 - \big(z_1 + (z_3 - z_2)\big) = 2z_2 - z_1 - z_3$

ANSWER C

61 Roots are $\omega = \text{cis}\frac{\pi}{3}, \overline{\omega} = \frac{1}{\omega} = \text{cis}\frac{\pi}{3}$ and -1

A true: complex roots of unity occur in conjugate pairs

B true: $\omega^2 + 1 - \omega = \text{cis}\frac{2\pi}{3} + 1 - \text{cis}\frac{\pi}{3}$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i + 1 - \frac{1}{2} - \frac{\sqrt{3}}{2}i = 1$

$\therefore \omega^2 + 1 - \omega \neq 0$

C true: when $|z| = 1$ $\bar{z} = \frac{1}{z}$, so $\frac{1}{\omega} = \overline{w}$

D false: $(\omega - 1)^3 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - 1\right)^3$

$= \left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$

$= \text{cis}\, 2\pi = 1$

ANSWER D

62 **i** Let $(a + ib)^2 = 16 - 30i$

$\therefore a^2 - b^2 = 16 \qquad 2ab = -30$

by inspection $a = 5, b = -3$ or $a = -5, b = 3$

the roots are $\pm(5 - 3i)$

Alternatively:

$a = \pm\sqrt{\frac{|z| + \text{Re}(z)}{2}}$

$= \pm\sqrt{\frac{\sqrt{(16)^2 + (-30)^2} + 16}{2}} = \pm 5$

$b = \frac{\text{Im}(z)}{2a} = \frac{-30}{2(\pm 5)} = \mp 3$

the roots are $\pm(5 - 3i)$

ii $z = \frac{2 \pm \sqrt{(-2)^2 + 4(1)(15 - 30i)}}{2(1)}$

$= \frac{2 \pm \sqrt{64 - 120i}}{2} = 1 \pm \sqrt{16 - 30i}$

$= 1 \pm (5 - 3i) = 6 - 3i, -4 + 3i$

63 $e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \qquad (2)$

$(1) + (2)$: $e^{i\theta} + e^{-i\theta} = 2\cos\theta \ \rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

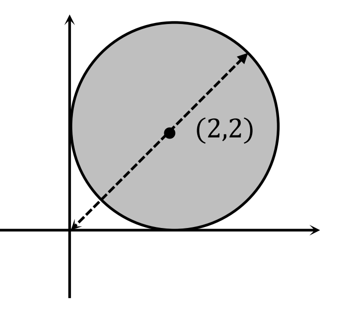
$(1) - (2)$: $e^{i\theta} - e^{-i\theta} = 2i\sin\theta \ \rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

64 **i** Inside the circle centred at $(2,2)$ with radius 2. The maximum modulus occurs on the far side of the circle on a line through the centre.

$|z|_{\text{max}} = \sqrt{(2)^2 + (2)^2} + 2 = 2\sqrt{2} + 2$

ii ΔOPT is right angled isosceles, so

Area $= \frac{1}{2} \times OP \times PT = \frac{1}{2}(OP)^2 = \frac{1}{2}\left(2\sqrt{2} + 2\right)^2 = \frac{1}{2}\left(8 + 8\sqrt{2} + 4\right) = 6 + 4\sqrt{2}$



65 A: $\arg(z_1 + z_2) = \arg\left(1 - \left(2 + \sqrt{3}\right)i\right) = \tan^{-1}\left(-\left(2 + \sqrt{3}\right)\right) = -1.309$

$-\frac{5\pi}{12} = -1.309 \therefore$ true

ANSWER A

66

i $z+1=(\cos\theta+1)+i\sin\theta=\left(2\cos^2\frac{\theta}{2}-1+1\right)+i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$

$=2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$

$z-1=(\cos\theta-1)+i\sin\theta$

$=\left(1-2\sin^2\frac{\theta}{2}-1\right)+i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$

$=-2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)$

ii $\frac{z-1}{z+1}=\frac{-2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}\times\frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}$

$=-\frac{\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}-\sin^2\frac{\theta}{2}i-\cos^2\theta\, i-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)}$

$=-\frac{\sin\frac{\theta}{2}(-i)}{\cos\frac{\theta}{2}}=i\tan\frac{\theta}{2}$

$\therefore \operatorname{Re}\left(\frac{z-1}{z+1}\right)=0$

67

$\sqrt{3}+i=2\operatorname{cis}\frac{\pi}{6}\ \therefore (\sqrt{3}+i)^4=\left(2\operatorname{cis}\frac{\pi}{6}\right)^4$

$=16\operatorname{cis}\frac{2\pi}{3}$

$=16\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-8+8\sqrt{3}i$

ANSWER D

68

i $\frac{1}{\sqrt{2}}+i\left(\frac{1}{\sqrt{2}}\right)=\operatorname{cis}\frac{\pi}{4}$

$\beta=\alpha\operatorname{cis}\frac{\pi}{3}=\operatorname{cis}\frac{\pi}{4}\times\operatorname{cis}\frac{\pi}{3}=\operatorname{cis}\frac{7\pi}{12}$

ii Area $=\frac{1}{2}\times|\alpha|\times|\beta|\times\sin\angle AOB$

$=\frac{1}{2}\times1\times1\times\sin\frac{\pi}{3}=\frac{\sqrt{3}}{4}$ u^2

69

$\operatorname{Im}\left(\frac{1}{\bar{z}-i}\right)=\operatorname{Im}\left(\frac{1}{x-iy-i}\times\frac{x+i(y+1)}{x+i(y+1)}\right)$

$=\operatorname{Im}\left(\frac{x+i(y+1)}{x^2+(y+1)^2}\right)$

$=\frac{y+1}{x^2+(y+1)^2}$

$\frac{y+1}{x^2+(y+1)^2}=1$

$x^2+y^2+2y+1=y+1$

$x^2+y^2+y+\frac{1}{4}=\frac{1}{4}$

$x^2+\left(y+\frac{1}{2}\right)^2=\frac{1}{4}$

The locus is a circle centres at $(0,-\frac{1}{2})$ with radius $\frac{1}{2}$.

70

$x+iy=\frac{1-t^2+2it}{1+t^2}\ \Rightarrow\ x=\frac{1-t^2}{1+t^2}\quad y$

$=\frac{2t}{1+t^2}$

$\therefore\ x^2+y^2$

$=\frac{(1-t^2)^2+(2t)^2}{(1+t^2)^2}$

$=\frac{1-2t^2+t^4+4t^2}{(1+t^2)^2}$

$=\frac{(1+t^2)^2}{(1+t^2)^2}$

$=1$

$\therefore z$ lies on the unit circle

71

$2\bar{z}-iz=2(a-bi)-i(a+bi)$

$=2a-2bi-ai+b$

$=(2a+b)-i(2b+a)$

$\therefore 2a+b=1\quad(1)\qquad 2b+a=-4\qquad(2)$

$2\times(1)-(2)\quad 3a=6\ \Rightarrow\ a=2\ \Rightarrow\ b=-3$

$\therefore z=2-3i$

72

Let $z=\operatorname{cis}\frac{\pi}{9}$

$z^6-z^3+1=\left(\operatorname{cis}\frac{\pi}{9}\right)^6-\left(\operatorname{cis}\frac{\pi}{9}\right)^3+1$

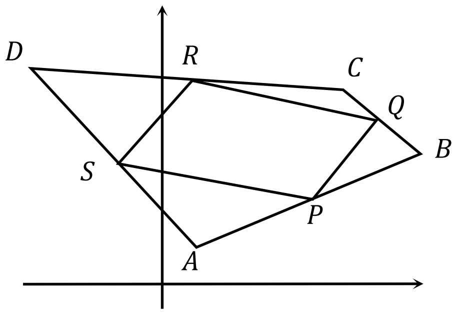
$=\operatorname{cis}\frac{2\pi}{3}-\operatorname{cis}\frac{\pi}{3}+1$

$=-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}+1$

$=0$

$\therefore z=\operatorname{cis}\frac{\pi}{9}$ satisfies $z^6-z^3+1=0$

73



P represents $\frac{a+b}{2}$, Q represents $\frac{b+c}{2}$, R represents $\frac{c+d}{2}$, S represents $\frac{a+d}{2}$.

$\therefore \overrightarrow{OM} = \frac{1}{2}\left(\frac{a+b}{2} + \frac{c+d}{2}\right)$

$= \frac{1}{4}(a+b+c+d)$

$\overrightarrow{ON} = \frac{1}{2}\left(\frac{b+c}{2} + \frac{a+d}{2}\right)$

$= \frac{1}{4}(a+b+c+d)$

Since $\overrightarrow{OM} = \overrightarrow{ON}$ the diagonals bisect each other, so $PQRS$ is a parallelogram.

74

$|2+2\sqrt{3}i| = \sqrt{(2)^2 + \left(2\sqrt{3}\right)^2} = 4$

$\arg\left(2+2\sqrt{3}i\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\therefore 2+2\sqrt{3}i = 4\ \text{cis}\frac{\pi}{3}$

$(r\ \text{cis}\ \theta)^4 = 4\ \text{cis}\frac{\pi}{3}$

$r^4 = 4 \qquad 4\theta = \left(2k\pi + \frac{\pi}{3}\right) \qquad k$

$= -2, -1, 0, 1$

$r = \sqrt{2} \qquad \theta = \frac{(6k+1)\pi}{12}$

$= -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

The roots are

$\sqrt{2}\ \text{cis}\left(-\frac{11\pi}{12}\right), \sqrt{2}\text{cis}\left(-\frac{5\pi}{12}\right), \sqrt{2}\ \text{cis}\left(\frac{\pi}{12}\right),$
$\sqrt{2}\text{cis}\left(\frac{7\pi}{12}\right)$

75

$(z-\bar{z})^2 + 18(z+\bar{z}) = 36$

$(2iy)^2 + 18(2x) = 36$

$-4y^2 + 36x = 36$

$4y^2 = 36x - 36$

$y^2 = 9x - 9$

which is a concave right parabola.

76

$\frac{z}{z^2+a^2}$

$= \frac{a(\cos\theta + i\sin\theta)}{[a(\cos\theta + i\sin\theta)]^2 + a^2}$

$= \frac{a(\cos\theta + i\sin\theta)}{a^2(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta) + a^2}$

$= \frac{a(\cos\theta + i\sin\theta)}{a^2(\cos^2\theta - \sin^2\theta + 1 + 2i\sin\theta\cos\theta)}$

$= \frac{a(\cos\theta + i\sin\theta)}{a^2(2\cos^2\theta + 2i\sin\theta\cos\theta)}$

$= \frac{a(\cos\theta + i\sin\theta)}{2a^2\cos\theta\,(\cos\theta + i\sin\theta)}$

$= \frac{1}{2\,a\cos\theta}$

77

$\omega_1 = cis\frac{2\pi}{3}, w_2 = cis\left(-\frac{2\pi}{3}\right)$

$1 + \omega_1^k + \omega_1^{2k}$

$= 1 + cis\left(\frac{2k\pi}{3}\right) + cis\left(\frac{4k\pi}{3}\right)$

$= 1 + \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right) + \cos\left(\frac{4k\pi}{3}\right) + i\sin\left(\frac{4k\pi}{3}\right)$

$= 1 + 1 + 0 + 1 + 0 = 3$ if k is a multiple of 3

$= 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = 0$ if k is one more than a multiple of 3

$= 1 + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = 0$ if k is two more than a multiple of 3

Similarly substituting ω_2 we get 0 or 3

ANSWER A

78

$$\frac{a}{1+i}+\frac{b}{1+2i}=1$$
$$(1+2i)a+(1+i)b=(1+i)(1+2i)$$
$$a+2ai+b+bi=1+2i+i-2$$
$$(a+b)+i(2a+b)=-1+3i$$
$$\therefore a+b=-1 \qquad 2a+b=3$$
$a=4, b=-5$ by inspection

79

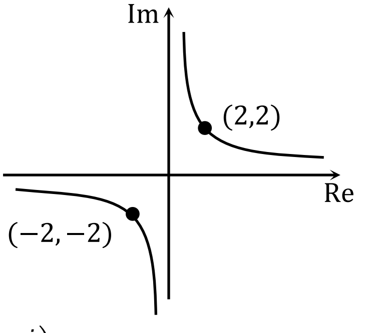
i $z^5=1 \;\Rightarrow\; z^5-1=0$
$\therefore (z-1)(z^4+z^3+z^2+z+1)=0$
$\therefore z^4+z^3+z^2+z+1=0$ since $z\neq 1$
dividing by z^2:
$$z^2+z+1+\frac{1}{z}+\frac{1}{z^2}=0$$
$$\therefore \left(z^2+2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)-1=0$$
$x^2+x-1=0$
$\therefore \alpha\beta=-1$
ii now $\operatorname{cis}\frac{2\pi}{5}$ and $\operatorname{cis}\frac{4\pi}{5}$ are solutions of $z^5=1$
$$\therefore \left(\operatorname{cis}\frac{2\pi}{5}+\frac{1}{\operatorname{cis}\frac{2\pi}{5}}\right)\left(\operatorname{cis}\frac{4\pi}{5}+\frac{1}{\operatorname{cis}\frac{4\pi}{5}}\right)=-1$$
$$\left(2\cos\frac{2\pi}{5}\right)\left(2\cos\frac{4\pi}{5}\right)=-1$$
$$\therefore \cos\frac{2\pi}{5}\cos\frac{4\pi}{5}=-\frac{1}{4}$$

80

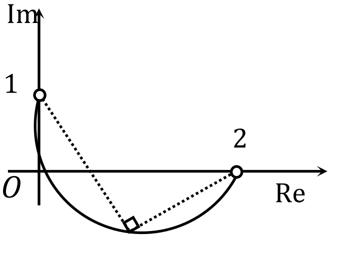
i $\angle AOB=\angle BOC=\angle COD=\frac{\pi}{4}$ (angle in a right-angled isosceles triangle)
$$\frac{|OA|}{|OB|}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}} \quad \therefore |OB|=\sqrt{2}|OA|$$
$$\arg\beta=\arg\alpha+\frac{\pi}{4}$$
$$\beta=|OB|\operatorname{cis}\left(\arg\alpha+\frac{\pi}{4}\right)=\sqrt{2}\operatorname{cis}\frac{\pi}{4}(|OA|\operatorname{cis}\alpha)$$
$$=\sqrt{2}\operatorname{cis}\frac{\pi}{4}\alpha$$
Similarly $\overrightarrow{OD}=\sqrt{2}\operatorname{cis}\frac{\pi}{4}\overrightarrow{OC}$; $\overrightarrow{OC}=\sqrt{2}\operatorname{cis}\frac{\pi}{4}\overrightarrow{OB}$
$$\therefore \overrightarrow{OD}=\left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)\left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)\left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)\alpha$$
$$=2\sqrt{2}\operatorname{cis}\frac{3\pi}{4}\,\alpha$$
$$=-2\sqrt{2}\,\frac{\operatorname{cis}\pi}{4}\,i\alpha$$
ii $64\alpha^4+\beta^4$
$$=64\alpha^4+\left(2\sqrt{2}\operatorname{cis}\frac{\pi}{4}\,i\alpha\right)^4$$
$$=64\alpha^4+64\operatorname{cis}\pi(i^4)\alpha^4$$
$$=64\alpha^4-64\alpha^4$$
$$=0$$

81

i $z^2-(\bar{z})^2=16i$
$(x+iy)^2-(x-iy)^2=16i$
$x^2+2xyi-y^2-x^2+2xyi+y^2=16i$
$4xy=16$
$xy=4$



ii $\arg\left(\frac{z-i}{z-2}\right)=\frac{\pi}{2}$



82

i $\left|\frac{z}{\bar{z}}\right|=\frac{|z|}{|\bar{z}|}=1$
$$\arg\left(\frac{z}{\bar{z}}\right)=\arg z-\arg\bar{z}=\arg z-(-\arg z)=2\arg z$$
$$\frac{z}{\bar{z}}=i$$
ii $\therefore 2\arg z=2k\pi+\frac{\pi}{2} \;\Rightarrow\; \arg z=\frac{\pi}{4},-\frac{3\pi}{4}$
$x=y$ for $\arg z=\frac{\pi}{4},-\frac{3\pi}{4}$
$\therefore z=(1+i)r=r+ir$ for real r

83 **i**

$$\begin{aligned} e^{n\theta i} + e^{-n\theta i} &= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta) \\ &= 2\cos n\theta \end{aligned}$$

ii

$$\begin{aligned} \left(e^{i\theta} + e^{-i\theta}\right)^3 &= \left(e^{i\theta}\right)^3 + 3\left(e^{i\theta}\right)^2\left(e^{-i\theta}\right) + 3\left(e^{i\theta}\right)\left(e^{-i\theta}\right)^2 + \left(e^{-i\theta}\right)^3 \\ &= e^{3\theta i} + 3e^{2\theta i}e^{-i\theta} + 3e^{i\theta}e^{-2\theta i} + e^{-3\theta i} \\ &= \left(e^{3\theta i} + e^{-3\theta i}\right) + 3\left(e^{i\theta} + e^{-i\theta}\right) \end{aligned}$$

iii

$$\begin{aligned} \left(e^{i\theta} + e^{-i\theta}\right)^3 &= \left(e^{3\theta i} + e^{-3\theta i}\right) + 3\left(e^{i\theta} + e^{-i\theta}\right) \\ (2\cos\theta)^3 &= (2\cos 3\theta) + 3(2\cos\theta) \\ 8\cos^3\theta &= 2\cos 3\theta + 6\cos\theta \\ \cos^3\theta &= \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta \end{aligned}$$

84 $wz = \dfrac{3+4i}{5} \times \dfrac{5+12i}{13} = \dfrac{15+36i+20i-48}{65} = \dfrac{-33}{65} + \dfrac{56i}{65}$

Since $|w| = |z| = 1$, $|wz|^2 = |w\bar{z}|^2 = |\bar{w}z|^2 = |\bar{w}\bar{z}|^2 = 1$. Choosing say the first two:

$|wz| = \left(-\dfrac{33}{65}\right)^2 + \left(\dfrac{56}{65}\right)^2 = 1 \Rightarrow \therefore 33^2 + 56^2 = 65^2$

$w\bar{z} = \dfrac{3+4i}{5} \times \dfrac{5-12i}{13} = \dfrac{15-36i+20i+48}{65} = \dfrac{63}{65} - \dfrac{16i}{65}$

$\therefore 63^2 + 16^2 = 65^2$

85

$$\begin{aligned} |(1+i)z^3 + iz| &\le |(1+i)z^3| + |iz| \text{ (by the triangle inequality)} \\ &\le |1+i||z|^3 + |z| \\ &< \sqrt{2}\left(\frac{1}{2}\right)^3 + \frac{1}{2} < \frac{\sqrt{2}+4}{8} < \frac{6}{8} < \frac{3}{4} \end{aligned}$$

86 **i** Produce AZ to N, BZ to M, and construct ZP parallel ot the x-axis.

$\angle NZP = \arg(Z - 2i)$ (corresponding angles to parallel lines)

$\angle MZP = \arg(Z - 2\sqrt{3})$ (corresponding angles to parallel lines)

$\therefore \angle MZN = \arg(Z - 2\sqrt{3}) - \arg(Z - 2i)$

$\angle MZN = \angle AZB = \alpha$ (vertically opposite)

$\therefore \alpha = \dfrac{\pi}{3}$

ii The centre of the circle lies on the perpendicular bisector of AB. Let the centre of the circle be C. The midpoint of AB is $\left(\sqrt{3}, 1\right)$, and interval AB has gradient $-\frac{1}{\sqrt{3}}$, so the perpendicular bisector has gradient $\sqrt{3}$, so equation

$y - 1 = \sqrt{3}(x - \sqrt{3})$.

$\angle ACB = 2\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}$ (angle at the centre is twice the angle at the circumference)

$\therefore \angle ACM = \dfrac{\pi}{3}$

$\therefore \angle CAM = \frac{\pi}{6}$ (angle sum ΔAOM)

$\therefore \angle CAM = \angle ABO = \dfrac{\pi}{6}$

$\therefore AC||OB$ $\therefore$ the y-value of C is 2. Sub in the equation:

$2 - 1 = \sqrt{3}(x - \sqrt{3})$ $\therefore x = \dfrac{4}{\sqrt{3}}$

The centre is $\left(\frac{4}{\sqrt{3}}, 2\right)$ and the radius is $\frac{4}{\sqrt{3}}$ (since AC is horizontal it equals the radius).

87 **i** $z^5+1=0 \quad \Rightarrow \quad z^5=1 \quad \Rightarrow \quad (r\ \text{cis}\theta)^5=1$

$\therefore\ r^5\text{cis}5\theta=\text{cis}\,\pi$

$r^5=1 \qquad 5\theta=(2k+1)\pi$

$r=1 \qquad 5\theta=-3\pi,-\pi,\pi,3\pi,5\pi$

$\theta=\pm\frac{3\pi}{5},\pm\frac{\pi}{5},\pi$

$z=\text{cis}\left(-\frac{3\pi}{5}\right),\text{cis}\left(-\frac{\pi}{5}\right),1,\text{cis}\left(\frac{\pi}{5}\right),\text{cis}\left(\frac{3\pi}{5}\right)$

ii $z^5+1=0$

$(z+1)(z^4-z^3+z^2-z+1)=0$

$\therefore z=-1\quad \text{or}\quad z^4-z^3+z^2-z+1=0$

Since $z=-1$ is real, the non-real solutions of $z^5+1=0$ must be the solutions of $z^4-z^3+z^2-z+1=0$.

iii $z^4-z^3+z^2-z+1=0$

$\therefore z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=0$

$z^2+z^{-2}-(z+z^{-1})+1=0$

$2\cos 2\theta-2\cos\theta+1=0$

$2(2\cos^2\theta-1)-2\cos\theta+1=0$

$4\cos^2\theta-2\cos\theta-1=0$

$z=\text{cis}\left(\frac{3\pi}{5}\right)$ is solution of $z^5+1=0$ from (i)

$\therefore\ \cos\left(\frac{3\pi}{5}\right)=\frac{2\pm\sqrt{(-2)^2-4(4)(-1)}}{2(4)}=\frac{2\pm2\sqrt{5}}{8}=\frac{1-\sqrt{5}}{4} \qquad \text{since } \cos\left(\frac{3\pi}{5}\right)<0$

$\therefore\ \sec\left(\frac{3\pi}{5}\right)=\frac{4}{1-\sqrt{5}}\times\frac{1+\sqrt{5}}{1+\sqrt{5}}=\frac{4+4\sqrt{5}}{1-5}=-1-\sqrt{5}$

88 **i** Let $c=\cos\theta\,, s=\sin\theta.$

$(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta,$ and also

$=c^5+5c^4si+10c^3s^2i^2+10c^2s^3i^3+5cs^4i^4+s^5i^5$

$=c^5+5c^4si-10c^3s^2-10c^2s^3i+5cs^4+s^5i$

$\therefore \cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$

$\sin 5\theta=5\cos^4\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$

$\therefore\ \tan 5\theta=\frac{5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta}{\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta}\div\frac{\cos^5\theta}{\cos^5\theta}$

$=\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}$

ii $\tan 5\theta=0$ occurs when $\sin 5\theta=0$

$\therefore 5\theta=k\pi \quad\Rightarrow\quad \theta=\frac{k\pi}{5}=0,\pm\frac{\pi}{5},\pm\frac{2\pi}{5}$

also $5\tan\theta-10\tan^3\theta+\tan^5\theta=0$

$\tan\theta\,(5-10\tan^2\theta+\tan^4\theta)=0$

Let $x=\tan\theta$

$\therefore x(x^4-10x^2+5)=0$

$\therefore x=0\ \text{ or }\ x^4-10x^2+5=0$

$x=\tan 0$ is the solution to $x=0$, so the other solutions to $\tan 5\theta=0$ must be the solutions to $x^4-10x^2+5=0$, so its solutions are $x=\tan\left(\pm\frac{\pi}{5}\right),\tan\left(\pm\frac{2\pi}{5}\right)=\pm\tan\frac{\pi}{5},\pm\tan\frac{2\pi}{5}$

iii $\prod\alpha=\frac{e}{a}$

$\therefore\ \tan\frac{\pi}{5}.\tan\frac{2\pi}{5}.\left(-\tan\frac{2\pi}{5}\right).\left(-\tan\frac{\pi}{5}\right)=\frac{5}{1}$

$\tan\frac{\pi}{5}.\tan\frac{2\pi}{5}.\tan\frac{3\pi}{5}.\tan\frac{4\pi}{5}=5$

89 **i** $(r \text{ cis } \theta)^5 = 1$

$r^5 \text{cis } 5\theta = \text{cis } 2k\pi$

$\therefore r = 1 \quad 5\theta = 2k\pi \quad k = -2, -1, 0, 1, 2$

$$\theta = \frac{2k\pi}{5} = 0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}$$

The roots of $z^5 - 1 = 0$ are $\text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right), 1$

ii The sum of the roots is zero, so:

$$\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right) + 1 + \text{cis}\left(-\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) = 0$$

equating real parts

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + 1 + \cos\left(-\frac{4\pi}{5}\right) + \cos\left(-\frac{2\pi}{5}\right) = 0$$

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = -1$$

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$$

90 **i** $z^n - z^{-n}$

$= (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= 2i\sin n\theta$

$$\therefore \sin n\theta = \frac{z^n - z^{-n}}{2i}$$

$z^n + z^{-n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$$\therefore \cos n\theta = \frac{z^n + z^{-n}}{2}$$

ii $32\sin^4\theta\cos^2\theta$

$= 32\sin^4\theta - 32\sin^6\theta$

$= 32\left(\frac{z - z^{-1}}{2i}\right)^4 - 32\left(\frac{z - z^{-1}}{2i}\right)^6$

$= 2(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4})$

$+\frac{1}{2}(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2}$

$- 6z^{-4} + z^{-6})$

$= \frac{1}{2}(z^6 + z^{-6}) - (z^4 + z^{-4}) - \frac{1}{2}(z^2 + z^{-2})$

$+ 22$

$= \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

91 **i** Roots are separated by $\frac{2\pi}{5}$

$\omega_1 = \text{cis}\left(-\frac{3\pi}{5}\right), \omega_2 = \text{cis}\left(-\frac{\pi}{5}\right),$

$\omega_3 = \text{cis}\left(\frac{\pi}{5}\right), \omega_4 = \text{cis}\left(\frac{3\pi}{5}\right)$

$\overline{\omega_4} = \overline{\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}}$

$= \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

$= \cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)$

$= \text{cis}\left(-\frac{3\pi}{5}\right)$

$= \omega_1$

ii $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$

Since $\omega \neq 1$, ω solves $z^4 - z^3 + z^2 - z + 1 = 0$

$\therefore \omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$

$\therefore \omega^4 + \omega^2 + 1 = \omega^3 + \omega$

iii $\omega_1^3 = \left(\text{cis}\left(-\frac{3\pi}{5}\right)\right)^3$

$= \text{cis}\left(-\frac{9\pi}{5}\right)$

$= \text{cis}\left(\frac{\pi}{5}\right)$

$= \omega_3$

iv $\omega_1^3 + \omega_2^3 + \omega_3^3 + \omega_4^3$

$= \omega_3 + \omega_1 + \omega_4 + \omega_2$

$= \sum\alpha - (-1)$

$= 0 - (-1)$

$= 1$

v $\omega^4 = \omega^3 - \omega^2 + \omega - 1$ (from ii)

$\therefore \omega_1^4 + \omega_2^4 + \omega_3^4 + \omega_4^4 = \omega_1^3 + \omega_2^3 + \omega_3^3 + \omega_4^3 - \omega_1^2 - \omega_2^2 - \omega_3^2 - \omega_4^2 + \omega_1 + \omega_2 + \omega_3 + \omega_4 - 4$

$\left(\text{cis}\left(-\frac{3\pi}{5}\right)\right)^4 + \left(\text{cis}\left(-\frac{\pi}{5}\right)\right)^4 + \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^4 + \left(\text{cis}\left(\frac{3\pi}{5}\right)\right)^4$

$= \omega_3 + \omega_1 + \omega_4 + w_2 - \text{cis}\left(\frac{4\pi}{5}\right) - \text{cis}\left(-\frac{2\pi}{5}\right) - \text{cis}\left(\frac{2\pi}{5}\right) - \text{cis}\left(-\frac{4\pi}{5}\right) + \omega_1 + \omega_2 + \omega_3 + \omega_4 - 4$

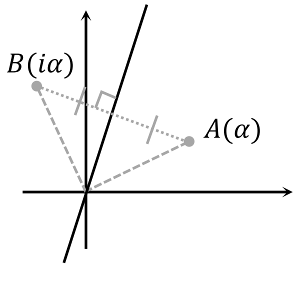
$2\left(\text{cis}\left(-\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(\frac{2\pi}{5}\right)\right) = 2(\omega_1 + \omega_2 + \omega_3 + \omega_4) - 4$

$\text{cis}\left(-\frac{4\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) = (1) - 2$

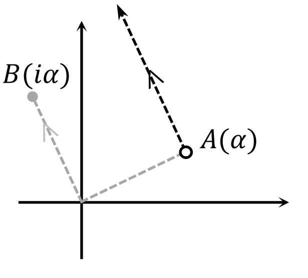
$2\cos\frac{4\pi}{5} + 2\cos\frac{2\pi}{5} = -1$

$\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = -\frac{1}{2}$

92 **i** The perpendicular bisector of AB



ii A ray from A parallel to $\overrightarrow{OB}$



The point forms a right angled isosceles triangle with O and A, so is $\sqrt{2}\text{cis}\frac{\pi}{4}\alpha = (1+i)\alpha$

93 **i** $(1+ic)^5 = 1 + 5ic + 10i^2c^2 + 10i^3c^3 + 5i^4c^4 + i^5c^5$
$= 1 + 5ic - 10c^2 - 10c^3i + 5c^4 + c^5i$
$\text{Im}\left((1+ic)^5\right) = 0$ since $(1+ic)^5$ is real,
$\therefore 5c - 10c^3 + c^5 = 0$
$c(c^4 - 10c^2 + 5) = 0$
$\therefore c^4 - 10c^2 + 5 = 0$ since $c \neq 0$
$c^2 = \dfrac{10 \pm \sqrt{(-10)^2 - 4(1)(5)}}{2(1)} = \dfrac{10 \pm 4\sqrt{5}}{2} = 5 \pm 2\sqrt{5}$
$\therefore c = \sqrt{5-2\sqrt{5}}, -\sqrt{5-2\sqrt{5}}, \sqrt{5+2\sqrt{5}}, -\sqrt{5+2\sqrt{5}}$
ii $1 + ic = r\,\text{cis}\,\theta$
$\therefore (1+ic)^5 = r^5\text{cis}\,5\theta$
since $1 + ic$ is real, $\sin 5\theta = 0$
$\therefore 5\theta = k\pi \qquad k$ integral
$\therefore \theta = \dfrac{k\pi}{5}$
The smallest positive value of θ is $\frac{\pi}{5}$
For the smallest possible value of c
$1 + ic = r\left(\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right)$
$\therefore \cos\dfrac{\pi}{5} = \dfrac{1}{r} \qquad \sin\dfrac{\pi}{5} = \dfrac{c}{r}$
$\therefore \tan\frac{\pi}{5} = \frac{\frac{c}{r}}{\frac{1}{r}} = c = \sqrt{5-2\sqrt{5}}$ since the smallest value of c corresponds to the smallest possible value of θ.

94 **i** $z^n + z^{-n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$
$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$
$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$
$= 2\cos n\theta$
ii The roots of $x^5 - 1 = 0$ are $1, \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$
$\therefore x^5 - 1$
$= (x-1)\left(x - \text{cis}\frac{2\pi}{5}\right)\left(x - \text{cis}\left(-\frac{2\pi}{5}\right)\right)\left(x - \text{cis}\frac{4\pi}{5}\right)\left(x - \text{cis}\left(-\frac{4\pi}{5}\right)\right)$
$= (x-1)\left(x^2 - 2\cos\dfrac{2\pi}{5} + 1\right)\left(x^2 - 2\cos\dfrac{4\pi}{5} + 1\right)$
iii now $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$
$\left(x^2 - 2\cos\frac{2\pi}{5} + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5} + 1\right) = x^4 + x^3 + x^2 + x + 1$
Let $x = 1$
$\left(1 - 2\cos\dfrac{2\pi}{5} + 1\right)\left(1 - 2\cos\dfrac{4\pi}{5} + 1\right) = 5$
$4\left(1 - \cos\dfrac{2\pi}{5}\right)\left(1 - \cos\dfrac{4\pi}{5}\right) = 5$
$\left(1 - \cos\dfrac{2\pi}{5}\right)\left(1 - \cos\dfrac{4\pi}{5}\right) = \dfrac{5}{4}$

95 **i** $v = \text{cis}\dfrac{\pi}{6} \quad w = 2\,\text{cis}\left(-\dfrac{2\pi}{3}\right)$

$wv = 2\,\text{cis}\left(\dfrac{\pi}{6} - \dfrac{2\pi}{3}\right) = 2\,\text{cis}\left(-\dfrac{\pi}{2}\right) = -2i$

$iv = \text{cis}\dfrac{\pi}{2} \times \text{cis}\dfrac{\pi}{6} = \text{cis}\dfrac{2\pi}{3}$

ii sub $v = \text{cis}\,\frac{\pi}{6}$ into $Z^4 = iZ$

$\left(\text{cis}\dfrac{\pi}{6}\right)^4 = i\,\text{cis}\dfrac{\pi}{6}$

$\text{cis}\dfrac{2\pi}{3} = \text{cis}\dfrac{\pi}{2} \times \text{cis}\dfrac{\pi}{6}$

$\text{cis}\dfrac{2\pi}{3} = \text{cis}\dfrac{2\pi}{3}$

$\therefore v$ satisfies $Z^4 = iZ$

$\therefore Z(Z^3 - i) = 0$

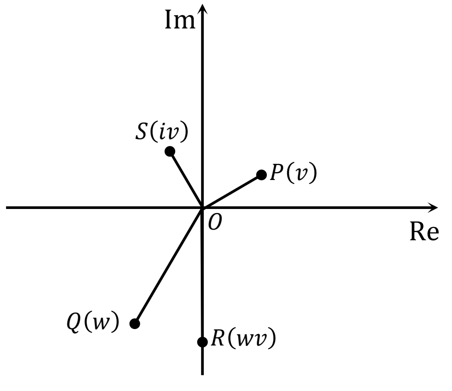
$\therefore Z = 0 \text{ or } Z = \sqrt[3]{i} = \text{cis}\left(\dfrac{2k\pi + \frac{\pi}{2}}{3}\right)$

$= \text{cis}\dfrac{(4k+1)\pi}{6}$

$= \text{cis}\left(-\dfrac{\pi}{2}\right), \text{cis}\dfrac{\pi}{6}, \text{cis}\left(\dfrac{5\pi}{6}\right)$

The other two non-zero solutions are $\text{cis}\left(-\frac{\pi}{2}\right) = -i$ and $\text{cis}\left(\frac{5\pi}{6}\right)$.

iii



iv OQR is isosceles, $\angle QOR = \frac{\pi}{6}, \angle QRO = \frac{1}{2}\left(\pi - \frac{\pi}{6}\right) = \frac{5\pi}{12}, \therefore \arg RQ = \frac{\pi}{2} + \frac{5\pi}{12} = \frac{11\pi}{12}$.

$\overrightarrow{PS} = iv - v = (-1+i)v = \sqrt{2}\text{cis}\left(\dfrac{3\pi}{4}\right)\text{cis}\dfrac{\pi}{6}$

$= \sqrt{2}\text{cis}\left(\dfrac{11\pi}{12}\right)$

$\therefore PS || RQ$ since $\arg\overrightarrow{RQ} = \arg\overrightarrow{PS} = \frac{11\pi}{12}$

$iv - v = u(w - wv)$

v $u = \dfrac{|iv - v|}{|w - wv|} = \dfrac{SP}{QR}$

$SP = \sqrt{OP^2 + OS^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$RQ = \sqrt{OR^2 + OQ^2 - 2 \times OR \times OQ \times \cos\dfrac{\pi}{6}}$

$= \sqrt{2^2 + 2^2 - 2 \times 2 \times 2\cos\dfrac{\pi}{6}} = \sqrt{8 - \dfrac{8\sqrt{3}}{2}}$

$= \sqrt{8 - 4\sqrt{3}}$

$u = \dfrac{\sqrt{2}}{\sqrt{8 - 4\sqrt{3}}} \times \dfrac{\sqrt{8 + 4\sqrt{3}}}{\sqrt{8 + 4\sqrt{3}}} = \dfrac{\sqrt{2}\sqrt{8 + 4\sqrt{3}}}{\sqrt{64 - 48}}$

$= \sqrt{\dfrac{16 + 8\sqrt{3}}{16}} = \dfrac{\sqrt{4 + 2\sqrt{3}}}{2} = \dfrac{\sqrt{\left(1 + \sqrt{3}\right)^2}}{2}$

$= \dfrac{1 + \sqrt{3}}{2}$

96 **i** $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n$

$= \left(\dfrac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\dfrac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$

$= \left(\dfrac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n$

$+ \left(\dfrac{\cos(-\theta) + i\sin(-\theta)}{\cos\theta}\right)^n$

$= \dfrac{\cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)}{\cos^n\theta}$

$= \dfrac{\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta}{\cos^n\theta}$

$= \dfrac{2\cos n\theta}{\cos^n\theta}$

ii $(1 + z)^4 + (1 - z)^4 = 0$

Let $z = i\tan\theta$ since z is purely imaginary

$(1 + i\tan\theta)^4 + (1 - i\tan\theta)^4 = 0$

$\dfrac{2\cos 4\theta}{\cos^4\theta} = 0 \qquad$ from (i)

$\therefore \cos 4\theta = 0$

$4\theta = k\pi + \dfrac{\pi}{2} \quad \text{for } k = -2, -1, 0, 1$

$\theta = \dfrac{(2k+1)\pi}{8}$

$= \pm\dfrac{\pi}{8}, \pm\dfrac{3\pi}{8}$

$\therefore z = \tan\left(\pm\dfrac{\pi}{8}\right), \tan\left(\pm\dfrac{3\pi}{8}\right)$

97 **i** $(w^n)^9 - 1$
$= (w^9)^n - 1$
$= \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^n - 1$
$= \cos 2\pi + i\sin 2\pi - 1$
$= 0$
ii $w + w^8$
$= \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9} + \cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}$
$= \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9} + \cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}$
$= 2\cos\frac{2\pi}{9}$

iii $(w^3 + w^6)(w^2 + w^7)$
$= w^5 + w^{10} + w^8 + w^{13}$
$= w^5 + w + w^8 + w^4$
iv the roots of $z^9 - 1 = 0$ are $w^k = \text{cis}\left(\frac{2k\pi}{9}\right)$ $k = -4, -3, \ldots 4$
$w^2 + w^7 = 2\cos\frac{4\pi}{9}, w^3 + w^6$
$= 2\cos\frac{6\pi}{9}$, and $w^4 + w^5$
$= 2\cos\frac{8\pi}{9}$ (similarly to (ii))
$\therefore 2\cos\frac{6\pi}{9} \times 2\cos\frac{4\pi}{9} = 2\cos\frac{8\pi}{9} + 2\cos\frac{2\pi}{9}$
$\therefore 2\cos\left(-\frac{\pi}{3}\right) \times 2\cos\frac{4\pi}{9} = -2\cos\frac{\pi}{9} + 2\cos\frac{2\pi}{9}$
$-2\cos\frac{4\pi}{9} = -2\cos\frac{\pi}{9} + 2\cos\frac{2\pi}{9}$
$\therefore \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$

98 Let $w = x + iy$
$\therefore (1 - i)w = (1 - i)(x + iy) = x + y + i(y - x)$
$\therefore y = x$ since $(1 - i)w$ is real
$\therefore w = x + ix = (1 + i)x \ \left\{= \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)x\right\}$
$w^2 = 2ix^2; w^3 = (2i - 2)x^3$
$\therefore (2i - 2)x^3 + i(2ix^2) + ik\big((1 + i)x\big) + 2i = 0$
$\therefore 2x^3 + 2x^2 + kx = 0 \ \ (1)$
$2x^3 + kx + 2 = 0 \quad (2)$
$(1) - (2) \ \ 2x^2 - 2 = 0$
$\therefore x = 1$ or $x = -1$
sub in (1) $k = -4$ $\quad k = 0$

99 **i**
$z^n + z^{-n} = z^n + \overline{z^n}$ since $|z| = 1$
$= 2\text{Re}(z^n)$
$= 2\text{Re}(\cos(n\theta) + i\sin(n\theta))$
$= 2\cos(n\theta)$
Similarly $z^n - z^{-n} = 2\text{Im}(z^n) = 2i\sin(n\theta)$

ii
$2^5 \sin^4\theta\cos^2\theta$
$= 32\left(\frac{z - z^{-1}}{2i}\right)^4\left(\frac{z + z^{-1}}{2}\right)^2$
$= 32\left(\frac{z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}}{16i^4}\right)\left(\frac{z^2 + 2 + z^{-2}}{4}\right)$
$= \frac{1}{2}(z^6 + 2z^4 + z^2 - 4z^4 - 8z^2 - 4 + 6z^2 + 12 + 6z^{-2} - 4 - 8z^{-2} - 4z^{-4} + z^{-2} + 2z^{-4} + z^{-6})$
$= \frac{1}{2}\big((z^6 + z^{-6}) - 2(z^4 + z^{-4}) - (z^2 + z^{-2}) + 4\big)$
$= \frac{1}{2}(2\cos 6\theta - 2(2\cos 4\theta) - (2\cos 2\theta) + 4)$
$= \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

100

i $\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}$

$$=\frac{1+\frac{1-t^2}{1+t^2}+i\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}-i\frac{2t}{1+t^2}}$$

$$=\frac{1+t^2+1-t^2+2ti}{1+t^2-1+t^2-2ti}$$

$$=\frac{2+2ti}{2t^2-2ti}$$

$$=\frac{1+ti}{t(t-i)}\times\frac{t+i}{t+i}$$

$$=\frac{t+i+t^2i-t}{t(t^2+1)}$$

$$=\frac{i(1+t^2)}{t(1+t^2)}$$

$$=\frac{i}{t}$$

$$=i\cot\frac{\theta}{2}$$

ii

$$\left(\frac{z-1}{z+1}\right)^8=-1$$

$$\therefore\frac{z-1}{z+1}=\sqrt[8]{-1}$$

$$=\text{cis}\frac{(2k+1)\pi}{8}\quad k=0,1,\ldots,7$$

$$\therefore z-1=z\text{cis}\frac{(2k+1)\pi}{8}+\text{cis}\frac{(2k+1)\pi}{8}$$

$$z\left(1-\text{cis}\frac{(2k+1)\pi}{8}\right)=\left(1+\text{cis}\frac{(2k+1)\pi}{8}\right)$$

$$z=\frac{1+\text{cis}\frac{(2k+1)\pi}{8}}{1-\text{cis}\frac{(2k+1)\pi}{8}}$$

$$=i\cot\frac{(2k+1)\pi}{16}\quad\text{from (i)}$$

$$=i\cot\left(\frac{\pi}{16}\right),i\cot\left(\frac{3\pi}{16}\right),\ldots i\cot\left(\frac{15\pi}{16}\right)$$

BASIC

Prove the following by mathematical induction unless otherwise stated.

1 $7^n - 2^n$ is divisible by 5 for $n \geq 1$

2 $2n + 1 < 2^n$ for $n \geq 3$

3 $n! > 3^n$ for $n \geq 7$

4 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for $n \geq 1$

5 A sequence is defined by letting $a_1 = 3$ and $a_n = 7a_{n-1}$ for $n \geq 2$. Prove that $a_n = 3 \times 7^{n-1}$ for $n \geq 1$.

6 $35^n + 3 \times 7^n + 2 \times 5^n + 6$ is divisible by 12 for $n \geq 0$

7 $2n + 7 < 2^{n+3}$ for $n \geq 0$

8 $5^n > 4n + 12$ for $n \geq 2$

MEDIUM

9 $8^n + 2 \times 7^n - 1$ is divisible by 7 for $n \geq 1$

10 The Fibonacci numbers f_n are defined by $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Prove that $f_n < \left(\frac{5}{3}\right)^n$ for $n \geq 1$. Hint: this is a second order recursive relation so you must prove two base cases and make two inductive hypotheses.

11 Prove $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$ for $n \geq 1$

12 $n(n^2 - 3n + 5)$ is divisible by 3 for $n \geq 1$

13 $x^n - y^n$ is divisible by $x^2 - y^2$ for even integers, where x, y are integers.

14 $11^{n+1} + 12^{2n-1}$ is divisible by 133 for $n \geq 1$

15 $x^{n+2} + (x+1)^{2n+1}$ is divisible by $x^2 + x + 1$ for $n \geq 0$ and integral x.

16 $\dfrac{d}{dx}([f(x)]^n) = nf'(x)[f(x)]^{n-1}$ for $n \geq 1$

17 $\dfrac{d^n}{dx^n}(x^n) = n!$ for $n \geq 1, x \geq 0$

18 Show that n lines in the plane, no two of which are parallel and no three meeting in a point, divide the plane into $\dfrac{n^2 + n + 2}{2}$ regions, for $n \geq 1$. You may assume that when the k^{th} line is added it passes through k of the regions.

19 A convex n-sided polygon has $\dfrac{n(n-3)}{2}$ diagonals for $n \geq 3$.

20 **i** Use calculus to show that $x > \ln(1+x)$ for all $x > 0$.
ii Use the inequality in part (i) and the principle of mathematical induction to prove that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} > \ln(1+n)$ for all positive integers, n.

21 $\sum_{r=1}^{n} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$ for $n \geq 1$.

22 Given that for $k > 0$, $2k+3 > 2\sqrt{(k+1)(k+2)}$, prove that $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > 2\left(\sqrt{n+1}-1\right)$ for all positive integers n.

23 **i** Prove that $(1+x)^n > 1+nx$ for $n \geq 1$ and $x > -1$.
ii Hence, deduce that $\left(1-\frac{1}{2n}\right)^n > \frac{1}{2}$ for $n > 1$.

24 Given $0 \leq x_1, x_2, x_3, \ldots, x_n < 1$
i Prove that $2(1+x_1x_2) \geq (1+x_1)(1+x_2)$ given $(1-x_1)(1-x_2) \geq 0$ (direct proof)
ii Prove that $2^{n-1}(1+x_1 \times x_2 \times \ldots \times x_n) \geq (1+x_1)(1+x_2)\ldots(1+x_n)$ for $n \geq 1$

25 **i** Show that for each positive integer n there are positive integers p_n and q_n such that $\left(1+\sqrt{2}\right)^n = p_n + q_n\sqrt{2}$
ii Hence show that $p_n^2 - 2q_n^2 = (-1)^n$

26 $\cos x + \cos 2x + \cos 3x + \ldots + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$ for $n \geq 1$

CHALLENGING

27 $n^4 - 1$ is divisible by 16 for odd integers $n \geq 3$

28 $2^{3^n} + 1$ is divisible by 3^{n+1} for $n \geq 1$

29 **i** If $x_1 > 1$ and $x_2 > 1$ show that $x_1 + x_2 > \sqrt{x_1x_2}$
ii Use the principle of mathematical induction to show that, for $n \geq 2$, if $x_j > 1$ where $j = 1, 2, 3, \ldots, n$ then, $\ln(x_1 + x_2 + x_3 + \ldots + x_n) > \frac{1}{2^{n-1}}(\ln x_1 + \ln x_2 + \ln x_3 + \ldots + \ln x_n)$

30 $n^5 - n$ is divisible by $2, 3$ and 5 for $n \geq 2$, given the product of n consecutive numbers is divisible by $n!$.

31 $2\left(\sqrt{n+1}-1\right) < 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ for $n \geq 1$

32 $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n-1} + \frac{1}{2^n} \geq 1 + \frac{n}{2}$ for $n \geq 1$

33 $\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \ldots \times \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$ for $n \geq 1$

34 **i** By considering $f'(x)$ where $f(x) = e^x - x$, show that $e^x > x$ for $x \geq 0$
ii Hence, use Mathematical Induction to show that for $x \geq 0$, $e^x > \frac{x^n}{n!}$ for all positive integers $n \geq 1$.

35 Show that n lines in the plane, no two of which are parallel and no three meeting in a point, divide the plane into $\frac{n^2+n+2}{2}$ regions, for $n \geq 1$.

36 $\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$ for $n \geq 1$

37 $(x+a)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}a^r$ for $n \geq 1$

38 $\binom{n}{r}$ is even if n is even and r is odd, for $n \geq 2$ and r is an odd constant less than n, k.

39 i prove, for any integer $k \geq 0$, that $\dfrac{2k+1}{2k+2} < \dfrac{\sqrt{2k+1}}{\sqrt{2k+3}}$ (Direct proof)

ii Prove, by induction on $n \geq 0$, that the central binomial coefficient $\binom{2n}{n}$ satisfies
$\binom{2n}{n} \leq \dfrac{4^n}{\sqrt{2n+1}}$

40 $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \ldots = \begin{cases} \dfrac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \dfrac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$

1 Let $P(n)$ represent the proposition.

$P(1)$ is true since $7^1 - 2^1 = 5$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $7^k - 2^k = 5m$ for integral m

RTP $P(k+1)$ $7^{k+1} - 2^{k+1} = 5p$ for integral p

$$\begin{aligned}
\text{LHS} &= 7(7^k) - 2(2^k) \\
&= 7(7^k - 2^k) + 5(2^k) \\
&= 7(5m) + 5(2^k) \quad \text{from } P(k) \\
&= 5(7m + 2^k) \\
&= 5p \qquad \text{for integral } p \text{ since } m, k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

2 Let $P(n)$ represent the proposition.

$P(3)$ is true since $\text{LHS} = 2(3) + 1 = 7, \text{RHS} = 2^3 = 8$

If $P(k)$ is true for some arbitrary $k \geq 3$ then $2k - 1 < 2^k$

RTP $P(k+1)$ $2k + 1 < 2^{k+1}$

$$\begin{aligned}
\text{LHS} &= 2k - 1 + 2 \\
&< 2^k + 2 \quad \text{from } P(k) \\
&< 2^k + 2^k \quad \text{for } k \geq 3 \\
&= 2^{k+1} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 3$ by induction □

3 Let $P(n)$ represent the proposition.

$P(7)$ is true since $\text{LHS} = 7! = 5040, \text{RHS} = 3^7 = 2187$

If $P(k)$ is true for some arbitrary $k \geq 7$ then $k! > 3^k$

RTP $P(k+1)$ $(k+1)! > 3^{k+1}$

$$\begin{aligned}
\text{LHS} &= (k+1)(k!) \\
&> (k+1)3^k \quad \text{from } P(k) \\
&> 3^{k+1} \quad \text{for } k \geq 7 \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 7$ by induction □

4 Let $P(n)$ represent the proposition.

$P(1)$ is true since $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta = \cos 1\theta + i\sin 1\theta$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

RTP $P(k+1)$ $\quad (\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

$$\begin{aligned}
\text{LHS} &= (\cos\theta + i\sin\theta)^k \cdot \cos\theta + i\sin\theta \\
&= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \quad \text{from } P(k) \\
&= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta \\
&= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i\,(\cos k\theta\sin\theta + \sin k\theta\cos\theta) \\
&= \cos(k\theta + \theta) + i\sin(k\theta + \theta) \\
&= \cos(k+1)\theta + i\sin(k+1)\theta \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

5 Let $P(n)$ represent the proposition.

$P(1)$ is true since $a_1 = 3 \times 7^{1-1} = 3$ as given

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = 3 \times 7^{k-1}$

RTP $P(k+1)$ $\quad a_{k+1} = 3 \times 7^k$

$$\begin{aligned}
\text{LHS} &= 7a_k \\
&= 7\left(3 \times 7^{k-1}\right) \quad \text{from } P(k) \\
&= 3 \times 7^k \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

6 Let $P(n)$ represent the proposition

$P(0)$ is true since $35^0 + 3 \times 7^0 + 2 \times 5^0 + 6 = 12$

If $P(k)$ is true for some arbitrary $k \geq 0$ $\quad 35^k + 3 \times 7^k + 2 \times 5^k + 6 = 12m$ for integral m

RTP: $P(k+1)$ $\quad 35^{k+1} + 3 \times 7^{k+1} + 2 \times 5^{k+1} + 6 = 12p$ for integral p

$$\begin{aligned}
\text{LHS} &= 35\left(35^k\right) + 7\left(3 \times 7^k\right) + 5\left(2 \times 5^k\right) + 6 \\
&= 35\left(35^k + 3 \times 7^k + 2 \times 5^k + 6\right) - 28\left(3 \times 7^k\right) - 30\left(2 \times 5^k\right) - 34 \times 6 \\
&= 35(12m) - 84 \times 7^k - 60 \times 5^k - 204 \\
&= 12\left(35m - 7^{k+1} - 5^{k+1} - 17\right) \\
&= 12p \text{ for integral } p \text{ since } m, k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 0$ by induction □

7 Let $P(n)$ represent the proposition
$P(0)$ is true since $\text{LHS} = 2(0) + 7 = 7; \text{RHS} = 2^{0+3} = 8$
If $P(k)$ is true for some arbitrary $k \geq 0$ $\quad 2k + 7 < 2^{k+3}$
RTP: $\quad P(k+1) \quad 2k + 9 < 2^{k+4}$

$$\begin{aligned} \text{LHS} &= 2k + 7 + 2 \\ &< 2^{k+3} + 2 \quad \text{from } P(k) \\ &< 2^{k+3} + 2^{k+3} \\ &< 2(2^{k+3}) \\ &< 2^{k+4} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$
$\therefore P(n)$ is true for $n \geq 0$ by induction □

8 Let $P(n)$ represent the proposition
$P(2)$ is true since $\text{LHS} = 5^2 = 32, \text{RHS} = 4(2) + 12 = 20$
If $P(k)$ is true for some arbitrary $k \geq 2$ $\quad 5^k > 4k + 12$
RTP: $\quad P(k+1) \quad 5^{k+1} > 4k + 16$

$$\begin{aligned} \text{LHS} &= 5(5^k) \\ &> 5(4k + 12) \quad \text{from } P(k) \\ &> 20k + 60 \\ &> 4k + 16 \quad \text{since } k \geq 2 \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$
$\therefore P(n)$ is true for $n \geq 2$ by induction □

9 Let $P(n)$ represent the proposition.

$P(1)$ is true since $8^1 + 2 \times 7^1 - 1 = 21 = 3 \times 7$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $8^k + 2 \times 7^k - 1 = 7m$ for integral m

RTP $\quad P(k+1) \quad 8^{k+1} + 2 \times 7^{k+1} - 1 = 7p$ for integral p

$$\begin{aligned} \text{LHS} &= 8(8^k) + 7(2 \times 7\hat{}k) - 1 \\ &= 8(8^k + 2 \times 7^k - 1) - 2 \times 7^k + 7 \\ &= 8(7m) - 2 \times 7^k + 7 \quad \text{from } P(k) \\ &= 7(8m - 2 \times 7^{k-1} + 1) \\ &= 7p \quad \text{for integral } p \text{ since } m,k \text{ integral} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

10 Let $P(n)$ represent the proposition

$P(1)$ is true since $1 < \left(\frac{5}{3}\right)^1$ and $P(2)$ is true since $1 < \left(\frac{5}{3}\right)^2$

If $P(k-1)$ and $P(k)$ are true for some arbitrary $k \geq 1$ $\quad f_{k-1} < \left(\frac{5}{3}\right)^{k-1} \quad f_k < \left(\frac{5}{3}\right)^k$

RTP: $P(k+1) \quad f_{k+1} < \left(\frac{5}{3}\right)^{k+1}$

$$\begin{aligned}
\text{LHS} &= f_{k-1} + f_k \\
&< \left(\frac{5}{3}\right)^{k-1} + \left(\frac{5}{3}\right)^k \quad \text{from } P(k-1) \text{ and } P(k) \\
&< \left(\frac{3}{5}\right)^2 \left(\frac{5}{3}\right)^{k+1} + \frac{3}{5}\left(\frac{5}{3}\right)^{k+1} \\
&< \left(\frac{9}{25} + \frac{3}{5}\right)\left(\frac{5}{3}\right)^{k+1} \\
&< \frac{24}{25}\left(\frac{5}{3}\right)^{k+1} \\
&< \left(\frac{5}{3}\right)^{k+1} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

11 Let $P(n)$ represent the proposition

$P(1)$ is true since $\text{LHS} = \frac{1}{(1+1)(1+2)} = \frac{1}{6}$, $\text{RHS} = \frac{1}{2(1+2)} = \frac{1}{6}$

If $P(k)$ is true for some arbitrary $k \geq 1$ $\quad \sum_{r=1}^{k} \frac{1}{(r+1)(r+2)} = \frac{k}{2(k+2)}$

RTP: $P(k+1) \quad \sum_{r=1}^{k+1} \frac{1}{(r+1)(r+2)} = \frac{k+1}{2(k+3)}$

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k} \frac{1}{(r+1)(r+2)} + \frac{1}{(k+2)(k+3)} \\
&= \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)} \quad \text{from } P(k) \\
&= \frac{1}{k+2}\left(\frac{k}{2} + \frac{1}{k+3}\right) \\
&= \frac{1}{k+2}\left(\frac{k^2+3k+2}{2(k+3)}\right) \\
&= \frac{1}{2(k+2)}\left(\frac{(k+1)(k+2)}{k+3}\right) \\
&= \frac{k+1}{2(k+3)} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

12 Let $P(n)$ represent the proposition.

$P(1)$ is true since $1(1^2 - 3(1) + 5) = 3$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $k(k^2 - 3k + 5) = 3m$ for integral m

RTP $P(k+1)$ $\quad (k+1)\big((k+1)^2 - 3(k+1) + 5\big) = 3p$ for integral p

$$\begin{aligned}
\text{LHS} &= (k+1)(k^2 + 2k + 1 - 3k - 3 + 5) \\
&= (k+1)((k^2 - 3k + 5) + 2(k-1)) \\
&= k(k^2 - 3k + 5) + 2k(k-1) + (k^2 - 3k + 5) + 2(k-1) \\
&= 3m + 2k^2 - 2k + k^2 - 3k + 5 + 2k - 2 \quad \text{from } P(k) \\
&= 3m + 3k^2 - 3k + 3 \\
&= 3(m + k^2 - k + 1) \\
&= 3p \quad \text{for integral } p \text{ since } m, k \text{ integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

13 Let $P(n)$ represent the proposition.

$P(2)$ is true since $x^2 - y^2$ is divisible by $x^2 - y^2$

If $P(k)$ is true for some arbitrary even $k \geq 2$ then $x^k - y^k = m(x^2 - y^2)$ for integral m

RTP $P(k+2)$ $\quad x^{k+2} - y^{k+2} = p(x^2 - y^2)$ for integral p

$$\begin{aligned}
\text{LHS} &= x^2(x^k) - y^2(y^k) \\
&= x^2(x^k - y^k) + (x^2 - y^2)(y^k) \\
&= x^2\big(m(x^2 - y^2)\big) + (x^2 - y^2)(y^k) \quad \text{from } P(k) \\
&= (x^2 - y^2)(mx^2 + y^k) \\
&= p(x^2 - y^2) \quad \text{for integral } p \text{ since } m, x, y, k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for even $n \geq 2$ by induction □

14 Let $P(n)$ represent the proposition.

$P(1)$ is true since $11^{1+1} + 12^{2(1)-1} = 133$

If $P(k)$ is true for some arbitrary even $k \geq 1$ then $11^{k+1} + 12^{2k-1} = 133m$ for integral m

RTP $P(k+1)$ $\quad 11^{k+2} + 12^{2k+1} = 133p$ for integral p

$$\begin{aligned}
\text{LHS} &= 11(11^{k+1}) + 12^2(12^{2k-1}) \\
&= 11(11^{k+1} + 12^{2k-1}) + (12^2 - 11)(12^{2k-1}) \\
&= 11(133m) + 133(12^{2k-1}) \quad \text{from } P(k) \\
&= 133(11m + 12^{2k-1}) \\
&= 133p \text{ for integral } p \text{ since } m, k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

15 Let $P(n)$ represent the proposition.

$P(0)$ is true since $x^2+(x+1)^1=x^2+x+1$

If $P(k)$ is true for some arbitrary $k\ge 0$ then $x^{k+2}+(x+1)^{2k+1}=m(x^2+x+1)$ for integral m

RTP $P(k+1)$ $\quad x^{k+3}+(x+1)^{2k+3}=p(x^2+x+1)$ for integral p

$$\begin{aligned}\text{LHS } &= x\left(x^{k+2}\right)+(x+1)^2(x+1)^{2k+1}\\ &= x\left(x^{k+2}\right)+(x^2+2x+1)(x+1)^{2k+1}\\ &= x\left(x^{k+2}+(x+1)^{2k+1}\right)+(x+2^x+1)(x+1)^{2k+1}\\ &= x\left(m(x^2+x+1)\right)+(x+2^x+1)(x+1)^{2k+1} \quad \text{from } P(k)\\ &= (x^2+x+1)\left(x+(x+1)^{2k+1}\right)\\ &= p(x^2+x+1) \text{ for integral } p \text{ since } x,k \text{ are integral}\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\ge 1$ by induction □

16 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\text{LHS}=\frac{d}{dx}\left(f(x)\right)=f'(x)$, $\text{RHS}=1\times f'(x)[f(x)]^0=f'(x)$

If $P(k)$ is true for some arbitrary $k\ge 1$ then $\frac{d}{dx}\left([f(x)]^k\right)=kf'(x)[f(x)]^{k-1}$

RTP $P(k+1)$ $\quad \frac{d}{dx}\left([f(x)]^{k+1}\right)=(k+1)f'(x)[f(x)]^k$

$$\begin{aligned}\text{LHS } &= \frac{d}{dx}\left([f(x)]^k\times f(x)\right)\\ &= f(x)\times\frac{d}{dx}\left([f(x)]^k\right)+[f(x)]^k\times\frac{d}{dx}\left(f(x)\right)\\ &= f(x)\times kf'(x)[f(x)]^{k-1}+[f(x)]^k\times f'(x)\\ &= k[f(x)]^k\times f'(x)+[f(x)]^k\times f'(x)\\ &= (k+1)f'(x)[f(x)]^k\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\ge 1$ by induction □

17 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{d}{dx}(x) = 1 = 1!$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{d^k}{dx^k}\left(x^k\right) = k!$

RTP $P(k+1)$ $\qquad \frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\right) = (k+1)!$

$$\text{LHS} = \frac{d^k}{dx^k}\left(\frac{d}{dx}\left(x^k \cdot x\right)\right)$$

$$= \frac{d^k}{dx^k}\left(x \cdot kx^{k-1} + x^k \cdot 1\right)$$

$$= \frac{d^k}{dx^k}\left(kx^k + x^k\right)$$

$$= \frac{d^k}{dx^k}\left((k+1)x^k\right)$$

$$= (k+1)\frac{d^k}{dx^k}\left(x^k\right)$$

$$= (k+1) \cdot k! \quad \text{from } P(k)$$

$$= (k+1)!$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

18 Let $P(n)$ represent the proposition.

$P(1)$ is true since 1 line divides the plane into 2 regions and $\frac{1^2+1+2}{2}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then k lines divide the plane into $\frac{k^2+k+2}{2}$ regions

RTP $P(k+1)$ $\qquad k+1$ lines divide the plane into $\frac{(k+1)^2+(k+1)+2}{2} = \frac{k^2+3k+4}{2}$ regions

The $k+1^{th}$ line passes through $k+1$ of the regions, so adding $k+1$ regions.

The number of regions with $k+1$ lines is

$$\frac{k^2+k+2}{2} + k + 1 \quad \text{from } P(k)$$

$$= \frac{k^2+k+2+2k+2}{2}$$

$$= \frac{k^2+3k+4}{2}$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

19 Let $P(n)$ represent the proposition.

$P(3)$ is true since a triangle has no diagonals and $\frac{3(3-3)}{2} = 0$

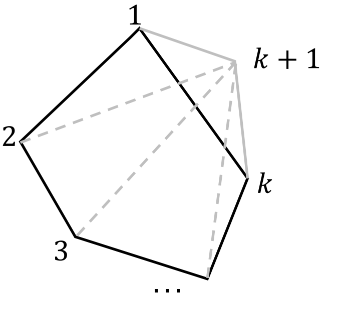
If $P(k)$ is true for some arbitrary $k \geq 1$ then a k-sided polygon has $\frac{k(k-3)}{2}$ diagonals

RTP $P(k+1)$ a k−sided polygon has $\frac{(k+1)(k-2)}{2}$ diagonals

When the $k+1^{th}$ vertex is added to a polygon it creates $k-1$ new diagonals

The number of diagonals on a $k+1$-sided polygon is

$$\frac{k(k-3)}{2} + k - 1$$
$$= \frac{k^2 - 3k + 2k - 2}{2}$$
$$= \frac{k^2 - k - 2}{2}$$
$$= \frac{(k+1)(k-2)}{2}$$
$$= \text{RHS}$$



$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

20 **i** Let $f(x) = x - \ln(1+x)$

$f(0) = 0 - \ln 1 = 0$

$$f'(x) = 1 - \frac{1}{1+x}$$

> 0 for $x > 0$

$\therefore$ $f(x)$ is an increasing function for $x > 0$

$\therefore f(x) > 0$ for $x > 0$ since $f(0) = 0$ and $f(x)$ is an increasing function

$\therefore x - \ln(1+x) > 0$

$x > \ln(1+x)$

ii Let $P(n)$ be the given proposition.

$P(1)$ is true since LHS $= \frac{1}{1} = 1$; RHS $= \ln(1+1) = \ln 2 \approx 0.693$

If $P(k)$ is true for some arbitrary $k \geq 1$ $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} > \ln(1+k)$

RTP: $P(k+1)$ $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} + \frac{1}{k+1} > \ln(2+k)$

$$\text{LHS} > \ln(1+k) + \frac{1}{k+1} \qquad \text{from } P(k)$$
$$> \ln(1+k) + \ln\left(1 + \frac{1}{k+1}\right) \qquad \text{from (i)}$$
$$> \ln(1+k) + \ln\left(\frac{k+2}{k+1}\right)$$
$$> \ln(k+2)$$
$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

21 Let $P(n)$ represent the proposition

$P(1)$ is true since LHS $= 1\ln\frac{1+1}{1} = \ln 2$; RHS $= \ln\left(\frac{(1+1)^1}{1!}\right) = \ln 2 \therefore P(1)$ is true

If $P(k)$ is true for some arbitrary $k \geq 1$ $\sum_{r=1}^{k} r\,ln\left(\frac{r+1}{r}\right) = ln\left(\frac{(k+1)^k}{k!}\right)$

RTP: $P(k+1)$ $\qquad \sum_{r=1}^{k+1} r\,ln\left(\frac{r+1}{r}\right) = ln\left(\frac{(k+2)^{k+1}}{(k+1)!}\right)$

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right) + (k+1)\ln\left(\frac{k+2}{k+1}\right)\\
&= ln\left(\frac{(k+1)^k}{k!}\right) + (k+1)\,ln\left(\frac{k+2}{k+1}\right)\\
&= ln\left(\frac{(k+1)^k}{k!}\right) + \ln\left(\frac{k+2}{k+1}\right)^{k+1}\\
&= \ln\left(\frac{(k+1)^k(k+2)^{k+1}}{k!\,(k+1)^{k+1}}\right)\\
&= \ln\left(\frac{(k+2)^{k+1}}{(k+1)!}\right)\\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

22 Let $P(n)$ represent the proposition

$P(1)$ is true since LHS $= \frac{1}{\sqrt{1}} = 1$; RHS $= 2\left(\sqrt{1+1}-1\right) \approx 0.828 \quad P(1)$

If $P(k)$ is true for some arbitrary $k \geq 1$ $\sum_{r=1}^{k}\frac{1}{\sqrt{r}} > 2\left(\sqrt{k+1}-1\right)$

RTP: $P(k+1)$ $\qquad \sum_{r=1}^{k+1}\frac{1}{\sqrt{r}} > 2\left(\sqrt{k+2}-1\right)$

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k}\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{k+1}}\\
&> 2\left(\sqrt{k+1}-1\right) + \frac{1}{\sqrt{k+1}}\\
&> \frac{2\left(k+1-\sqrt{k+1}\right)+1}{\sqrt{k+1}}\\
&> \frac{2k+3}{\sqrt{k+1}} - 2\\
&> \frac{2\sqrt{(k+1)(k+2)}}{\sqrt{k+1}} - 2 \qquad \text{given from the question}\\
&> 2\left(\sqrt{k+2}-1\right)\\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

23 i Let $P(n)$ represent the proposition

$P(1)$ is true since $\text{LHS} = (1+x)^1 = 1+x\,;\text{RHS} = 1+1\times x = 1+x$

If $P(k)$ is true for some arbitrary $k \geq 2$ $(1+x)^k > 1+kx$

RTP: $P(k+1)$ $\quad (1+x)^{k+1} > 1+(k+1)x$

$$\begin{aligned}\text{LHS} &= (1+x)^{k+1}\\ &= (1+x)(1+x)^k\\ &> (1+x)(1+kx) \quad \text{from } P(k)\\ &> 1+kx+x+kx^2\\ &> 1+(k+1)x \quad \text{since } k>0\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

ii Let $x = -\frac{1}{2n}$

$$\begin{aligned}\therefore \left(1-\frac{1}{2n}\right)^n &> 1+n\left(-\frac{1}{2n}\right)\\ &> 1-\frac{1}{2}\\ &> \frac{1}{2}\end{aligned}$$

24 **i**

$$\begin{aligned}(1-x_1)(1-x_2) &\geq 0\\ 1-x_2-x_1+x_1x_2 &\geq 0\\ 2+2x_1x_2 &\geq 1+x_1+x_2+x_1x_2\\ 2(1+x_1x_2) &\geq (1+x_1)(1+x_2)\end{aligned}$$

ii

Let $P(n)$ represent the proposition

$P(1)$ is true since $\text{LHS} = 2^{1-1}(1+x_1) = 1+x_1, \text{RHS} = 1+x_1$

If $P(k)$ is true for some arbitrary $k \geq 1$ $\quad 2^{k-1}(1+x_1\times x_2\times\ldots\times x_k) \geq (1+x_1)(1+x_2)\ldots(1+x_k)$

RTP: $P(k+1)$ $\quad 2^k(1+x_1\times x_2\times\ldots\times x_k\times x_{k+1}) \geq (1+x_1)(1+x_2)\ldots(1+x_k)(1+x_{k+1})$

$$\begin{aligned}\text{LHS} &= 2^{k-1}\left(2\left(1+(x_1\times x_2\times\ldots\times x_k)\times x_{k+1}\right)\right)\\ &\geq 2^{k-1}(1+x_1\times x_2\times\ldots\times x_k)(1+x_{k+1}) \quad \text{from (i) since } 0 \leq x_1\times x_2\times\ldots\times x_k < 1\\ &\geq (1+x_1)(1+x_2)\ldots(1+x_k)(1+x_{k+1}) \quad \text{from } P(k)\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

25 i Let $P(n)$ represent the proposition

$P(1)$ is true since LHS $= \left(1+\sqrt{2}\right)^1 = 1+\sqrt{2}$; RHS $= p_n + q_n\sqrt{2}$ $\therefore p_1 = 1, q_1 = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ $\left(1+\sqrt{2}\right)^k = p_k + q_k\sqrt{2}$ where p_k, q_k are integers.

RTP: $P(k+1)$ $\left(1+\sqrt{2}\right)^{k+1} = p_{k+1} + q_{k+1}\sqrt{2}$ where p_{k+1}, q_{k+1} are integers.

$$\begin{aligned}
\text{LHS} &= \left(1+\sqrt{2}\right)^{k+1} \\
&= \left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right)^k \\
&= \left(1+\sqrt{2}\right)\left(p_k + q_k\sqrt{2}\right) \quad \text{from } P(k) \\
&= (p_k + 2q_k) + (p_k + q_k)\sqrt{2} \\
&= p_{k+1} + q_{k+1}\sqrt{2} \quad \text{since } p_k, q_k \text{ are integers} \qquad (*) \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

ii Let $P(n)$ represent the proposition

$P(1)$ is true since $1^2 - 2(1)^2 = -1 = (-1)^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ $p_k^2 - 2q_k^2 = (-1)^k$

RTP: $P(k+1)$ $p_{k+1}^2 - 2q_{k+1}^2 = (-1)^{k+1}$

$$\begin{aligned}
\text{LHS} &= p_{k+1}^2 - 2q_{k+1}^2 \\
&= (p_k + 2q_k)^2 - 2(p_k + q_k)^2 \qquad \text{from } (*) \\
&= p_k^2 + 4p_kq_k + 4q_k^2 - 2p_k^2 - 4p_kq_k - 2q_k^2 \\
&= 2q_k^2 - p_k^2 \\
&= -\left(p_k^2 - 2q_k^2\right) \\
&= (-1)(-1)^k \qquad \text{from } P(k) \\
&= (-1)^{k+1} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

26 Let $P(n)$ represent the proposition

$P(1)$ is true since LHS $= \cos x$, RHS $= \frac{\sin\frac{3}{2}x - \sin\frac{x}{2}}{2\sin\frac{x}{2}} = \frac{2\cos x\sin\frac{x}{2}}{2\sin\frac{x}{2}} = \cos x$

If $P(k)$ is true for some arbitrary $k \geq 1$ $\cos x + \cos 2x + \cos 3x + \ldots + \cos kx = \frac{\sin\left(k+\frac{1}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$

RTP: $P(k+1)$ $\cos x + \cos 2x + \cos 3x + \ldots + \cos kx + \cos(k+1)x = \frac{\sin\left(k+\frac{3}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$

$$\begin{aligned}
\text{LHS} &= \frac{\sin\left(k+\frac{1}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x} + \cos(k+1)x \quad \text{from } P(k) \\
&= \frac{\sin\left(k+\frac{1}{2}\right)x - \sin\frac{1}{2}x + 2\sin\frac{1}{2}x\cos(k+1)x}{2\sin\frac{1}{2}x} \\
&= \frac{\sin\left(k+\frac{1}{2}\right)x - \sin\frac{1}{2}x + \sin\left((k+1)x + \frac{1}{2}x\right) - \sin\left((k+1)x - \frac{1}{2}x\right)}{2\sin\frac{1}{2}x} \\
&= \frac{\sin\left(k+\frac{1}{2}\right)x - \sin\frac{1}{2}x + \sin\left(\left(k+\frac{3}{2}\right)x\right) - \sin\left(\left(k+\frac{1}{2}\right)x\right)}{2\sin\frac{1}{2}x} \\
&= \frac{\sin\left(k+\frac{3}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

27 Let $P(n)$ represent the proposition.

$P(3)$ is true since $3^4 - 1 = 80 = 5 \times 16$

If $P(k)$ is true for some arbitrary odd $k \geq 3$ then $k^4 - 1 = 16m$ for integral m

RTP $P(k+2)$ $(k+2)^4 - 1 = 16p$ for integral p

$$\begin{aligned}
\text{LHS} &= k^4 + 8k^3 + 24k^2 + 32k + 16 - 1 \\
&= (k^4 - 1) + (8k^3 + 24k^2 + 32k + 16) \\
&= 16m + 8(k^3 + 3k^2 + 4k + 2) \quad \text{from } P(k) \\
&= 16m + 8\big(k^2(k+3) + 2(2k+1)\big) \\
&= 16m + 8(2q) \text{ for integral } q \text{ since } k^2(k+3) \text{ is even for } k \text{ odd or even as is } 2(2k+1) \\
&= 16(m+q) \\
&= 16p \text{ for integral } p \text{ since } m, q \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd $n \geq 1$ by induction □

28 Let $P(n)$ represent the proposition.

$P(1)$ is true since $2^{3^1} + 1 = 9 = 3^{1+1}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $2^{3^k} + 1 = m(3^{k+1})$ for integral m

RTP $P(k+1)$ $2^{3^{k+1}} + 1 = p(3^{k+2})$ for integral p

$$\begin{aligned}
\text{LHS} &= \left(2^{3^k}\right)^3 + 1 \\
&= \left(2^{3^k} + 1 - 1\right)^3 + 1 \\
&= \left(m(3^{k+1}) - 1\right)^3 + 1 \text{ from } P(k) \\
&= m^3(3^{k+1})^3 - 3m^2(3^{k+1})^2 + 3m(3^{k+1}) - 1 + 1 \\
&= 3^{3k+3}m^3 - 3^{2k+3}m^2 + 3^{k+2}m \\
&= 3^{k+2}(3^{2k+1}m^3 - 3^{k+1}m^2 + m) \\
&= p(3^{k+2}) \text{ for integral } p \text{ since } m, k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

29 **i**

$(\sqrt{x_1}-\sqrt{x_2})^2 \geq 0$

$x_1 - 2\sqrt{x_1 x_2} + x_2 \geq 0$

$x_1 + x_2 \geq 2\sqrt{x_1 x_2}$

$\therefore x_1 + x_2 > \sqrt{x_1 x_2}$ since $x_1 > 1, x_2 > 1$

ii Let $P(n)$ be the given proposition.

$P(2)$ is true since

$$\begin{aligned}\text{LHS} &= \ln(x_1+x_2)\\ &> \ln\sqrt{x_1x_2} \text{ from (i)}\\ &> \frac{1}{2}(\ln x_1 + \ln x_2)\\ &> \frac{1}{2^2-2}(\ln x_1 + \ln x_2)\\ &> \text{RHS}\end{aligned}$$

If $P(k)$ is true for some arbitrary $k \geq 2$

$$\ln(x_1+x_2+x_3+\ldots+x_k) > \frac{1}{2^{k-1}}(\ln x_1 + \ln x_2 + \ln x_3 + \ldots + \ln x_k)$$

RTP: $P(k+1)$

$$\ln(x_1+x_2+x_3+\ldots+x_k+x_{k+1}) > \frac{1}{2^{k-1}}(\ln x_1 + \ln x_2 + \ln x_3 + \ldots + \ln x_k + \ln x_{k+1})$$

$$\begin{aligned}\text{LHS} &= \ln(x_1+x_2+x_3+\ldots+x_k+x_{k+1})\\ &= \ln\big((x_1+x_2+x_3+\ldots+x_k)+(x_{k+1})\big)\\ &> \ln\sqrt{(x_1+x_2+x_3+\ldots+x_k)(x_{k+1})} \qquad \text{from (i)}\\ &> \frac{1}{2}\ln\big((x_1+x_2+x_3+\ldots+x_k)(x_{k+1})\big)\\ &> \frac{1}{2}\ln(x_1+x_2+x_3+\ldots+x_k) + \frac{1}{2}\ln(x_{k+1})\\ &> \frac{1}{2^k}(\ln x_1 + \ln x_2 + \ln x_3 + \ldots + \ln x_k) + \frac{1}{2}\ln(x_{k+1}) \qquad \text{from } P(k)\\ &> \frac{1}{2^k}(\ln x_1 + \ln x_2 + \ln x_3 + \ldots + \ln x_k + \ln x_{k+1}) + \frac{2^{k-1}-1}{2^k}\ln x_{k+1}\\ &> \frac{1}{2^k}(\ln x_1 + \ln x_2 + \ln x_3 + \ldots + \ln x_k + \ln x_{k+1}) \qquad \text{since } k \geq 2\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction □

30 If a number is divisible by $2, 3$ and 5 the it must be divisibly be $2 \times 3 \times 5 = 30$

Let $P(n)$ represent the proposition.

$P(2)$ is true since $2^5 - 2 = 30$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $k^5 - k = 30m$ for integral m

RTP $P(k+1)$ $(k+1)^5 - (k+1) = 30p$ for integral p

$$
\begin{aligned}
\text{LHS } &= (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1) \quad \text{from binomial theorem} \\
&= (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k) \\
&= 30m + 5k(k^3 + 2k^2 + 2k + 1) \quad \text{from } P(k) \\
&= 30m + 5k\big((k+1)^3 - k^2 - k\big) \\
&= 30m + 5k\,\big((k+1)^3 - k(k+1)\big) \\
&= 30m + 5k(k+1)(k^2 + k + 1) \\
&= 30m + 5k(k+1)(k^2 + 4k + 4 - 3k - 3) \\
&= 30m + 5k(k+1)(k+2)^2 + 5k(k+1)\big(-3(k+1)\big) \\
&= 30m + 5(k+2)\Big[k(k+1)(k+2)\Big] - 15(k+1)\Big[k(k+1)\Big] \\
&= 30m + 5(k+2)\Big[6q\Big] - 15(k+1)\Big[2r\Big]
\end{aligned}
$$

since the product of n consecutive numbers is divisible by n!

$$
\begin{aligned}
&= 30m + 30q(k+2) - 30r(k+1) \\
&= 30\Big[m + q(k+2) - r(k+1)\Big] \\
&= 30p \quad \text{since } m, q, k, r \text{ are integral} \\
&= \text{RHS}
\end{aligned}
$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

31 Let $P(n)$ represent the proposition.

$P(1)$ is true since $2\left(\sqrt{1+1}-1\right) \approx 0.828,\ 1+\frac{1}{\sqrt{2}} \approx 1.707, 2\sqrt{1}=2$

If $P(k)$ is true for some arbitrary even $k \geq 1$ then $2\left(\sqrt{k+1}-1\right) < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}} < 2\sqrt{k}$

RTP $P(k+1)$ $\quad 2\left(\sqrt{k+2}-1\right) < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$

$$2\left(\sqrt{k+1}-1\right) < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}} < 2\sqrt{k} \quad \text{from } P(k)$$

$$\therefore 2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}} < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} < 2\sqrt{k}+\frac{1}{\sqrt{k+1}}$$

$$\frac{2(k+1)+1}{\sqrt{k+1}}-2 < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} < \frac{2\left(\sqrt{k}\cdot\sqrt{k+1}+\frac{1}{2}\right)}{\sqrt{k+1}}$$

$$\frac{2\left(k+\frac{3}{2}\right)}{\sqrt{k+1}}-2 < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} < \frac{2(k+1)}{\sqrt{k+1}}$$

$$\frac{2\sqrt{k^2+3k+\frac{9}{4}}}{\sqrt{k+1}}-2 < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

$$\frac{2\sqrt{k^2+3k+2}}{\sqrt{k+1}}-2 < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

$$\frac{2\sqrt{(k+1)(k+2)}}{\sqrt{k+1}}-2 < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

$$2\left(\sqrt{k+2}-1\right) < 1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

$$\sqrt{k}\cdot\sqrt{k+1}+\frac{1}{2}-(k+1)$$

$$=\sqrt{k^2+k}-\left(k+\frac{1}{2}\right)$$

$$=\sqrt{k^2+k}-\sqrt{k^2+k+\frac{1}{4}}$$

$$<0$$

$$\therefore \sqrt{k}\cdot\sqrt{k+1}+\frac{1}{2} < k+1$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

32 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{1}{1}+\frac{1}{2} \geq 1+\frac{1}{2}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^{k}-1}+\frac{1}{2^{k}} \geq 1+\frac{k}{2}$

RTP $P(k+1)$ $\quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^{k}-1}+\frac{1}{2^{k}}+\frac{1}{2^{k}+1}+\ldots+\frac{1}{2^{k+1}-1}+\frac{1}{2^{k+1}} \geq 1+\frac{k+1}{2}$

$$\text{LHS} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^{k}-1}+\frac{1}{2^{k}}+\frac{1}{2^{k}+1}+\ldots+\frac{1}{2^{k+1}-1}+\frac{1}{2^{k+1}}$$

$$\geq 1+\frac{k}{2}+\frac{1}{2^{k}+1}+\ldots+\frac{1}{2^{k+1}-1}+\frac{1}{2^{k+1}}$$

$$\geq 1+\frac{k}{2}+\underbrace{\frac{1}{2^{k+1}}+\ldots+\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}}_{2^{k+1}-2^{k}\text{ times}}$$

$$\geq 1+\frac{k}{2}+\frac{2^{k+1}-2^{k}}{2^{k+1}}$$

$$\geq 1+\frac{k}{2}+\frac{2^{k}(2-1)}{2^{k+1}}$$

$$\geq 1+\frac{k}{2}+\frac{1}{2}$$

$$\geq 1+\frac{k+1}{2}$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

33 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\text{LHS} = \frac{1}{2}, \text{RHS} = \frac{1}{\sqrt{3(1)+1}} = \frac{1}{2}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \ldots \times \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}}$

RTP $P(k+1)$ $\quad \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \ldots \times \frac{2k-1}{2k} \times \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}$

$$\text{LHS} = \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \ldots \times \frac{2k-1}{2k} \times \frac{2k+1}{2k+2}$$

$$\leq \frac{1}{\sqrt{3k+1}} \times \frac{2k+1}{2k+2}$$

$$\leq \frac{1}{\sqrt{3k+1}} \times \frac{\sqrt{(2k+1)(2k+1)}}{\sqrt{(2k+2)(2k+2)}}$$

$$\leq \frac{\sqrt{(2k+1)(2k+1)}}{\sqrt{(3k+1)(2k+2)(2k+2)}}$$

$$\leq \frac{\sqrt{(2k+1)(2k+1)}}{\sqrt{12k^2+28k^2+20k+4}}$$

$$\leq \frac{\sqrt{(2k+1)(2k+1)}}{\sqrt{12k^2+28k^2+19k+4}}$$

$$\leq \frac{\sqrt{(2k+1)(2k+1)}}{\sqrt{(3k+4)(2k+1)(2k+1)}}$$

$$\leq \frac{1}{\sqrt{3k+4}}$$

$= \text{RHS}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

$$(3k+1)(2k+2)^2$$
$$= (3k+1)(4k^2+8k+4)$$
$$= 12k^2+28k^2+20k+4$$

$$(3k+1)(2k+1)^2$$
$$= (3k+4)(4k^2+4k+1)$$
$$= 12k^2+28k^2+19k+4$$

34 i

$f(x) = e^x - x$ $\qquad f(0) = e^0 - 0 = 1 > 0$
$f'(x) = e^x - 1$ $\qquad f'(x) > 0$ for $x > 0$ since $e^x > 1$
$\therefore f(x) > 0$ for $x \geq 0$, since $f(0) > 0$ and $f'(x) > 0$
$\therefore e^x - x > 0$
$e^x > x$

ii

Let $P(n)$ represent the proposition.
$P(1)$ is true since $e^x > x$ from (i)
If $P(k)$ is true for some arbitrary $k \geq 1$ then $e^x > \frac{x^k}{k!}$
RTP $P(k+1)$ $\qquad e^x > \frac{x^{k+1}}{(k+1)!}$

$$e^x > \frac{x^k}{k!} \qquad \text{from } P(k)$$
$$\int_0^x e^x\,dx > \int_0^x \frac{x^k}{k!}\,dx$$
$$\left[e^x\right]_0^x > \left[\frac{x^{k+1}}{(k+1)!}\right]_0^x$$
$$e^x - 1 > \frac{x^{k+1}}{(k+1)!} - 0$$
$$e^x > \frac{x^{k+1}}{(k+1)!} + 1$$
$$e^x > \frac{x^{k+1}}{(k+1)!}$$
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

35 Let $P(n)$ represent the proposition.

$P(1)$ is true since 1 line divides the plane into 2 regions and $\frac{1^2+1+2}{2}$
If $P(k)$ is true for some arbitrary $k \geq 1$ then k lines divide the plane into $\frac{k^2+k+2}{2}$ regions
RTP $P(k+1)$ $\qquad k+1$ lines divide the plane into $\frac{(k+1)^2+(k+1)+2}{2} = \frac{k^2+3k+4}{2}$ regions
By observation, when the k^{th} line is added it passes through k of the regions, so adding k regions.
The number of regions with $k+1$ lines is

$$\frac{k^2+k+2}{2} + k + 1 \quad \text{from } P(k)$$
$$= \frac{k^2+k+2+2k+2}{2}$$
$$= \frac{k^2+3k+4}{2}$$
$$= \text{RHS}$$
$\therefore P(k) \Rightarrow P(k+1)$
$\therefore P(n)$ is true for $n \geq 1$ by induction □

36 Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= \sin^2 x$, RHS $= \frac{1}{2}(1-\cos 2x) = \frac{1}{2}\left(1-(1-2\sin^2 x)\right) = \sin^2 x$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\sin^2 kx = \frac{1}{2}(1-\cos 2kx)$

RTP $P(k+1)$ $\quad \sin^2(k+1)x = \frac{1}{2}(1-\cos 2(k+1)x)$

$$\begin{aligned}
\text{LHS} &= \sin^2(kx+x)\\
&= 1-\cos^2(kx+x)\\
&= 1-(\cos kx\cos x-\sin kx\sin x)^2\\
&= 1-\cos^2 kx\cos^2 x+2\cos kx\cos x\sin kx\sin x-\sin^2 kx\sin^2 x\\
&= 1-(1-\sin^2 kx)\cos^2 x+2\sin kx\cos kx\sin x\cos x-\sin^2 kx\sin^2 x\\
&= 1-\cos^2 x+\sin^2 kx\cos^2 x+\frac{1}{2}\sin 2kx\sin 2x-\sin^2 kx\sin^2 x\\
&= \sin^2 x+\sin^2 kx\,(\cos^2 x-\sin^2 x)+\frac{1}{2}\sin 2x\sin 2kx\\
&= \sin^2 x+\frac{1}{2}(1-\cos 2kx)\cos 2x+\frac{1}{2}\sin 2x\sin 2kx \qquad \text{from } P(k)\\
&= \sin^2 x+\frac{1}{2}\cos 2x-\frac{1}{2}\cos 2kx\cos 2x+\frac{1}{2}\sin 2x\sin 2kx\\
&= \sin^2 x+\frac{1}{2}(1-2\sin^2 x)-\frac{1}{2}\cos 2kx\cos 2x+\frac{1}{2}\sin 2x\sin 2kx\\
&= \frac{1}{2}(1-\cos 2kx\cos 2x+\sin 2kx\sin 2x)\\
&= \frac{1}{2}(1-\cos(2kx+2x))\\
&= \frac{1}{2}(1-\cos 2(k+1)x)\\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

37 Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS = $(x+a)^1 = x+a$, RHS = $x^1 + a^1 = x+a$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $(x+a)^k = \sum_{r=0}^{k}\binom{k}{r}x^{k-r}a^r$

RTP $P(k+1)$ $\quad (x+a)^{k+1} = \sum_{r=0}^{k+1}\binom{k+1}{r}x^{k-r+1}a^r$

$$\begin{aligned}
\text{LHS} &= (x+a)^k(x+a)\\
&= \left(\sum_{r=0}^{k}\binom{k}{r}x^{k-r}a^r\right)\left(x+a\right)\\
&= \sum_{r=0}^{k}\binom{k}{r}x^{k-r+1}a^r+\sum_{r=0}^{k}\binom{k}{r}x^{k-r}a^{r+1}\\
&= x^{k+1}+\left(\binom{k}{0}+\binom{k}{1}\right)x^k a+\left(\binom{k}{1}+\binom{k}{2}\right)x^{k-1}a^2+\ldots+\left(\binom{k}{r-1}+\binom{k}{r}\right)x^{k-r+1}a^r+\ldots+a^{k+1}\\
&= \binom{k+1}{0}x^{k+1}+\binom{k+1}{1}x^k a+\binom{k+1}{2}x^{k-1}a^2+\ldots+\binom{k+1}{r}x^{k-r+1}a^r+\ldots+\binom{k+1}{k+1}a^{k+1}\\
&= \sum_{r=0}^{k+1}\binom{k+1}{r}x^{k-r+1}a^r\\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

38 Let $P(n)$ represent the proposition.

$P(2)$ is true since $\binom{2}{1} = 2$ which is even

If $P(k)$ is true for some arbitrary even $k > r \geq 1$ $\binom{k}{r} = 2m$ for integral m

RTP $P(k+2)$ $\binom{k+2}{r} = 2p$ for integral p

$$\begin{aligned}
\text{LHS} &= \frac{(k+2)!}{r!\,(k-r+2)!} \\
&= \frac{k!}{r!\,(k-r)!} \times \frac{(k+1)(k+2)}{(k-r+1)(k-r+2)} \\
&= \binom{k}{r} \times \frac{(k+1)(k+2)}{(k-r+1)(k-r+2)} \\
&= 2m \times \frac{(k+1)(k+2)}{(k-r+1)(k-r+2)} \qquad \text{from } P(k) \\
&= 2m \times \frac{2s}{2t} \quad \text{for integral } s, t \text{ since the product of consecutive numbers is even} \\
&= 2p \text{ for integral } p \text{ since } m, s, t \text{ are integral and } \binom{n}{r} \text{ is integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for $n \geq 2$ by induction □

39

i Consider $\dfrac{\left(\frac{2k+1}{2k+2}\right)^2}{\frac{2k+1}{2k+3}} = \dfrac{(2k+1)(2k+3)}{(2k+2)^2} = \dfrac{4k^2+8k+3}{4k^2+8k+4} < 1$ since $k \geq 0$

$\therefore \dfrac{\left(\frac{2k+1}{2k+2}\right)^2}{\frac{2k+1}{2k+3}} < 1$

$\therefore \left(\dfrac{2k+1}{2k+2}\right)^2 < \dfrac{2k+1}{2k+3}$

$\therefore \dfrac{2k+1}{2k+2} < \dfrac{\sqrt{2k+1}}{\sqrt{2k+3}}$

ii Let $P(n)$ be the given proposition.

$P(0)$ is true since LHS $= \binom{0}{0} = 1$; RHS $= \frac{4^0}{\sqrt{0+1}} = 1$ $\therefore P(0)$ is true

If $P(k)$ is true for some arbitrary $k \geq 0$ $\binom{2k}{k} \leq \frac{4^k}{\sqrt{2k+1}}$

RTP $P(k+1)$ $\dbinom{2k+2}{k+1} \leq \dfrac{4^{k+1}}{\sqrt{2k+3}}$

$$\begin{aligned}
\text{LHS} &= \frac{(2k+2)!}{(k+1)!\,(k+1)!} \\
&= \frac{(2k+2)(2k+1)}{(k+1)^2} \times \frac{(2k)!}{k!\,k!} \\
&= \frac{(2k+2)(2k+1)}{(k+1)^2} \times \binom{2k}{k} \\
&\leq \frac{(2k+2)(2k+1)}{(k+1)^2} \times \frac{4^k}{\sqrt{2k+1}} \qquad \text{from } P(k) \\
&\leq \frac{2(k+1)}{k+1} \times 2 \times \frac{2k+1}{2k+2} \times \frac{4^k}{\sqrt{2k+1}} \\
&\leq \frac{2k+1}{2k+2} \times \frac{4^{k+1}}{\sqrt{2k+1}} \\
&\leq \frac{\sqrt{2k+1}}{\sqrt{2k+3}} \times \frac{4^{k+1}}{\sqrt{2k+1}} \qquad \text{from (i)} \\
&\leq \frac{4^{k+1}}{\sqrt{2k+3}} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 0$ by induction □

40 Let $P(n)$ represent the proposition
Case 1 - if n is odd
$P(1)$ is true since $1^1 = \frac{1^2(1+1)}{2} = 1$
If $P(k)$ is true for some arbitrary odd $k \geq 1$

$$1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + \ldots + k^2 = \frac{k^2(k+1)}{2} \text{ since } k \text{ is odd}$$

RTP $P(k+1)$

$$1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + \ldots + k^2 + 2\times(k+1)^2 = \frac{(k+1)(k+2)^2}{2} \text{ since } k+1 \text{ is even}$$

$$\begin{aligned}\text{LHS} &= \frac{k^2(k+1)}{2} + 2\times(k+1)^2\\ &= \frac{k+1}{2}(k^2+4k+4)\\ &= \frac{(k+1)(k+2)^2}{2}\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$ if k is odd
$\therefore P(n)$ is true for odd $n \geq 1$ by induction

Case 2 - if n is even
$P(2)$ is true since $1^1 + 2\times 2^2 = \frac{2(2+1)^2}{2} = 9$
If $P(k)$ is true for some arbitrary even $k \geq 2$

$$1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + \ldots + 2k^2 = \frac{k(k+1)^2}{2} \text{ since } k \text{ is even}$$

RTP $P(k+1)$

$$1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + \ldots + 2k^2 + (k+1)^2 = \frac{(k+1)^2(k+2)}{2} \text{ since } k+1 \text{ is odd}$$

$$\begin{aligned}\text{LHS} &= \frac{k(k+1)^2}{2} + (k+1)^2\\ &= \frac{(k+1)^2}{2}(k+2)\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$ if k is even
$\therefore P(n)$ is true for even $n \geq 2$ by induction

$\therefore P(n)$ is true for $n \geq 1$ by induction □

BASIC

1 Find $\int \frac{dx}{x^2+6x+10}$

2 Find $\int \frac{1}{x^2+2x+2}\,dx$

3 Let $I_n = \int_1^e (\ln t)^n\,dt$ for $n = 1, 2, 3, \ldots$

i Show that $I_n = e - nI_{n-1}$ for $n = 1, 2, 3, \ldots$

ii Hence, or otherwise, find the exact value I_3.

4 Find $\int x \tan^{-1} x\,dx$

5 Find $\int \sin x \cos(\cos x)dx$

6 Which of the following is an expression for $\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$?

A $\sin^{-1}\left(\frac{x-3}{2}\right)+c$ **B** $\sin^{-1}\left(\frac{x+3}{2}\right)+c$ **C** $\sin^{-1}\left(\frac{x-3}{4}\right)+c$ **D** $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

7 Using the substitution $u = e^x + 1$ or otherwise, evaluate $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$

8 Find $\int \frac{1}{x\ln x}\,dx$

9 What is the value of $\int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}}\,dx$?

A $\frac{\pi^2}{4}$ **B** $-\frac{\pi^2}{4}$ **C** $\frac{\pi^2}{8}$ **D** $-\frac{\pi^2}{8}$

10 Evaluate $\int_0^{\frac{\pi}{3}} \sec^3 x \tan x\,dx$

11 If n is a non-negative integer, then for what values of n is $\int_0^1 x^n\,dx = \int_0^1 (1-x)^n\,dx$?

A no solution **B** non zero n, only **C** n even, only **D** all values of n

12 $\int \frac{dx}{(x-1)(x+2)} =$

13 Evaluate $\int_0^1 xe^{-x^2}\,dx$

14 **i** Find real numbers a, b and c such that $\frac{10}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$

ii Hence find $\int \frac{10}{(x+1)(x^2+4)}\,dx$

15 Find $\int \frac{t^2-1}{t^3}\,dt$

16 Find $\int \frac{dx}{\sqrt{6-x-x^2}}$

17 Find $\int x^2 \log_e(3x)\,dx$

18 Find $\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$

19 Find $\int \frac{dx}{\sqrt{2-4x-2x^2}}$

20 The value of $\int_0^{\pi} 5\sin x\cos^4 x\,dx$ is:

A 0 **B** 2 **C** -2 **D** 20

21 Find $\int x\cos x\,dx$

22 Find $\int \frac{x+1}{x-2}\,dx$

23 Which of the following is an expression for $\int \frac{x}{\sqrt{16-x^2}}\,dx$

A $-2\sqrt{16-x^2}+c$ **B** $-\sqrt{16-x^2}+c$ **C** $\frac{1}{2}\sqrt{16-x^2}+c$ **D** $-\frac{1}{2}\sqrt{16-x^2}+c$

24 Using $t=\tan\frac{x}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$

25 Use the substitution $u=e^x$, or otherwise, find $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$

26 Find $\int \frac{4x^3-2x^2+1}{2x-1}\,dx$

27 Find $\int \frac{dx}{\sqrt{3-4x-4x^2}}$

28 Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\sin^5 x\,dx$

29 Evaluate $\int_0^1 \tan^{-1} x\,dx$

30 Find $\int \frac{dx}{x^2-6x+13}$

31 **i** Find the real numbers a and b such that $\frac{1}{x(2x+1)}=\frac{a}{x}+\frac{b}{2x+1}$

ii Hence evaluate $\int_{\frac{1}{2}}^{1} \frac{dx}{x(2x+1)}$

32 Which of the following is an expression for $\int xe^{\frac{x}{2}}\,dx$?

A $\frac{1}{2}xe^{\frac{x}{2}}-\frac{1}{4}e^{\frac{x}{2}}+c$ **B** $\frac{1}{2}xe^{\frac{x}{2}}-\frac{1}{2}e^{\frac{x}{2}}+c$ **C** $2xe^{\frac{x}{2}}-2e^{\frac{x}{2}}+c$ **D** $2xe^{\frac{x}{2}}-4e^{\frac{x}{2}}+c$

33 Evaluate $\int_0^{\frac{\pi}{4}} \cos\theta \sin^3\theta \, d\theta$

34 Use the substitution $u = 1 + x^2$ to evaluate $\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}} dx$

35 Find $\int x\sqrt{x}\left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right) dx$

36 Evaluate $\int_0^{\log_3 2} 3^{x \log_3 2} \, dx$

MEDIUM

37 Evaluate $\int_0^{\frac{\pi}{6}} \frac{d\theta}{9 - 8\cos^2\theta}$ using the substitution $t = \tan\theta$.

38 Which expression is equal to $\int 3\sqrt{x} \ln x \, dx$

A $2x\sqrt{x}\left(\ln x - \frac{2}{3}\right) + c$

B $2x\sqrt{x}\left(\ln x + \frac{2}{3}\right) + c$

C $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x - 1\right) + c$

D $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x + 1\right) + c$

39 Find $\int \frac{e^{2x}}{e^x + 1} dx$

40 **i** Find real numbers a, b and c such that $\frac{8 - 2x}{(1+x)(4+x^2)} = \frac{a}{1+x} + \frac{bx + c}{4 + x^2}$

ii Hence evaluate in simplest form $\int_0^4 \frac{8 - 2x}{(1+x)(4+x^2)} dx$

41 The algebraic fraction $\frac{x+1}{5(x+h)^2}$, where h is a non-zero real number can be written in partial fraction form, where A and B are real numbers, as

A $\frac{A}{x+h} + \frac{B}{x+h}$

B $\frac{A}{5x+h} + \frac{B}{(x+h)^2}$

C $\frac{A}{x+h} + \frac{B}{(x+h)^2}$

D $\frac{A}{5(x+h)} + \frac{B}{x+h}$

42 The value of $\int_{-1}^{1} \frac{1}{1 + e^{-x}} dx$ is

A $\frac{1}{2}$

B 1

C $\ln(1+e)$

D $2\ln(1+e)$

43 Find $\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}} dx$

44 Consider the integral $I_n = \int_0^1 \sqrt{x}(1-x)^n \, dx$, $n = 0, 1, 2, 3, \ldots$

i Show that $I_n = \left(\frac{2n}{2n+3}\right) I_{n-1}$, $n = 1, 2, 3$

ii Hence evaluate $I_3 = \int_0^1 \sqrt{x}(1-x)^3 \, dx$

45 The value of $\int_0^2 |x - 1| \, dx$ is

A -1

B 1

C 2

D 3

46 **i** Find a, b and c such that $\dfrac{16}{(x^2+4)(2-x)} = \dfrac{ax+b}{x^2+4} + \dfrac{c}{2-x}$

ii Find $\displaystyle\int \frac{16}{(x^2+4)(2-x)}\,dx$

47 Evaluate $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

48 Evaluate $\displaystyle\int_0^1 \frac{x}{(x+1)(2x+1)}\,dx$

49 Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$

A $\dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$ **B** $\dfrac{\pi}{4} + \dfrac{1}{2}\ln 2$ **C** $\dfrac{\pi}{4} + 2\ln\sqrt{2}$ **D** $\dfrac{7}{3}$

50 Find $\displaystyle\int (\tan^3 2x + \tan 2x)\,dx$

A $\dfrac{1}{4}\tan^4 2x + \dfrac{1}{2}\sec^2 2x + c$ **B** $\tan^2 2x + c$

C $\dfrac{1}{4}\tan^2 2x + c$ **D** $\dfrac{1}{2}\tan^2 2x + c$

51 **i** Find the numbers a and b such that $\dfrac{3x^2+x}{(x+1)(x^2+1)} \equiv \dfrac{a}{x+1} + \dfrac{2x+b}{x^2+1}$

ii Find $\displaystyle\int \frac{3x^2+x}{(x+1)(x^2+1)}\,dx$

52 Use the substitution $t = \tan\dfrac{\theta}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin\theta}\,d\theta$

53 Find $\displaystyle\int \sqrt{\frac{5-x}{5+x}}\,dx$

54 **i** Find the real numbers A, B and C such that $\dfrac{10}{(3+x)(1+x^2)} = \dfrac{A}{3+x} + \dfrac{Bx+C}{1+x^2}$

ii Hence use the substitution $t = \tan\theta$ to find $\displaystyle\int \frac{10}{3+\tan\theta}\,d\theta$

55 Consider the integral $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$, for $n \geq 0$.

i Show that $I_0 = 2\sqrt{2} - 2$

ii Given that $I_n + I_{n-1} = \displaystyle\int_0^1 x^{n-1}\sqrt{1+x}\,dx$, show that $I_n = \dfrac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$

iii Hence evaluate I_2 in exact form.

56 Evaluate $\displaystyle\int_1^2 \frac{dx}{x(1+x^2)}$

57 Find $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$

58 Find $\displaystyle\int e^{-2x}\cos x\,dx$

59 Consider the following two statements:

$$\textbf{I}: \int_0^1 \frac{dx}{1+x^n} < \int_0^1 \frac{dx}{1+x^{n+1}} \quad (n > 1)$$

$$\textbf{II}: \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$$

Which of these statements is correct?

A Neither statement
B Statement I only
C Statement II only
D Both statements

60 Evaluate $\int_0^1 \cos^{-1} x\, dx$

61 Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}} dx$

i Show that $I_1 = \frac{4}{3}$

ii Show that $I_{n-1} - I_n = \int_0^1 x^{n-1}\sqrt{1-x}\, dx$

iii Use integration by parts on the result of part (ii) to show that $I_n = \frac{2n}{2n+1} I_{n-1}$

62 Use the substitution $t = \tan\frac{x}{2}$ to simplify $\int \sec x\, dx$

A $\ln|(t+1)(t-1)| + c$
B $\ln\left|\frac{1+t}{1-t}\right| + c$
C $\ln|(1+t)(1-t)| + c$
D $\ln\left|\frac{t+1}{t-1}\right| + c$

63 Evaluate $\int_0^{\frac{\pi}{4}} x \tan^2 x\, dx$

64 Find $\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$

A $\sin(\sqrt{x}) + c$
B $2\sin(\sqrt{x}) + c$
C $\frac{1}{\sin(\sqrt{x})} + c$
D $\frac{2}{\sin(\sqrt{x})} + c$

65 **i** Find real numbers A and B such that: $\frac{3x^2+3x-2}{(x-1)(x+1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{1}{(x+1)^2}$

ii Hence find $\int \frac{3x^2+3x-2}{(x-1)(x+1)^2} dx$

66 Use the substitution $x = 2\cos\theta$ to evaluate $\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}} dx$

67 **i** Find constants A, B and C such that $\frac{x^2-x+1}{(x+1)^2} = A + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

ii Hence find $\int \frac{x^2-x+1}{(x+1)^2} dx$

68 Show, using integration by parts, that $\int_0^{\frac{\pi}{3}} x \sec^2 x\, dx = \frac{\pi\sqrt{3}}{3} - \ln 2$

69 Let $I_n = \int_0^{\pi} x^n \sin x \, dx$, where $n = 0, 1, 2, \ldots$

i Use integration by parts to show that $I_n = \pi^n - n(n-1)I_{n-2}$ for $n = 2, 3, 4, \ldots$

ii Hence evaluate $\int_0^{\pi} x^4 \sin x \, dx$

70 If $I = \int_0^{\ln 2} \frac{e^x}{e^x + e^{-x}} dx$ and $J = \int_0^{\ln 2} \frac{e^{-x}}{e^x + e^{-x}} dx$, then the exact value of $I - J$ is:

A $\ln\left(\frac{5}{2}\right)$ **B** $\ln 2$ **C** $\ln 5$ **D** $\ln\left(\frac{5}{4}\right)$

71 Find $\int \frac{dx}{(x+1)(x^2+4)}$

72 **i** If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ show that $I_n = \frac{n-1}{n} I_{n-2}$

ii Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$

73 Find $\int \sin^3 x \, dx$

A $\frac{1}{4}\sin^4 x + c$ **B** $-\cos x + \frac{1}{3}\cos^3 x + c$

C $-\cos x - \frac{1}{3}\cos^3 x + c$ **D** $\cos x - \frac{1}{3}\cos^3 x + c$

74 **i** Find values of A, B and C so that $\frac{2x^2 + x + 9}{(x^2+4)(x+1)} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x+1}$

ii Hence find $\int_0^2 \frac{2x^2 + x + 9}{(x^2+4)(x+1)} dx$ giving your answer in exact form.

75 Show that $\int x\sqrt{x-1} \, dx = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + c$

76 Find $\int \frac{x}{\sqrt{2 - x^2}} dx$ using the substitution $x = \sqrt{2}\sin\theta$.

77 Which of the following is an expression for $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} dx$?

You are given that $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

A $x + \frac{1}{4}\cos 2x + c$ **B** $x - \frac{1}{4}\cos 2x + c$ **C** $x + \frac{1}{2}\sin 2x + c$ **D** $x - \frac{1}{2}\sin 2x + c$

78 Find $\int \frac{\sin 2x + \sin x}{\cos^2 x} dx$

79 Find $\int e^x \sin x \, dx$ by the method of integration of parts.

80 Prove $\int \sin 2x \sin 4x \, dx = \frac{\sin 2x}{4} - \frac{\sin 6x}{12} + c$

81 Prove $\int \frac{\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)}{\cos\left(x - \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)} dx = -\ln|\cos x| + c$

82 $\int \frac{x}{\sqrt{x+5}}\,dx =$

A $2\sqrt{x+5}+c$

B $\frac{2}{3}\sqrt{(x+5)^3}+c$

C $\frac{2}{3}\left\{\sqrt{(x+5)^3}-10\sqrt{x+5}\right\}+c$

D $\frac{2}{3}(x-10)\sqrt{x+5}+c$

83 By using the fact that $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x} = \ln 2$, evaluate $\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}$

84 Prove $\int \frac{dx}{x^3\sqrt{x^2-4}} = \frac{1}{16}\cos^{-1}\left(\frac{2}{x}\right)+\frac{\sqrt{x^2-4}}{8x^2}+c$

85 Using the substitution $u=\pi-x$,

i Show that $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$

ii Hence deduce that $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$

86 The value of $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}\,dx$ is equal to:

A 0

B π

C $\frac{\pi}{2}$

D $\frac{\pi}{4}$

87 **i** Use the substitution $u=\frac{\pi}{4}-x$, to show $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$

ii Hence find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

88 Find $\int \frac{e^x-e^{-x}}{(e^x+e^{-x})^2}\,dx$

89 Let $I_n = \int_0^{\frac{\pi}{6}} \sin^{2n}\theta\sec\theta\,d\theta$, for $n\geq 1$.

i Show that $I_n - I_{n-1} = -\frac{1}{2^{2n-1}(2n-1)}$

ii Hence, show that $I_n = \frac{1}{2}\ln 3 - \sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)}$

90 Given that $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x\,dx$, for $n = 1, 2, \ldots$

i Show that $I_1 = \frac{1}{2}\ln 2$

ii Show that $I_{n-2}+I_n = \frac{1}{n-1}\left(3^{\frac{1}{2}(n-1)}-1\right)$, for $n = 2, 3, 4, \ldots$

iii Find I_5

91 Use the substitution $t=\tan\frac{\theta}{2}$ to show that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta-2\cos\theta+6} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$

92 Show that $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$, using the substitution $x = \frac{1-u}{1+u}$

93 **i** Show that $\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx = \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$

ii Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1+2\sin 2x+\cos 2x}\,dx$ using the substitution $t = \tan x$.

94 **i** If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$, show that for $n > 1$, $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

ii Hence find the area of the finite region bounded by the curve $y = x^4\cos x$ and the x-axis for $0 \le x \le \frac{\pi}{2}$

95 Given $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$ for $n \ge 1$ show that $I_n = \frac{1}{2^n(1-2n)} + \frac{2n}{2n-1}I_{n+1}$

96 Let $I_n = \int_0^1 x^n\sqrt{1-x^3}\,dx$, for $n \ge 2$

i Show that $I_n = \frac{2n-4}{2n+5}I_{n-3}$ for $n \ge 5$.

ii Hence find I_8

97 Given that $\tan x + \cot x = \frac{1}{\sin x\cos x}$ then a primitive of $\frac{1}{\sin x\cos x}$ is:

A $\frac{1}{\cos^2 x}\ln\left|\sin x\right|$ **B** $\ln\left|\sin x\cos x\right|$ **C** $\ln\left|\tan x\right|$ **D** $\ln\left|\cot x\right|$

98 **i** Show that $(1+t^2)^{n-1} + t^2(1+t^2)^{n-1} = (1+t^2)^n$

ii $I_n = \int_0^x (1+t^2)^n\,dt$ for $n = 1, 2, 3, \ldots$

Use integration by parts, and part (i) above to show $I_n = \frac{1}{2n+1}(1+x^2)^n x + \frac{2n}{2n+1}I_{n-1}$

99 For $n = 0, 1, 2, 3, \ldots$, let $I_n = \int_{e^{-1}}^1 (1+\log_e x)^n\,dx$ and $J_n = \int_{e^{-1}}^1 (\log_e x)(1+\log_e x)^n\,dx$

i Show that $I_n = 1 - nI_{n-1}$ for $n = 1, 2, 3, \ldots$

ii Show that $J_n = 1 - (n+2)I_n$ for $n = 0, 1, 2, 3, \ldots$

iii Hence find the value of J_3 in simplest exact form.

100 Using the recurrence relation $U_n = \int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - U_{n-2}$, then $\int \tan^6 x\,dx = ?$

A $\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + x + c$ **B** $\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x + c$

C $\frac{\tan^6 x}{6} - \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} + x + c$ **D** $\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$

1

$$\int \frac{dx}{x^2+6x+10}$$
$$= \int \frac{dx}{(x+3)^2+1^2}$$
$$= \tan^{-1}(x+3)+c$$

2

$$\int \frac{1}{x^2+2x+2}\,dx$$
$$= \int \frac{1}{(x+1)^2+1^2}\,dx$$
$$= \tan^{-1}(x+1)+c$$

3

i $I_n = \int_1^e (\ln t)^n\,dt$

$u = (\ln t)^n \qquad \frac{dv}{dt} = 1$

$\frac{du}{dt} = \frac{n(\ln t)^{n-1}}{t} \qquad v = t$

$$= \Big[t(\ln t)^n\Big]_1^e - n\int (\ln t)^{n-1}\,dt$$
$$= e - n\int (\ln t)^{n-1}\,dt$$
$$= e - nI_{n-1}$$

ii $I_3 = e - 3I_2$
$$= e - 3(e - 2I_1)$$
$$= -2e + 6(e - I_0)$$
$$= 4e - 6\int_1^e dt$$
$$= 4e - 6\Big[\,t\,\Big]_1^e$$
$$= 4e - 6(e-1)$$
$$= -2e + 6$$

4

$$\int x\tan^{-1}x\,dx$$

$u = \tan^{-1}x \qquad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2}$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,dx$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$
$$= \frac{x^2\tan^{-1}x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c$$

5

$$\int \sin x\cos(\cos x)dx$$
$$= -\int -\sin x\cos(\cos x)dx$$
$$= -\sin(\cos x) + c$$

6

$$\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{16-(3+x)^2}}\,dx$$
$$= \sin^{-1}\left(\frac{x+3}{4}\right) + C$$

ANSWER (D)

7

$$\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$$

$u = e^x + 1$

$du = e^x\,dx$

$dx = \frac{du}{e^x}$

$$= \int_2^{e+1} \frac{e^x}{u^2}\times\frac{du}{e^x}$$
$$= \int_2^{e+1} u^{-2}du$$
$$= -\left[\frac{1}{u}\right]_2^{e+1}$$
$$= -\left[\frac{1}{u}\right]_2^{e+1}$$
$$= -\left(\frac{1}{e+1} - \frac{1}{2}\right)$$
$$= \frac{1}{2} - \frac{1}{e+1}$$

8

$$\int \frac{1}{x\ln x}\,dx$$
$$= \int \frac{1/x}{\ln x}\,dx$$
$$= \ln\left|\ln x\right| + c$$

9

$$\int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}}\,dx$$
$$= -\int_0^1 \left(-\frac{1}{\sqrt{1-x^2}}\right) \times (\cos^{-1} x)^1\,dx$$
$$= -\frac{1}{2}\left[(\cos^{-1} x)^2\right]_0^1$$
$$= -\frac{1}{2}\left(0 - \left(\frac{\pi}{2}\right)^2\right)$$
$$= \frac{\pi^2}{8}$$

ANSWER (C)

10

$$\int_0^{\frac{\pi}{3}} \sec^3 x \tan x\,dx$$
$$= \int_0^{\frac{\pi}{3}} \sec x \tan x\,(\sec x)^2\,dx$$
$$= \left[\frac{\sec^3 x}{3}\right]_0^{\frac{\pi}{3}}$$
$$= \frac{8}{3} - \frac{1}{3}$$
$$= \frac{7}{3}$$

11

The identity is true for all values of n since
$$\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$$

ANSWER (D)

12

$$\int \frac{dx}{(x-1)(x+2)}$$
$$= \frac{1}{3}\int \left(\frac{1}{x-1} - \frac{1}{x+2}\right)dx$$
$$= \frac{1}{3}(\ln|x-1| - \ln|x+2|) + c$$
$$= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + c$$

13

$$\int_0^1 xe^{-x^2}\,dx$$
$$= -\frac{1}{2}\int_0^1 (-2x)e^{-x^2}\,dx$$
$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$
$$= -\frac{1}{2}(e^{-1} - 1)$$
$$= \frac{1}{2} - \frac{1}{2e}$$

14

i $a = \dfrac{10}{(-1)^2 + 4} = 2$

equating coefficients of x^2:
$$a + b = 0 \Rightarrow b = -2$$
equating coefficients of x:
$$b + c = 0 \Rightarrow c = 2$$

ii $\displaystyle\int \left(\frac{2}{x+1} + \frac{-2x+2}{x^2+4}\right)dx$
$$= \int \left(\frac{2}{x+1} - \frac{2x}{x^2+4} + \frac{2}{x^2+4}\right)dx$$
$$= 2\ln|x+1| - \ln|x^2+4| + \tan^{-1}\frac{x}{2} + c$$

15

$$\int \frac{t^2-1}{t^3}\,dt$$
$$= \int \left(\frac{1}{t} - t^{-3}\right)dt$$
$$= \ln|t| - \frac{t^{-2}}{-2} + c$$
$$= \ln|t| + \frac{1}{2t^2} + c$$

16

$$\int \frac{dx}{\sqrt{6-x-x^2}}$$
$$= \int \frac{dx}{\sqrt{-\left(x^2+x+\frac{1}{4}-\frac{25}{4}\right)}}$$
$$= \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(\frac{1}{2}+x\right)^2}}$$
$$= \sin^{-1}\left(\frac{x+\frac{1}{2}}{5/2}\right) + c$$
$$= \sin^{-1}\left(\frac{2x+1}{5}\right) + c$$

17

$$\int x^2 \log_e(3x)\,dx$$

$u = \log_e 3x$	$\dfrac{dv}{dx} = x^2$
$\dfrac{du}{dx} = \dfrac{3}{3x} = \dfrac{1}{x}$	$v = \dfrac{x^3}{3}$

$$= \frac{x^3}{3}\log_e 3x - \frac{1}{3}\int x^2\,dx$$
$$\frac{x^3}{3}\log_e 3x - \frac{x^3}{9} + c$$

18

$$\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$
$$= \int \cos\theta(\sin\theta)^{-4}\,d\theta$$
$$= \frac{(\sin\theta)^{-3}}{-3} + c$$
$$= -\frac{1}{3\sin^3\theta} + c$$

19

$$\int \frac{dx}{\sqrt{2-4x-2x^2}}$$
$$= \int \frac{dx}{\sqrt{2(1-2x-x^2)}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{2-(1+x)^2}}$$
$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

20

$$\int_0^\pi 5\sin x\cos^4 x\,dx$$
$$= -5\int_0^\pi (-\sin x)(\cos x)^4\,dx$$
$$= -5\left[\frac{\cos^5 x}{5}\right]_0^\pi$$
$$= 1(-1-1)$$
$$= 2$$

ANSWER (B)

21

$$\int x\cos x\,dx$$

$u = x$	$\frac{dv}{dx} = \cos x$
$\frac{du}{dx} = 1$	$v = \sin x$

$$= x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + c$$

22

$$\int \frac{x+1}{x-2}\,dx$$
$$= \int \frac{x-2+3}{x-2}\,dx$$
$$= \int \left(1 + \frac{3}{x-2}\right)dx$$
$$= x + 3\ln|x-2| + c$$

23

$$\int \frac{x}{\sqrt{16-x^2}}\,dx$$
$$= -\frac{1}{2}\int (-2x)(16-x^2)^{-\frac{1}{2}}\,dx$$
$$= -\frac{1}{2}\times 2(16-x^2)^{\frac{1}{2}} + c$$
$$= -\sqrt{16-x^2} + c$$

ANSWER (B)

24

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$$

$t = \tan\frac{x}{2}$
$dx = \frac{2dt}{1+t^2}$

$$= \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int_0^1 \frac{1}{1+t^2+2t}\,dt$$
$$= 2\int_0^1 (t+1)^{-2}\,dt$$
$$= -2\left[\frac{1}{t+1}\right]_0^1$$
$$= -2\left(\frac{1}{2} - 1\right)$$
$$= 1$$

25

$$\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$$

$u = e^x$
$du = e^x dx$
$dx = \frac{du}{e^x}$

$$= \int \frac{e^x}{\sqrt{1-u^2}} \times \frac{du}{e^x}$$
$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$
$$= \sin^{-1} u + c$$
$$= \sin^{-1} e^x + c$$

26

$$\int \frac{4x^3 - 2x^2 + 1}{2x-1}\,dx$$
$$= \int \frac{2x^2(2x-1)+1}{2x-1}\,dx$$
$$= \int \left(2x^2 + \frac{1}{2x-1}\right)dx$$
$$= \frac{2}{3}x^3 + \frac{1}{2}\ln|2x-1| + c$$

27

$$\int \frac{dx}{\sqrt{3-4x-4x^2}}$$
$$= \int \frac{dx}{\sqrt{-(4x^2+4x+1-4)}}$$
$$= \int \frac{dx}{\sqrt{2^2-(2x+1)^2}}$$
$$= \frac{1}{2}\int \frac{2\,dx}{\sqrt{2^2-(2x+1)^2}}$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{2x+1}{2}\right) + c$$

28

$\cos^4 x\sin^5 x$ is the product of an even and an odd function which is odd.

$\int_{-a}^{a} f(x)\,dx = 0$ for odd functions

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\sin^5 x\,dx = 0$$

29

$$\int_0^1 \tan^{-1} x\,dx$$

$u = \tan^{-1} x$	$\frac{dv}{dx} = 1$
$\frac{du}{dx} = \frac{1}{1+x^2}$	$v = x$

$$= \left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$
$$= \frac{\pi}{4} - \frac{1}{2}\left[\ln|1+x^2|\right]_0^1$$
$$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0)$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

30

$$\int \frac{dx}{x^2-6x+13}$$
$$= \int \frac{dx}{(x-3)^2+2^2}$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + c$$

31

i $a = \frac{1}{2(0)+1} = 1$

$b = \frac{1}{-\frac{1}{2}} = -2$

$a = 1, b = -2$

ii $\int_{\frac{1}{2}}^{1} \frac{dx}{x(2x+1)}$

$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{x} - \frac{2}{2x+1}\right) dx$

$= \left[\ln|x| - \ln|2x+1|\right]_{\frac{1}{2}}^{1}$

$= (0 - \ln 3) - \left(\ln\frac{1}{2} - \ln 2\right)$

$= -\ln 3 + \ln 2 + \ln 2$

$= \ln\frac{4}{3}$

32

$\int xe^{\frac{x}{2}}\, dx$

$= 2xe^{\frac{x}{2}} - 2\int e^{\frac{x}{2}}\, dx$

$= 2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + c$

ANSWER (D)

$u = x$	$\frac{dv}{dx} = e^{\frac{x}{2}}$
$\frac{du}{dx} = 1$	$v = 2e^{\frac{x}{2}}$

33

$\int_0^{\frac{\pi}{4}} \cos\theta \sin^3\theta\, d\theta$

$= \left[\frac{\sin^4\theta}{4}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{4}\left(\left(\frac{1}{\sqrt{2}}\right)^4 - 0\right)$

$= \frac{1}{16}$

34

$\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}} dx$

$u = 1 + x^2$
$du = 2x\, dx$
$dx = \frac{du}{2x}$

$= \int_1^4 \frac{x^3}{u^{\frac{3}{2}}} \times \frac{du}{2x}$

$= \frac{1}{2}\int_1^4 (u-1)u^{-\frac{3}{2}}\, du$

$= \frac{1}{2}\int_1^4 \left(u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\right) du$

$= \frac{1}{2}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]_1^4$

$= \frac{1}{2}\left((4+1) - (2+2)\right)$

$= \frac{1}{2}$

35

$\int x\sqrt{x}\left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right) dx$

$= \int (x^2 + x^3)\, dx$

$= \frac{x^3}{3} + \frac{x^4}{4} + c$

36

$\int_0^{\log_3 2} 3^{x \log_3 2}\, dx$

$= \int_0^{\log_3 2} 3^x \times 2dx$

$= \frac{2}{\ln 3}\left[3^x\right]_0^{\log_3 2}$

$= \frac{2}{\ln 3}(2 - 1)$

$= \frac{2}{\ln 3}$

37

$\int_0^{\frac{\pi}{6}} \frac{d\theta}{9 - 8\cos^2\theta}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9 - \frac{8}{\sec^2\theta}} \times \frac{dt}{1+t^2}$

$t = \tan\theta$
$\frac{dt}{d\theta} = \sec^2\theta$
$= 1 + t^2$
$d\theta = \frac{dt}{1+t^2}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9 - \frac{8}{1+t^2}} \times \frac{dt}{1+t^2}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9 + 9t^2 - 8} dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1 + 9t^2} dt$

$= \frac{1}{3}\int_0^{\frac{1}{\sqrt{3}}} \frac{3}{1 + (3t)^2} dt$

$= \frac{1}{3}\left[\tan^{-1} 3t\right]_0^{\frac{1}{\sqrt{3}}}$

$= \frac{1}{3}\left(\frac{\pi}{3} - 0\right)$

$= \frac{\pi}{9}$

38

$\int 3\sqrt{x} \ln x\, dx$

$= 2x^{\frac{3}{2}} \ln x - \int 2x^{\frac{1}{2}}\, dx$

$= 2x^{\frac{3}{2}} \ln x - \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= 2x^{\frac{3}{2}}\left(\ln x - \frac{2}{3}\right) + c$

ANSWER (A)

$u = \ln x$	$\frac{dv}{dx} = 3x^{\frac{1}{2}}$
$\frac{du}{dx} = \frac{1}{x}$	$v = \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} = 2x^{\frac{3}{2}}$

39

$$\int \frac{e^{2x}}{e^x+1}\,dx$$

$u = e^x$
$du = e^x dx$
$dx = \dfrac{du}{u}$

$$= \int \frac{u^2}{u+1}\times\frac{du}{u}$$
$$= \int \frac{u}{u+1}\,du$$
$$= \int \left(\frac{u+1}{u+1}-\frac{1}{u+1}\right)du$$
$$= \int \left(1-\frac{1}{u+1}\right)du$$
$$= u - \ln(u+1) + c$$
$$= e^x - \ln|e^x+1| + c$$

40

i $a = \dfrac{8-2(-1)}{4+(-1)^2} = 2$

equating coefficients of x^2: $a+b=0 \quad \Rightarrow \quad b=-2$

equating constants: $4a+c=8 \quad \Rightarrow \quad c=0$

$\therefore a=2, b=-2, c=0$

ii $\displaystyle\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\,dx$

$$= \int_0^4 \left(\frac{2}{1+x}-\frac{2x}{4+x^2}\right)dx$$
$$= \Big[2\ln(1+x) - \ln(4+x^2)\Big]_0^4$$
$$= (2\ln 5 - \ln 20) - (2\ln 1 - \ln 4)$$
$$= \ln 25 - \ln 20 + \ln 4$$
$$= \ln\frac{25\times 4}{20}$$
$$= \ln 5$$

41

The 5 can be absorbed into the values of A and B, so is not needed in either denominator as a factor. The easiest way to get the required denominator of $(x+h)^2$ is to have a constant over $(x+h)^2$ and another constant over $x+h$.

ANSWER (C)

42

$$\int_{-1}^{1} \frac{1}{1+e^{-x}}\times\frac{e^x}{e^x}\,dx$$
$$= \int_{-1}^{1} \frac{e^x}{e^x+1}\,dx$$
$$= \Big[\ln(e^x+1)\Big]_{-1}^{1}$$
$$= \ln(e+1) - \ln(e^{-1}+1)$$
$$= \ln(e+1) - \ln\left(\frac{1+e}{e}\right)$$
$$= \ln\left(\frac{e+1}{(1+e)/e}\right)$$
$$= \ln e$$
$$= 1$$

ANSWER (B)

43

$$\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}\,dx$$
$$= \int \frac{\sqrt{(1+x)(1-x)}}{\sqrt{1-x}}\,dx$$
$$= \int \sqrt{1+x}\,dx$$
$$= \int (x+1)^{\frac{1}{2}}\,dx$$
$$= \frac{2}{3}(x+1)^{\frac{3}{2}} + c$$
$$= \frac{2\sqrt{(x+1)^3}}{3} + c$$

44

i $I_n = \displaystyle\int_0^1 \sqrt{x}(1-x)^n\,dx$

$u = (1-x)^n$	$\dfrac{dv}{dx} = x^{\frac{1}{2}}$
$\dfrac{du}{dx} = -n(1-x)^{n-1}$	$v = \dfrac{2}{3}x^{\frac{3}{2}}$

$$= \frac{2}{3}\left[x^{\frac{3}{2}}(1-x)^n\right]_0^1 + \frac{2n}{3}\int_0^1 x^{\frac{3}{2}}(1-x)^{n-1}\,dx$$
$$= 0 + \frac{2n}{3}\int_0^1 \sqrt{x}\cdot x(1-x)^{n-1}\,dx$$
$$= -\frac{2n}{3}\int_0^1 \sqrt{x}(1-x-1)(1-x)^{n-1}\,dx$$
$$= -\frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^n\,dx + \frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^{n-1}\,dx$$
$$\therefore\ I_n = -\frac{2n}{3}I_n + \frac{2n}{3}I_{n-1}$$
$$\frac{2n+3}{3}I_n = \frac{2n}{3}I_{n-1}$$
$$I_n = \left(\frac{2n}{2n+3}\right)I_{n-1}$$

ii $I_3 = \left(\dfrac{2(3)}{2(3)+3}\right)I_2$

$$= \frac{2}{3}\left(\frac{2(2)}{2(2)+3}\right)I_1$$
$$= \frac{8}{21}\left(\frac{2(1)}{2(1)+3}\right)I_0$$
$$= \frac{16}{105}\int_0^1 \sqrt{x}\,dx$$
$$= \frac{16}{105}\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^1$$
$$= \frac{32}{315}$$

45

$$\int_0^2 |x-1|\,dx$$
$$= \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx$$
$$= \frac{1}{2}\times 1\times 1 + \frac{1}{2}\times 1\times 1 \qquad \text{area of a triangle}$$
$$= 1$$

ANSWER (B)

46

i $c = \dfrac{16}{2^2+4} = 2$
equating coefficients of x^2: $\quad -a + c = 0 \quad \Rightarrow \quad a = 2$
equating constants: $2b + 4c = 16 \quad \Rightarrow \quad b = 4$
$a = 2, b = 4, c = 2$

ii $\displaystyle\int \frac{16}{(x^2+4)(2-x)}\,dx$
$= \displaystyle\int \left(\frac{2x+4}{x^2+4} + \frac{2}{2-x}\right) dx$
$= \displaystyle\int \left(\frac{2x}{x^2+4} + \frac{4}{x^2+4} + \frac{2}{2-x}\right) dx$
$= \ln|x^2+4| + 2\tan^{-1}\left(\dfrac{x}{2}\right) - 2\ln|2-x| + c$

47

$\displaystyle\int_0^1 \sin^{-1} x\, dx$

$u = \sin^{-1} x$	$\dfrac{dv}{dx} = 1$
$\dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}}$	$v = x$

$= \left[x \sin^{-1} x\right]_0^1 - \displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$
$= \dfrac{\pi}{2} - 0 + \dfrac{1}{2}\displaystyle\int_0^1 (-2x)(1-x^2)^{-\frac{1}{2}}\,dx$
$= \dfrac{\pi}{2} + \dfrac{1}{2} \times 2\left[(1-x^2)^{\frac{1}{2}}\right]_0^1$
$= \dfrac{\pi}{2} - 1$

48

$\displaystyle\int_0^1 \frac{x}{(x+1)(2x+1)}\,dx$
$= \displaystyle\int_0^1 \left(\frac{1}{x+1} - \frac{1}{2x+1}\right) dx$
$= \left[\ln|x+1| - \dfrac{1}{2}\ln|2x+1|\right]_0^1$
$= \left(\ln 2 - \dfrac{1}{2}\ln 3\right) - \left(\ln 1 - \dfrac{1}{2}\ln 1\right)$
$= \ln \dfrac{2}{\sqrt{3}}$

49

$\displaystyle\int_0^{\frac{\pi}{4}} x \sec^2 x\, dx$

$u = x$	$\dfrac{dv}{dx} = \sec^2 x$
$\dfrac{du}{dx} = 1$	$v = \tan x$

$= \left[x \tan x\right]_0^{\frac{\pi}{4}} - \displaystyle\int_0^{\frac{\pi}{4}} \tan x\, dx$
$= \dfrac{\pi}{4} - 0 + \left[\ln|\cos x|\right]_0^{\frac{\pi}{4}}$
$= \dfrac{\pi}{4} + \ln\left|\dfrac{1}{\sqrt{2}}\right|$
$= \dfrac{\pi}{4} + \ln 1 - \ln\sqrt{2}$
$= \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$
ANSWER (A)

50

$\displaystyle\int (\tan^3 2x + \tan 2x)\, dx$
$= \displaystyle\int \tan 2x\,(\tan^2 2x + 1)\, dx$
$= \dfrac{1}{2}\displaystyle\int 2\sec^2 2x\,(\tan 2x)^1\, dx$
$= \dfrac{\tan^2 2x}{2 \times 2} + c$
$= \dfrac{1}{4}\tan^2 2x + c$
ANSWER (C)

51

i $a = \dfrac{3(-1)^2 + (-1)}{(-1)^2 + 1} = 1$
equating constants: $a + b = 0 \quad \Rightarrow \quad b = -1$
$a = 1, b = -1$

ii $\displaystyle\int \frac{3x^2+x}{(x+1)(x^2+1)}\,dx$
$= \displaystyle\int \left(\frac{1}{x+1} + \frac{2x-1}{x^2+1}\right) dx$
$= \displaystyle\int \left(\frac{1}{x+1} + \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right) dx$
$= \ln|x+1| + \ln|x^2+1| - \tan^{-1} x + c$

52

$$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin\theta}\,d\theta$$

$$t = \tan\frac{\theta}{2}$$
$$dt = \frac{2dt}{1+t^2}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1-\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+t^2-2t}\,dt$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(t-1)^2}\,dt$$

$$= -2\left[\frac{1}{t-1}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= -2\left(\frac{1}{\frac{1}{\sqrt{3}}-1}+1\right)$$

$$= -2\left(\frac{\sqrt{3}}{1-\sqrt{3}}+\frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$$

$$= -2\left(\frac{1}{1-\sqrt{3}}\right)\times\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= -\frac{2(1+\sqrt{3})}{1-3}$$

$$= \sqrt{3}+1$$

53

$$\int\sqrt{\frac{5-x}{5+x}}\,dx$$

$$= \int\sqrt{\frac{5-x}{5+x}}\times\sqrt{\frac{5-x}{5-x}}\,dx$$

$$= \int\frac{5-x}{\sqrt{25-x^2}}\,dx$$

$$= \int\frac{5}{\sqrt{25-x^2}}\,dx + \frac{1}{2}\int(-2x)(25-x^2)^{-\frac{1}{2}}\,dx$$

$$= 5\sin^{-1}\left(\frac{x}{5}\right)+\frac{1}{2}\times 2(25-x^2)^{\frac{1}{2}}+c$$

$$= 5\sin^{-1}\left(\frac{x}{5}\right)+\sqrt{25-x^2}+c$$

54

$$A = \frac{10}{1+(-3)^2} = 1$$

equating coefficients of x^2: $A+B=0 \Rightarrow B=-1$

equating constants: $A+3C=10 \Rightarrow C=3$

$A=1, B=-1, C=3$

ii $$\int\frac{10}{3+\tan\theta}\,d\theta$$

$$t = \tan\theta$$
$$d\theta = \frac{dt}{1+t^2}$$

$$= \int\frac{10}{3+t}\times\frac{dt}{1+t^2}$$

$$= \int\frac{10}{(3+t)(1+t^2)}\,dt$$

$$= \int\left(\frac{1}{3+t}-\frac{t-3}{1+t^2}\right)dt$$

$$= \int\left(\frac{1}{3+t}-\frac{t}{1+t^2}+\frac{3}{1+t^2}\right)dt$$

$$= \ln|3+t| - \frac{1}{2}\ln|1+t^2| + 3\tan^{-1}t + c$$

$$= \ln|3+\tan\theta| - \frac{1}{2}\ln|1+\tan^2\theta| + 3\tan^{-1}(\tan\theta) + c$$

$$= \ln|\tan\theta+3| - \frac{1}{2}\ln|\sec^2\theta| + 3\theta + c$$

55

i $$I_0 = \int_0^1\frac{1}{\sqrt{1+x}}\,dx$$

$$= \int_0^1 (1+x)^{-\frac{1}{2}}\,dx$$

$$= \left[\frac{(1+x)^{\frac{1}{2}}}{1/2}\right]_0^1$$

$$= 2(\sqrt{2}-1)$$

$$= 2\sqrt{2}-2$$

ii $$I_n = \int_0^1\frac{x^n}{\sqrt{1+x}}\,dx$$

$$u = x^n \qquad \frac{dv}{dx} = \frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$$
$$\frac{du}{dx} = nx^{n-1} \qquad v = 2(1+x)^{\frac{1}{2}}$$

$$= 2\left[x^n(1+x)^{\frac{1}{2}}\right]_0^1 - 2n\int_0^1 x^{n-1}(1+x)^{\frac{1}{2}}\,dx$$

$$= 2\sqrt{2} - 0 - 2n\int_0^1 x^{n-1}\sqrt{1+x}\,dx$$

$$= 2\sqrt{2} - 2n(I_n + I_{n-1}) \quad \text{given}$$

$$\therefore (2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$$

$$I_n = \frac{2\sqrt{2}-2nI_{n-1}}{2n+1}$$

iii $$I_1 = \frac{2\sqrt{2}-2(1)I_0}{2(1)+1}$$

$$= \frac{2\sqrt{2}-2(2\sqrt{2}-2)}{3}$$

$$= \frac{4-2\sqrt{2}}{3}$$

$$I_2 = \frac{2\sqrt{2}-2(2)I_1}{2(2)+1}$$

$$= \frac{2\sqrt{2}-4\left(\frac{4-2\sqrt{2}}{3}\right)}{5}$$

$$= \frac{6\sqrt{2}-16+8\sqrt{2}}{15}$$

$$= \frac{14\sqrt{2}-16}{15}$$

56

$$\int_1^2 \frac{dx}{x(1+x^2)}$$
$$= \int_1^2 \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx$$
$$= \left[\ln|x| - \frac{1}{2}\ln|1+x^2|\right]_1^2$$
$$= \left(\ln 2 - \frac{1}{2}\ln 5\right) - \left(\ln 1 + \frac{1}{2}\ln 2\right)$$
$$= \frac{3}{2}\ln 2 - \frac{1}{2}\ln 5$$

57

$$\int \frac{x}{\sqrt{1-x}} dx$$
$$= \int \frac{(1-u^2)}{u}(-2u\, du)$$
$$= 2\int (u^2 - 1)\, du$$
$$= \frac{2u^3}{3} - 2u + C$$
$$= \frac{2\sqrt{(1-x)^3}}{3} - 2\sqrt{1-x} + c$$

$u^2 = 1 - x$
$2u\, du = -dx$
$dx = -2u\, du$

58

$u = \cos x \quad \frac{dv}{dx} = e^{-2x}$
$\frac{du}{dx} = -\sin x \quad v = -\frac{1}{2}e^{-2x}$

$$I = \int e^{-2x} \cos x\, dx$$

$u = \sin x \quad \frac{dv}{dx} = e^{-2x}$
$\frac{du}{dx} = \cos x \quad v = -\frac{1}{2}e^{-2x}$

$$I = -\frac{1}{2}e^{-2x}\cos x - \frac{1}{2}\int e^{-2x}\sin x\, dx$$
$$= -\frac{1}{2}e^{-2x}\cos x - \frac{1}{2}\left(-\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\int e^{-2x}\cos x\, dx\right)$$
$$= -\frac{1}{2}e^{-2x}\cos x + \frac{1}{4}e^{-2x}\sin x - \frac{1}{4}I + c$$
$$\frac{5}{4}I = -\frac{1}{2}e^{-2x}\cos x + \frac{1}{4}e^{-2x}\sin x + c$$
$$I = \frac{e^{-2x}(\sin x - 2\cos x)}{5} + c$$

59

for $0 < x < 1 \quad x^n > x^{n+1}$
$\therefore \frac{1}{1+x^n} < \frac{1}{1+x^{n+1}}$ ∴ statement I is true
$\sin x$ and $\cos x$ are symmetric
so $\int_0^{\frac{\pi}{2}} \sin x\, dx = \int_0^{\frac{\pi}{2}} \cos x\, dx$
ANSWER (D)

60

$u = \cos^{-1} x \quad \frac{dv}{dx} = 1$
$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad v = x$

$$\int_0^1 \cos^{-1} x\, dx$$
$$= \left[x\cos^{-1} x\right]_0^1 + \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$
$$= 0 - 0 - \frac{1}{2}\int_0^1 (-2x)(1-x^2)^{-\frac{1}{2}} dx$$
$$= -\frac{1}{2} \times 2\left[(1-x^2)^{\frac{1}{2}}\right]_0^1$$
$$= -(0-1)$$
$$= 1$$

61

$u^2 = 1 - x$
$2u\, du = -dx$
$dx = -2u\, du$

i $I_1 = \int_0^1 \frac{x}{\sqrt{1-x}} dx$
$$= \int_1^0 \frac{1-u^2}{u} \times (-2u\, du)$$
$$= 2\int_0^1 (1-u^2)\, du$$
$$= 2\left[u - \frac{u^3}{3}\right]_0^1$$
$$= 2\left(1 - \frac{1}{3}\right)$$
$$= \frac{4}{3}$$

ii $I_{n-1} - I_n$
$$= \int_0^1 \frac{x^{n-1}}{\sqrt{1-x}} dx - \int_0^1 \frac{x^n}{\sqrt{1-x}} dx$$
$$= \int_0^1 \frac{x^{n-1}(1-x)}{\sqrt{1-x}} dx$$
$$= \int_0^1 x^{n-1}\sqrt{1-x}\, dx$$

$u = \sqrt{1-x} \quad \frac{dv}{dx} = x^{n-1}dx$
$\frac{du}{dx} = -\frac{1}{2}(1-x)^{-\frac{1}{2}} \quad v = \frac{x^n}{n}$
$= -\frac{1}{2\sqrt{1-x}}$

iii $I_{n-1} - I_n = \int_0^1 x^{n-1}\sqrt{1-x}\, dx$
$$= \frac{1}{n}\left[x^n\sqrt{1-x}\right]_0^1 + \frac{1}{2n}\int_0^1 \frac{x^n}{\sqrt{1-x}} dx$$
$$I_{n-1} - I_n = \frac{1}{2n}I_n$$
$$2nI_{n-1} - 2nI_n = I_n$$
$$(2n+1)I_n = 2nI_{n-1}$$
$$I_n = \frac{2n}{2n+1}I_{n-1}$$

62

$t = \tan\frac{x}{2}$
$dx = \frac{2dt}{1+t^2}$

$$\int \sec x\, dx$$
$$= \int \frac{1+t^2}{1-t^2}\left(\frac{2dt}{1+t^2}\right)$$
$$= 2\int \frac{1}{1-t^2} dt$$
$$= \int \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt$$
$$= \ln|1+t| - \ln|1-t| + c$$
$$= \ln\left|\frac{1+t}{1-t}\right| + c$$

63

$$\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx$$

$u = x \qquad \dfrac{dv}{dx} = \tan^2 x = \sec^2 x - 1$

$\dfrac{du}{dx} = 1 \qquad v = \tan x - x$

$$= \Big[x(\tan x - x)\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (\tan x - x)\,dx$$
$$= \left(\frac{\pi}{4}\left(1-\frac{\pi}{4}\right)\right) - 0 - \left[-\ln|\cos x| - \frac{x^2}{2}\right]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{4}\left(1-\frac{\pi}{4}\right) - \left(\left(-\ln\frac{1}{\sqrt{2}} - \frac{\pi^2}{32}\right) - (0-0)\right)$$
$$= \frac{\pi}{4} - \frac{\pi^2}{16} + \frac{\pi^2}{32} - \frac{1}{2}\ln 2$$
$$= \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\ln 2$$

64

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$$
$$= 2\int \left(\frac{1}{2}x^{-\frac{1}{2}}\right)\cos\sqrt{x}\,dx$$
$$= 2\sin\sqrt{x} + c$$

ANSWER (B)

65

i $B = \dfrac{3(1)^2 + 3(1) - 2}{(1+1)^2} = 1$

equating coefficients of x^2: $\quad A + B = 3 \quad \Rightarrow \quad A = 2$

$A = 2, B = 1$

ii $\displaystyle\int \frac{3x^2 + 3x - 2}{(x-1)(x+1)^2}\,dx$
$$= \int \left(\frac{2}{x+1} + \frac{1}{x-1} + \frac{1}{(x+1)^2}\right)dx$$
$$= 2\ln|x+1| + \ln|x-1| - \frac{1}{x+1} + c$$
$$= \ln\left|\frac{(x+1)^2}{x-1}\right| - \frac{1}{x+1} + c$$

66

$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$$

$x = 2\cos\theta$

$\dfrac{dx}{d\theta} = -2\sin\theta$

$dx = -2\sin\theta\,d\theta$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{1}{4\cos^2\theta\sqrt{4 - 4\cos^2\theta}}(-2\sin\theta\,d\theta)$$
$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos^2\theta\sin\theta} \times \sin\theta\,d\theta$$
$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2\theta\,d\theta$$
$$= \frac{1}{4}\Big[\tan\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \frac{1}{4}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{4}\left(\frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{2\sqrt{3}}$$
$$= \frac{\sqrt{3}}{6}$$

67

i $\dfrac{x^2 - x + 1}{x^2 + 2x + 1} = \dfrac{1(x^2 + 2x + 1) - 3(x+1) + 3}{x^2 + 2x + 1}$
$$= 1 - \frac{3}{x+1} + \frac{3}{(x+1)^2}$$

$A = 1, B = -3, C = 3$

ii $\displaystyle\int \frac{x^2 - x + 1}{(x+1)^2}\,dx$
$$= \int \left(1 - \frac{3}{x+1} + \frac{3}{(x+1)^2}\right)dx$$
$$= x - 3\ln|x+1| - \frac{3}{x+1} + c$$

68

$$\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx$$

$u = x \qquad \dfrac{dv}{dx} = \sec^2 x$

$\dfrac{du}{dx} = 1 \qquad v = \tan x$

$$= \Big[x\tan x\Big]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan x\,dx$$
$$= \frac{\pi\sqrt{3}}{3} + \Big[\ln|\cos x|\Big]_0^{\frac{\pi}{3}}$$
$$= \frac{\pi\sqrt{3}}{3} + \ln\frac{1}{2} - \ln 1$$
$$= \frac{\pi\sqrt{3}}{3} - \ln 2$$

69

i $I_n = \displaystyle\int_0^{\pi} x^n \sin x\,dx$

$u = x^n \qquad \dfrac{dv}{dx} = \sin x$

$\dfrac{du}{dx} = nx^{n-1} \qquad v = -\cos x$

$$= -\Big[x^n\cos x\Big]_0^{\pi} + n\int_0^{\pi} x^{n-1}\cos x\,dx$$
$$= -(-\pi^n - 0) + n\int_0^{\pi} x^{n-1}\cos x\,dx$$

$u = x^{n-1} \qquad \dfrac{dv}{dx} = \cos x$

$\dfrac{du}{dx} = (n-1)x^{n-2} \qquad v = \sin x$

$$= \pi^n + n\left\{\Big[x^{n-1}\sin x\Big]_0^{\pi} - (n-1)\int x^{n-2}\sin x\,dx\right\}$$
$$I_n = \pi^n + n\{0 - (n-1)I_{n-2}$$
$$= \pi^n - n(n-1)I_{n-2}$$

ii $\displaystyle\int_0^{\pi} x^4\sin x\,dx$
$$= I_4$$
$$= \pi^4 - 4(3)I_2$$
$$= \pi^4 - 12(\pi^2 - 2(1)I_0)$$
$$= \pi^4 - 12\pi^2 + 24\int_0^{\pi}\sin x\,dx$$
$$= \pi^4 - 12\pi^2 + 24\Big[-\cos x\Big]_0^{\pi}$$
$$= \pi^4 - 12\pi^2 + 24(1+1)$$
$$= \pi^4 - 12\pi^2 + 48$$

70

$I - J$
$$= \int_0^{\ln 2} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$
$$= \Big[\ln|e^x + e^{-x}|\Big]_0^{\ln 2}$$
$$= \ln\left(2 + \frac{1}{2}\right) - \ln(1+1)$$
$$= \ln\left(\frac{5}{4}\right)$$

ANSWER (D)

71

$$\frac{1}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$$

$$a = \frac{1}{(-1)^2+4} = \frac{1}{5}$$

equating coefficients of x^2: $a + b = 0 \quad \Rightarrow \quad b = -\frac{1}{5}$

equating constants: $4a + c = 1 \quad \Rightarrow \quad c = \frac{1}{5}$

$$a = \frac{1}{5}, b = -\frac{1}{5}, c = \frac{1}{5}$$

$$\int \frac{dx}{(x+1)(x^2+4)}$$

$$= \frac{1}{5}\int \left(\frac{1}{x+1} - \frac{x-1}{x^2+4}\right) dx$$

$$= \frac{1}{5}\int \left(\frac{1}{x+1} - \frac{x}{x^2+4} + \frac{1}{x^2+4}\right) dx$$

$$= \frac{1}{5}\ln|x+1| - \frac{1}{10}\ln|x^2+4| - \frac{1}{10}\tan^{-1}\frac{x}{2} + c$$

72

i $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$u = \sin^{n-1} x$	$\frac{dv}{dx} = \sin x$
$\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x$	$v = -\cos x$

$$= -\left[\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\, dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,(1 - \sin^2 x)\, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

$$\therefore I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = (n-1)I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

ii $\int_0^{\frac{\pi}{2}} \sin^5 x\, dx$

$$= I_5$$

$$= \frac{4}{5} I_3$$

$$= \frac{4}{5} \times \frac{2}{3} I_1$$

$$= \frac{8}{15}\int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= \frac{8}{15}\left[-\cos x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{15}(0+1)$$

$$= \frac{8}{15}$$

73

$$\int \sin^3 x\, dx$$

$$= \int \sin^2 x \sin x\, dx$$

$$= \int (1 - \cos^2 x)\sin x\, dx$$

$$= \int \sin x\, dx - \int \sin x \cos^2 x\, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + c$$

ANSWER (B)

74

i $C = \frac{2(-1)^2 + (-1) + 9}{(-1)^2 + 4} = 2$

equating coefficients of x^2: $A + C = 2 \quad \Rightarrow \quad A = 0$

equating constants: $B + 4C = 9 \quad \Rightarrow \quad B = 1$

$A = 0, B = 1, C = 2$

ii $\int_0^2 \frac{2x^2 + x + 9}{(x^2+4)(x+1)} dx$

$$= \int_0^2 \left(\frac{1}{x^2+4} + \frac{2}{x+1}\right) dx$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{x}{2} + 2\ln|x+1|\right]_0^2$$

$$= \left(\frac{1}{2} \times \frac{\pi}{4} + 2\ln 3\right) - (0+0)$$

$$= \frac{\pi}{8} + 2\ln 3$$

75

$$\int x\sqrt{x-1}\, dx$$

$u^2 = x - 1$
$2u\, du = dx$

$$= \int (u^2+1)u \times 2u\, du$$

$$= 2\int (u^4 + u^2)\, du$$

$$= \frac{2u^5}{5} + \frac{2u^3}{3} + c$$

$$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + c$$

76

$$\int \frac{x}{\sqrt{2-x^2}} dx$$

$x = \sqrt{2}\sin\theta$
$dx = \sqrt{2}\cos\theta\, d\theta$

$$= \int \frac{\sqrt{2}\sin\theta}{\sqrt{2 - 2\sin^2\theta}} \times \sqrt{2}\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{2}\sin\theta}{\sqrt{2} \times \sqrt{1 - \sin^2\theta}} \times \sqrt{2}\cos\theta\, d\theta$$

$$= \sqrt{2}\int \frac{\sin\theta}{\cos\theta} \times \cos\theta\, d\theta$$

$$= \sqrt{2}\int \sin\theta\, d\theta$$

$$= -\sqrt{2}\cos\theta + c$$

$$= -\sqrt{2}\sqrt{1 - \sin^2\theta} + c$$

$$= -\sqrt{2} \times \sqrt{1 - \frac{x^2}{2}} + c$$

$$= -\sqrt{2 - x^2} + c$$

77

$$\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} dx$$

$$= \int \frac{(\cos x + \sin x)(\cos^2 - \cos x \sin x + \sin^2 x)}{\cos x + \sin x} dx$$

$$= \int (1 - \cos x \sin x)\, dx$$

$$= \int \left(1 - \frac{1}{2}\sin 2x\right) dx$$

$$= x + \frac{1}{4}\cos 2x + c$$

ANSWER (A)

78

$$\int \frac{\sin 2x + \sin x}{\cos^2 x}\,dx$$
$$= \int \left(\frac{2\sin x\cos x}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right)dx$$
$$= \int (2\tan x + \sin x\,(\cos x)^{-2})\,dx$$
$$= -2\ln|\cos x| + \frac{1}{\cos x} + c$$
$$= -2\ln|\cos x| + \sec x + c$$

79

$u = \sin x$	$\frac{dv}{dx} = e^x$
$\frac{du}{dx} = \cos x$	$v = e^x$

$u = \cos x$	$\frac{dv}{dx} = e^x$
$\frac{du}{dx} = -\sin x$	$v = e^x$

$$\int e^x \sin x\,dx$$
$$= e^x \sin x - \int e^x \cos x\,dx$$
$$= e^x \sin x - \left\{e^x \cos x + \int e^x \sin x\,dx\right\}$$
$$\therefore 2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$$
$$\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2}$$

80

$$\int \sin 2x \sin 4x\,dx$$
$$= \frac{1}{2}\int \Big[\cos(2x-4x) - \cos(2x+4x)\Big]dx$$
$$= \frac{1}{2}\int \Big[\cos(-2x) - \cos 6x\Big]dx$$
$$= \frac{1}{2}\int \Big[\cos 2x - \cos 6x\Big]dx$$
$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{6}\sin 6x\right] + c$$
$$= \frac{\sin 2x}{4} - \frac{\sin 6x}{12} + c$$

81

$$\int \frac{\sin\left(x+\frac{\pi}{4}\right) + \sin\left(x-\frac{\pi}{4}\right)}{\cos\left(x-\frac{\pi}{4}\right) + \cos\left(x+\frac{\pi}{4}\right)}dx$$
$$= \int \frac{2\sin x\cos\frac{\pi}{4}}{2\cos x\cos\frac{\pi}{4}}dx$$
$$= \int \frac{\sin x}{\cos x}dx$$
$$= -\ln|\cos x| + c$$

82

$u^2 = x + 5$
$2u\,du = dx$

$$\int \frac{x}{\sqrt{x+5}}dx$$
$$= \int \frac{u^2-5}{u} \times 2u\,du$$
$$= 2\int (u^2-5)\,du$$
$$= \frac{2u^3}{3} - 10u + C$$
$$= \frac{2\sqrt{(x+5)^3}}{3} - 10\sqrt{x+5} + c$$
$$= \frac{2(x+5)\sqrt{x+5}}{3} - 10\sqrt{x+5} + c$$
$$= \frac{2}{3}\sqrt{x+5}(x+5-15) + c$$
$$= \frac{2}{3}(x-10)\sqrt{x+5} + c$$

ANSWER (D)

83

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}$$
$$= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{1+\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}dx$$
$$= \int_o^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{1+\sin x+\cos x}dx$$
$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x} - \int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}$$
$$\therefore 2\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x} = \frac{\pi}{2}\ln 2$$
$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x} = \frac{\pi}{4}\ln 2$$

84

$x = 2\sec\theta$
$dx = 2\sec\theta\tan\theta\,d\theta$

$$\int \frac{dx}{x^3\sqrt{x^2-4}}$$
$$= \int \frac{1}{8\sec^3\theta\sqrt{4\sec^2\theta-4}} \times 2\sec\theta\tan\theta\,d\theta$$
$$= \int \frac{2\sec\theta\tan\theta}{8\sec^3\theta\,2\tan\theta}d\theta$$
$$= \frac{1}{8}\int \cos^2\theta\,d\theta$$
$$= \frac{1}{8}\int \left(\frac{1}{2}(1+\cos 2\theta)\right)d\theta$$
$$= \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + c$$
$$= \frac{1}{16}\sec^{-1}\frac{x}{2} + \frac{1}{16} \times \frac{\sqrt{x^2-4}}{x} \times \frac{2}{x} + c$$
$$= \frac{1}{16}\cos^{-1}\frac{2}{x} + \frac{\sqrt{x^2-4}}{8x^2} + c$$

85

$u = \pi - x$
$du = -dx$
$dx = -du$

i $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$
$$= \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}(-du)$$
$$= \int_0^{\pi} \frac{(\pi-u)(\sin u)}{1+(-\cos u)^2}du$$
$$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}du \quad \text{since } u \text{ is a dummy variable}$$

ii $\displaystyle\therefore \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \int_0^{\pi}\left(\frac{\pi\sin x}{1+\cos^2 x} - \frac{x\sin x}{1+\cos^2 x}\right)du$

$v = \cos x$
$\frac{dv}{dx} = -\sin x$
$dx = -\frac{dv}{\sin x}$

$$2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$
$$= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$
$$= \pi\int_1^{-1} \frac{\sin x}{1+v^2} \times \left(-\frac{dv}{\sin x}\right)$$
$$= \pi\int_{-1}^{1} \frac{1}{1+v^2}dv$$
$$= \pi\Big[\tan^{-1} v\Big]_{-1}^{1}$$
$$= \pi\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$
$$= \frac{\pi^2}{2}$$
$$\therefore \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$$

86

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}\,dx$$

$$\therefore\ 2\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$= \Big[\,x\,\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$$

ANSWER (D)

88

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\,dx$$

$$= \int (e^x - e^{-x})(e^x + e^{-x})^{-2}\,dx$$

$$= -\frac{1}{e^x + e^{-x}} + \boldsymbol{c}$$

87

$$u = \frac{\pi}{4} - x \qquad dx = -du$$

i $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\,dx$

$$= \int_{\frac{\pi}{4}}^{0} \ln\left(1 + \tan\left(\frac{\pi}{4} - u\right)\right)(-du)$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{\tan\frac{\pi}{4} - \tan u}{1 + \tan\frac{\pi}{4}\tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1 - \tan u}{1 + \tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1 + \tan u + 1 - \tan u}{1 + \tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan u}\right) du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan x}\right) dx \qquad \text{swapping variables}$$

ii $\therefore \int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\,dx = \int_0^{\frac{\pi}{4}} \Big(\ln 2 - \ln(1 + \tan x)\Big)\,dx$

$$2\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,dx$$

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\,dx = \frac{1}{2}\ln 2\Big[\,x\,\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\ln 2 \times \frac{\pi}{4}$$

$$= \frac{\pi\ln 2}{8}$$

89

i $I_n - I_{n-1}$

$$= \int_0^{\frac{\pi}{6}} (\sin^{2n}\theta\sec\theta - \sin^{2(n-1)}\sec\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sin^{2n-2}\,\theta\sec\theta\,(\sin^2\theta - 1)\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sin^{2n-2}\,\theta\sec\theta\,(-\cos^2\theta)\,d\theta$$

$$= -\int_0^{\frac{\pi}{6}} \cos\theta\sin^{2n-2}\theta\,d\theta$$

$$= -\left[\frac{\sin^{2n-1}\theta}{2n-1}\right]_0^{\frac{\pi}{6}}$$

$$= -\frac{\left(\frac{1}{2}\right)^{2n-1}}{2n-1} - 0$$

$$= -\frac{1}{2^{2n-1}(2n-1)}$$

ii $\therefore \sum_{k=1}^{n}(I_k - I_{k-1}) = \sum_{k=1}^{n}\frac{-1}{2^{2k-1}(2k-1)}$

$$(I_1 - I_0) + (I_2 - I_1) + (I_3 - I_2) + \ldots + (I_n - I_{n-1}) = -\sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$

$$I_n - I_0 = -\sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$

$$I_n = I_0 - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$

$$= \int_0^{\frac{\pi}{6}} \sec\theta\,d\theta - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$

$$= \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\frac{\pi}{6}} - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$

$$= \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \ln|1 + 0| - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$

$$= \ln\sqrt{3} - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$

$$= \frac{1}{2}\ln 3 - \sum_{k=1}^{n}\frac{1}{2^{2k-1}(2k-1)}$$

90

i $I_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot x \, dx$

$= \Big[\ln|\sin x|\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \ln\frac{1}{\sqrt{2}} - \ln\frac{1}{2}$

$= \ln 1 - \ln\sqrt{2} - \ln 1 + \ln 2$

$= 0 - \frac{1}{2}\ln 2 - 0 + \ln 2$

$= \frac{1}{2}\ln 2$

ii $I_{n-2} + I_n$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cot^{n-2} x + \cot^n x)\, dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^{n-2} x(1 + \cot^2 x)\, dx$

$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (-\operatorname{cosec}^2 x)\cot^{n-2} x\, dx$

$= -\left[\frac{\cot^{n-1} x}{n-1}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= -\frac{1}{n-1}\left(1 - (\sqrt{3})^{n-1}\right)$

$= \frac{(\sqrt{3})^{n-1} - 1}{n-1}$

$= \frac{1}{n-1}\left(3^{\frac{1}{2}(n-1)} - 1\right)$

iii $I_5 = \frac{1}{5-1}\left(3^{\frac{1}{2}(5-1)} - 1\right) - I_3$

$= 2 - \left(\frac{1}{3-1}\left(3^{\frac{1}{2}(3-1)} - 1\right) - I_1\right)$

$= 1 + \frac{1}{2}\ln 2$

91

$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6}$

$t = \tan\frac{\theta}{2}$ $d\theta = \frac{2dt}{1+t^2}$

$= \int_0^1 \frac{1}{4\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) + 6} \times \frac{2dt}{1+t^2}$

$= 2\int_0^1 \frac{1}{8t - 2 + 2t^2 + 6 + 6t^2}\, dt$

$= 2\int_0^1 \frac{1}{8t^2 + 8t + 4}\, dt$

$= \int_0^1 \frac{1}{4t^2 + 4t + 2}\, dt$

$= \frac{1}{2}\int_0^1 \frac{2}{(2t+1)^2 + 1^2}\, dt$

$= \frac{1}{2}\Big[\tan^{-1}(2t+1)\Big]_0^1$

$= \frac{1}{2}(\tan^{-1} 3 - \tan^{-1} 1)$

$= \frac{1}{2}\tan^{-1}(\tan(\tan^{-1} 3 - \tan^{-1} 1))$

$= \frac{1}{2}\tan^{-1}\left(\frac{\tan(\tan^{-1} 3) - \tan(\tan^{-1} 1)}{1 + \tan(\tan^{-1} 3)\tan(\tan^{-1} 1)}\right)$

$= \frac{1}{2}\tan^{-1}\left(\frac{3-1}{1+(3)(1)}\right)$

$= \frac{1}{2}\tan^{-1}\frac{1}{2}$

92

$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx$

$x = \frac{1-u}{1+u}$ $\frac{dx}{du} = \frac{(1+u)(-1) - (1-u)(1)}{(1+u)^2}$ $= -2(1+u)^{-2}$ $dx = -\frac{2du}{(1+u)^2}$

$= \int_1^0 \frac{\ln\left(1 + \frac{1-u}{1+u}\right)}{1 + \frac{(1-u)^2}{(1+u)^2}} \times \left(-\frac{2du}{(1+u)^2}\right)$

$= 2\int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{1 + 2u + u^2 + 1 - 2u + u^2}\, du$

$= \int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{u^2+1}\, du$

$= \int_0^1 \left(\frac{\ln 2}{u^2+1} - \frac{\ln(1+u)}{u^2+1}\right) du$

$\therefore I = \ln 2\int_0^1 \frac{1}{u^2+1}\, du - I$

$2I = \ln 2\Big[\tan^{-1} u\Big]_0^1$

$I = \frac{\ln 2}{2}\left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi \ln 2}{8}$

93

i $\frac{5 - 5x^2}{(1+2x)(1+x^2)} = \frac{a}{1+2x} + \frac{bx+c}{1+x^2}$

$a = \frac{5 - 5\left(-\frac{1}{2}\right)^2}{1 + \left(-\frac{1}{2}\right)^2} = 3$

equating coefficients of x^2: $a + 2b = -5 \quad \Rightarrow \quad b = -4$

equating constants: $a + c = 5 \quad \Rightarrow \quad c = 2$

$\int_0^1 \frac{5 - 5x^2}{(1+2x)(1+x^2)}\, dx$

$= \int_0^1 \left(\frac{3}{1+2x} - \frac{4x-2}{1+x^2}\right) dx$

$= \int_0^1 \left(\frac{3}{1+2x} - \frac{4x}{1+x^2} + \frac{2}{1+x^2}\right) dx$

$= \left[\frac{3}{2}\ln|1+2x| - 2\ln|1+x^2| + 2\tan^{-1} x\right]_0^1$

$= \left(\frac{3}{2}\ln 3 - 2\ln 2 + \frac{\pi}{2}\right) - \left(\frac{3}{2}\ln 1 - 2\ln 1 + 0\right)$

$= \frac{1}{2}(\ln 3^3 - \ln 2^4 + \pi)$

$= \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$

93 ...

ii $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1+2\sin 2x+\cos 2x}dx$

$t=\tan x$

$\frac{dt}{dx}=\sec^2 x$

$dx=\frac{dt}{\sec^2 x}=\frac{dt}{1+\tan^2 x}=\frac{dt}{1+t^2}$

$=\int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+\frac{4t}{1+t^2}+\frac{1-t^2}{1+t^2}}\times\frac{dt}{1+t^2}$

$=\int_0^1 \frac{1-t^2}{1+t^2+4t+1-t^2}\times\frac{dt}{1+t^2}$

$=\int_0^1 \frac{1-t^2}{2(1+2t)(1+t^2)}dt$

$=\frac{1}{10}\int_0^1 \frac{5-5t^2}{(1+2t)(1+t^2)}dt$

$=\frac{1}{10}\times\frac{1}{2}\left(\pi+\ln\frac{27}{16}\right)$

$=\frac{1}{20}\left(\pi+\ln\frac{27}{16}\right)$

95

$I_n=\int_0^1 \frac{dx}{(1+x^2)^n}$

$u=(1+x^2)^{-n}$, $\frac{dv}{dx}=1$

$\frac{du}{dx}=-n(1+x^2)^{-n-1}(2x)$, $v=x$

$=\left[\frac{x}{(1+x^2)^n}\right]_0^1+2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}dx$

$=\frac{1}{2^n}-0+2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}dx$

$=\frac{1}{2^n}+2n\int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}dx$

$=\frac{1}{2^n}+2n\int_0^1 \frac{1}{(1+x^2)^n}dx-2n\int_0^1 \frac{1}{(1+x^2)^{n+1}}dx$

$\therefore I_n=\frac{1}{2^n}+2nI_n-2nI_{n+1}$

$(1-2n)I_n=\frac{1}{2^n}-2nI_{n+1}$

$I_n=\frac{1}{2^n(1-2n)}+\frac{2n}{2n-1}I_{n+1}$

96

i $I_n=\int_0^1 x^n\sqrt{1-x^3}\,dx$

$u=x^{n-2}$, $\frac{dv}{dx}=x^2\sqrt{1-x^3}$

$\frac{du}{dx}=(n-2)x^{n-3}$, $v=-\frac{2}{9}(1-x^3)^{\frac{3}{2}}$

$=-\frac{2}{9}\left[x^{n-2}(1-x^3)^{\frac{3}{2}}\right]_0^1+\frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)^{\frac{3}{2}}dx$

$=0+\frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)\sqrt{1-x^3}\,dx$

$=\frac{2}{9}(n-2)\int_0^1 \left(x^{n-3}\sqrt{1-x^3}-x^n\sqrt{1-x^3}\right)dx$

$I_n=\frac{2n-4}{9}I_{n-3}-\frac{2n-4}{9}I_n$

$\frac{2n+5}{9}I_n=\frac{2n-4}{9}I_{n-3}$

$I_n=\frac{2n-4}{2n+5}I_{n-3}$

ii $I_8=\frac{2(8)-4}{2(8)+5}I_5$

$=\frac{4}{7}\left(\frac{2(5)-4}{2(5)+5}\right)I_2$

$=\frac{8}{35}\int_0^1 x^2\sqrt{1-x^3}\,dx$

$=\frac{8}{35}\int_1^0 x^2\times u\times\left(-\frac{2u\,du}{3x^2}\right)$

$=\frac{16}{105}\int_0^1 u^2\,du$

$=\frac{16}{105}\left[\frac{u^3}{3}\right]_0^1$

$=\frac{16}{...}$

94

i $I_n=\int_0^{\frac{\pi}{2}} x^n\cos x\,dx$

$u=x^n$, $\frac{dv}{dx}=\cos x$

$\frac{du}{dx}=nx^{n-1}$, $v=\sin x$

$=\left[x^n\sin x\right]_0^{\frac{\pi}{2}}-n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$

$=\left(\frac{\pi}{2}\right)^n-n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$

$u=x^{n-1}$, $\frac{dv}{dx}=\sin x$

$\frac{du}{dx}=(n-1)x^{n-2}$, $v=-\cos x$

$=\left(\frac{\pi}{2}\right)^n-n\left\{\left[-x^{n-1}\cos x\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x\,dx\right\}$

$\therefore I_n=\left(\frac{\pi}{2}\right)^n-n\{0+(n-1)I_{n-2}\}$

$I_n=\left(\frac{\pi}{2}\right)^n-n(n-1)I_{n-2}$

ii $A=\int_0^{\frac{\pi}{2}} x^4\cos x\,dx$

$=I_4$

$=\left(\frac{\pi}{2}\right)^4-4\times3\times I_2$

$=\frac{\pi^4}{16}-12\left(\left(\frac{\pi}{2}\right)^2-2\times1\times I_0\right)$

$=\frac{\pi^4}{16}-3\pi^2+24\int_0^{\frac{\pi}{2}}\cos x\,dx$

$=\frac{\pi^4}{16}-3\pi^2+24\left[\sin x\right]_0^{\frac{\pi}{2}}$

$=\frac{\pi^4}{16}-3\pi^2+24$

97

$\int \frac{1}{\sin x\cos x}dx$

$=\int(\tan x+\cot x)\,dx$

$=-\ln|\cos x|+\ln|\sin x|+C$

$=\ln\left|\frac{\sin x}{\cos x}\right|+C$

$=\ln|\tan x|+C$

ANSWER (C)

98

i $(1+t^2)^{n-1}+t^2(1+t^2)^{n-1}$

$=(1+t^2)^{n-1}(1+t^2)$

$=(1+t^2)^{n-1}$

ii $I_n=\int_0^x (1+t^2)^n\,dt$

$u=(1+t^2)^n$, $\frac{dv}{dt}=1$

$\frac{du}{dx}=n(1+t^2)^{n-1}.2t$, $v=t$

$=\left[t(1+t^2)^n\right]_0^x-2n\int t^2(1+t^2)^{n-1}dt$

$=x(1+x^2)^n-2n\int\left((1+t^2)^n-(1+t^2)^{n-1}\right)dt$

$\therefore\ I_n=x(1+x^2)^n-2nI_n+2n\,I_{n-1}$

$(2n+1)I_n=x(1+x^2)^n+2nI_{n-1}$

$I_n=\frac{x(1+x^2)^n}{2n+1}+\frac{2n}{2n+1}I_{n-1}$

99

i $I_n = \int_{e^{-1}}^{1} (1 + \log_e x)^n \, dx$

$u = (1 + \ln x)^n$	$\frac{dv}{dx} = 1$
$\frac{du}{dx} = n(1 + \ln x)^{n-1} \times \frac{1}{x}$	$v = x$

$= \left[x(1 + \ln x)^n\right]_{e^{-1}}^{1} - n\int_{e^{-1}}^{1} (1 + \ln x)^{n-1} \, dx$
$= 1 - 0 - nI_{n-1}$
$= 1 - nI_{n-1}$

ii $J_n = \int_{e^{-1}}^{1} (\log_e x)(1 + \log_e x)^n \, dx$
$= \int_{e^{-1}}^{1} (1 + \log_e x - 1)(1 + \log_e x)^n \, dx$
$= \int_{e^{-1}}^{1} (1 + \log_e x)^{n+1} \, dx - \int_{e^{-1}}^{1} (1 + \log_e x)^n \, dx$
$= I_{n+1} - I_n$
$= 1 - (n + 1)I_n - I_n$
$= 1 - (n + 2)I_n$

iii $J_3 = 1 - (3 + 2)I_3$
$= 1 - 5(1 - 3I_2)$
$= -4 + 15(1 - 2I_1)$
$= 11 - 30(1 - I_0)$
$= -19 + 30\int_{e^{-1}}^{1} dx$
$= -19 + 30\left[x\right]_{e^{-1}}^{1}$
$= -19 + 30\left(1 - \frac{1}{e}\right)$
$= 11 - \frac{30}{e}$

100

$\int \tan^6 x \, dx = U_6$
$= \frac{\tan^5 x}{5} - U_4$
$= \frac{\tan^5 x}{5} - \left(\frac{\tan^3 x}{3} - U_2\right)$
$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - U_0$
$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - \int dx$
$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$

ANSWER (D)

BASIC

1 If $\underset{\sim}{a} = (2,4,1)$ and $\underset{\sim}{b} = (-4,6,4)$ find $\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$

2 Sketch $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

3 What type of triangle is formed by the points $A(1,2,3), B(1,5,7)$ and $C(13,5,7)$?

4 Simplify $\underset{\sim}{i} \cdot \left(\underset{\sim}{i} + \underset{\sim}{j}\right)$

5 Rewrite $y = 2x - 1$ as a vector equation.

6 If $\left|\overrightarrow{AB}\right| = 5$ where $A(0,1,2)$ and $B(1,3,z)$, find the possible values of z.

7 Sketch the interval $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ for $-2 \le \lambda \le 2$

8 Find the equation of a sphere with centre $(1,2,3)$ and radius 2 units.

9 $\underset{\sim}{a} = \underset{\sim}{b} + \underset{\sim}{c}$ and $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right| + \left|\underset{\sim}{c}\right|$ only occur when the vectors ______

10 Consider the points $A\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $B\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$.

a) Find a vector equation of the line through A and B.
b) Find a vector equation for the interval from A to B.

11 Show that the points $A(0,2,-1), B(3,1,3)$ and $C(-6,4,-9)$ are collinear.

12 Find the vector equation of the line through $A\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ parallel to $\overrightarrow{BC}$ with $B\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $C\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

13 Two sides of a parallelogram are formed by the vectors $(1,2,3)$ and $(2,-1,3)$, Find vectors representing the diagonals of the parallelogram.

14 Find the vector equation of the line through $A\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ parallel to $\overrightarrow{BC}$ with $B\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $C\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$.

15 Given $A(6,0,0), B(0,8,0)$ and $C(x,3,0)$ are such that triangle ABC is right angled at A, find x.

16 Which vector is not parallel to the others?

$$\underset{\sim}{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \underset{\sim}{b} = \begin{pmatrix} -4 \\ -2 \\ -2 \end{pmatrix}, \underset{\sim}{c} = \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}, \underset{\sim}{d} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

17 Find a vector equation for the line through $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ with gradient $m = 4$

18 Find a if the following pair of vectors is orthogonal: $\underset{\sim}{u} = a\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

19 Prove the following lines are parallel:

$$\underset{\sim}{r} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -8 \\ 12 \end{pmatrix} \text{ and } \underset{\sim}{q} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}.$$

20 Sketch $x = \cos t, y = \sin t, z = 0$

21 If $\underset{\sim}{u} = (\lambda, 2\lambda, 2\lambda)$ and $\left|\underset{\sim}{u}\right| = 6$ find λ.

22 Prove the lines $\underset{\sim}{r} = \binom{6}{1} + \lambda\binom{1}{2}$ and $\underset{\sim}{q} = \binom{4}{1} + \lambda\binom{-4}{2}$ are perpendicular

23 A rectangular prism has a vertex at the origin and its diagonally opposite vertex at $(3,4,5)$, with one face lying along the xy plane. What are the coordinates of the other vertices?

24 If $\underset{\sim}{u} = (1,2,3)$, $\underset{\sim}{w} = (-1,3,-2)$ and $\underset{\sim}{u} + 2\underset{\sim}{v} = \underset{\sim}{w}$ find $\left|\underset{\sim}{v}\right|$

25 The lines $\underset{\sim}{r} = \lambda\binom{1}{3}$ and $\underset{\sim}{q} = \binom{4}{1} + \lambda\binom{p}{1}$ are perpendicular. Find p.

26 Find the unit vector having the same direction as $\underset{\sim}{u} = (1,2,3)$

27 The lines $\underset{\sim}{r} = \lambda \begin{pmatrix} 1 \\ 0 \\ -p \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ are perpendicular. Find p.

28 Sketch $x = t - 1, y = 2t^2$

29 Find the longest diagonal of a rectangular prism with dimensions $2 \times 2 \times 2\sqrt{2}$ metres

30 Prove the point $\binom{3}{4}$ lies on the line $\underset{\sim}{r} = \binom{6}{1} + \lambda\binom{-1}{1}$

31 If vectors $3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $\lambda\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$ are parallel find the value of λ.

32 Prove the point $\binom{1}{2}$ does not lie on $\underset{\sim}{r} = \binom{3}{4} + \lambda\binom{-1}{2}$

33 Describe geometrically the set of points (x, y, z) that satisfy $x = 2$.

34 Sketch $x = t + 3, y = \frac{-2}{t}$

35 Use vectors to find the point that divides $P(1,2,3)$ and $Q(6,-3,8)$ in the ratio $2:3$.

36 Prove the point $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ does not lie on $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

37 Find the angle between the vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$ to the nearest degree.

38 Prove that the curve $\underset{\sim}{r}(t) = 2\sin t\,\underset{\sim}{i} + 2\cos t\,\underset{\sim}{j}$ has Cartesian form $x^2 + y^2 = 4$

39 For what value of λ do the vectors $(3,-2,1)$ and $(\lambda,-1,1)$ have the same magnitude?

40 Rewrite the following vector equations in Cartesian form.

a $\underset{\sim}{r} = \binom{1}{2} + \lambda\binom{3}{-3}$

b $\underset{\sim}{r} = \binom{-2}{3} + \lambda\binom{4}{2}$

MEDIUM

41 Find the value of λ for which $\lambda\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$ is a unit vector

42 Parallelogram $ABCD$ has $A(2,4,5), B(1,0,2), C(x,y,z)$ and $D(p,r,s)$, If the diagonals AC and BD intersect at $X(3,3,3)$ find the coordinates of C and D.

43 Find the point of intersection of the lines

$\underset{\sim}{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\2\end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix}1\\3\\6\end{pmatrix} + \mu\begin{pmatrix}-2\\1\\3\end{pmatrix}$

44 If $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k},\ \underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ find the unit vector in the direction of $2\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}$

45 Sketch $x = \cos t, y = \sin t$ by first converting it into Cartesian form

46 Prove that the angle in a semicircle is a right angle.

47 Given $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ find the vector $\underset{\sim}{u}$ parallel to $\underset{\sim}{a} + \underset{\sim}{b}$ with magnitude 2.

48 If the diagonals of a rectangle are perpendicular, prove the rectangle must be a square.

49 Prove that the vector equations $\underset{\sim}{r} = \binom{2}{4} + \lambda\binom{-1}{2}$ and $\underset{\sim}{q} = \binom{0}{8} + \lambda\binom{2}{-4}$ have the same Cartesian equation.

50 The point $B\begin{pmatrix}x\\y\\z\end{pmatrix}$ is on the interval AC and is twice as far from $A\begin{pmatrix}1\\2\\2\end{pmatrix}$ as it is from $C\begin{pmatrix}4\\-1\\5\end{pmatrix}$. Use vectors to find the coordinates of B.

51 Prove that the diagonals of a parallelogram bisect each other.

52 Sketch $x = t, y = t^2$ by first converting it into Cartesian form.

53 If $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k},\ \underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ find $\underset{\sim}{d}$ given that it is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and $\underset{\sim}{c} \cdot \underset{\sim}{d} = 12$

54 In trapezium $ABCD$ $AD||BC$ and the midpoints of the non-parallel sides AB and CD are M and N respectively. The midsegment of the trapezium is the interval joining the midpoints of the two non-parallel sides, so MN. Prove that the midsegment MN is parallel to AD and BC and the average of the lengths of AD and BC.

55 Given $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = 3\underset{\sim}{a} + 2\underset{\sim}{b}$, prove that if $\overrightarrow{OD} = \frac{1}{5}\overrightarrow{OC}$ that D lies on AB.

56 The scalar product of $\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ and the sum of $\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $\lambda\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ is 5. Find λ.

57 Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be 2 dimensional unit vectors, inclined to the x-axis at angles of α and β respectively. You may assume $\underset{\sim}{a} = \cos\alpha\, \underset{\sim}{i} + \sin\alpha\, \underset{\sim}{j}$ and $\underset{\sim}{b} = \cos\beta\, \underset{\sim}{i} + \sin\beta\, \underset{\sim}{j}$. Prove that $\cos(a - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

58 Sketch $x = \cos t\,, y = 1, z = \sin t$

59 If $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$ are mutually perpendicular vectors of equal magnitude, show that the sum of the vectors is equally inclined to $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$.

60 Show that the points $A(2,-1,1), B(1,-3,-5)$ and $C(3,-4,-4)$ form the vertices of a right angled triangle.

61 Find the point of intersection of the line through $(1,2,3)$ and $(3,2,0)$ and the xy plane.

62 Prove that for non-zero vectors $\underset{\sim}{a}, \underset{\sim}{b}$ that $\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{a} + \underset{\sim}{b}\right) = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2$ only if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular

63 If the diagonals of a quadrilateral bisect each other at right angles, prove the quadrilateral must be a rhombus.

64 The parameterised equation of a sphere is $x = r\sin\alpha\cos\beta\,, y = r\sin\alpha\sin\beta\,, z = r\cos\alpha$. Prove that it satisfies $x^2 + y^2 + z^2 = r^2$

65 If the angle between two vectors is obtuse what can we say about their dot product?

66 Given $\left|\underset{\sim}{a}\right| = 2, \left|\underset{\sim}{b}\right| = 4$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 4$, prove that $\left|\underset{\sim}{a} - \underset{\sim}{b}\right| = 2\sqrt{3}$.

67 Find the point of intersection of the line through $A(1,2,3)$ parallel to the line through $B(2,-1,1)$ and $C(0,2,3)$ with the line through $D(1,3,3)$ and $E(5,-1,-1)$

68 The sum of two unit vectors is also a unit vector if the angle between the two vectors is what angle?

69 $ABCD$ is a trapezium with $\overrightarrow{AD} = 2\overrightarrow{BC}$. Given $A(-1,6,-3), B(-1,5,1)$ and $D(3,-2,9)$ find the coordinates of C.

70 Sketch $x = \sqrt{t-1}, y = \frac{1}{t-1}$

71 The points A, B and C have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\lambda(\underset{\sim}{a} + 2\underset{\sim}{b})$ respectively, relative to the origin O, with O, A and B being non-collinear. Find λ if:
a) $\overrightarrow{BC}$ is parallel to $\overrightarrow{OA}$ b) A, B and C are collinear

72 The position of two ships, P and Q, at time t hours, is given by $\underset{\sim}{r_A} = 20(t-1)\underset{\sim}{i} + 30t\underset{\sim}{j}$ and $\underset{\sim}{r_B} = 10t\underset{\sim}{i} - 30(t+1)\underset{\sim}{j}$ respectively. Prove that the ships will not collide.

73 Find the equation of a sphere with centre $(1,2,3)$ that goes through the point $(3,-1,1)$.

74 A cube has one vertex at the origin and three of the sides run along the x, y and z axes. Find the angle, to the nearest minute, that the diagonal from the origin to the furthest vertex makes with each of the axes.

75 Prove the lines $y = -3x + 1$ and $y = \frac{1}{3}x$ are perpendicular
a Using the product of their gradients
b By first converting them to vector form

76 Describe geometrically the set of points (x, y, z) that satisfy $x + y = 2$.

77 Describe geometrically the set of points (x, y, z) that satisfy $x + y + z = 2$.

78 A hot air balloon exerts an upward force of 1000 N. It is being held in a steady position by four people holding ropes, exerting the forces in Newtons of $(25,25,-250)$, $(30,-30,-300), (-15,15,-150)$ and (a, b, c). Find a, b and c.

79 Find the intersection of the sphere with centre $(2,1,3)$ and radius 5 with the xy plane, and describe it geometrically.

80 Any three dimensional vector can be expressed as scalar multiples of any three non-parallel vectors. Find λ, μ, ν such that $\underset{\sim}{u} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{b} + \nu\underset{\sim}{c}$ where $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{u} = 5\underset{\sim}{i} + 5\underset{\sim}{j} + 5\underset{\sim}{k}$

81 Prove the Cosine Rule $c^2 = a^2 + b^2 - 2ab\cos C$ for ΔABC using vectors.

82 Find the shortest distance from the origin to the line through $A(1,3,1)$ and $B(0,1,-1)$

83 Sketch $x = 2\sin t\,, y = 1 + \sin^2 t$

84 The three altitudes of a triangle are the intervals passing through each vertex and perpendicular to the opposite side. Prove that the altitudes of a triangle are concurrent.

85 The spheres $(x-2)^2 + (y+1)^2 + (z+2)^2 = 169$ and $(x-2)^2 + (y+1)^2 + (z-12)^2 = 225$ intersect in a circle. Find the centre and radius of the circle.

86 The three medians of a triangle are the intervals passing through each vertex and the midpoint of the opposite side. Prove that the medians of a triangle are concurrent.

87 Sketch $x = \cos t\,, y = \sin t\,, z = \sin t$

88 In ΔABC the midpoint M of AC is equidistant from A, B and C. Use vectors to show that ΔABC is right angled at B.

89 Find the parametric equations of a circle where as t increases from zero the point moves anticlockwise from $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, centred about the origin.

90 The faces of tetrahedron $ODEF$ comprise equilateral triangles of side length 1 unit. Its base ODE lies flat on the xy plane with two vertices at O and $\mathrm{D}(1,0,0)$, with F above the xy plane. Prove the coordinates of the vertex F are $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}\right)$.

91 Prove that the square of the hypotenuse equals the sum of the squares of the other two sides in a right angled triangle.

92 In ΔABC $\overrightarrow{\mathrm{OA}} = \underset{\sim}{a}$ and $\overrightarrow{(OB)} = \underset{\sim}{b}$. C lies on OB so that $OC\!:\!CB = 3\!:\!1$. P lies on AC so that . Q lies on AB so that O, P and Q are collinear. Determine the ratio $AQ\!:\!QB$.

93 A triangle has vertices $A(0,0,0), B(0,4,2)$ and $C(6,4,0)$. Find the equations of the three medians and show that they are concurrent.

94 A missile follows a path of $\underset{\sim}{r}_M = (100 - 10t)\underset{\sim}{i} + (50 - 5t)\underset{\sim}{j} + (10 - t)\underset{\sim}{k}$ towards its target, where t is the time in minutes from when it was first detected, and units are in kilometres. An anti-missile rocket is fired to intercept the missile and follows a path of $\underset{\sim}{r}_R = (9(t-T)+2)\underset{\sim}{i} + (4(t-T)+2)\underset{\sim}{j} + (t-T)\underset{\sim}{k}$, where T is the time in minutes from when the missile is first detected until the rocket is fired. Find T if the rocket is to intercept the missile.

95 Sketch $x = \cos t\,, y = t, z = \sin t$ for $t \geq 0$.

96 Three vertices of a parallelogram are $O(0,0,0), A(2,2,1)$ and $B(1,2,2)$. Find the possible positions of the fourth vertex.

97 A cube has diagonally opposite vertices at the origin and $(2a, 4a, 6a)$. Note that the edges of the cube are not parallel to the axes. Prove that the vertices of the cube lie on the sphere $x^2 + y^2 + z^2 = 2ax + 4ay + 6az$

98 Prove the spheres $x^2 + y^2 + z^2 - 16 = 0$ and $x^2 - 8x + y^2 - 8y + z^2 - 14z - 40 = 0$ intersect at a point.

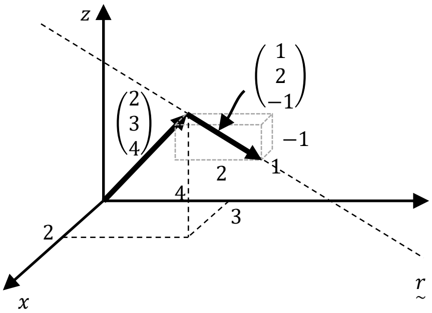
99 Sketch $x = 2\sec\theta + 1, y = 3\tan\theta - 2$.

100 Three points in space are such that $\overrightarrow{KB} = 2\overrightarrow{AK}$. Prove that if M is a fourth distinct point in space then $2\left|\overrightarrow{MA}\right|^2 + \left|\overrightarrow{MB}\right|^2 - 3\left|\overrightarrow{MK}\right|^2$ is constant.

1

$$\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b} = (2,4,1) - \frac{1}{2}(-4,6,4)$$
$$= (4,1,-1)$$

2



$$\underset{\sim}{r} = \begin{pmatrix}2\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

3

$$AB = \sqrt{(1-1)^2 + (5-2)^2 + (7-3)^2} = 5$$
$$BC = \sqrt{(13-1)^2 + (5-5)^2 + (7-7)^2} = 12$$
$$AC = \sqrt{(13-1)^2 + (5-2)^2 + (7-3)^2} = 13$$

$5:12:13$ is a Pythagorean Triad ΔABC is right angled

4

$$\underset{\sim}{i} \cdot \left(\underset{\sim}{i} + \underset{\sim}{j}\right)$$
$$= 1 + 0$$
$$= 1$$

5

The y-intercept is $(0,-1)$ so let $\underset{\sim}{a} = \binom{0}{-1}$.

The gradient is $\frac{2}{1}$ so let $\underset{\sim}{b} = \binom{1}{2}$.

$y = 2x - 1$ is equivalent to $\underset{\sim}{r} = \binom{0}{-1} + \lambda\binom{1}{2}$

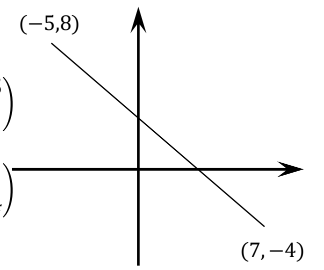
6

$$\sqrt{(1-0)^2 + (3-1)^2 + (z-2)^2} = 5$$
$$1 + 4 + (z-2)^2 = 25$$
$$(z-2)^2 = 20$$
$$z - 2 = \pm 2\sqrt{5}$$
$$z = 2 \pm 2\sqrt{5}$$

7

Substitute $\lambda = -2$ and $\lambda = 2$ to find the end points of the interval.

$$\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} - 2\begin{pmatrix}3\\-3\end{pmatrix} = \begin{pmatrix}-5\\8\end{pmatrix}$$
$$\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + 2\begin{pmatrix}3\\-3\end{pmatrix} = \begin{pmatrix}7\\-4\end{pmatrix}$$



8 $(x-1)^2 + (y-2)^2 + (z-3)^2 = 4$

9 Have the same direction

10

a)
$$\begin{pmatrix}5\\3\\2\end{pmatrix} - \begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}3\\0\\-2\end{pmatrix}$$

$\therefore \underset{\sim}{r} = \begin{pmatrix}2\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\0\\-2\end{pmatrix}$ is one correct answer.

b) Letting $\lambda = 0$ in $\underset{\sim}{r} = \begin{pmatrix}2\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\0\\-2\end{pmatrix}$ gives us $\begin{pmatrix}2\\3\\4\end{pmatrix}$, and $\lambda = 1$ gives $\begin{pmatrix}5\\3\\2\end{pmatrix}$, so one vector equation for the interval is

$$\underset{\sim}{r} = \begin{pmatrix}2\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\0\\-2\end{pmatrix}, \qquad 0 \le \lambda \le 1$$

11

$$\overrightarrow{AB} = (3,-1,4)$$
$$\overrightarrow{AC} = (-6,2,-8)$$
$$\therefore \overrightarrow{AC} = -2\overrightarrow{AB}$$

$\therefore A, B$ and C are collinear

12

$$\overrightarrow{BC} = \begin{pmatrix}1\\-2\end{pmatrix} - \begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}-3\\-3\end{pmatrix}$$
$$\therefore \underset{\sim}{r} = \begin{pmatrix}2\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\-3\end{pmatrix}$$

13

$(1,2,3) + (2,-1,3) = (3,1,6)$
$(1,2,3) - (2,-1,3) = (-1,3,0)$ or $(1,-3,0)$

14

$$\overrightarrow{BC} = \begin{pmatrix}2\\4\\1\end{pmatrix} - \begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}3\\1\\-1\end{pmatrix}$$
$$\therefore \underset{\sim}{r} = \begin{pmatrix}1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\-1\end{pmatrix}$$

15

$$\overrightarrow{AC} \cdot \overrightarrow{AB} = 0$$
$$\begin{pmatrix}x-6\\3\\0\end{pmatrix} \cdot \begin{pmatrix}-6\\8\\0\end{pmatrix} = 0$$
$$-6x + 36 + 24 + 0 = 0$$
$$6x = 60$$
$$x = 10$$

16 $\underset{\sim}{b}$, as $\underset{\sim}{c} = 3\underset{\sim}{a}$ and $\underset{\sim}{d} = -\underset{\sim}{a}$

17 The gradient of 4 can be represented by the vector $\binom{1}{4}$, or any vector where the y-value is four times the x-value.

$$\therefore \underset{\sim}{r} = \begin{pmatrix}1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\4\end{pmatrix}$$

18

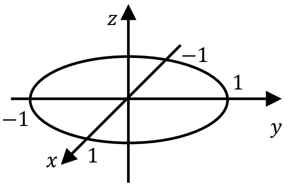
$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$$
$$2a - 1 + 3 = 0$$
$$2a = -2$$
$$a = -1$$

19 Two vectors $\underset{\sim}{a_1} + \lambda \underset{\sim}{b_1}$ and $\underset{\sim}{a_2} + \lambda \underset{\sim}{b_2}$ are parallel if $\underset{\sim}{b_1} = k\underset{\sim}{b_2}$.

$$\begin{pmatrix}4\\-8\\12\end{pmatrix} = -4\begin{pmatrix}-1\\2\\-3\end{pmatrix}$$

$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are parallel

20 We see that in the xy plane this is the unit circle, and its elevation is constant at $z = 0$, so it the unit circle.



21

$$\sqrt{\lambda^2 + (2\lambda)^2 + (2\lambda)^2} = 6^2$$
$$9\lambda^2 = 36$$
$$\lambda^2 = 4$$
$$\lambda = \pm 2$$

22

$$\begin{pmatrix}1\\2\end{pmatrix} \cdot \begin{pmatrix}-4\\2\end{pmatrix} = -4 + 4 = 0$$

$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are perpendicular

23 $(3,0,0), (3,4,0), (0,4,0), (0,0,5), (3,0,5), (0,4,5)$

24

$$(1,2,3) + 2\underset{\sim}{v} = (-1,3,-2)$$
$$2\underset{\sim}{v} = (-2,1,-5)$$
$$\underset{\sim}{v} = \left(-1, \frac{1}{2}, -\frac{5}{2}\right)$$
$$\left|\underset{\sim}{v}\right| = \sqrt{(-1)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{5}{2}\right)^2}$$
$$= \sqrt{\frac{4 + 1 + 25}{4}}$$
$$= \frac{\sqrt{30}}{2}$$

25

$$\begin{pmatrix}1\\3\end{pmatrix} \cdot \begin{pmatrix}p\\1\end{pmatrix} = 0$$
$$p + 3 = 0$$
$$p = -3$$

26

$$\left|\underset{\sim}{u}\right| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$
$$\hat{\underset{\sim}{u}} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

27

$$\begin{pmatrix}1\\0\\-p\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix} = 0$$
$$1(1) + 0(-1) - p(-1) = 0$$
$$1 + p = 0$$
$$\therefore p = -1$$

28 With practice we can note that this is the basic parabola moved 1 unit to the left and stretched vertically by a factor of 2.

Alternatively we could find the Cartesian equation:

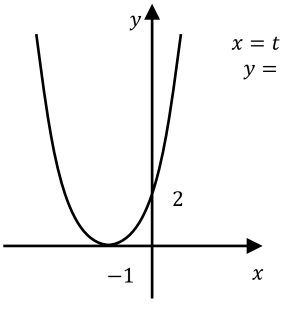
$x = t - 1 \rightarrow t = x + 1 \quad (1)$

$y = 2t^2 \quad (2)$

sub (1) in (2)

$y = 2(x+1)^2$

Again this is the basic parabola moved 1 unit to the left and stretched vertically by a factor of 2.



29

$$d = \sqrt{2^2 + 2^2 + \left(2\sqrt{2}\right)^2}$$
$$= 4$$

30

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}6-\lambda\\1+\lambda\end{pmatrix}$$
$$3 = 6 - \lambda \rightarrow \lambda = 3$$
$$4 = 1 + \lambda \rightarrow \lambda = 3$$

$\therefore \binom{3}{4}$ lies on the line $\underset{\sim}{r} = \binom{6}{1} + \lambda\binom{-1}{1}$

31

$\mu\begin{pmatrix}3\\1\\-1\end{pmatrix}=\begin{pmatrix}\lambda\\-4\\4\end{pmatrix}$

$\mu=-4$ by inspection

$\lambda=3\times(-4)=-12$

32

$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3-\lambda\\4+2\lambda\end{pmatrix}$

$1=3-\lambda\ \rightarrow\lambda=2$

$2=4+2\lambda\ \rightarrow\lambda=-1$

$\therefore\binom{1}{2}$ does not lie on $\underset{\sim}{r}=\binom{3}{4}+\lambda\binom{-1}{2}$

33

A vertical plane passing through the line $x=2$.

34

This is the basic rectangular hyperbola moved 3 units to the right, reflected about the x-axis and stretched vertically by a factor of 2.

Alternatively we could find the Cartesian equation:

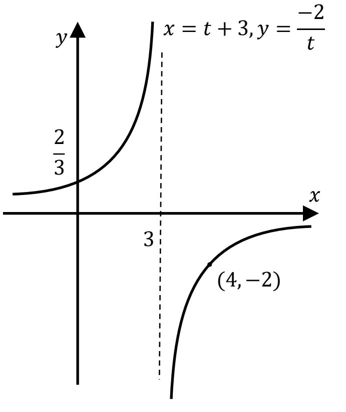
$x=t+3\ \rightarrow t=x-3\quad(1)$

$y=-\frac{2}{t}\quad(2)$

sub (1) in (2)

$y=-\frac{2}{x-3}$

Again this is the basic hyperbola moved 3 units to the right, reflected about the x-axis and stretched vertically by a factor of 2.



35

$\begin{pmatrix}a-1\\b-2\\c-3\end{pmatrix}=\frac{2}{5}\begin{pmatrix}6-1\\-3-2\\8-3\end{pmatrix}$

$\begin{pmatrix}a-1\\b-2\\c-3\end{pmatrix}=\begin{pmatrix}2\\-2\\2\end{pmatrix}$

$X\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}3\\0\\5\end{pmatrix}$

36

$\begin{pmatrix}3\\4\\2\end{pmatrix}=\begin{pmatrix}1+2\lambda\\3+\lambda\\1+3\lambda\end{pmatrix}$

$3=1+2\lambda\ \rightarrow\lambda=1$

$4=3+\lambda\ \rightarrow\lambda=1$

$2=1+3\lambda\ \rightarrow\ \lambda=-\frac{1}{3}$

$\therefore\begin{pmatrix}3\\4\\2\end{pmatrix}$ does not lie on $\underset{\sim}{r}=\begin{pmatrix}1\\3\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\3\end{pmatrix}$

37

$$\cos\theta=\frac{x_1x_2+y_1y_2+z_1z_2}{\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|}$$

$$=\frac{(1)(1)+(1)(0)+(1)(1)}{\sqrt{1^2+1^2+1^2}\times\sqrt{1^2+0^2+1^2}}$$

$$=\frac{2}{\sqrt{6}}$$

$\theta=35°$ (nearest degree)

38

$x=2\sin t\rightarrow\sin t=\frac{x}{2}$

$y=2\cos t\rightarrow\cos t=\frac{y}{2}$

$\sin^2 t+\cos^2 t=1$

$\left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2=1$

$\frac{x^2}{4}+\frac{y^2}{4}=1$

$x^2+y^2=4$

39

$\sqrt{3^2+2^2+1^2}=\sqrt{\lambda^2+1^2+1^2}$

$14=\lambda^2+2$

$\lambda^2=12$

$\lambda=\pm2\sqrt{3}$

40 **a** Gradient-intercept method

$m=\frac{-3}{3}=-1$

$b=2-1\left(\frac{-3}{3}\right)=3$

$\therefore y=-x+3$

Point-gradient method

$y-2=\frac{-3}{3}(x-1)$

$y-2=-x+1$

$y=-x+3$

b

$m=\frac{2}{4}=\frac{1}{2}$

$b=3-(-2)\left(\frac{1}{2}\right)=4$

$\therefore y=\frac{1}{2}x+4$

$y-3=\frac{2}{4}(x-(-2))$

$y-3=\frac{1}{2}x+1$

$y=\frac{1}{2}x+4$

41

$\sqrt{\lambda^2+\lambda^2+\lambda^2}=1$

$\sqrt{3}\lambda=1$

$\lambda=\frac{1}{\sqrt{3}}$

$=\frac{\sqrt{3}}{3}$

42

$\frac{1}{2}\big((2,4,5)+(x,y,z)\big)=(3,3,3)$

$(x+2,y+4,z+5)=(6,6,6)$

$(x,y,z)=(4,2,1)$

$\frac{1}{2}\big((1,0,2)+(p,r,s)\big)=(3,3,3)$

$(p+1,r,s+2)=(6,6,6)$

$(p,r,s)=(5,6,4)$

43

$\begin{pmatrix}1+\lambda\\2\\-1+2\lambda\end{pmatrix}=\begin{pmatrix}1-2\mu\\3+\mu\\6+3\mu\end{pmatrix}$

$2=3+\mu\rightarrow\mu=-1$

The point of intersection is $\begin{pmatrix}1-2(-1)\\3+(-1)\\6+3(-1)\end{pmatrix}=\begin{pmatrix}3\\2\\3\end{pmatrix}$

44

$\underset{\sim}{u}=2\underset{\sim}{a}+\underset{\sim}{b}-\underset{\sim}{c}$

$=2\left(\underset{\sim}{i}-\underset{\sim}{j}-\underset{\sim}{k}\right)+\left(2\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right)-\left(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right)$

$=3\underset{\sim}{i}-2\underset{\sim}{j}-2\underset{\sim}{k}$

$\left|\underset{\sim}{u}\right|=\sqrt{3^2+2^2+2^2}$

$=\sqrt{17}$

$\hat{\underset{\sim}{u}}=\frac{3}{\sqrt{17}}\underset{\sim}{i}-\frac{2}{\sqrt{17}}\underset{\sim}{j}-\frac{2}{\sqrt{17}}\underset{\sim}{k}$

45

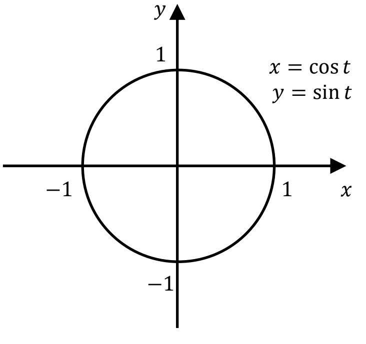
$x=\cos t\quad(1)$

$y=\sin t\quad(2)$

$\sin^2\theta+\cos^2\theta=1$ Pythagorean Identity

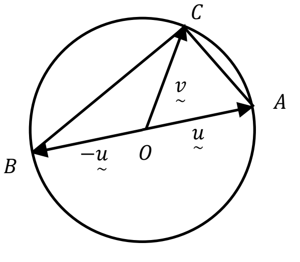
$x^2+y^2=1$

Which is the unit circle - a circle centred at the origin with radius 1.



46 Let AB be the diameter of a circle of centre O, and C be a point on the circumference. $\therefore\ \angle ACB$ is the angle in a semicircle.

Let $\overrightarrow{OA}=\underset{\sim}{u}, \overrightarrow{OB}=-\underset{\sim}{u}$ and $\overrightarrow{OC}=\underset{\sim}{v}$.



$\overrightarrow{AC}\cdot\overrightarrow{BC}=\left(\overrightarrow{OC}-\overrightarrow{OA}\right)\cdot\left(\overrightarrow{OC}-\overrightarrow{OB}\right)$

$=\left(\underset{\sim}{v}-\underset{\sim}{u}\right)\cdot\left(\underset{\sim}{v}-\left(-\underset{\sim}{u}\right)\right)$

$=\left(\underset{\sim}{v}-\underset{\sim}{u}\right)\cdot\left(\underset{\sim}{v}+\underset{\sim}{u}\right)$

$=\underset{\sim}{v}\cdot\underset{\sim}{v}-\underset{\sim}{u}\cdot\underset{\sim}{u}$

$=\left|\underset{\sim}{v}\right|^2-\left|\underset{\sim}{u}\right|^2$

$=0$ since $\left|\underset{\sim}{u}\right|=\left|\underset{\sim}{v}\right|=r$

$\therefore\ AC\perp BC$ so the angle in a semicircle is a right angle.

47

$\underset{\sim}{a}+\underset{\sim}{b}=3\underset{\sim}{i}+3\underset{\sim}{j}+0\underset{\sim}{k}$

$\left|\underset{\sim}{a}+\underset{\sim}{b}\right|=\sqrt{3^2+3^2+0^2}$

$=3\sqrt{2}$

$\underset{\sim}{u}=\pm2\left(\frac{3}{3\sqrt{2}}\underset{\sim}{i}+\frac{3}{3\sqrt{2}}\underset{\sim}{j}+0\underset{\sim}{k}\right)$

$=\sqrt{2}\underset{\sim}{i}+\sqrt{2}\underset{\sim}{j}+0\underset{\sim}{k},\qquad-\sqrt{2}\underset{\sim}{i}-\sqrt{2}\underset{\sim}{j}+0\underset{\sim}{k}$

48

Let the sides of the rectangle be represented by the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

The diagonals are $\underset{\sim}{u}+\underset{\sim}{v}$ and $\underset{\sim}{u}-\underset{\sim}{v}$

If the diagonals are perpendicular then

$\left(\underset{\sim}{u}+\underset{\sim}{v}\right)\cdot\left(\underset{\sim}{u}-\underset{\sim}{v}\right)=0$

$\underset{\sim}{u}\cdot\underset{\sim}{u}-\underset{\sim}{v}\cdot\underset{\sim}{v}=0$

$\left|\underset{\sim}{u}\right|^2-\left|\underset{\sim}{v}\right|^2=0$

$\left|\underset{\sim}{u}\right|^2=\left|\underset{\sim}{v}\right|^2$

$\left|\underset{\sim}{u}\right|=\left|\underset{\sim}{v}\right|$

$\therefore$ the rectangle must be a square if the diagonals are perpendicular

49

For $\underset{\sim}{r}$:

$y-4=-2(x-2)$

$y=-2x+8$

For $\underset{\sim}{q}$:

$y-8=-2(x-0)$

$y=-2x+8$

Both vector equations have the same Cartesian equation.

50

$$\overrightarrow{AB}=\frac{2}{3}\overrightarrow{(AC)}$$

$$\begin{pmatrix}x-1\\y-2\\z-2\end{pmatrix}=\frac{2}{3}\begin{pmatrix}4-1\\-1-2\\5-2\end{pmatrix}$$

$$\begin{pmatrix}x-1\\y-2\\z-2\end{pmatrix}=\frac{2}{3}\begin{pmatrix}3\\-3\\3\end{pmatrix}$$

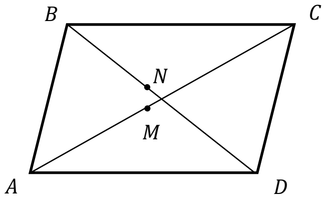
$$\begin{pmatrix}x-1\\y-2\\z-2\end{pmatrix}=\begin{pmatrix}2\\-2\\2\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\0\\4\end{pmatrix}$$

51

We will find the midpoint of each diagonal and prove they are the same, so the diagonals bisect each other.

In parallelogram $ABCD$ let M be the midpoint of AC and N be the midpoint of BD



$$\begin{aligned}\overrightarrow{AM}&=\frac{1}{2}\overrightarrow{AC}\\&=\frac{1}{2}\left(\overrightarrow{AB}+\overrightarrow{BC}\right)\\\overrightarrow{BN}&=\frac{1}{2}\overrightarrow{BD}\\&=\frac{1}{2}\left(\overrightarrow{AD}-\overrightarrow{AB}\right)\\&=\frac{1}{2}\left(\overrightarrow{BC}-\overrightarrow{AB}\right)\\\overrightarrow{AN}&=\overrightarrow{AB}+\overrightarrow{BN}\\&=\overrightarrow{AB}+\frac{1}{2}\left(\overrightarrow{BC}-\overrightarrow{AB}\right)\\&=\frac{1}{2}\left(\overrightarrow{AB}+\overrightarrow{BC}\right)\\&=\overrightarrow{AM}\end{aligned}$$

$\therefore$ $M$and N coincide, so the diagonals of a parallelogram bisect each other.

52

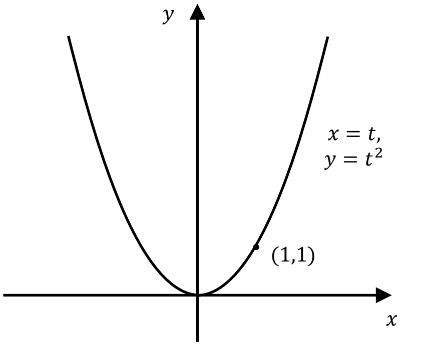
$x=t$ (1)

$y=t^2$ (2)

sub (1) in (2)

$y=x^2$

This is the basic parabola which is concave up with vertex at the origin.



53

Let $\underset{\sim}{d}=(x,y,z)$

$\underset{\sim}{a}\perp\underset{\sim}{d}\rightarrow\ x-y-z=0$ (1)

$\underset{\sim}{b}\perp\underset{\sim}{d}\rightarrow 2x-y+z=0$ (2)

$\underset{\sim}{c}\cdot\underset{\sim}{d}=10\rightarrow\ x+y-z=12$ (3)

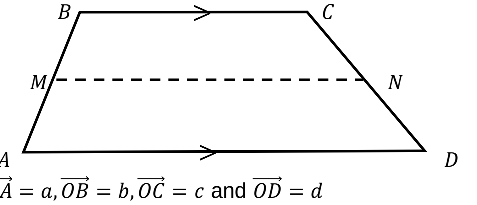
$(2)+(3)$: $3x=12\rightarrow x=4$

$(1)+(3)$: $2x-2z=12\rightarrow 8-2z=12\rightarrow z=-2$

sub in (1): $4-y+2=0\rightarrow\ y=6$

$\therefore\underset{\sim}{d}=4\underset{\sim}{i}+6\underset{\sim}{j}-2\underset{\sim}{k}$

54



Let $\overrightarrow{OA}=\underset{\sim}{a},\overrightarrow{OB}=\underset{\sim}{b},\overrightarrow{OC}=\underset{\sim}{c}$ and $\overrightarrow{OD}=\underset{\sim}{d}$

$\therefore\ \overrightarrow{OM}=\frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})$ and $\overrightarrow{ON}=\frac{1}{2}(\underset{\sim}{c}+\underset{\sim}{d})$

$$\begin{aligned}\overrightarrow{MN}&=\frac{1}{2}\left(\underset{\sim}{c}+\underset{\sim}{d}\right)-\frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}\right)\\&=\frac{1}{2}\left(\underset{\sim}{c}-\underset{\sim}{b}\right)+\frac{1}{2}\left(\underset{\sim}{d}-\underset{\sim}{a}\right)\quad(1)\\&=\frac{1}{2}\left(\overrightarrow{BC}+\overrightarrow{AD}\right)\end{aligned}$$

$\therefore\left|\overrightarrow{MN}\right|=\frac{1}{2}\left(\left|\overrightarrow{BC}\right|+\left|\overrightarrow{AD}\right|\right)$

$BC||AD\ \rightarrow\ \underset{\sim}{c}-\underset{\sim}{b}=\lambda(\underset{\sim}{d}-\underset{\sim}{a})$

In (1):

$$\begin{aligned}\overrightarrow{MN}&=\frac{1}{2}\left(\lambda\left(\underset{\sim}{d}-\underset{\sim}{a}\right)\right)+\frac{1}{2}\left(\underset{\sim}{d}-\underset{\sim}{a}\right)\\&=\frac{\lambda+1}{2}\left(\underset{\sim}{d}-\underset{\sim}{a}\right)\\&=\frac{\lambda+1}{2}\overrightarrow{AD}\end{aligned}$$

$\therefore MN||AD$ and so $MN||BC$

$\therefore$ the midsegment MN is parallel to AD and BC and the average of the lengths of AD and BC.

55

for AB $\underset{\sim}{r}=\underset{\sim}{a}+\lambda(\underset{\sim}{b}-\underset{\sim}{a})$

for $\overrightarrow{OD}$

$$\begin{aligned}\frac{1}{5}\left(3\underset{\sim}{a}+2\underset{\sim}{b}\right)&=\underset{\sim}{a}+\lambda\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\\\frac{3}{5}\underset{\sim}{a}+\frac{2}{5}\underset{\sim}{b}&=(1-\lambda)\underset{\sim}{a}+\lambda\underset{\sim}{b}\\1-\lambda&=\frac{3}{5}\quad\lambda=\frac{2}{5}\\\lambda&=\frac{2}{5}\end{aligned}$$

$\therefore D$ lies on AB with $\lambda=\frac{2}{5}$

56

$$\begin{pmatrix}1\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1+\lambda\\1+2\\-1-1\end{pmatrix}=5$$

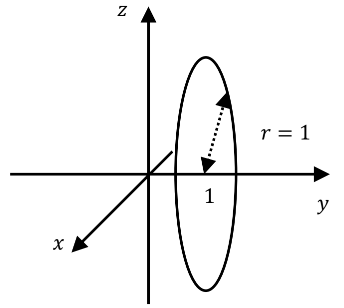
$$1+\lambda-3+2=5$$

$$\lambda=5$$

57 The angle between the vectors is $\alpha - \beta$, and using the angle between two vectors:

$$\cos(\alpha-\beta) = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\left|\underset{\sim}{a}\right|\times\left|\underset{\sim}{b}\right|}$$
$$= \frac{\left(\cos\alpha\,\underset{\sim}{i}+\sin\alpha\,\underset{\sim}{j}\right)\left(\cos\beta\,\underset{\sim}{i}+\sin\beta\,\underset{\sim}{j}\right)}{1\times 1}$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

58 Here we have a unit circle in the xz plane, but moved 1 unit to the right as y value is constant at 1.



59 Let $\underset{\sim}{a} = \underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}$, which is inclined to $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$ at α, β and γ respectively.

$$\cos\alpha = \frac{\underset{\sim}{u}\cdot\left(\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}\right)}{\left|\underset{\sim}{u}\right|\times\left|\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}\right|}$$
$$= \frac{\underset{\sim}{u}\cdot\underset{\sim}{u}+0+0}{\left|\underset{\sim}{u}\right|\times\left|\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}\right|} \qquad \underset{\sim}{u}\perp\underset{\sim}{v}, \underset{\sim}{u}\perp\underset{\sim}{w}$$
$$= \frac{\left|\underset{\sim}{u}\right|^2}{\left|\underset{\sim}{u}\right|\times\left|\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}\right|}$$
$$= \frac{\left|\underset{\sim}{u}\right|}{\left|\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}\right|}$$

Similarly

$$\cos\beta = \frac{\left|\underset{\sim}{v}\right|}{\left|\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}\right|}, \cos\gamma = \frac{\left|\underset{\sim}{w}\right|}{\left|\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}\right|}$$

Now $\left|\underset{\sim}{u}\right| = \left|\underset{\sim}{v}\right| = \left|\underset{\sim}{w}\right|$

$\therefore \cos\alpha = \cos\beta = \cos\gamma$

$\therefore \alpha = \beta = \gamma$

$\therefore$ $\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}$ is equally inclined to $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$

60 $\overrightarrow{AB} = (1-2, -3+1, -5-1) = (-1,-2,-6)$

$\overrightarrow{BC} = (3-1, -4+3, -4+5) = (2,-1,1)$

$\overrightarrow{AC} = (3-2, -4+1, -4-1) = (1,-3,-5)$

$\overrightarrow{AC}\cdot\overrightarrow{BC} = 2+3-5 = 0$

$\therefore$ ΔABC is right angled at C.

61 $\underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}3-1\\2-2\\0-3\end{pmatrix} = \begin{pmatrix}1+2\lambda\\2\\3-3\lambda\end{pmatrix}$

On the xy plane $z = 0$

$\therefore 3-3\lambda = 0 \to \lambda = 1$

$\underset{\sim}{r} = \begin{pmatrix}1+2\\2\\3-3\end{pmatrix} = \begin{pmatrix}3\\2\\0\end{pmatrix}$

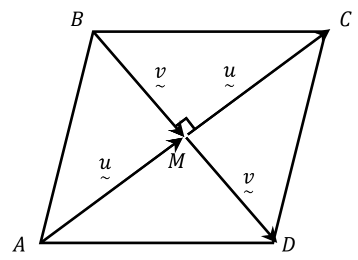
62 $\left(\underset{\sim}{a}+\underset{\sim}{b}\right)\cdot\left(\underset{\sim}{a}+\underset{\sim}{b}\right) = \underset{\sim}{a}\cdot\underset{\sim}{a} + 2\,\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b}$

$= \left|\underset{\sim}{a}\right|^2 + 2\,\underset{\sim}{a}\cdot\underset{\sim}{b} + \left|\underset{\sim}{b}\right|^2$

$\therefore$ if $\left(\underset{\sim}{a}+\underset{\sim}{b}\right)\cdot\left(\underset{\sim}{a}+\underset{\sim}{b}\right) = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2$ then $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$

$\therefore$ $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular since $\underset{\sim}{a}, \underset{\sim}{b}$ are non-zero vectors

63 Let the diagonals of quadrilateral $ABCD$ bisect each other at right angles at M, and $\overrightarrow{AM} = \overrightarrow{MC} = \underset{\sim}{u}, \overrightarrow{BM} = \overrightarrow{MD} = \underset{\sim}{v}$.



Since ΔABM is right angled

$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AM}\right|^2 + \left|\overrightarrow{BM}\right|^2$

$= \left|\underset{\sim}{u}\right|^2 + \left|\underset{\sim}{v}\right|^2$

Similarly $\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{CD}\right|^2 = \left|\overrightarrow{AD}\right|^2 = \left|\underset{\sim}{u}\right|^2 + \left|\underset{\sim}{v}\right|^2$

$\therefore \left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right| = \left|\overrightarrow{CD}\right| = \left|\overrightarrow{AD}\right|$

$\therefore AB = BC = CD = AD$

$\therefore$ $ABCD$ is a rhombus if it diagonals bisect at right angles.

64 $x^2+y^2+z^2$

$= (r\sin\alpha\cos\beta)^2 + (r\sin\alpha\sin\beta)^2 + (r\cos\alpha)^2$

$= r^2\sin^2\alpha\cos^2\beta + r^2\sin^2\alpha\sin^2\beta + r^2\cos^2\alpha$

$= r^2(\sin^2\alpha\,(\cos^2\beta+\sin^2\beta) + \cos^2\alpha)$

$= r^2(\sin^2\alpha\times(1) + \cos^2\alpha)$

$= r^2(\sin^2\alpha + \cos^2\alpha)$

$= r^2$

$\therefore$ $x = r\sin\alpha\cos\beta, y = r\sin\alpha\sin\beta, z = r\cos\alpha$ satisfies $x^2+y^2+z^2 = r^2$

65 Their dot product must be negative (cosine is negative in the second quadrant)

66 $\left|\underset{\sim}{a}-\underset{\sim}{b}\right|^2 = \left(\underset{\sim}{a}-\underset{\sim}{b}\right)\cdot\left(\underset{\sim}{a}-\underset{\sim}{b}\right)$

$= \underset{\sim}{a}\cdot\underset{\sim}{a} - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b}$

$= \left|\underset{\sim}{a}\right|^2 - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \left|\underset{\sim}{b}\right|^2$

$= (2)^2 - 2(4) + (4)^2$

$= 12$

$\therefore \left|\underset{\sim}{a}-\underset{\sim}{b}\right| = 2\sqrt{3}$

67 $\underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}0-2\\2-(-1)\\3-1\end{pmatrix} = \begin{pmatrix}1-2\lambda\\2+3\lambda\\3+2\lambda\end{pmatrix}$

$\underset{\sim}{q} = \begin{pmatrix}1\\3\\3\end{pmatrix} + \mu\begin{pmatrix}5-1\\-1-3\\-1-3\end{pmatrix} = \begin{pmatrix}1+4\mu\\3-4\mu\\3-4\mu\end{pmatrix}$

$\therefore 1-2\lambda = 1+4\mu \quad (1)$

$2+3\lambda = 3-4\mu \quad (2)$

$3+2\lambda = 3-4\mu \quad (3)$

from (2) and (3): $2+3\lambda = 3+2\lambda \to \lambda = 1$

$\underset{\sim}{r} = \begin{pmatrix}1-2(1)\\2+3(1)\\3+2(1)\end{pmatrix} = \begin{pmatrix}-1\\5\\5\end{pmatrix}$

68 $\frac{2\pi}{3}$ or $120°$, since the unit vectors and their sum form an equilateral triangle

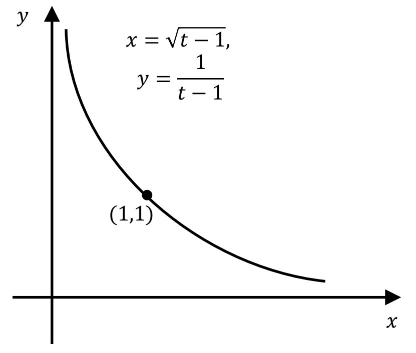
69 Let $C(x,y,z)$

$$\begin{pmatrix} x+1 \\ y-5 \\ z-1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3-(-1) \\ -2-6 \\ 9-(-3) \end{pmatrix}$$
$$\begin{pmatrix} x+1 \\ y-5 \\ z-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$$

70 Here it is safest to rewrite in Cartesian form, as the results will not quite be what we expect. Note $t > 1$ due to the square root and denominator.

$$x = \sqrt{t-1} \;\rightarrow\; t = x^2 + 1$$
$$y = \frac{1}{t-1} = \frac{1}{x^2+1-1} = \frac{1}{x^2} \quad \text{for } x > 0$$

Since $t > 1$, so $x > 0$ and $y > 0$, so the sketch is in the 1st quadrant only.



71 a)

$$\lambda(\underset{\sim}{a} + 2\underset{\sim}{b}) - \underset{\sim}{b} = \mu\underset{\sim}{a}$$
$$\lambda\underset{\sim}{a} + (2\lambda - 1)\underset{\sim}{b} = \mu\underset{\sim}{a}$$
$\therefore \lambda = \mu$ and $2\lambda - 1 = 0 \rightarrow \lambda = \frac{1}{2}$

b)

$$\underset{\sim}{b} - \underset{\sim}{a} = k\left(\lambda\left(\underset{\sim}{a} + 2\underset{\sim}{b}\right) - \underset{\sim}{a}\right)$$
$$\underset{\sim}{b} - \underset{\sim}{a} = k(\lambda - 1)\underset{\sim}{a} + 2k\lambda\underset{\sim}{b}$$
$\therefore 2k\lambda = 1$ and $k(\lambda - 1) = -1$

$\therefore k = \frac{1}{2\lambda} \qquad k = -\frac{1}{\lambda - 1}$

$$\therefore \frac{1}{2\lambda} = -\frac{1}{\lambda - 1}$$
$$\lambda - 1 = -2\lambda$$
$$3\lambda = 1$$
$$\lambda = \frac{1}{3}$$

72 The ships will collide if $\underset{\sim}{r_A} = \underset{\sim}{r_B}$ for any value of t.

$$20(t-1)\underset{\sim}{i} + 30t\underset{\sim}{j} = 10t\underset{\sim}{i} - 30(+1)t\underset{\sim}{j}$$

$\therefore 20(t-1) = 10t$ and $30t = -30(t+1)$

$20t - 20 = 10t \qquad 30t = -30t - 30$

$10t = 20 \qquad 60t = -30$

$t = 2 \qquad t = -\frac{1}{2}$

Since the x and y values of the vectors are not equal for the same value of t the ships do not collide.

73

$$\begin{aligned} r &= |(3,-1,1) - (1,2,3)| \\ &= |(2,-3,-2)| \\ &= \sqrt{2^2 + (-3)^2 + (-2)^2} \\ &= \sqrt{17} \end{aligned}$$

$$\therefore (x-1)^2 + (y-2)^2 + (z-3)^2 = 17$$

74 The diagonal passes through $(0,0,0)$ in the direction of $(1,1,1)$, so is $\underset{\sim}{r} = \lambda(1,1,1)$.

Finding the angle that it makes with $\underset{\sim}{i}$ we have:

$$\begin{aligned} \cos\theta &= \frac{(1,0,0)\cdot(1,1,1)}{1 \times \sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \\ \theta &= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= 54°44' \end{aligned}$$

75 **a**

$$m_1 = -3, m_2 = \frac{1}{3}$$
$$m_1 \times m_2 = -3 \times \left(\frac{1}{3}\right) = -1$$

$\therefore$ The two lines are perpendicular

b

In vector form $y = -3x + 1 \;\rightarrow\; \underset{\sim}{r_1} = \binom{0}{1} + \lambda\binom{1}{-3}$ and

$y = \frac{1}{3}x \;\rightarrow\; \underset{\sim}{r_2} = \lambda\binom{3}{1}$

$$\binom{1}{-3}\cdot\binom{3}{1} = 3 - 3 = 0$$

$\therefore$ The two lines are perpendicular

76 A vertical plane passing through the line $x + y = 2$.

77 A plane passing through $(2,0,0), (0,2,0)$ and $(0,0,2)$

78

$$\begin{pmatrix} 25+30-15+a \\ 25-30+15+b \\ -250-300-150+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1000 \end{pmatrix}$$
$$\begin{pmatrix} a+40 \\ b+10 \\ c-700 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1000 \end{pmatrix}$$
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -40 \\ -10 \\ -300 \end{pmatrix}$$

79

$$(x-2)^2 + (y-1)^2 + (0-3)^2 = 5^2$$
$$(x-2)^2 + (y-1)^2 = 16$$

Which is a circle on the xy plane centred at $(2,1,0)$ with radius 4.

80

$$\underset{\sim}{u} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{b} + \nu\underset{\sim}{c}$$

$$\begin{pmatrix}5\\5\\5\end{pmatrix} = \lambda\begin{pmatrix}2\\3\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix} + \nu\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

$$\begin{pmatrix}5\\5\\5\end{pmatrix} = \begin{pmatrix}2\lambda+\mu-\nu\\3\lambda-\mu+2\nu\\\lambda+2\mu-\nu\end{pmatrix}$$

$$2\lambda+\mu-\nu = 5 \quad (1)$$
$$3\lambda-\mu+2\nu = 5 \quad (2)$$
$$\lambda+2\mu-\nu = 5 \quad (3)$$
$$(1)+(2): 5\lambda+\nu = 10 \quad (4)$$
$$(3)-2(2): -3\lambda+\nu = -5 \quad (5)$$
$$(4)-(5): 8\lambda = 15 \ \rightarrow\ \lambda = \frac{15}{8}$$
$$\text{sub in (4): } 5\left(\frac{15}{8}\right)+\nu = 10 \ \rightarrow \nu = \frac{5}{8}$$
$$\text{sub in (3): } \frac{15}{8}+2\mu-\frac{5}{8} = 5 \ \rightarrow \mu = \frac{15}{8}$$

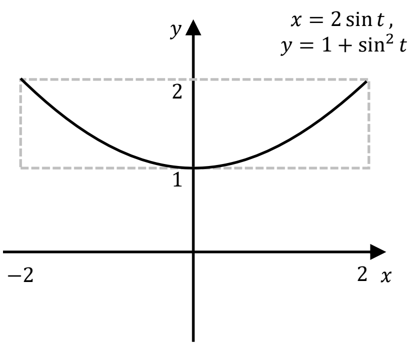
$$\therefore \lambda = \frac{15}{8}, \mu = \frac{15}{8} \text{ and } \nu = \frac{5}{8}$$

83

domain: $[-2,2]$ since $-1 \le \sin t \le 1$
range: $[1,2]$ since $0 \le \sin^2 t \le 1$

The sketch must fit within the rectangle as shown.

$$\sin t = \frac{x}{2}$$
$$\therefore y = 1 + \left(\frac{x}{2}\right)^2$$
$$= \frac{x^2}{4} + 1 \ \text{ for } -2 \le x \le 2$$



85

$$(x-2)^2 + (y+1)^2 + (z+2)^2 = 169 \quad (1)$$
$$(x-2)^2 + (y+1)^2 + (z-12)^2 = 225 \quad (2)$$
$$(2)-(1): (z-12)^2 - (z+2)^2 = 56$$
$$z^2 - 24z + 144 - z^2 - 4z - 4 = 56$$
$$-28z + 140 = 56$$
$$28z = 84$$
$$z = 3$$
$$\text{sub in (1): } (x-2)^2 + (y+1)^2 + (3+2)^2 = 169$$
$$(x-2)^2 + (y+1)^2 = 144$$

Which is a circle parallel to the xy plane centred at $(2,-1,3)$ with radius 12.

81

$$c^2 = \left|\overrightarrow{AB}\right|^2$$
$$= \overrightarrow{AB}\cdot\overrightarrow{AB}$$
$$= \left(\overrightarrow{CB}-\overrightarrow{CA}\right)\cdot\left(\overrightarrow{CB}-\overrightarrow{CA}\right)$$
$$= \overrightarrow{CB}\cdot\overrightarrow{CB} - 2\overrightarrow{CB}\cdot\overrightarrow{CA} + \overrightarrow{CA}\cdot\overrightarrow{CA}$$
$$= \left|\overrightarrow{CB}\right|^2 + \left|\overrightarrow{CA}\right|^2 - 2\left|\overrightarrow{CB}\right| \times \left|\overrightarrow{CA}\right| \times \cos\angle ACB$$
$$= a^2 + b^2 - 2ab\cos C$$

82

The shortest vector from the origin to the line $\overrightarrow{AB}$ is $\overrightarrow{AX} - \overrightarrow{AO}$ where $\overrightarrow{AX}$ is the projection of $\overrightarrow{AO}$ onto $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \begin{pmatrix}2-1\\1-3\\-1-1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$

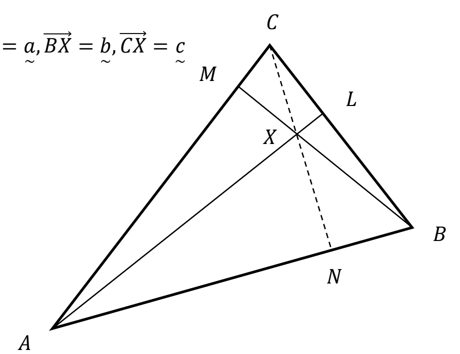
$$\overrightarrow{AX} = \frac{\begin{pmatrix}1\\3\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\-2\end{pmatrix}}{(1^2+2^2+2^2)}\begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$
$$= \frac{-1-6-2}{9}\begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$
$$= -\begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$
$$= \begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\overrightarrow{OX} = \begin{pmatrix}1\\2\\2\end{pmatrix} + \begin{pmatrix}1\\3\\1\end{pmatrix}$$
$$= \begin{pmatrix}2\\5\\3\end{pmatrix}$$
$$\left|\overrightarrow{OX}\right| = \sqrt{2^2+5^2+3^2} = \sqrt{38}$$

84

Let the altitudes from A, B meet the triangle at L, M respectively, and altitudes AL and BM intersect at X. Produce CX to meet AB at N

Let $\overrightarrow{AX} = \underset{\sim}{a}, \overrightarrow{BX} = \underset{\sim}{b}, \overrightarrow{CX} = \underset{\sim}{c}$



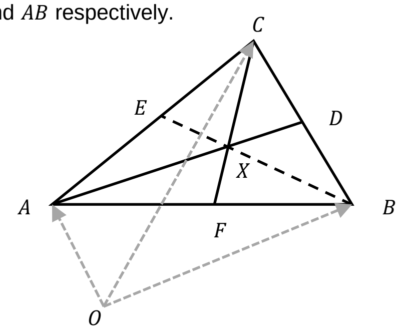
$$AL \perp BC \ \therefore \underset{\sim}{a}\cdot\left(\underset{\sim}{c}-\underset{\sim}{b}\right) = 0 \ \rightarrow \ \underset{\sim}{a}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \ (1)$$
$$BM \perp AC \ \therefore \underset{\sim}{b}\cdot\left(\underset{\sim}{c}-\underset{\sim}{a}\right) = 0 \ \rightarrow \ \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \ (2)$$
$$(1)-(2): \quad \underset{\sim}{a}\cdot\underset{\sim}{c} - \underset{\sim}{b}\cdot\underset{\sim}{c} = 0$$
$$\underset{\sim}{c}\cdot\left(\underset{\sim}{a}-\underset{\sim}{b}\right) = 0$$
$$\therefore CX \perp BA$$
$$\therefore CN \perp \text{AB}$$

$\therefore$ CN is an altitude of the triangle
$\therefore$ the three altitudes of a triangle are concurrent (at X).

86 We will let X be the intersection of AD and CF and prove that it lies on BE.

Let $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

In ΔABC let D, E, F be the midpoints of the sides BC, AC and AB respectively.



$\therefore \overrightarrow{OE} = \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2}, \overrightarrow{OD} = \frac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$ and $\overrightarrow{OF} = \frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX}\\ &= \underset{\sim}{a} + \lambda\overrightarrow{AD}\\ &= \underset{\sim}{a} + \lambda\left(\frac{\underset{\sim}{b}+\underset{\sim}{c}}{2} - \underset{\sim}{a}\right)\\ &= (1-\lambda)\underset{\sim}{a} + \frac{\lambda}{2}\underset{\sim}{b} + \frac{\lambda}{2}\underset{\sim}{c} \quad (1)\end{aligned}$$

Also

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OC} + \overrightarrow{CX}\\ &= \underset{\sim}{c} + \mu\overrightarrow{CF}\\ &= \underset{\sim}{c} + \mu\left(\frac{\underset{\sim}{a}+\underset{\sim}{b}}{2} - \underset{\sim}{c}\right)\\ &= \frac{\mu}{2}\underset{\sim}{a} + \frac{\mu}{2}\underset{\sim}{b} + (1-\mu)\underset{\sim}{c} \quad (2)\end{aligned}$$

From (1) and (2):

$1-\lambda = \frac{\mu}{2} \quad \frac{\lambda}{2} = \frac{\mu}{2} \quad \frac{\lambda}{2} = 1-\mu$

$\lambda = \mu$

$$\begin{aligned}1-\lambda &= \frac{\lambda}{2}\\ 1 &= \frac{3\lambda}{2}\\ \lambda = \mu &= \frac{2}{3}\end{aligned}$$

$\therefore \overrightarrow{OX} = \frac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3}$

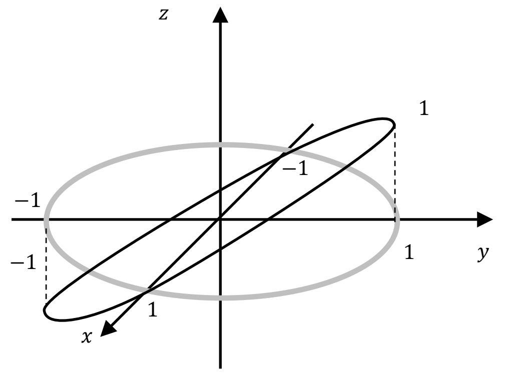
$$\begin{aligned}\overrightarrow{BX} &= \overrightarrow{OX} - \overrightarrow{OB}\\ &= \frac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3} - \underset{\sim}{b}\\ &= \frac{\underset{\sim}{a}+\underset{\sim}{c}-2\underset{\sim}{b}}{3}\\ \overrightarrow{BE} &= \overrightarrow{OE} - \overrightarrow{(OB)}\\ &= \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2} - \underset{\sim}{b}\\ &= \frac{\underset{\sim}{a}+\underset{\sim}{c}-2\underset{\sim}{b}}{2}\\ &= \frac{3}{2}\overrightarrow{BX}\end{aligned}$$

$\therefore B, E, X$ are collinear

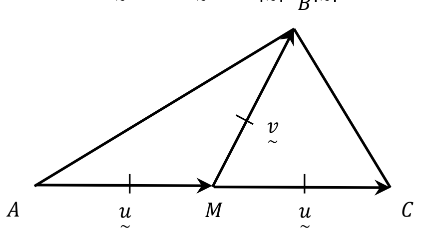
$\therefore$ the medians of a triangle are concurrent

87 As we move around the unit circle the elevation increases from $\sin 0 = 0$ to $\sin\frac{\pi}{2} = 1$, then descends to $\sin\pi = 0$ then $\sin\frac{3\pi}{2} = -1$, then climbs back to $\sin 2\pi = 0$.

The shape is an ellipse tilted $45°$ about the x-axis. By Pythagoras we could see that $a = \sqrt{2}$ and $b = 1$.



88 Let $\overrightarrow{AM} = \overrightarrow{MC} = \underset{\sim}{u}, \overrightarrow{MB} = \underset{\sim}{v}$ with $\left|\underset{\sim}{u}\right| = \left|\underset{\sim}{v}\right|$



$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AM} + \overrightarrow{MB} = \underset{\sim}{u} + \underset{\sim}{v}\\ \overrightarrow{BC} &= \overrightarrow{BM} + \overrightarrow{MC} = -\underset{\sim}{v} + \underset{\sim}{u} = \underset{\sim}{u} - \underset{\sim}{v}\\ \overrightarrow{AB}\cdot\overrightarrow{BC} &= \left(\underset{\sim}{u}+\underset{\sim}{v}\right)\left(\underset{\sim}{u}-\underset{\sim}{v}\right)\\ &= \underset{\sim}{u}\cdot\underset{\sim}{u} - \underset{\sim}{v}\cdot\underset{\sim}{v}\\ &= \left|\underset{\sim}{u}\right|^2 - \left|\underset{\sim}{v}\right|^2\\ &= 0\end{aligned}$$

$\therefore AB \perp BC$

$\therefore \Delta ABC$ is right angled at B

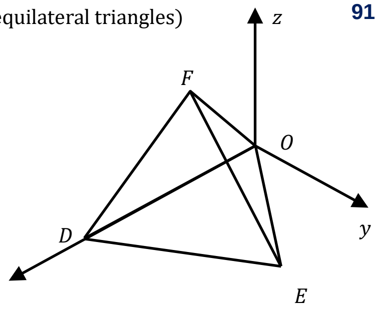
89 The radius of the new circle is $r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$.

For the point to move anticlockwise $x = \cos(f(t)), y = \sin(f(t))$.

To move the starting position to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, or $\left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right)$ we use $t + \frac{\pi}{4}$.

$\therefore x = \cos\left(t + \frac{\pi}{4}\right), y = \sin\left(t + \frac{\pi}{4}\right)$

90 $\angle FOD = \angle FOE = \frac{\pi}{3}$ (equilateral triangles)



In ΔFOD:

$$\cos\angle FOD = \frac{\overrightarrow{OF}\cdot\overrightarrow{OD}}{|\overrightarrow{OF}||\overrightarrow{OD}|}$$

$$\cos\frac{\pi}{3} = \frac{\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{1\times 1}$$

$$\frac{1}{2} = \frac{a+0+0}{1}$$

$$a = \frac{1}{2}$$

We have $E(\frac{1}{2},\frac{\sqrt{3}}{2},0)$ using exact triangles.

In ΔFOE:

$$\cos\angle FOE = \frac{\overrightarrow{OF}\cdot\overrightarrow{OE}}{|\overrightarrow{OF}||\overrightarrow{OE}|}$$

$$\cos\frac{\pi}{3} = \frac{\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\\0\end{pmatrix}}{1\times 1}$$

$$\frac{1}{2} = \frac{\frac{a}{2}+\frac{b\sqrt{3}}{2}+0}{1}$$

$$\frac{a}{2}+\frac{b\sqrt{3}}{2} = \frac{1}{2}$$

$$a + b\sqrt{3} = 1$$

$$\frac{1}{2} + b\sqrt{3} = 1$$

$$b\sqrt{3} = \frac{1}{2}$$

$$b = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

$$\sqrt{a^2+b^2+c^2} = 1$$

$$\therefore \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2 + c^2 = 1$$

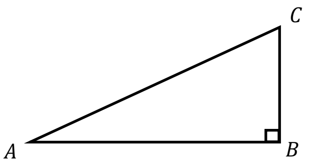
$$c^2 = 1 - \frac{1}{4} - \frac{1}{12}$$

$$c^2 = \frac{2}{3}$$

$$c = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$$

$$\therefore F\left(\frac{1}{2},\frac{\sqrt{3}}{6},\frac{\sqrt{6}}{3}\right)$$

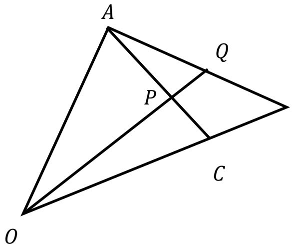
91 Let ΔABC be right angled at B.



$$\begin{aligned}AC^2 &= \left|\overrightarrow{AC}\right|^2\\ &= \left|\overrightarrow{AB}+\overrightarrow{BC}\right|^2\\ &= \left(\overrightarrow{AB}+\overrightarrow{BC}\right)\cdot\left(\overrightarrow{AB}+\overrightarrow{BC}\right)\\ &= \overrightarrow{AB}\cdot\overrightarrow{AB} + 2\left(\overrightarrow{AB}\cdot\overrightarrow{BC}\right) + \overrightarrow{BC}\cdot\overrightarrow{BC}\\ &= \overrightarrow{AB}\cdot\overrightarrow{AB} + \overrightarrow{BC}\cdot\overrightarrow{BC} \qquad \text{since } \overrightarrow{AB}\perp\overrightarrow{BC} \ \therefore \overrightarrow{AB}\cdot\overrightarrow{BC}\\ &= \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2\\ &= AB^2 + BC^2\end{aligned}$$

∴ The square on the hypotenuse of a right angled triangle equals the sum of the squares on the other two sides.

92



$\overrightarrow{OC} = \frac{3}{4}\overrightarrow{OB} = \frac{3}{4}\underset{\sim}{b}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \frac{3}{4}\underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \underset{\sim}{a} + \frac{2}{3}\overrightarrow{AC} = \underset{\sim}{a} + \frac{2}{3}\left(\frac{3}{4}\underset{\sim}{b} - \underset{\sim}{a}\right) = \frac{1}{3}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

$\overrightarrow{OQ} = \lambda\left(\overrightarrow{OP}\right) = \frac{\lambda}{3}\underset{\sim}{a} + \frac{\lambda}{2}\underset{\sim}{b}(1)$

$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \underset{\sim}{a} + \mu\overrightarrow{AB} = \underset{\sim}{a} + \mu\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

$= (1-\mu)\underset{\sim}{a} + \mu\underset{\sim}{b}$ (2)

From (1) and (2):

$\frac{\lambda}{3} = 1 - \mu \qquad \frac{\lambda}{2} = \mu \rightarrow \lambda = 2\mu$

$\frac{2\mu}{3} = 1 - \mu$

$\frac{5\mu}{3} = 1$

$\mu = \frac{3}{5}$

$AQ : QB = \frac{3}{5}\left(\underset{\sim}{b} - \underset{\sim}{a}\right) : \frac{2}{5}\left(\underset{\sim}{b} - \underset{\sim}{a}\right) = 3:2$

93 The midpoints are $M_{AB} = (0,2,1), M_{AC} = (3,2,0), M_{BC} = (3,4,1)$. The equations of the medians through each vertex are:

$$\underset{\sim}{r_A} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}3-0\\4-0\\1-0\end{pmatrix} = \begin{pmatrix}3\lambda\\4\lambda\\\lambda\end{pmatrix} \qquad (1)$$

$$\underset{\sim}{r_B} = \begin{pmatrix}0\\4\\2\end{pmatrix} + \lambda\begin{pmatrix}3-0\\2-4\\0-2\end{pmatrix} = \begin{pmatrix}3\lambda\\4-2\lambda\\2-2\lambda\end{pmatrix} \qquad (2)$$

$$\underset{\sim}{r_C} = \begin{pmatrix}6\\4\\0\end{pmatrix} + \lambda\begin{pmatrix}0-6\\2-4\\1-0\end{pmatrix} = \begin{pmatrix}6-6\lambda\\4-2\lambda\\\lambda\end{pmatrix} \qquad (3)$$

For (1) and (2):

$3\lambda = 3\mu \qquad 4\lambda = 4 - 2\mu \qquad \lambda = 2 - 2\mu$

$\lambda = \mu \qquad 3\lambda = 2 \ \rightarrow \lambda = \frac{2}{3}$

∴ (1) and (2) intersect at $\left(2,\frac{8}{3},\frac{2}{3}\right)$

sub $\lambda = \frac{2}{3}$ in (3)

$$\underset{\sim}{r_C} = \begin{pmatrix}6-6\times\frac{2}{3}\\4-2\times\frac{2}{3}\\\frac{2}{3}\end{pmatrix} = \begin{pmatrix}2\\\frac{8}{3}\\\frac{2}{3}\end{pmatrix}$$

∴ all three medians intersect at $\left(2,\frac{8}{3},\frac{2}{3}\right)$

94 $(100-10t)\underset{\sim}{i}+(50-5t)\underset{\sim}{j}+(10-t)\underset{\sim}{k}$

$= (9(t-T)+2)\underset{\sim}{i}+(4(t-T)+2)\underset{\sim}{j}+(t-T)\underset{\sim}{k}$

$10-t=t-T$

$2t=10+T$

$t=5+\frac{T}{2}\quad(1)$

$50-5t=4(t-T)+2$

$50-5t=4t-4T+2$

$4T=9t-48\quad(2)$

sub (1) in (2):

$4T=9\left(5+\frac{T}{2}\right)-48$

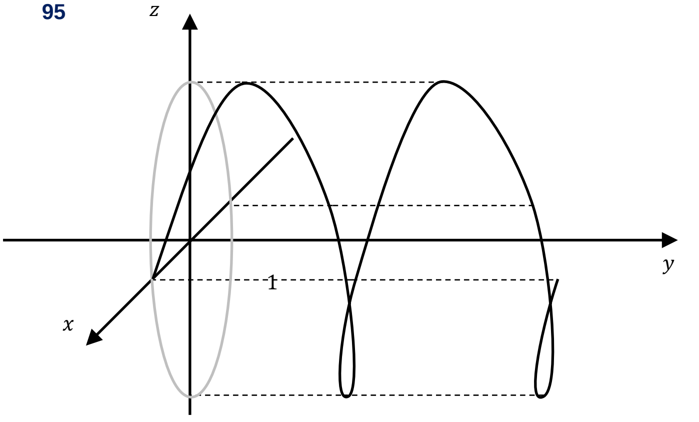
$4T=45+\frac{9T}{2}-48$

$\frac{T}{2}=3$

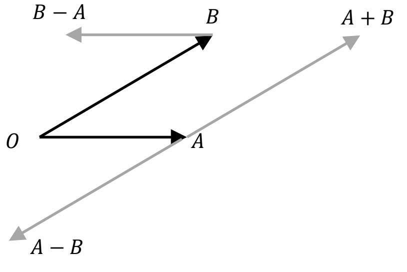
$T=6$

The rocket needs to be fired 6 seconds after the missile is first detected.

95



96 There are three possible vectors for $\overrightarrow{OD}$ that would create a parallelogram: $\overrightarrow{OA}+\overrightarrow{OB}, \overrightarrow{OA}-\overrightarrow{OB}$ and $\overrightarrow{OB}-\overrightarrow{OA}$.



$\therefore \overrightarrow{OD}=(2,2,1)+(1,2,2)=(3,4,3)$ or

$=(2,2,1)-(1,2,2)=(1,0,-1)$ or

$=(1,2,2)-(2,2,1)=(-1,0,1)$

The fourth vertex is at $(3,4,3),(1,0,-1)$ or $(-1,0,1)$.

97 The long diagonal of the cube is $\sqrt{(2a)^2+(4a)^2+(6a)^2}=2\sqrt{14}a$.

Each vertex is $\sqrt{14}a$ units from the centre of the cube.

The sphere is centred at the midpoint of the diagonal $(a,2a,3a)$ and has radius $\sqrt{14}a$.

The equation of the sphere is

$(x-a)^2+(y-2a)^2+(z-3a)^2=14a^2$

$x^2-2ax+a^2+y^2-4ay+4a^2+z^2-6az+9a^2=14a^2$

$x^2+y^2+z^2=2ax+4ay+6az$

98

$x^2+y^2+z^2-16=0$

$x^2+y^2+z^2=16$

Which is a sphere centred at the origin with radius 4

$x^2-8x+y^2-8y+z^2-14z+56=0$

$x^2-8x+16+y^2-8y+16+z^2-14z+49=25$

$(x-4)^2+(y-4)^2+(z-7)^2=25$

Which is a sphere centred at $(4,4,7)$ with radius 5

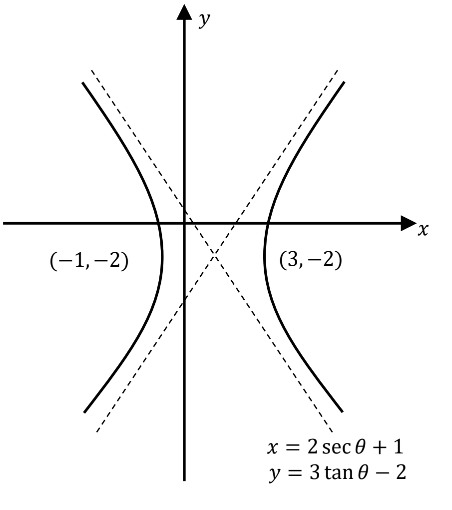
The distance between the centres of the spheres is $\sqrt{4^2+4^2+7^2}=9$ which matches the sum of the radii, so the spheres intersect at a point.

99 This is the hyperbola $\frac{x^2}{2^2}-\frac{y^2}{3^2}=1$ moved 1 unit to the right and 2 units down.

The asymptotes also move 1 to the right and down 2, so we get:

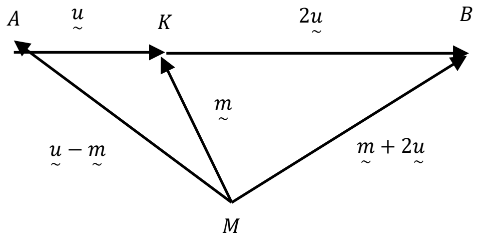
$$y+2=\pm\frac{3}{2}(x-1)$$

$$y=\frac{3}{2}x-\frac{7}{2}\text{ and }y=-\frac{3}{2}x-\frac{1}{2}$$



100

Let $\overrightarrow{AK} = \underset{\sim}{u}, \overrightarrow{KB} = 2\underset{\sim}{u}$ and $\overrightarrow{MK} = \underset{\sim}{m}$,so$\overrightarrow{MA} = \underset{\sim}{u} - \underset{\sim}{m}$, $\overrightarrow{MB} = \underset{\sim}{m} + 2\underset{\sim}{u}$.



$2\left|\overrightarrow{MA}\right|^2 + \left|\overrightarrow{MB}\right|^2 - 3\left|\overrightarrow{MK}\right|^2$
$= 2\left|\underset{\sim}{u} - \underset{\sim}{m}\right|^2 + \left|\underset{\sim}{m} + 2\underset{\sim}{u}\right|^2 - 3\left|\underset{\sim}{m}\right|^2$
$= 2\left(\underset{\sim}{u} - \underset{\sim}{m}\right)\cdot\left(\underset{\sim}{u} - \underset{\sim}{m}\right) + \left(\underset{\sim}{m} + 2\underset{\sim}{u}\right)\cdot\left(\underset{\sim}{m} + 2\underset{\sim}{u}\right) - 3\left|\underset{\sim}{m}\right|^2$
$= 2\left(\underset{\sim}{u}\cdot\underset{\sim}{u} - 2\,\underset{\sim}{u}\cdot\underset{\sim}{m} + \underset{\sim}{m}\cdot\underset{\sim}{m}\right) + \underset{\sim}{m}\cdot\underset{\sim}{m} + 4\underset{\sim}{m}\cdot\underset{\sim}{u} + 4\underset{\sim}{u}\cdot\underset{\sim}{u}$
$- 3\left|\underset{\sim}{m}\right|^2$
$= 2\left|\underset{\sim}{u}\right|^2 - 4\underset{\sim}{u}\cdot\underset{\sim}{m} + 2\left|\underset{\sim}{m}\right|^2 + \left|\underset{\sim}{m}\right|^2 + 4\underset{\sim}{u}\cdot\underset{\sim}{m} + 4\left|\underset{\sim}{u}\right|^2$
$- 3\left|\underset{\sim}{m}\right|^2$
$= 6\left|\underset{\sim}{u}\right|^2$
which is constant for any given A, K

BASIC

1 If the velocity v of a particle moving on the x-axis is given by $v^2 = -3x^2 + 20x + 7$, which of the following expresses its acceleration in terms of x?

A $\ddot{x} = -3\left(x - 3\frac{1}{3}\right)$ **B** $\ddot{x} = -3(x - 2)$

C $\ddot{x} = -3(x - 3)$ **D** $\ddot{x} = -2(x - 2)$

2 A particle moves in SHM about the centre of motion $x = 2$, with amplitude 3 and period π. Find a possible equation of motion.

3 A particle of unit mass is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $\mu(v^2 + v)$ Newtons, its speed is v ms^{-1} and μ is a positive constant. At time t seconds the particle has displacement x metres from a fixed point O on the line. If the particle has initial velocity v_0 at the origin, prove

$$x = \frac{1}{\mu}\ln\left(\frac{v_0 + 1}{v + 1}\right)$$

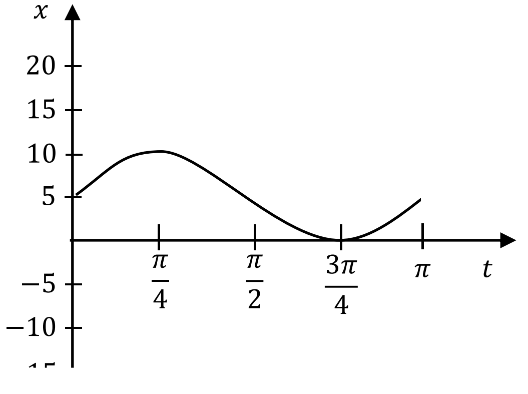
4 A stone is thrown from the top of a cliff. Its parametric equations of motion are $x = 3t$ and $y = 50 + 4t - 5t^2$. What is its Cartesian equation?

5 The velocity of a particle moving in a straight line is given by $v = 2x + 3$ where x metres is the distance from a fixed point O and v is the velocity in metres per second. What is the acceleration of the particle when it is 4 metres from O?

A $\ddot{x} = 11 \text{ ms}^{-2}$ **B** $\ddot{x} = 22 \text{ ms}^{-2}$

C $\ddot{x} = 19 \text{ ms}^{-2}$ **D** $\ddot{x} = 23.5 \text{ ms}^{-2}$

6 The graph below shows the displacement of a particle in SHM. Its equation of motion is given by $x = a\cos(nt + \alpha) + c$. Find the values of a, n, α and c.



7 A particle oscillates in simple harmonic motion between $x = -1$ and $x = 3$. Find the position of the particle when its acceleration is half its maximum acceleration.

8 A body is moving in a horizontal straight line. At time t seconds, its displacement is x metres from a fixed point O on the line, and its acceleration is $-0.2\sqrt{v}$ where $v \geq 0$ is its velocity. The body is initially at O with velocity $v = 9$. Show that $t = 30 - 10\sqrt{v}$

9 If a particle moves in a straight line so that its velocity at any particular time is given by $v = \sin^{-1} x$, then the acceleration is given by

A $-\cos^{-1} x$ **B** $\cos^{-1} x$

C $-\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$ **D** $\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

10 The displacement, in metres, of a particle in SHM at time t seconds, $t \geq 0$, is given by $x = 2\cos\left(\pi t + \frac{\pi}{4}\right) + 2$. How many oscillations does the particle make per second?

11 A rock is thrown at an angle of $20°$ to the horizontal, in a medium where air resistance is proportional to velocity squared. Let A, B and C be successive points on the trajectory in close proximity, where B is at the top of the trajectory, and A and C have equal height. Let the horizontal velocity at the points be $\dot{x}_a$, $\dot{x}_b$ and $\dot{x}_c$ respectively. Which statement lists the correct order of the magnitudes of the horizontal velocities?

A $\dot{x}_a < \dot{x}_b < \dot{x}_c$ **B** $\dot{x}_c < \dot{x}_b < \dot{x}_a$

C $\dot{x}_b < \dot{x}_c < \dot{x}_a$ **D** $\dot{x}_b < \dot{x}_a < \dot{x}_c$

12 A ball is thrown up in the air and follows a parabolic path. Ignoring air resistance, at any point on the upward part of the trajectory the vector representing acceleration on the particle would point:

A towards the point of projection (approx.) **B** straight down

C towards the point of impact (approx.) **D** towards the top of the trajectory (appx.)

13 A particle of unit mass is moving in a straight line under the action of a force, $F = \lambda(x+2)$, where λ is a positive constant. Given the particle starts from rest at the origin, prove $v = \sqrt{\lambda(x^2+4x)}$.

14 A particle of unit mass is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $2(v+v^2)$ Newtons, where its speed is v ms^{-1}. At time t seconds the particle has displacement x metres from a fixed point O on the line. Given it is initially at the origin with velocity 2 m/s, which of the following is an expression for x in terms of v?

A $x = -\dfrac{1}{2}\displaystyle\int_v^2 \dfrac{1}{1+v}\,dv$ **B** $x = \dfrac{1}{2}\displaystyle\int_2^v \dfrac{1}{1+v}\,dv$

C $x = \dfrac{1}{2}\displaystyle\int_v^2 \dfrac{1}{1+v}\,dv$ **D** $x = \dfrac{1}{2}\displaystyle\int_v^2 \dfrac{1}{1-v}\,dv$

15 Two rocks are thrown into the air with the same speed, with Rock A having a higher angle of projection than Rock B. Which statement is always true?

A They will have the same range, if air resistance is negligible, and the angles of projection are complementary.

B They will have the same time of flight, if air resistance is negligible, and the angles of projection are complementary.

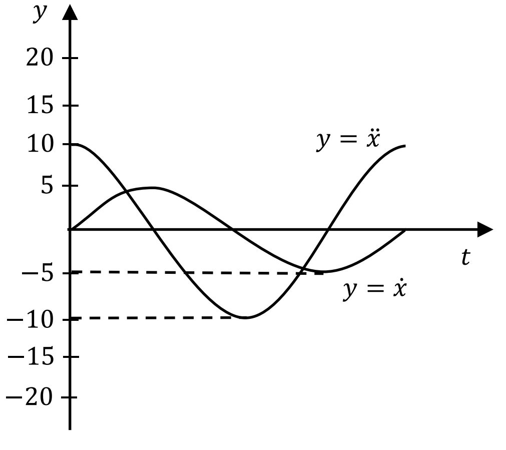
C They will have the same range, if air resistance is linear, and the angles of projection are complementary.

D They will have the same time of flight, if air resistance is linear, and the angles of projection are complementary.

16 Two particles oscillate horizontally. The displacement of the first is given by $x = 12\cos 2t$ and the displacement of the second is given by $x = a\sin nt$. In one oscillation, the second particle covers one quarter the distance of the first particle, in half the time. What are the values of a and n?

17 A 30 kilogram box sits on a slippery ramp which is inclined at an angle of $30°$ to the horizontal. If the box starts from rest, find as functions of g and t:
i its velocity after t seconds.
ii its displacement after t seconds

18 The graph below shows part of the graph of the velocity and acceleration of a particle in SHM, with the horizontal scale missing.



i Find the value of n
ii Sketch the displacement of the particle onto the graph.

19 A particle of mass 10 kg is projected along a horizontal path by a force of 100 newtons, and resisted by friction of $\frac{v^2}{4}$. What is the terminal velocity of the particle?

20 A golfer hits a ball at a velocity of $50\ \mathrm{ms^{-1}}$ and the ball hits the top of a 1.5 metre high flag which is $200\ \mathrm{m}$ away. Find the two possible angles at which the ball could have been hit, to the nearest degree. Assume there is no air resistance and that $g = 10\ \mathrm{ms^{-2}}$.

The equation of motion is

$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$$

21 A particle moves in a straight line along the x-axis so that its acceleration is given by $\ddot{x} = x + 3$ where x is the displacement from the origin. Initially the particle is at the origin and has velocity $v = 3$.

i Show that $v = x + 3$

ii Find x as a function of t.

22 A particle moves around the unit circle, with parametric equations $x = \cos t\,, y = \sin t$. Prove that the x component is in simple harmonic motion.

23 A particle moves with equation of motion $x = 2\sin t - 2\cos t + 1$ metres. Prove that the particle is in SHM, and find the centre and amplitude of its motion.

24 A particle moves in a straight line. At time t seconds where $t \geq 0$, its displacement x metres from the origin and its velocity v metres per second are such that $v = 25 + x^2$. If $x = 5$ initially, then t is equal to:

A $25x + \dfrac{x^3}{3}$

B $25x + \dfrac{x^3}{3} + \dfrac{500}{3}$

C $\tan^{-1}\left(\dfrac{x}{5}\right) - \dfrac{\pi}{4}$

D $\dfrac{1}{5}\tan^{-1}\left(\dfrac{x}{5}\right) - \dfrac{\pi}{20}$

25 A particle is moving in simple harmonic motion. Which of the following is true?

A the speed is zero at the centre of motion

B the speed is a maximum at the centre of motion

C the acceleration is zero at the extremities of motion

D the acceleration is a maximum at the centre of motion

26 A particle is projected with velocity 20 ms^{-1} at an angle of $50°$ to the horizontal in a resistive medium. It lands 29.8 m away from its projection point. Which of the following statements cannot be true?

A The maximum height occurs when the particle has travelled 14.9 m horizontally

B The velocity at impact is 18 ms^{-1}.

C The angle to the horizontal at impact is $58°$

D It reached a maximum height of 10.3 m

27 A rock is thrown horizontally from the top of a cliff with speed v and lands in the water a distance d from the base of the cliff. How far from the base of the cliff would the rock have landed if it was initially thrown horizontally with speed $2v$? Ignore air resistance.

A d

B $\sqrt{2}d$

C $2d$

D $4d$

28 A particle of unit mass is moving in a straight line under the action of a force.

$$F = \frac{5 - 7x}{x^3}$$

Which of the following equations is the representation of its velocity, if the particle starts from rest at $x = 1$?

A $v = \pm\frac{3}{x}\sqrt{x^2 - 7x + 5}$

B $v = \pm\frac{1}{x}\sqrt{-9x^2 + 14x - 5}$

C $v = \pm 3x\sqrt{x^2 - 7x + 5}$

D $v = \pm x\sqrt{9x^2 + 14x - 5}$

29 A particle of unit mass is projected along a horizontal path with velocity 10 m/s, with a force F newtons acting with its motion and a friction of $\frac{v}{5}$ newtons acting against its motion. If the velocity stays constant, find the value of F.

30 A particle of unit mass is projected in a medium where air resistance is proportional to velocity, at 20 ms^{-1} at an angle of $\frac{\pi}{6}$ to the horizontal. The vertical equation of motion of is $\dot{y} = 60e^{-0.2t} - 50$, where $\dot{y}$ is in metres per second. Find the time taken to reach maximum height, to 1 decimal place.

31 A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} given by $v = \frac{2}{3\sqrt{x}}$ and acceleration a ms^{-2}. Initially the particle is 1 metre to the right of O.

i Show that $a = -\frac{2}{9x^2}$

ii Show that $x = (t + 1)^{\frac{2}{3}}$

32 A particle is moving in simple harmonic motion according to the equation

$$x = 2 - 3\cos\left(2t + \frac{\pi}{3}\right)$$

In which interval does the particle oscillate?

A $[-3,3]$

B $\left[\frac{1}{2}, 3\frac{1}{2}\right]$

C $[-1,5]$

D $[1,5]$

33 Two particles travel horizontally through a resistive medium. Particle A experiences linear drag, $\ddot{x}_A = F - kv$, while Particle B experiences quadratic drag, $\ddot{x}_B = F - kv^2$. The particles are of such shape and mass that F and k are equal for both particles. If both particles are moving at velocity v, where $v > 0$, for what values of v will Particle A have greater acceleration than Particle B?

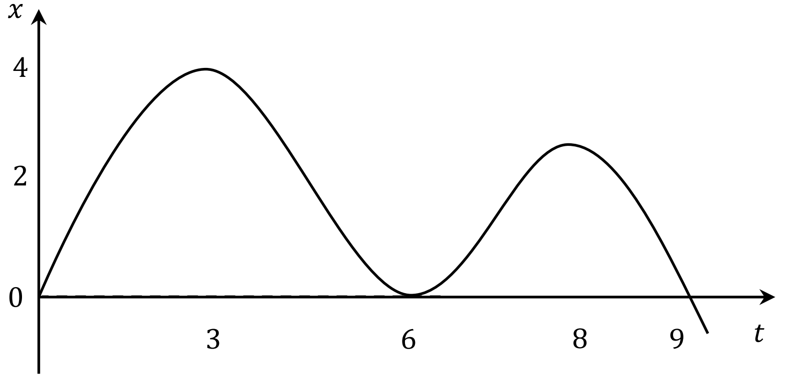
34 Given the parametric equations

$$x = Vt\cos\theta \text{ and } y = -\frac{gt^2}{2} + Vt\sin\theta$$

prove the Cartesian equation of motion of a projectile fired from the Origin is

$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$$

35 The graph below shows the displacement of a particle moving horizontally along the x-axis over time.



i When does the resultant force on the particle equal zero?

ii When is the force directed to the right?

For the rest of the question assume that the displacement function is a polynomial of degree 4.

iii What are the degrees of the functions of velocity and acceleration?

iv How many times is the particle at the origin?

v How many times is the particle at rest?

36 A particle is moving in simple harmonic motion according to the equation

$$x = 2\cos\left(3t + \frac{\pi}{6}\right)$$

What is the maximum speed of the particle?

A 2 m/s

B 6 m/s

C 0 m/s

D $\frac{\pi}{9}$ m/s

37 A particle is moving in SHM with $v^2 = 48 - 16x^2 - 32x$.

i Find an expression for the acceleration of the particle in terms of x, in the form $\ddot{x} = -n^2(x - c)$.

ii Find the centre of motion and period

38 The acceleration of a particle is given by $\ddot{x} = 2v^2 + 4v$. If initially the particle has a velocity of $4\ \text{ms}^{-1}$ at the origin find an expression for the velocity in terms of displacement.

39 A particle is oscillating between A and B, 10 m apart, in Simple Harmonic Motion. The time for a particle to travel from B to A and back is 5 seconds. Find the velocity and acceleration at M, the midpoint of OB where O is the centre of AB where M is to the right of O.

40 A particle is moving with equation of motion $v^2 + 16x^2 = 4$. Show that the particle is in SHM with period $\frac{\pi}{2}$.

41 A particle is projected at an angle to the horizontal in a medium where resistance is proportional to velocity. Its equations of motion are $\dot{x} = 16\sqrt{2}e^{-2t}$ and $\dot{y} = 16\left(1+\sqrt{2}\right)e^{-2t} - 5$. Find the angle, to the nearest minute, of its motion to the horizontal after $t = \ln 2$ seconds.

42 The acceleration of a particle moving on the x-axis is given by $\ddot{x} = x - 2$ where x is the displacement from the origin O after t seconds. Initially, the particle is at rest at $x = 3$.

i Show that its velocity at any position x is $v^2 = (x-1)(x-3)$

ii Find its acceleration when its velocity is $2\sqrt{6}\ \text{ms}^{-1}$.

43 A particle moves in SHM with $x = \sin 2t + 1$. Sketch displacement and acceleration on the same axes.

44 A particle is moving with equation of motion $x = a\sin(nt + \alpha)$. Prove that $v^2 + n^2x^2$ is constant.

45 A particle which starts at the origin with velocity $v = 2\text{ms}^{-1}$, has its acceleration described as $\frac{1}{1+9x^2}\ \text{ms}^{-2}$. Find an expression for v^2 as a function of x.

46 A particle starts from rest, 2 metres to the right of the origin, and moves along the x-axis in simple harmonic motion with a period of 2 seconds. Which equation could represent the motion of the particle?

A $x = 2\cos \pi t$

B $x = 2\cos 2t$

C $x = 2 + 2\sin \pi t$

D $x = 2 + 2\sin 2t$

47 A stone is thrown from the top of a $100\ \text{m}$ high cliff and lands in the sea $40\ \text{m}$ from the base. If the stone was thrown at a velocity of $30\ \text{ms}^{-1}$ what are the possible angles of projection, to the nearest degree? Assume $g = -9.8\ \text{ms}^{-2}$ and that air resistance is negligible.

48 A particle moves horizontally, propelled by a force of 2000 N, experiencing quadratic drag, with a terminal velocity of $50\ \mathrm{m/s}$. If the car has a mass of 1500 kg and is dropped from a helicopter, what terminal velocity could it approach if it falls nose first and experiences the same drag? Assume $g = 10\ \mathrm{ms}^{-2}$.

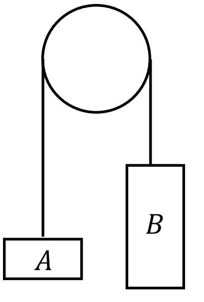
49 A particle moves in a straight line so that its velocity at any particular time is given by $v = k(a - x)$, where x is its displacement from a given point O. The particle is initially at O. Which of the following gives an expression for x:

A $x = a(1 - e^{kt})$ **B** $x = a(1 + e^{kt})$

C $x = a(1 - e^{-kt})$ **D** $x = a(1 + e^{-kt})$

50 Particle A of mass $2m$ kg and Particle B of mass $3m$ kg are connected by a light inextensible string passing over a frictionless pulley. Initially the particles are at rest. After Particle A has travelled x metres in an upwards direction it is travelling at v metres per second.

Prove $v = \sqrt{\dfrac{2gx}{5}}$



51 A ball is kicked on level ground to clear a fence 3 metres high and 20 metres away. The initial velocity is 20 metres per second and the angle of projection is α. The displacement equations are $x = 20t\cos\alpha$ and $y = -5t^2 + 20t\sin\alpha$. (Do NOT prove these.)

i Show that $y = -\dfrac{x^2}{80}\sec^2\alpha + x\tan\alpha$

ii Hence, or otherwise, find the angles of projection that allow the ball to clear the fence. Answer to the nearest degree.

52 The velocity $v\ \mathrm{ms}^{-1}$ of a particle is given by $v = 1 + e^{-x}$. Initially the particle is at the origin and its velocity is $2\ \mathrm{ms}^{-1}$. Find the time taken by the particle to reach a velocity of $1.5\ \mathrm{ms}^{-1}$.

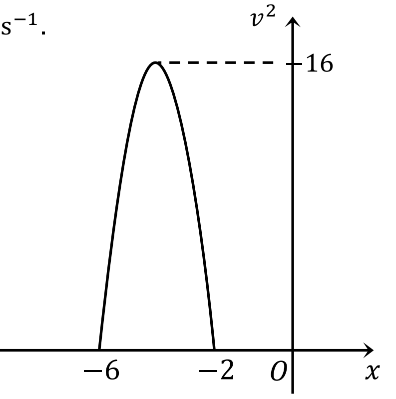
53 A particle of unit mass is moving in a straight line under the action of a force, $F = \frac{5+x}{x^2}$. Find an expression for velocity as a function of its displacement x, if the particle starts from rest at $x = 2$.

54 A particle is moving along the x-axis in simple harmonic motion.
The displacement of the particle is x metres and its velocity is v ms^{-1}.
The parabola at right shows v^2 as a function of x.

i For what value(s) of x is the particle at rest?

ii What is the maximum speed of the particle?

iii The velocity of the particle is given by the equation $v^2 = n^2(a^2 - (x - c)^2)$, where a, c and n are positive constants.
What are the values of a, c and n?



55 A particle of mass 2 kg is projected vertically upwards with a velocity of U ms^{-1} in a medium which exerts a resistive force of $\frac{v}{10}$ Newtons. Assume $g = 10$ ms^{-2} and ignore air resistance.

i Show that the maximum height H metres reached by the particle is given by:

$$H = 20U + 4000 \ln\left(\frac{200}{200 + U}\right)$$

ii Find the time taken for the particle to reach the maximum height H.

iii If $U = 400$, show that the average speed during the ascent is:

$$200\left(\frac{2}{\ln 3} - 1\right) \text{ ms}^{-1}$$

56 A particle moving in a straight line has acceleration given by $\ddot{x} = x^2$ where its displacement is x metres from the origin. If initially the particle is at rest 2 metres to the right of the origin, find its velocity when it is 4 metres from the origin.

57 The rise and fall of a tide approximates simple harmonic motion. In a harbour, low tide is at 7 am and high tide is at 1:40 pm. The corresponding depths are 20 m and 40 m. Find the first time after 7 am that a ship which requires $(5\sqrt{3} + 30)$ metres of water is able to enter the harbour.

58 A particle is moving in SHM between the points $x = 0$ and $x = 4$ with period π, and initially the particle is at rest at the origin. Derive the equation for $v^2 = 2^2(2^2 - (x - 2)^2)$.

59

i When considering motion in a straight line, prove that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$

ii An object moving in a straight line has an acceleration given by $\ddot{x} = x^2(4 - x^{-3})$ where x is the displacement in metres. When the object is 1 metre to the right of the origin, it has a speed of 3 m/s. Find its speed to 2 decimal places when it is 5 metres to the right of the origin.

60 A particle moves in SHM centred about the origin. When $x = -3$ the particle is at rest. When $x = 1$ the velocity of the particle is 2. Given the equation of motion is $x = a\sin(nt)$ find the values of a and n.

61 A parachutist of mass 80 kg jumps from a plane carrying a sandbag of mass 20 kg. They approach a terminal velocity of 60 m/s, then drop the sandbag. If everything else remains equal, describe what then happens to their velocity. Assume $g = 10$ ms^{-2}.

62 A 10 kg trolley is pushed with a force of 200 N. Friction causes a resistive force which is proportional to the trolley's velocity.

i Show that $\ddot{x} = 20 - \frac{kv}{10}$ where k is a positive constant.

ii If the trolley is initially stationary at the origin, show that the distance travelled when its speed is V is given by

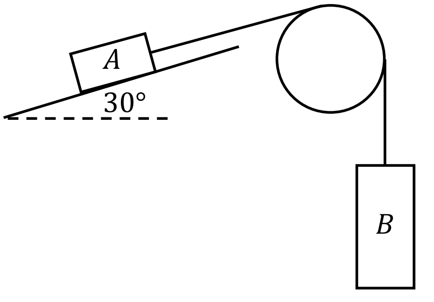
$$x = \frac{2000}{k^2}\ln\left(\frac{200}{200 - kV}\right) - \frac{10V}{k}$$

63 Two particles moving in a straight line are initially at the origin. The velocity of one particle is $\frac{2}{\pi}$ ms^{-1} and the velocity of the other particle at time t seconds is given by $v = -2\cos t$ ms^{-1}.

i Determine equations that give the displacements, x_1 and x_2 metres, of the particles from the origin at time t seconds.

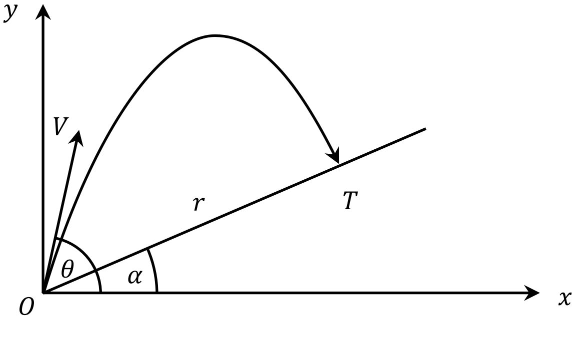
ii Show that the particles will never meet again.

64 Particle A and Particle B, both of mass m kg, are connected by a light inextensible string passing over a frictionless pulley. Particle A is on a frictionless surface inclined at $30°$ to the horizontal. Initially the particles are at rest. After Particle A has travelled x metres towards the pulley it is travelling at v metres per second.



Find an expression for x as a function of t.

65 The diagram shows an inclined road which makes an angle of α with the horizontal.



A projectile is fired from O, at the bottom of the inclined road, with a speed of V m/s at an angle of elevation θ to the horizontal as shown above. Using the axes above, you may assume that the position of the projectile is given by

$$x = Vt\cos\theta \text{ and } y = Vt\sin\theta - \frac{1}{2}gt^2$$

where t is the time, in seconds, after firing, and g is the acceleration due to gravity. For simplicity assume that $\frac{2V^2}{g} = 1$. Air resistance is negligible.

i Show that the path of the trajectory of the projectile is $y = x\tan\theta - x^2\sec^2\theta$

ii Show that the range of the projectile $r = OT$ metres, up the inclined road is given by

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2\alpha}$$

66 Prove

$$\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

67 A particle moves in a straight line and its position at a time t is given by $x = a\cos(9t + \theta)$. The particle is initially at the origin moving with a velocity of 15 m/s in the negative direction.

i Show that the particle is undergoing simple harmonic motion

ii Find the period of motion

iii Find the value of the constants a and θ

iv Find the position of the particle after 6 seconds, correct to two decimal places.

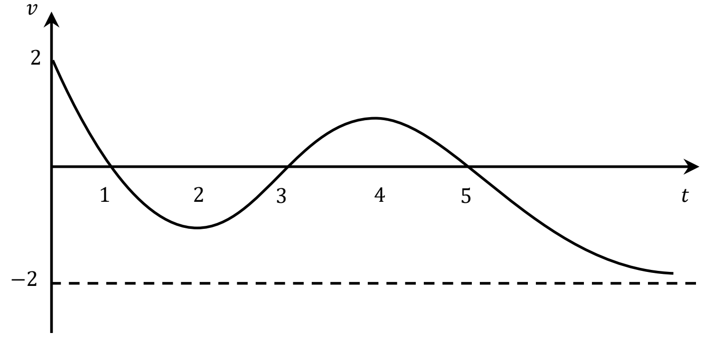
68 A particle is moving in a straight line according to the equation $x = 13 + 12\cos 2t + 5\sin 2t$, where x is the displacement in metres and t is the time in seconds.

i Prove that the particle is moving in simple harmonic motion by showing that x satisfies an equation of the form $\ddot{x} = -n^2(x - c)$.

ii When is the displacement of the particle zero for the first time? Answer to one decimal place.

69 Two particles are projected in a medium where air resistance is proportional to the square of velocity. Particle A is projected vertically, while Particle B is at an angle to the horizontal. They have equal initial vertical velocities. Compare the time and vertical displacement to the top of their trajectory for both particles.

70 The graph below shows the velocity of a particle moving horizontally along the x-axis over time.



i What is the initial velocity of the particle, and is the acceleration positive or negative?

ii When is the particle furthest to the right?

iii If the particle was initially at the origin, what would the graph of the displacement of the particle look like as $t \to \infty$?

71 Prove that a particle moving with equation of motion $x = \sin(at) + \cos(bt)$, cannot be moving in simple harmonic motion, assuming that a and b are unequal positive numbers.

72 A particle moves in Simple Harmonic Motion, the period being 2 seconds and the amplitude 3 metres. Find the maximum speed and the maximum acceleration during the motion.

73 The acceleration of a particle P is given by $\ddot{x} = 8x(x^2 + 4)$ where x is the displacement of P from a fixed point O after t seconds. Initially the particle is at O and has a velocity of 8 m/s in the positive direction.

i Show that the speed of the particle is given by $2(x^2 + 4)$ m/s.

ii Explain why the velocity of the particle is always positive.

iii Hence find the time taken for the particle to travel 2 metres from O.

74 A body of unit mass is projected vertically upwards in a medium that has a constant gravitational force g and a resistance $\frac{v}{10}$, where v is the velocity of the projectile at a given time t. The initial velocity is $10(20 - g)$.

i Show that the time T for the particle to reach its greatest height is given by $T = 10\ln\left(\frac{20}{g}\right)$.

ii Show that the maximum height H is given by $H = 2000 - 10g[10 + T]$

iii If the particle then falls from this height, find the terminal velocity in this medium.

75 A particle of unit mass is set in motion, with speed u ms^{-1}and moves in a straight line before coming to rest. At time t seconds the particle has displacement x metres from its starting point O, velocity v ms^{-1} and acceleration a ms^{-2}.

The resultant force acting on the particle directly opposes its motion and has magnitude $(1 + v)$ Newtons.

i Find expressions for

$\boldsymbol{\alpha}$ x in terms of v

$\boldsymbol{\beta}$ v in terms of t

$\boldsymbol{\gamma}$ x in terms of t

ii Show that $x + v + t = u$

iii Find the distance travelled and time taken by the particle in coming to rest.

76 A 100gm bullet is fired vertically into the air from the ground with an initial velocity of 1200 m/s. The bullet is subject to air resistance of $\frac{v}{10g}$ newtons in the opposite direction to the motion. The bullet is also subject to a downwards gravitational force of $\frac{g}{10}$ newtons. Assume $g = 10\ \mathrm{m/s^2}$.

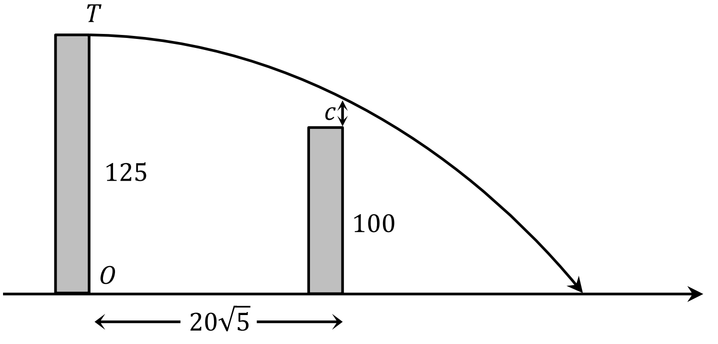
i Prove $x = 12000 + 1000\ln\left(\frac{v+100}{1300}\right) - 10v$.

ii Find the time taken to reach maximum height (to 2 dp)

iii Find the maximum height, to the nearest metre.

iv After reaching maximum height the bullet returns to the ground. Find the terminal velocity.

77 A projectile is thrown horizontally from the top of a 125 m tower with velocity V metres per second. It clears a second tower of height 100 m by a distance of c metres, as shown. The two towers are $20\sqrt{5}$ metres apart. Air resistance is negligible.



The equations of motion for this system are $x = Vt$ and $y = -5t^2 + 125$. Do NOT prove this

i Show that $V = \dfrac{100}{\sqrt{25-c}}$

ii Prove that the minimum initial speed of the projectile to just clear the 100 m tower is 20 m/s.

iii If $V = 20$ m/s, find how far past the 100m tower will the projectile strike the ground.

iv Determine the vertical component of the velocity of the projectile when it strikes the ground.

78 A skydiver of mass 60 kg jumps out of a stationary balloon and starts falling freely to Earth. She experiences gravity of mg downwards and air resistance of $\frac{v^2}{6}$ upwards. Given that down is positive, and $x = t = 0$ at the balloon:

i Show that $\ddot{x} = g - \dfrac{v^2}{360}$ and find her terminal velocity, given that $g = 9.8$ ms^{-2}.

ii Show that $x = 180 \ln\left(\frac{360g}{360g - v^2}\right)$ and find its distance fallen when the skydiver reaches 50 ms^{-1}.

iii Find the time taken for the skydiver to reach this speed.

79 The paddle wheel of a river boat rotates once every twenty seconds. The wheel has a diameter of 4 metres, and is centred 1.5 above the water surface. Prove that each paddle is under water for $10 - \frac{20}{\pi}\sin^{-1}\left(\frac{3}{4}\right)$ seconds in each rotation of the paddle wheel, given that the height of each paddle is in SHM.

80 A particle Q of mass 0.2 kg is released from rest at a point 7.2 m above the surface of the liquid in a container. The particle Q falls through the air and into the liquid. There is no air resistance and there is no instantaneous change of speed as Q enters the liquid.

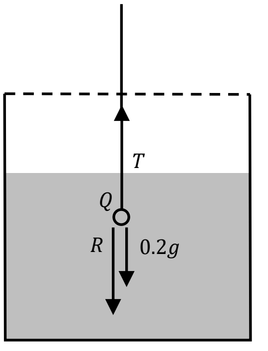
When Q is at a distance of 0.8 m below the surface of the liquid, Q's speed is 6 ms^{-1}. The only force on Q due to the liquid is a constant resistance to motion of magnitude R newtons. Take g, the acceleration due to gravity, to be 10 ms^{-2}.

i Show that prior to entering the liquid that $\frac{dv}{dx} = \frac{10}{v}$.

ii Hence find the speed as Q enters the liquid. Answer correct to 2 decimal places.

iii Find the value of R.

The depth of the liquid in the container is 3.6 m. Q is taken from the container and attached to one end of a light inextensible string. Q is placed at the bottom of the container and then pulled vertically upwards with constant acceleration. The resistance to motion of R newtons continues to act.



The diagram shows the forces acting on Q as it is being pulled out of the container.

The particle reaches the surface 4 seconds after leaving the bottom of the container.

iv By resolving the forces and finding an expression for $\frac{dv}{dt}$, find the tension in the string.

81 Use integration to prove for a particle in SHM about a point c with amplitude a that $v^2 = n^2(a^2 - (x-c)^2)$. You may assume $\ddot{x} = -n^2(x-c)$.

82 A particle of unit mass is dropped from rest in a medium where the resistance to the motion has magnitude $\frac{1}{40}v^2$ when the speed of the particle is v ms^{-1}. After t seconds the particle has fallen x metres. The acceleration due to gravity is 10 ms^{-2}.

i Explain why $\ddot{x} = \frac{1}{40}(400 - v^2)$.

ii Find an expression for t in terms of v.

iii Show that $v = 20\left(1 - \frac{2}{1+e^t}\right)$

iv Show that $x = \left[t + 2\ln\left(\frac{1+e^{-t}}{2}\right)\right]$

83 A particle with mass m is fired vertically upwards from the Earth's surface at U ms^{-1}. Ignoring air resistance, the particle is under the influence of gravity, which is inversely proportional to the square of the distance of the particle from the centre of Earth. At the Earth's surface the force of gravity acting on the particle is mg. If the Earth's radius is R:

i Show that $v^2 = \frac{2gR^2}{x} + U^2 - 2gR$

ii If $U^2 = gR$ show that the particle reaches a height of R above the Earth 's surface.

iii Also if $U^2 = gR$ show that the time taken to reach a height of R above the Earth's surface is given by

$$t = \int_R^{2R} \frac{1}{\sqrt{\frac{2gR^2}{x} - gR}}\, dx$$

and find this time in terms of R and g.

84 A particle P of unit mass is projected vertically upwards from the ground, with an initial velocity of u m/s, in a medium of resistance kv^2, where k is a positive constant and v is the velocity of the particle.

i Show that the maximum height H, from the ground, attained by the particle P is given by

$$H = \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right)$$

where g is the acceleration due to gravity.

ii At the same time that P is projected upwards, another particle of unit mass, Q, initially at rest, is allowed to fall downwards in the same medium, from a height of H metres from the ground, along the same vertical path as P. Show that at the time of collision of P and Q,

$$\frac{1}{v_2^2} - \frac{1}{v_1^2} = \frac{1}{V^2}$$

where v_1 and v_2 are the velocities of particles P and Q respectively, at the time of collision, and $V = \sqrt{\frac{g}{k}}$.

85 A rubber ball of mass 7 kg, falls from rest, from the top of a building. While falling the ball experiences a resistive force $\frac{7v^2}{10}$, where v is the velocity of the ball. Take g, acceleration due to gravity, as $g = 10$ ms^{-2}.

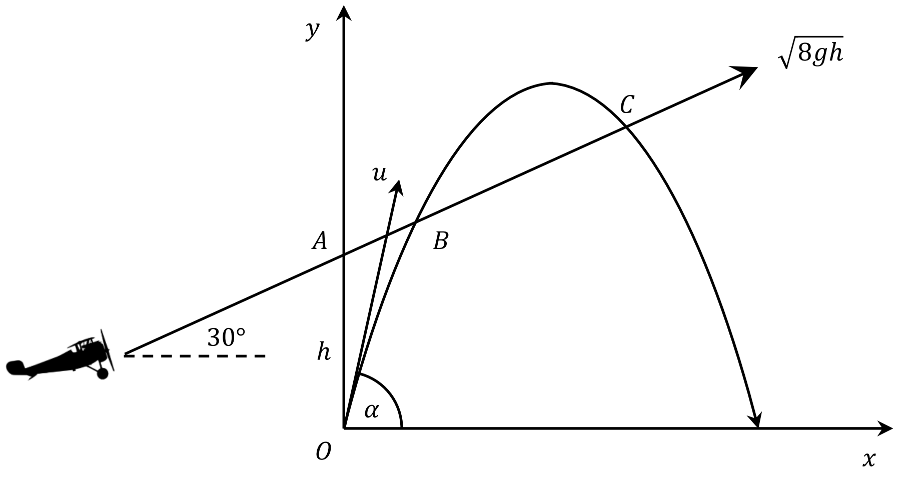
i Show that $\ddot{x} = 10 - \frac{v^2}{10}$, where x is the distance the ball has fallen.

ii Find the terminal velocity of the ball as it falls.

iii Show that $v^2 = 100\left(1 - e^{-\frac{x}{5}}\right)$

iv After hitting the ground the ball rises vertically such that $\ddot{X} = -10 - \frac{V^2}{10}$, where V is the velocity of the ball as it rises and X is the distance the ball rises. Find the time that it takes for the ball to rise to its maximum height if initially $V = \frac{10}{\sqrt{3}}$ ms^{-1}.

86 The diagram below shows a plane P which is flying at a constant speed of $\sqrt{8gh}$ m/s upwards at an angle of elevation of $30°$, passing through A, B and C. At the instant when the plane is at a height h metres vertically above a missile silo located at a point O on the ground, a missile from the silo is launched at an angle of elevation α to hit the plane where $0° < \alpha < 90°$. Air resistance is negligible.



With the axes shown in the diagram above, you may assume that the position of the missile is given by (DO NOT prove this)

$$x = ut\cos\alpha$$
$$y = ut\sin\alpha - \frac{1}{2}gt^2$$

where the launching speed of the missile is u m/s, t is the time in seconds after launch and g is the acceleration due to gravity.

i Show that the trajectory of the plane is given by

$$y = \frac{x}{\sqrt{3}} + h$$

ii Assuming that the missile can hit the plane, hence, from part (i), show that the x-coordinates of the points of collision must satisfy

$$\frac{x^2}{12} + \left(\frac{1}{\sqrt{3}} - \tan\alpha\right)hx + h^2 = 0$$

iii Suppose that $\tan\alpha > \frac{2}{\sqrt{3}}$

α Show that there are two possible points of collision B and C between the plane and the missile.

β Show that the time T (in seconds) elapsed between the two points of collision is given by

$$T = \sqrt{\frac{8h\tan\alpha}{g}\left(3\tan\alpha - 2\sqrt{3}\right)}$$

87 Prove that a particle where $x = a\cos^2(nt + \alpha) + c$ is in Simple Harmonic Motion.

88 A particle is fired vertically upwards with initial velocity V metres per second, and is subject both to constant gravity, and to air resistance proportional to speed, so that its equation of motion is: $\ddot{x} = -g - kv$, where $k > 0$ is a constant, and g is acceleration due to gravity.

Prove that the projectile reaches a maximum height H given by:

$$H = \frac{V}{k} - \frac{g}{k^2}\ln\left(1 + \frac{kV}{g}\right)$$

89 A particle of mass one kg is moving in a straight line. It is initially at the origin and is travelling with velocity 1 ms^{-1}. The particle is moving against a resisting force $v + v^3$, where v is the velocity.

i Show that the displacement x of the particle from the origin is given by

$$x = \tan^{-1}\left(\frac{1-v}{1+v}\right)$$

ii Show that the time t which has elapsed when the particle is travelling with velocity V is given by

$$t = \ln\sqrt{\frac{1+V^2}{2V^2}}$$

90 Consider a particle of unit mass falling through a fluid. The resistive frictional force on the particle is proportional to its velocity. That is, the resistance force may be written as $R = -kv$ where k is a constant and the particles velocity is v ms^{-1}.

i If the particle falls vertically from rest, show that the terminal velocity V_T is given by $V_T = \frac{g}{k}$, where g ms^{-2} is the acceleration due to gravity.

ii If the particle is projected upwards into the resistive fluid with speed V_T, show that after t seconds.

$\boldsymbol{\alpha}$ its speed v ms^{-1}is given by $v = V_T\left(2e^{-kt} - 1\right)$

$\boldsymbol{\beta}$ its height, x m is given by $x = \frac{V_T}{k}\left(2 - kt - 2e^{-kt}\right)$.

iii Hence, show that the greatest height that the particle can reach is

$$x_{\max} = \frac{V_T}{k}(1 - \ln 2)$$

91 A particle P is projected vertically upwards from the surface of the Earth with initial velocity u. The acceleration due to gravity at any point on its path is given by $-\frac{k}{x^2}$, where x is the distance of the particle from the centre of the Earth and k is a constant.

i Explain why $k = gR^2$, where R is the radius of the Earth and g is the acceleration due to gravity at the Earth's surface.

ii Neglecting air resistance, show that the velocity v of the particle is given by

$$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$$

iii If the initial velocity of the particle is given by $u = \sqrt{2gR}$, show that the time taken to reach a height $3R$ above the Earth's surface is given by $\frac{7\sqrt{2R}}{3\sqrt{g}}$.

92 A particle is projected from the origin with an initial velocity $20\sqrt{2}\text{ ms}^{-1}$ at $45°$ to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -\frac{\dot{x}}{20} \quad \text{and} \quad \ddot{y} = -\frac{\dot{y}}{20} - 10,$$

(You are NOT required to show these.)

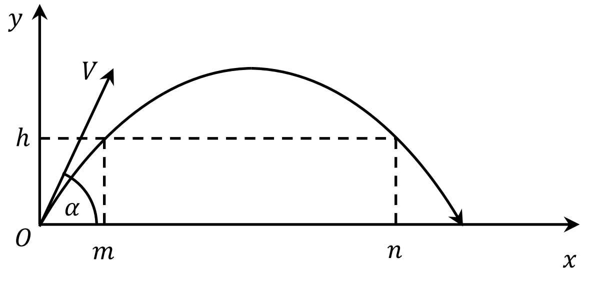
i Prove $x = 400\left(1 - e^{-\frac{t}{20}}\right)$.

ii Prove $y = 4400\left(1 - e^{-\frac{t}{20}}\right) - 200t$.

iii Prove $y = 11x + 4000\ln\left(\frac{400-x}{400}\right)$

iv After the particle has travelled 50 metres horizontally, prove that the angle of its trajectory is $23°12'$ below the horizontal.

93 An object is projected with velocity V ms^{-1} from a point O at an angle of elevation α. Axes x and y are taken horizontally and vertically through O. The object just clears two vertical chimneys of height h metres at horizontal distance of m metres and n metres from O. The acceleration due to gravity is taken as 10 ms^{-2} and air resistance is ignored.



i Show that the expression of the particle after time t seconds for the horizontal displacement is $x = Vt\cos\alpha$ and the vertical displacement is $y = Vt\sin\alpha - 5t^2$.

ii Show that

$$V^2 = \frac{5m^2(1 + \tan^2\alpha)}{m\tan\alpha - h}$$

iii Show that

$$\tan\alpha = \frac{h(m+n)}{mn}$$

94 A food parcel of mass 15 kg is dropped vertically from a helicopter which is hovering 2000 metres above a group of stranded bushwalkers. After 10 seconds a parachute of mass 5 kg opens automatically. Air resistance is neglected for the first 10 seconds but then the effect of the open parachute is to supply a resistance of $40v$ Newtons where V ms^{-1} is the velocity after t seconds ($t \geq 10$ seconds).

Take the position of the helicopter to be the origin, the downwards direction as positive and the value of g, the acceleration due to gravity, as 10 ms^{-2}.

i Use calculus to find the equations of motion in terms of t for the parcel before the parachute opens and prove that the velocity at the end of 10 seconds is 100 ms^{-1} and the distance fallen at the end of 10 seconds is 500 metres.

ii Show that the velocity of the parcel after the parachute opens is given by $v = 5 + 95e^{-2(t-10)}$ for $t \geq 10$.

iii Find x, the distance fallen as a function of t and calculate the height of the parcel above the bushwalkers 2 minutes after it leaves the helicopter.

iv Calculate the terminal velocity of the parcel.

95 The tide level in a harbour oscillates according to simple harmonic motion. At 5 am the tide is at its lowest level at 3 m and at 11 am the tide rises to its peak at 6 m.

i Calculate the amplitude and period of motion

ii The motion can be written in the form $x - b = a\cos(nt)$ where $a,\ b$ and n are constants, x represents the tide level in metres and t is the number of hours after 5am. Explain why $a = -1.5$, $b = 4.5$ and $n = \frac{\pi}{6}$.

iii Show that it satisfies the condition $\ddot{x} = -n^2(x - b)$

iv Between which two times (to the nearest minute) during the same day that a boat can enter then leave the harbour, if the hull requires a minimum depth of 4 m for safe passage?

96 In an aerobatics display, Cynthia (mass 50 kg) and Rebel (mass 60 kg) jump from a great height and go through a period of free fall before opening their parachutes. While in free fall at speed v ms^{-1} Cynthia experiences air resistance $100v$ newtons, but Rebel, who spread eagles, experiences air resistance $120v + \frac{480}{g}v^2$ newtons.

i Using a force diagram, show that Cynthia's terminal velocity is $\frac{g}{2}$ ms^{-1}.

ii Find Rebel's terminal velocity.

iii Cynthia opens her parachute when her speed is $\frac{g}{6}$ ms^{-1}. Find the time she has been in free fall.

97 A car, starting from rest, moves along a straight horizontal road. The car's engine produces a constant horizontal force of magnitude 5000 Newtons. At time t seconds, the speed of the car is v ms^{-1} and a resistance force of magnitude $10v^2$ Newtons acts upon the car. The mass of the car is 2000 kg.

i Show that $\frac{dv}{dt} = \frac{500-v^2}{200}$

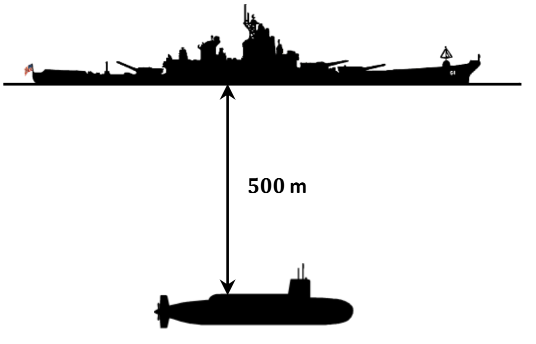
ii Prove

$$v = \sqrt{500}\left(\frac{e^{\frac{t}{2\sqrt{5}}} - 1}{e^{\frac{t}{2\sqrt{5}}} + 1}\right)$$

iii What is the maximum speed (terminal velocity) of the car? Answer correct to 2 decimal places.

iv How long does it take for the car to reach 99% of its terminal velocity? Leave in exact form.

98 A stationary submarine fires a missile of mass 40 kg with a speed of 500 ms^{-1} at a ship at rest 500 m above it.



The missile is subject to a downward gravitational force of 400 N and a water resistance of $\frac{3v^2}{100}$ N , where v is the velocity of the missile.

i Show that while the missile is rising, its displacement from the submarine is given by

$$x = \frac{2000}{3}\ln\left(\frac{790000}{40000 + 3v^2}\right)$$

ii Show that the velocity of the missile at the time of impact with the ship is approximately 333 ms^{-1}.

99 A body is projected vertically upwards from the surface of the Earth with initial speed u. The acceleration due to gravity at any point on its path is inversely proportional to the square of its distance from the centre of the Earth. The radius of the Earth is R.

i Prove that the speed at any position x is given by $v^2 = u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

ii Prove that the greatest height H above the Earth's surface is given by $H = \frac{u^2R}{2gR-u^2}$.

iii Show that the body will escape from the Earth if $u \geq \sqrt{2gR}$

100 A particle of mass m is projected from the origin with an initial velocity $V\ \mathrm{ms}^{-1}$ at an angle of θ to the horizontal. The particle experiences the effect of gravity and a resistance proportional to its velocity in both the horizontal and vertical directions.

Prove the following results, where k is the coefficient of drag and g is gravitational acceleration.

i $\dot{x} = V\cos\theta\, e^{-\frac{k}{m}t}$

ii $x = \frac{mV\cos\theta}{k}\left(1 - e^{-\frac{k}{m}t}\right)$

iii $\dot{y} = \left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$

iv $y = \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right)\left(1 - e^{-\frac{k}{m}t}\right) - \frac{mgt}{k}$

v $y = \left(\frac{mg}{kV\cos\theta} + \tan\theta\right)x + \frac{m^2g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)$

1

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}(-6x+20) = -3x+10$$

$$= -3\left(x - 3\frac{1}{3}\right)$$

ANSWER (A)

2

$c = 2, a = 3$

$T = \frac{2\pi}{n} = \pi \;\rightarrow\; n = 2$

One possible equation of motion is

$x = 3\sin(2t) + 2.$

Alternative solutions would swap the sine for cosine, and replace $2t$ with $2t + \alpha$ for any angle α

3

$$v\frac{dv}{dx} = -\mu(v^2+v)$$

$$\frac{dv}{dx} = -\mu(v+1)$$

$$\frac{dx}{dv} = -\frac{1}{\mu}\times\frac{1}{v+1}$$

$$x = -\frac{1}{\mu}\int_{v_0}^{v}\frac{1}{v+1}dv$$

$$= -\frac{1}{\mu}\Big[\ln(v+1)\Big]_{v_0}^{v}$$

$$= -\frac{1}{\mu}(\ln(v+1) - \ln(v_0+1))$$

$$= \frac{1}{\mu}\ln\left(\frac{v_0+1}{v+1}\right)$$

4

$x = 3t \qquad (1)$

$y = 50 + 4t - 5t^2 \qquad (2)$

$t = \frac{x}{3} \qquad$ from (1)

sub in (2):

$$y = 50 + 4\left(\frac{x}{3}\right) - 5\left(\frac{x}{3}\right)^2$$

$$= 50 + \frac{4x}{3} - \frac{5x^2}{9}$$

5

$$\ddot{x} = v\frac{dv}{dx} = (2x+3)(2) = 4x+6$$

Let $x = 4 \rightarrow \ddot{x} = 4(4) + 6 = 22$

ANSWER (B)

6

$T = \frac{2\pi}{n} = \pi \quad \rightarrow \quad n = 2$

$a = \frac{10-0}{2} = 5$

$c = \frac{10+0}{2} = 5$

The curve is a cosine curve shifted vertically by 5 and right by $\frac{\pi}{4}$, so:

$$x = 5\cos\left[2\left(t - \frac{\pi}{4}\right)\right] + 5 = 5\cos\left(2t - \frac{\pi}{2}\right) + 5$$

7

The maximum acceleration occurs at the left hand extremity of motion, so at $x = -1$. Since acceleration is proportional to the distance from the centre of motion, the acceleration will be half the maximum acceleration when the particle is halfway from the centre to $x = -1$, so at $x = 0$.

8

$$\frac{dv}{dt} = -0.2\sqrt{v}$$

$$\frac{dt}{dv} = -5v^{-\frac{1}{2}}$$

$$t = -5\int_9^v v^{-\frac{1}{2}}dv$$

$$= -10\Big[\sqrt{v}\Big]_9^v$$

$$= -10(\sqrt{v} - 3)$$

$$= 30 - 10\sqrt{v}$$

9

$$\ddot{x} = v\frac{dv}{dx}$$

$$= \sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$$

ANSWER (D)

10

$$f = \frac{1}{T} = \frac{n}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$$

11

The particle's horizontal velocity decreases as it moves. The height of the three points is irrelevant.

ANSWER (B)

12

The only acceleration is due to gravity, which points straight down.

ANSWER (B)

13

$$\ddot{x} = \lambda(x+2)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \lambda(x+2)$$

$$\frac{1}{2}v^2 = \lambda\int_0^x (x+2)\,dx$$

$$v^2 = 2\lambda\left(\frac{x^2}{2} + 2x\right)$$

$$= \lambda x^2 + 4\lambda x$$

$$v = \sqrt{\lambda(x^2+4x)} \quad v > 0 \text{ given initial conditions}$$

14

$$v\frac{dv}{dx} = -2(v+v^2)$$

$$\frac{dv}{dx} = -2(1+v)$$

$$\frac{dx}{dv} = -\frac{1}{2}\times\frac{1}{1+v}$$

$$x = -\frac{1}{2}\int_2^v \frac{1}{1+v}dv$$

$$= \frac{1}{2}\int_v^2 \frac{1}{1+v}dv$$

ANSWER (C)

15 If air resistance is negligible, then particles with equal velocity and complementary angles of projection will have the same range, but unequal times of flight. If air resistance is taken into account, the particle with the smaller angle of projection has a longer range but shorter time of flight.

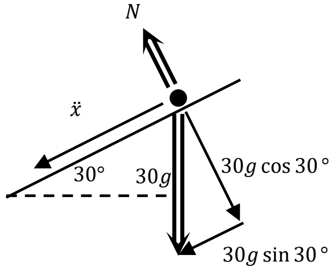
ANSWER (A)

16 One quarter the distance means that the amplitude is quartered. Half the time means that the angular velocity (n) is doubled.

$\therefore a_2 = \frac{1}{4}a_1 = \frac{1}{4}(12) = 3$

$T_2 = \frac{1}{2}T_1 = \frac{1}{2} \times \frac{2\pi}{2} = \frac{\pi}{2} \to \frac{2\pi}{n_2} = \frac{\pi}{2} \to n_2 = 4$

17 **i**



$\ddot{x} = 30g \sin 30°$

$\frac{dv}{dt} = 15g$

$v = 15g\int_0^t dt$

$= 15gt$

ii

$\therefore \frac{dx}{dt} = 15gt$

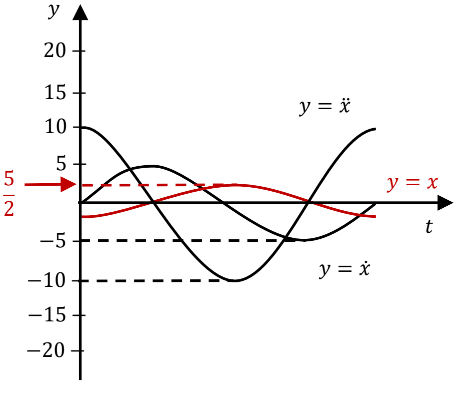
$x = 15g\int_0^t t\,dt$

$= \frac{15g}{2}\left[t^2\right]_0^t$

$= \frac{15gt^2}{2}$

18 **i** The amplitudes are 5 and 10 respectively, and the amplitude of $\ddot{x}$ is the amplitude of $\dot{x}$ multiplied by n, so $n = 2$

ii The graph of displacement is the reflection of the graph of acceleration over the t-axis, but with amplitude $\frac{1}{n}$ that of velocity, so $\frac{5}{2} = 2.5$.



19

$0 = 100 - \frac{{v_T}^2}{4}$

${v_T}^2 = 400$

$v_T = 20$ m/s.

Note that the mass is irrelevant.

20

$y = -\frac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$

$1.5 = -\frac{10 \times 200^2}{2 \times 50^2}(1 + \tan^2\theta) + 200\tan\theta$

$1.5 = -80(1 + \tan^2\theta) + 200\tan\theta$

$80\tan^2\theta - 200\tan\theta + 81.5 = 0$

$\tan\theta = \frac{200 \pm \sqrt{200^2 - 4(80)(81.5)}}{2(80)}$

$= 0.512605\ldots, \quad 1.987394$

$\theta = 27°, \quad 63°$

21 **i**

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x + 3$

$\frac{1}{2}v^2 = \int(x+3)\,dx$

$v^2 = 2\left(\frac{x^2}{2} + 3x\right) + c$

$= x^2 + 6x + c$

Let x=0, v=3

$3^3 = 0 + 0 + c$

$c = 9$

$v^2 = x^2 + 6x + 9$

$= (x+3)^2$

$v = x + 3 \quad v > 0$ given initial conditions

ii

$\therefore \frac{dx}{dt} = x + 3$

$\frac{dt}{dx} = \frac{1}{x+3}$

$t = \int_0^x \frac{1}{x+3}dx$

$= \left[\ln(x+3)\right]_0^x$ since x>0 given initial

$t = \ln(x+3) - \ln 3$

$\ln 3 + t = \ln(x+3)$

$3e^t = x + 3$

$x = 3e^t - 3$

22
$x = \cos t$
$\dot{x} = -\sin t$
$\ddot{x} = -\cos t$
$= -x$
$\therefore$ the x component is in SHM.

23
Let $\sqrt{2}\cos t - \sin t = R\sin(t-\alpha)$
$\therefore R = \sqrt{(2)^2+(2)^2} = 2\sqrt{2}$
$\alpha = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$
$\therefore x = 2\sqrt{2}\sin\left(t - \frac{\pi}{4}\right) + 1$
This is in the form $x = a\sin(nt+\alpha)+c$, so the particle is in SHM. The centre of motion is 1 and the amplitude is $2\sqrt{2}$.

24
$\frac{dx}{dt} = 25 + x^2$
$\frac{dt}{dx} = \frac{1}{25+x^2}$
$t = \int_5^x \frac{1}{25+x^2}\,dx$
$= \frac{1}{5}\left[\tan^{-1}\left(\frac{x}{5}\right)\right]_5^x$
$= \frac{1}{5}\left(\tan^{-1}\left(\frac{x}{5}\right) - \frac{\pi}{4}\right)$
$= \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) - \frac{\pi}{20}$
ANSWER (D)

25 **ANSWER (B)**

26 The maximum height occurs closer to the point of impact, so A is incorrect.
ANSWER (A)

27 Both times the rock will be in the air for the same time, so it will travel twice as far.
ANSWER (C)

28
$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 5x^{-3} - 7x^{-2}$
$\frac{1}{2}v^2 = \int_1^x (5x^{-3} - 7x^{-2})\,dx$
$= \left[-\frac{5}{2}x^{-2} + 7x^{-1}\right]_1^x$
$= \left(-\frac{5}{2}x^{-2} + 7x^{-1}\right) - \left(-\frac{5}{2} + 7\right)$
$v^2 = \frac{-9x^2 + 14x - 5}{x^2}$
$v = \pm\frac{1}{x}\sqrt{-9x^2 + 14x - 5}$
ANSWER (B)

29
$m\ddot{x} = F - \frac{v}{5}$
$1(0) = F - \frac{10}{5}$
$F = 2\text{ N}$

30
Let $\dot{y} = 0$
$\therefore 60e^{-0.2t} - 50 = 0$
$60e^{-0.2t} = 50$
$e^{-0.2t} = \frac{5}{6}$
$e^{0.2t} = \frac{6}{5}$
$0.2t = \ln\frac{6}{5}$
$t = 5\ln\frac{6}{5}$
$= 0.9$ seconds (1 dp)

31
i
$a = v\frac{dv}{dx}$
$= \frac{2}{3}x^{-\frac{1}{2}}\left(\frac{2}{3}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}\right)$
$= \frac{2}{3\sqrt{x}} \times \left(-\frac{1}{3\sqrt{x^3}}\right)$
$= -\frac{2}{9x^2}$

ii
$\frac{dx}{dt} = \frac{2}{3\sqrt{x}}$
$\frac{dt}{dx} = \frac{3}{2}x^{\frac{1}{2}}$
$t = \frac{3}{2}\int_1^x x^{\frac{1}{2}}\,dx$
$\frac{2t}{3} = \frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^x$
$t = x^{\frac{3}{2}} - 1$
$x^{\frac{3}{2}} = t + 1$
$x = (t+1)^{\frac{2}{3}}$

32 The centre of motion is 2 and the amplitude is 3, so $[2-3, 2+3] = [-1, 5]$.

33
$\ddot{x}_A > \ddot{x}_B$
$mg - kv > mg - kv^2$
$kv^2 > kv$
$v^2 - v > 0$
$v(v-1) > 0$
$v > 1 \quad (v > 0)$

34
$x = Vt\cos\theta \qquad (1)$
$y = -\frac{gt^2}{2} + Vt\sin\theta \qquad (2)$

From (1):
$t = \frac{x}{V\cos\theta}$
Substituting into (2):
$y = -\frac{g\left(\frac{x}{V\cos\theta}\right)^2}{2} + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$
$y = -\frac{gx^2}{2V^2\cos^2\theta} + x\tan\theta$
$y = -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta$
$y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$

35 i After approximately 4.5 and 7 seconds at the points of inflexion, since the curve is neither concave up or down so the net force is zero.

ii From approx $t = 4.5$ to approx $t = 7$ seconds, since the curve is concave up.

iii The velocity is of degree 3 and acceleration of degree 2, since they are the first and second derivative of displacement.

iv Only the three t-intercepts shown, as the particle will continue to move to the left.

v Only the three turning points shown, as the particle will continue moving to the left.

36 $\dot{x} = -6\cos\left(3t + \frac{\pi}{6}\right)$

Since $-1 \leq \cos(\theta) \leq 1$ the maximum speed is 6 m/s

ANSWER(B)

37 **i**

$$\begin{aligned}\ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\\ &= \frac{d}{dx}\left(\frac{1}{2}(48 - 16x^2 - 32x)\right)\\ &= -16x - 16\\ &= -16(x-1)\\ &= -4^2(x - (-1))\end{aligned}$$

ii

$c = -1, n = 4$ from (i)

$T = \frac{2\pi}{4} = \frac{\pi}{2}$

38

$$\begin{aligned}v\frac{dv}{dx} &= 2v^2 + 4v\\ \frac{dv}{dx} &= 2v + 4\\ \frac{dx}{dv} &= \frac{1}{2v+4}\\ x &= \int_4^v \frac{dv}{2v+4}\\ &= \frac{1}{2}\Big[\ln(2v+4)\Big]_4^v\\ &= \frac{1}{2}\left(\ln(2v+4) - \ln 12\right)\\ &= \frac{1}{2}\ln\frac{v+2}{6}\\ e^{2x} &= \frac{v+2}{6}\\ v &= 6e^{2x} - 2\end{aligned}$$

39 $a = \frac{10}{2} = 5, \frac{2\pi}{n} = 5 \rightarrow n = \frac{2\pi}{5}$, at M $x = \frac{10}{4} = 2.5$.

Let $x = 5\sin\left(\frac{2\pi t}{5}\right)$

$\dot{x} = 2\pi\cos\left(\frac{2\pi t}{5}\right)$

At M:

$$\begin{aligned}5\sin\left(\frac{2\pi t}{5}\right) &= 2.5\\ \sin\left(\frac{2\pi t}{5}\right) &= \frac{1}{2}\\ \frac{2\pi t}{5} &= \frac{\pi}{6}\\ t &= \frac{5}{12}\end{aligned}$$

$$\begin{aligned}\dot{x}_M &= 2\pi\cos\left(\frac{2\pi\left(\frac{5}{12}\right)}{5}\right)\\ &= 2\pi\cos\frac{\pi}{6}\\ &= 2\pi \times \frac{\sqrt{3}}{2}\\ &= \sqrt{3}\pi \text{ ms}^{-1}\\ \ddot{x}_M &= -n^2x\\ &= -\left(\frac{2\pi}{5}\right)^2 \times \frac{5}{2}\\ &= -\frac{2\pi^2}{5}\text{ ms}^{-2}\end{aligned}$$

40

$$\begin{aligned}v^2 &= 4 - 16x^2\\ \ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\\ &= \frac{1}{2}\times\frac{d}{dx}\left(4 - 16x^2\right)\\ &= \frac{1}{2}(-32x)\\ &= -16x\end{aligned}$$

∴ the particle is in SHM with $n = 4$, so the period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

41

$$\begin{aligned}\dot{x} &= 16\sqrt{2}e^{-2\ln 2}\\ &= \frac{16\sqrt{2}}{e^{\ln 4}}\\ &= 4\sqrt{2}\\ \dot{y} &= 16\left(1+\sqrt{2}\right)e^{-2\ln 2} - 5\\ &= \frac{16\left(1+\sqrt{2}\right)}{e^{\ln 4}} - 5\\ &= 4\sqrt{2} - 1\end{aligned}$$

$$\begin{aligned}\tan\theta &= \frac{\dot{y}}{\dot{x}}\\ \theta &= \tan^{-1}\left(\frac{4\sqrt{2}-1}{4\sqrt{2}}\right)\\ &= 39°28'\end{aligned}$$

The particle is moving upwards at an angle of $39°28'$ to the horizontal.

42 **i**

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= x - 2\\ \frac{1}{2}v^2 &= \left[\frac{x^2}{2} - 2x\right]_3^x\\ v^2 &= 2\left(\left(\frac{x^2}{2} - 2x\right) - \left(\frac{(3)^2}{2} - 2(3)\right)\right)\\ &= x^2 - 4x + 3\\ &= (x-1)(x-3)\end{aligned}$$

ii

Given initial conditions $v > 0$ for all x

Let $v = 2\sqrt{6} \rightarrow v^2 = 24$

$$\begin{aligned}24 &= x^2 - 4x + 3\\ x^2 - 4x - 21 &= 0\\ (x-7)(x+3) &= 0\\ x &= 7 \quad x > \text{ given initial conditions}\end{aligned}$$

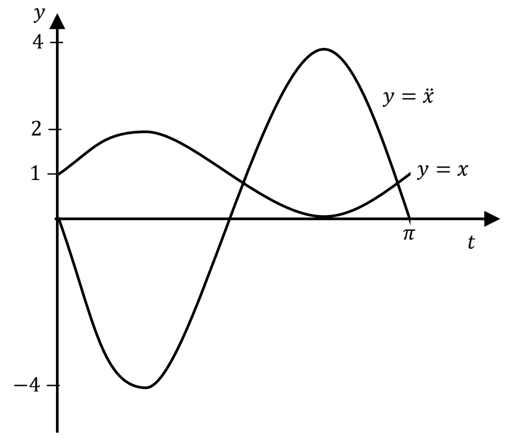
$\ddot{x} = x - 2$

Let $x = 7$

$\ddot{x} = 7 - 2 = 5\text{ ms}^{-2}$

43 Displacement is a sine curve with an amplitude of 1, centre of motion 1 and $T = \pi$ since $n = 2$. So it is a sine curve moved up 1 unit.

Acceleration is given by $\ddot{x} = -2^2(x-1) = -4x+4$ since $n = 2$ and $c = 1$, so it is the displacement curve reflected over the x-axis, vertically dilated by a factor of 4 then moved up 4 units. Notice that it is centred about 0, not 1, as acceleration is proportional to the distance from the centre of motion.



44

$$
\begin{aligned}
x &= a\sin(nt+\alpha)\\
v &= an\cos(nt+\alpha)\\
v^2+n^2x^2 &= (an\cos(nt+\alpha))^2+n^2(a\sin(nt+\alpha))^2\\
&= a^2n^2\cos^2(nt+\alpha)+a^2n^2\sin^2(nt+\alpha)\\
&= a^2n^2(\cos^2(nt+\alpha)+\sin^2(nt+\alpha))\\
&= a^2n^2
\end{aligned}
$$

which is constant

45

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{1}{1+9x^2}\\
\frac{1}{2}(v^2-2^2) &= \int_0^x \frac{1}{1+9x^2}\,dx\\
v^2-4 &= \frac{2}{3}\Big[\tan^{-1}(3x)\Big]_0^x\\
v^2 &= \frac{2}{3}\tan^{-1}(3x)+4
\end{aligned}
$$

46

$\frac{2\pi}{n} = 2 \quad \rightarrow \quad n = \pi \quad \therefore A, C$

$x_0 = 2$ for A, B, C, D

$\dot{x}_0 = 0$ for A, B

ANSWER (A)

47 The stone is thrown from $(0,100)$ with velocity 30 ms^{-1} at an angle of θ and lands at $(40,0)$ at time t.

$$
\begin{aligned}
\int_0^t V\,dt &= \int_0^t 30\cos\theta\,dt\\
\Big[x\Big]_0^t &= 30\cos\theta\Big[t\Big]_0^t\\
40-0 &= 30t\cos\theta\\
t &= \frac{4}{3\cos\theta}\\
\int_0^t \ddot{y}\,dt &= \int_0^t (-9.8)\,dt\\
\Big[\dot{y}\Big]_0^t &= -9.8\Big[t\Big]_0^t\\
\dot{y}-30\sin\theta &= -9.8t\\
\dot{y} &= -9.8t+30\sin\theta\\
\int_0^t \dot{y}\,dt &= \int_0^t (-9.8t+30\sin\theta)\,dt\\
\Big[y\Big]_0^t &= \Big[-4.9t^2+30t\sin\theta\Big]_0^t\\
y-100 &= -4.9t^2+30t\sin\theta\\
y &= -4.9t^2+30t\sin\theta+100
\end{aligned}
$$

Let $t = \frac{4}{3\cos\theta}, y = 0$

$$
\begin{aligned}
0 &= -4.9\left(\frac{4}{3\cos\theta}\right)^2+30\left(\frac{4}{3\cos\theta}\right)\sin\theta\\
0 &= -\frac{4.9\times 16}{9}\sec^2\theta+40\tan\theta+100\\
0 &= -\frac{392}{45}(\tan^2\theta+1)+40\tan\theta+100\\
0 &= 392\tan^2\theta-1800\tan\theta-4108\\
\tan\theta &= \frac{1800\pm\sqrt{1800^2-4(392)(-4108)}}{2\times 392}\\
\tan\theta &= -1.672813\ldots, 6.264650\ldots,\\
\theta &= -59°, 81°
\end{aligned}
$$

The stone can be thrown up at an angle of 81° or down at an angle of 59°.

48

$$
\begin{aligned}
0 &= 2000-k(50)^2\\
k &= \frac{2000}{2500}\\
&= 0.8\\
0 &= 1500\times 10-0.8V_T^2\\
0.8V_T^2 &= 15000\\
V_T &= \sqrt{\frac{15000}{0.8}}\\
&= 136.9\text{ m/s}
\end{aligned}
$$

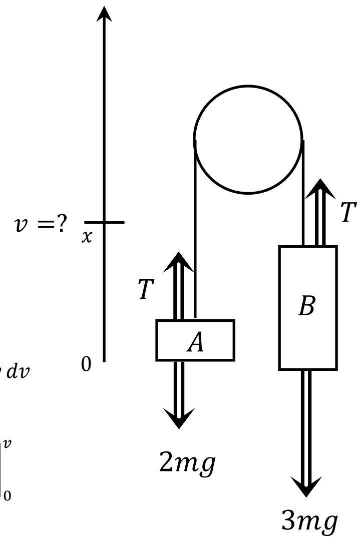
49

$$
\begin{aligned}
\frac{dx}{dt} &= k(a-x)\\
\frac{dt}{dx} &= \frac{1}{k}\times\frac{1}{a-x}\\
t &= \frac{1}{k}\int_0^x \frac{1}{a-x}\,dx\\
kt &= \Big[\ln(a-x)\Big]_x^0\\
kt &= \ln a-\ln(a-x)\\
kt &= \ln\left(\frac{a}{a-x}\right)\\
e^{kt} &= \frac{a}{a-x}\\
a-x &= ae^{-kt}\\
x &= a\left(1-e^{-kt}\right)
\end{aligned}
$$

ANSWER (C)

50

$$(2m+3m)\ddot{x} = (3mg-T)-(2mg-T)$$
$$5m\ddot{x} = mg$$
$$\ddot{x} = \frac{g}{5}$$
$$v\frac{dv}{dx} = \frac{g}{5}$$
$$\frac{dv}{dx} = \frac{g}{5v}$$
$$\frac{dx}{dv} = \frac{5v}{g}$$
$$x = \frac{5}{g}\int_0^v v\,dv$$
$$= \frac{5}{g}\left[v^2\right]_0^v$$
$$= \frac{5v^2}{2g}$$
$$v^2 = \frac{2gx}{5}$$
$$v = \sqrt{\frac{2gx}{5}} \quad \text{since the particle only moves up}$$



51

i

$$x = 20t\cos\alpha \;\rightarrow\; t = \frac{x}{20\cos\alpha}$$
$$\therefore y = -5\left(\frac{x}{20\cos\alpha}\right)^2 + 20\left(\frac{x}{20\cos\alpha}\right)\sin\alpha$$
$$= -\frac{x^2}{80}\sec^2\alpha + x\tan\alpha$$

ii

Let $x = 20, y = 3$

$$3 = -\frac{20^2}{80}(\tan^2\alpha+1) + 20\tan\alpha$$
$$3 = -5\tan^2\alpha - 5 + 20\tan\alpha$$
$$5\tan^2\alpha - 20\tan\alpha + 8 = 0$$
$$\tan\alpha = \frac{20 \pm \sqrt{(-20)^2-4(5)(8)}}{2(5)}$$
$$= 0.450806\ldots, 3.549193$$
$$= 24°, 74°$$

The ball will clear the fence for any angle from $24°$ to $74°$.

52

$$\frac{dx}{dt} = 1 + e^{-x} = \frac{e^x+1}{e^x}$$
$$\frac{dt}{dx} = \frac{e^x}{e^x+1}$$
$$t = \int_0^x \frac{e^x}{e^x+1}\,dx$$
$$= \left[\ln(e^x+1)\right]_0^x$$
$$t = \ln(e^x+1) - \ln 2$$
$$\ln 2 + t = \ln(e^x+1)$$
$$2e^t = e^x + 1$$
$$e^x = 2e^t - 1$$
$$x = \ln(2e^t - 1)$$
$$v = \frac{dx}{dt} = \frac{2e^t}{2e^t-1}$$
$$\text{Let } v = \frac{3}{2}$$
$$\frac{3}{2} = \frac{2e^t}{2e^t-1}$$
$$6e^t - 3 = 4e^t$$
$$2e^t = 3$$
$$e^t = \frac{3}{2}$$
$$t = \ln\left(\frac{3}{2}\right)$$

53

$$\ddot{x} = \frac{5+x}{x^2}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 5x^{-2} + \frac{1}{x}$$
$$\frac{1}{2}v^2 = \int_{-2}^x \left(5x^{-2} + \frac{1}{x}\right)dx$$
$$v^2 = 2\left[-\frac{5}{x} + \ln x\right]_2^x$$
$$= 2\left(\left(-\frac{5}{x} + \ln x\right) - \left(-\frac{5}{2} + \ln 2\right)\right)$$
$$= -\frac{10}{x} + 2\ln x + 5 - 2\ln 2$$
$$v = \sqrt{2\ln\left(\frac{x}{2}\right) - \frac{10}{x} + 5}$$

since $v > 0$ given initial conditions

54

i $x = -6$ or -2

ii $s = \sqrt{16} = 4$ m/s

iii

$$c = \frac{-6+(-2)}{2} = -4 \qquad \text{(centre of oscillation)}$$
$$a = \frac{-2-(-6)}{2} = 2 \qquad \text{(the amplitude of the motion)}$$

$$v^2 = n^2(a^2 - (x-c)^2)$$

maximum velocity when $x = -4$

$$16 = n^2(2^2 - (-4+4)^2)$$
$$16 = 4n^2$$
$$n = \frac{\sqrt{16}}{2} = 2 \quad (n > 0)$$

55 **i**

$$m\ddot{x} = -mg - \frac{v}{10}$$
$$\ddot{x} = -\frac{10mg + v}{10m}$$
$$v\frac{dv}{dx} = -\frac{10 \times 2 \times 10 + v}{10 \times 2}$$
$$\frac{dv}{dx} = -\frac{200 + v}{20v}$$
$$\frac{dx}{dv} = -\frac{20v}{200 + v}$$
$$H = -20\int_U^0 \frac{v}{200 + v}\,dv$$
$$= 20\int_0^U \frac{200 + v - 200}{200 + v}\,dv$$
$$= 20\int_0^U \left(1 - \frac{200}{200 + v}\right)dv$$
$$= 20\Big[v - 200\ln(200 + v)\Big]_0^U$$
$$= 20\big((U - 200\ln(200 + U)) - (-200\ln 200)\big)$$
$$= 20U + 4000\ln\left(\frac{200}{200 + U}\right)$$

ii

$$\frac{dv}{dt} = -\frac{200 + v}{20}$$
$$\frac{dt}{dv} = -\frac{20}{200 + v}$$
$$t = -20\int_U^0 \frac{1}{200 + v}\,dv$$
$$= 20\Big[\ln(200 + v)\Big]_0^U$$
$$= 20(\ln(200 + U) - \ln 200)$$
$$= 20\ln\left(\frac{200 + U}{200}\right)$$

iii

If $U = 400$:

$$H = 20(400) + 4000\ln\left(\frac{200}{200 + 400}\right)$$
$$= 8000 + 4000\ln\left(\frac{1}{3}\right)$$
$$= 4000(2 - \ln 3)$$
$$t = 20\ln\left(\frac{200 + 400}{200}\right)$$
$$= 20\ln 3$$
$$\text{Average Speed} = \frac{H}{t}$$
$$= \frac{4000(2 - \ln 3)}{20\ln 3}$$
$$= 200\left(\frac{2}{\ln 3} - 1\right)$$

56

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x^2$$
$$\frac{1}{2}v^2 = \int_2^x x^2\,dx$$
$$v^2 = 2\left[\frac{x^3}{3}\right]_2^x$$
$$v^2 = 2\left(\frac{x^3}{3} - \frac{8}{3}\right)$$
$$v = \sqrt{\frac{2x^3 - 16}{3}}\ \text{given initial conditions}$$

Let $x = 4$

$$v = \sqrt{\frac{2(4)^3 - 16}{3}}$$
$$= \frac{4\sqrt{7}}{\sqrt{3}}$$
$$= \frac{4\sqrt{21}}{3}$$

57

$T = 2 \times \left(6\frac{2}{3}\right) = \frac{40}{3}$ hours

$n = \frac{2\pi}{T} = \frac{3\pi}{20}$

Equilibrium position is $\frac{20+40}{2} = 30$ m and amplitude is $\frac{40-20}{2} = 10$ m.

If we use $-\cos nt$ then the curve will start at its minimum, so we can take $t = 0$ as low tide.

$x = 30 - 10\cos\left(\frac{3\pi}{20}t\right)$

where t is the number of hours after 7 am.

Let $x = 5\sqrt{3} + 30$

$$\therefore 30 - 10\cos\left(\frac{3\pi}{20}t\right) = 5\sqrt{3} + 30$$
$$\cos\left(\frac{3\pi}{20}t\right) = -\frac{\sqrt{3}}{2}$$
$$\frac{3\pi}{20}t = \pi - \frac{\pi}{6}$$
$$t = \frac{5\pi}{6} \times \frac{20}{3\pi}$$
$$= \frac{50}{9}$$

= 5 h 33 m

The first time the ship can enter the harbour is 12:33 pm.

58

$$\frac{2\pi}{n} = \pi \ \rightarrow \ n = 2$$
$$c = \frac{0 + 4}{2} = 2$$
$$\therefore \ddot{x} = -2^2(x - 2)$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4(x - 2)$$
$$\frac{1}{2}v^2 = -4\int_0^x (x - 2)\,dx \qquad **$$
$$v^2 = -8\left[\frac{x^2}{2} - 2x\right]_0^x$$
$$v^2 = -4x^2 + 16x$$
$$= 2^2(-x^2 + 4x)$$
$$= 2^2(-(x - 2)^2 + 4)$$
$$= 2^2(2^2 - (x - 2)^2)$$

** at the origin $x = 0, v = 0$

59 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dx} \times \frac{d}{dv}\left(\frac{1}{2}v^2\right)$$
$$= \frac{dx}{dv} \times v$$
$$= \frac{dv}{dx} \times \frac{dx}{dt}$$
$$= \frac{dv}{dt}$$
$$= \frac{d}{dt}\left(\frac{dx}{dt}\right)$$
$$= \frac{d^2x}{dt^2}$$
$$= \ddot{x}$$

ii

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x^2(4 - x^{-3})$$
$$= 4x^2 - \frac{1}{x}$$
$$\frac{1}{2}(v^2 - 3^2) = \int_1^5 \left(4x^2 - \frac{1}{x}\right) dx$$
$$v^2 - 9 = 2\left[\frac{4x^3}{3} - \ln x\right]_1^5$$
$$v^2 = 2\left(\left(\frac{4(5)^3}{3} - \ln 5\right) - \left(\frac{4(1)^3}{3} - 0\right)\right) + 9$$
$$= 336.45\cdot$$
$$v = 18.34 \text{ ms}^{-1}$$

since $v > 0$

60

$a = 3$
$x = 3\sin(nt)$
$\dot{x} = 3n\cos(nt)$
Let $x = 1, \dot{x} = 2$

$$\therefore 1 = 3\sin(nt) \quad \rightarrow \sin(nt) = \frac{1}{3} \quad (1)$$
$$2 = 3n\cos(nt) \quad \rightarrow \cos(nt) = \frac{2}{3n} \quad (2)$$
$$\sin^2(nt) + \cos^2(nt) = 1 \quad \text{Pythagorean Identity}$$
$$\therefore \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3n}\right)^2 = 1$$
$$\frac{1}{9} + \frac{4}{9n^2} = 1$$
$$n^2 + 4 = 9n^2$$
$$8n^2 = 4$$
$$n^2 = \frac{1}{2}$$
$$n = \frac{1}{\sqrt{2}}$$

61

$$0 = (80 + 20) \times 10 - k \times 60^2$$
$$3600k = 1000$$
$$k = \frac{5}{18}$$

Let V_T be the new terminal velocity

$$0 = 80 \times 10 - \frac{5}{18} \times {V_T}^2$$
$$\frac{5}{18}V_T^2 = 800$$
$$V_T = \sqrt{800 \times \frac{18}{5}}$$
$$= 53.7 \text{ m/s}$$

The parachutist will slow towards a new terminal velocity of 53.7 m/s

62 **i**

$$10\ddot{x} = 200 - kv$$
$$\ddot{x} = 20 - \frac{kv}{10}$$

ii

$$v\frac{dv}{dx} = 20 - \frac{kv}{10}$$
$$\frac{dv}{dx} = \frac{200 - kv}{10v}$$
$$\frac{dx}{dv} = \frac{10v}{200 - kv}$$
$$x = \int_0^V \frac{10v}{200 - kv} dv$$
$$= \int_0^V \frac{-\frac{10}{k}(200 - kv) + \frac{2000}{k}}{200 - kv} dv$$
$$= \int_0^V \left(-\frac{10}{k} - \frac{2000}{k^2} \times \left(\frac{-k}{200 - kv}\right)\right) dv$$
$$= \left[-\frac{10v}{k} - \frac{2000}{k^2}\ln(200 - kv)\right]_0^V$$
$$= \left(-\frac{10V}{k} - \frac{2000}{k^2}\ln(200 - kV)\right)$$
$$= \frac{2000}{k^2}\ln\left(\frac{200}{200 - kV}\right) - \frac{10V}{k}$$

63 **i**

$$x_1 = \int \frac{2}{\pi} dt$$
$$= \frac{2}{\pi} t \text{ ms}^{-1}$$
$$x_2 = \int_0^t (-2\cos t)\, dt$$
$$= -2\left[\sin t\right]_0^t$$
$$= -2\sin t$$

ii

Initially the first particle moves to the right and the second particle moves to the left. The second particle returns to the origin at $t = \pi$ seconds, at which point the first particle is at $x = 2$ and continues to move to the right. Since the maximum displacement for the second particle is 2, the two particles never meet again

64

$$(m+m)\ddot{x} = (mg - T) - (mg\sin 30° - T)$$

$$2m\ddot{x} = \frac{mg}{2}$$

$$\ddot{x} = \frac{g}{4}$$

$$\frac{dv}{dt} = \frac{g}{4}$$

$$v = \frac{g}{4}\int_0^t dt$$

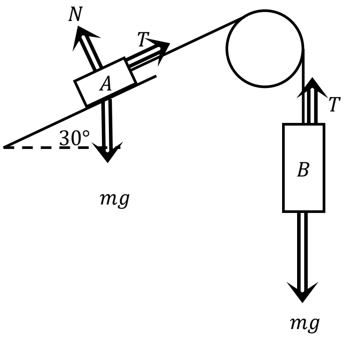
$$= \frac{gt}{4}$$

$$\therefore \frac{dx}{dt} = \frac{gt}{4}$$

$$x = \frac{g}{4}\int_0^t t\,dt$$

$$= \frac{g}{8}\Big[t^2\Big]_0^t$$

$$= \frac{gt^2}{8}$$



65

i

$$x = Vt\cos\theta \to t = \frac{x}{v\cos\theta}$$

$$y = V\left(\frac{x}{V\cos\theta}\right)\sin\theta - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{1}{2}\times\frac{gx^2}{V^2\cos^2\theta}$$

$$= x\tan\theta - \frac{g}{2V^2}\times x^2\sec^2\theta$$

$$= x\tan\theta - x^2\sec^2\theta \text{ since } \frac{2V^2}{g} = 1$$

ii

At $T(x,y)$:

$x = r\cos\alpha$

$y = r\sin\alpha$

$$\therefore r\sin\alpha = r\cos\alpha\tan\theta - (r\cos\alpha)^2\sec^2\theta \text{ from } (i)$$

$$r\sin\alpha\cos^2\theta = r\cos\alpha\sin\theta\cos\theta - r^2\cos^2\alpha$$

$$r^2\cos^2\alpha = r\cos\theta\,(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$

$$r^2\cos^2\alpha = r\cos\theta\sin(\theta-\alpha)$$

$$r = \frac{\sin(\theta-\alpha)\cos\theta}{\cos^2\alpha}$$

66

$$\ddot{x} = \frac{d^2x}{dt^2}$$

$$= \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \frac{d}{dt}(v)$$

$$= \frac{dv}{dt} \qquad (1)$$

$$= \frac{dv}{dx}\times\frac{dx}{dt}$$

$$= \frac{dv}{dx}\times v \qquad (2)$$

$$= \frac{dv}{dx}\times\frac{d}{dv}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \qquad (3)$$

$$\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) \text{ from (1), (2), (3)}$$

67

i

$x = a\cos(9t+\theta)$

$\dot{x} = -9a\sin(9t+\theta)$

$\ddot{x} = -81a\cos(9t+\theta)$

$= -9^2x$

$\therefore$ the particle is in SHM with $n = 9$

ii

$$T = \frac{2\pi}{n} = \frac{2\pi}{9}$$

iii

Let $t = 0, x = 0, \dot{x} = -15$

$$0 = a\cos\theta \to \cos\theta = 0 \to \theta = \frac{\pi}{2}$$

$$-15 = -9a\sin\left(\frac{\pi}{2}\right) \to a = \frac{15}{9} = \frac{5}{3}$$

iv

$$x_6 = \tfrac{5}{3}\cos\left(9\times 6 + \tfrac{\pi}{2}\right) = 0.94 \text{ m (2dp)}$$

68

i

$x = 13 + 12\cos 2t + 5\sin 2t$

$\dot{x} = -24\sin 2t + 10\cos 2t$

$\ddot{x} = -48\cos 2t - 20\sin 2t$

$= -4(12\cos 2t + 5\sin 2t)$

$= -4(13 + 12\cos 2t + 5\sin 2t - 13)$

$= -2^2(x - 13)$

ii

$13 + 12\cos 2t + 5\sin 2t = 0$

$12\cos 2t + 5\sin 2t = -13$

$$r = \sqrt{12^2+5^2} = 13$$

$$\alpha = \tan^{-1}\left(\frac{5}{12}\right) = 0.44258\ldots$$

$\therefore 13\cos(2t - 0.44258) = -13$

$\cos(2t - 0.44258) = -1$

$2t - 0.44258 = \pi$

$$t = \frac{1}{2}(\pi + 0.44258)$$

$= 1.792\ldots$

$t = 1.8\ s$ (1 dp)

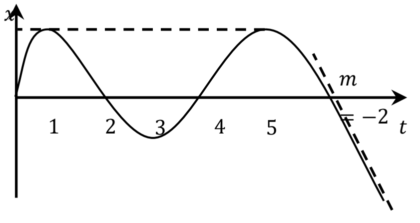
69

Particle B has a greater vertical resistance, since $R_y = -kv_yv$ as opposed to $R = -v_y^2$ for Particle A, and $v > v_y$ for Particle B. So Particle B will reach its maximum height sooner, and it will be lower, than for Particle A.

70 **i** The particle is initially at a velocity of 2 metres per second to the right (since the height is 2) with negative acceleration (since the curve then has a negative gradient)

ii The particle moves to the right from $t = 0$ to $t = 1$. It moves to the left from $t = 1$ to $t = 3$, then a similar amount to the right from $t = 3$ to $t = 5$, then returns to the left. It is furthest to the right at $t = 1$ and $t = 5$.

iii Equal heights at $t = 1$ and $t = 5$, minimum turning point at $t = 3$, and approaching an oblique asymptote with gradient -3 as $t \to \infty$.



71

$x = \sin(at) + \cos(bt)$

$\dot{x} = a\cos(at) - b\sin(bt)$

$\ddot{x} = -a^2\sin(at) - b^2\cos(bt)$

$\neq -n^2(\sin(at) + \cos(bt))$

since $a \neq b$ and a, b non-zero

∴ the particle is not in SHM.

72

$n = \frac{2\pi}{T} = \pi; a = 3$

Let $x = 3\sin(\pi t)$

$\frac{dx}{dt} = 3\pi\cos(\pi t)$

$\frac{d^2x}{dt^2} = -3\pi^2\sin(\pi t)$

Letting $\cos(\pi t) = 1$, then $\sin(\pi t) = -1$, we see that the maximum speed is 3π ms^{-1} and maximum acceleration is $3\pi^2$ ms^{-2}.

73 **i**

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8x(x^2 + 4)$

$\frac{1}{2}(v^2 - 8^2) = \int_0^x 8x(x^2 + 4)\,dx$

$v^2 - 64 = 2\left[2x^4 + 16x^2\right]_0^x$

$v^2 = 4x^4 + 32x^2 + 64$

$= 4(x^4 + 8x^2 + 16)$

$= 2^2(x^2 + 4)^2$

speed $= |v| = 2(x^2 + 4)$ m/s

ii

Initially the particle is at the origin moving to the right, so $v > 0$.Since $x^2 + 4 > 0$ the velocity can never be zero, so acceleration will never become negative, so the velocity is always positive.

iii

$\frac{dx}{dt} = 2(x^2 + 4)$

$\frac{dt}{dx} = \frac{1}{2(x^2 + 4)}$

$t = \frac{1}{2}\int_0^x \frac{1}{x^2 + 4}\,dx$

$= \frac{1}{4}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^x$

$= \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right)$

Let $x = 2$

$t = \frac{1}{4}\tan^{-1}(1)$

$= \frac{\pi}{16}$

74 **i**

$\ddot{x} = -g - \frac{v}{10}$

$\therefore \frac{dv}{dt} = -g - \frac{v}{10}$

$\frac{dt}{dv} = -\frac{10}{10g + v}$

$t = -\int_{10(20-g)}^{v} \frac{10}{10g + v}\,dv$

$= 10\left[\ln(10g + v)\right]_v^{10(20-g)}$

$= 10\left(\ln(10g + 200 - 10g) - \ln(10g + v)\right)$

$= 10\ln\left(\frac{200}{10g + v}\right)$

when $t = T, v = 0$

$\therefore T = 10\ln\left(\frac{200}{10g}\right)$

$= 10\ln\left(\frac{20}{g}\right)$

ii

$t = 10\ln\left(\frac{200}{10g + v}\right)$

$e^{\frac{t}{10}} = \frac{200}{10g + v}$

$10ge^{\frac{t}{10}} + ve^{\frac{t}{10}} = 200$

$ve^{\frac{t}{10}} = 200 - 10ge^{\frac{t}{10}}$

$v = 200e^{-\frac{t}{10}} - 10g$

$x = \int_0^T (200e^{-\frac{t}{10}} - 10g)\,dt$

$= \left[-2000e^{-\frac{t}{10}} - 10gt\right]_0^T$

$= \left(-2000e^{-\ln\left(\frac{20}{g}\right)} - 10gT\right) - (-2000 - 0)$

$= -2000\left(\frac{g}{20}\right) - 10gT + 2000$

$= 2000 - 10g[10 + T]$

iii

$\ddot{x} = g - \frac{v}{10}$

Let $\ddot{x} = 0, v = V_T$

$0 = g - \frac{V_T}{10}$

$V_T = 10g$

75 **iα**

$$v\frac{dv}{dx} = -(1+v)$$
$$\frac{dv}{dx} = -\frac{1+v}{v}$$
$$\frac{dx}{dv} = -\frac{v}{1+v}$$
$$x = -\int_u^v \frac{v}{1+v}\,dv$$
$$= -\int_u^v \frac{1+v-1}{1+v}\,dv$$
$$= \int_v^u \left(1 - \frac{1}{1+v}\right) dv$$
$$= \Big[v - \ln(1+v)\Big]_v^u$$
$$= \big((u - \ln(1+u)) - (v - \ln(1+v))\big)$$
$$= u - v + \ln\frac{1+v}{1+u}$$

β

$$\frac{dv}{dt} = -(1+v)$$
$$\frac{dt}{dv} = -\frac{1}{1+v}$$
$$t = -\int_u^v \frac{1}{1+v}\,dv$$
$$= -\Big[\ln(1+v)\Big]_u^v$$
$$= \ln\frac{1+u}{1+v}$$
$$\therefore e^t = \frac{1+u}{1+v}$$
$$e^t + ve^t = 1+u$$
$$ve^t = 1 + u - e^t$$
$$v = (1+u)e^{-t} - 1$$

γ

$$x = \int_0^t \big((1+u)e^{-t} - 1\big)\,dt$$
$$= \Big[-(1+u)e^{-t} - t\Big]_0^t$$
$$= (-(1+u)e^{-t} - t) - (-(1+u) - 0)$$
$$= -(1+u)(e^{-t} - 1) - t$$

ii

$$x + v + t$$
$$= -(1+u)(e^{-t} - 1) - t + (1+u)e^{-t} - 1 + t$$
$$= -e^{-t} + 1 - ue^{-t} + u - t + e^{-t} + ue^{-t} - 1 + t$$
$$= u$$

iii

Let $v = 0$

$$0 = (1+u)e^{-t} - 1$$
$$e^{-t} = \frac{1}{1+u}$$
$$e^t = 1 + u$$
$$t = \ln(1+u)\ \text{s}$$
$$x + v + t = u$$
$$x + 0 + \ln(u+1) = u$$
$$x = u - \ln(1+u)$$

76 **i**

$$m\ddot{x} = -\frac{g}{10} - \frac{v}{10g}$$
$$0.1\ddot{x} = -\frac{g^2 + v}{10g}$$
$$= -\frac{100 + v}{100}$$
$$v\frac{dv}{dx} = -\frac{100+v}{10}$$
$$\frac{dv}{dx} = -\frac{100+v}{10v}$$
$$\frac{dx}{dv} = -\frac{10v}{v+100}$$
$$x = -\int_{1200}^v \frac{10v}{v+100}\,dv$$
$$= \int_v^{1200} \frac{10(v+100) - 1000}{v+100}\,dv$$
$$= \int_v^{1200} \left(10 - \frac{1000}{v+100}\right) dv$$
$$= \Big[10v - 1000\ln(v+100)\Big]_v^{1200}$$
$$= (12000 - 1000\ln 1300) - (10v - 1000\ln(v$$
$$= 12000 + 1000\ln\left(\frac{v+100}{1300}\right) - 10v$$

ii

$$\frac{dv}{dt} = -\frac{100+v}{10}$$
$$\frac{dt}{dv} = -\frac{10}{100+v}$$
$$t = -\int_{1200}^0 \frac{10}{100+v}\,dv$$
$$= 10\Big[\ln(v+100)\Big]_0^{1200}$$
$$= 10(\ln 1300 - \ln 100)$$
$$= 10\ln 13$$
$$= 25.65 \text{ seconds}$$

iii

$$H = 12000 + 1000\ln\left(\frac{(0)+100}{1300}\right) - 10(0) \quad \text{from (i)}$$
$$= 9435 \text{ m}$$

iv

$$m\ddot{x} = mg - \frac{v}{10g}$$
$$m\ddot{x} = 0.1 \times 10 - \frac{v}{100}$$
$$= 1 - \frac{v}{100}$$

At terminal velocity $\ddot{x} = 0, v = v_T$

$$1 - \frac{v_T}{100} = 0$$
$$v_T = 100 \text{ m/s}$$

77 **i**

As the projectile clears the second building

$$20\sqrt{5} = Vt \to t = \frac{20\sqrt{5}}{V} \quad (1)$$
$$100 + c = -5t^2 + 125$$
$$5t^2 = 25 - c \quad (2)$$
$$\text{sub(1)in(2)}: 5\left(\frac{20\sqrt{5}}{V}\right)^2 = 25 - c$$
$$\frac{10000}{V^2} = 25 - c$$
$$V^2 = \frac{10000}{25 - c}$$
$$V = \frac{100}{\sqrt{25 - c}} \quad (V > 0)$$

ii

$$c \geq 0$$
$$\therefore V \geq \frac{100}{\sqrt{25}}$$
$$\geq 20$$

iii

$$-5t^2 + 125 = 0$$
$$5t^2 = 125$$
$$t^2 = 25$$
$$t = 5 \ \ (t > 5)$$
$$x = Vt$$
$$= 20 \times 5$$
$$= 100$$

The projectile strikes the ground $100 - 20\sqrt{5}$ metres beyond the second tower.

iv

$$\frac{dy}{dt} = -10t$$
$$= -10(5)$$
$$= -50 \text{ m/s}$$

78 **i**

$$m\ddot{x} = mg - \frac{mv^2}{360}$$
$$\ddot{x} = g - \frac{v^2}{360}$$

Let $\ddot{x} = 0, v = v_T, g = 9.8$

$$0 = 9.8 - \frac{v_T^2}{360}$$
$$v_T = \sqrt{9.8 \times 360}$$
$$= 59.4 \text{ m/s (1dp)}$$

ii

$$v\frac{dv}{dx} = \frac{360g - v^2}{360}$$
$$\frac{dv}{dx} = \frac{360g - v^2}{360v}$$
$$\frac{dx}{dv} = \frac{360v}{360g - v^2}$$
$$x = 360\int_0^v \frac{v}{360g - v^2}\,dv$$
$$= -180\left[\ln(360g - v^2)\right]_0^v$$
$$= -180\left(\ln(360g - v^2) - \ln 360g\right)$$
$$= 180\ln\left(\frac{360g}{360g - v^2}\right)$$

when $v = 50$

$$x = 180\ln\left(\frac{360 \times 9.8}{360 \times 9.8 - 50^2}\right)$$
$$= 221.96 \text{ m}$$

iii

$$\frac{dv}{dt} = \frac{360g - v^2}{360}$$
$$\frac{dt}{dv} = \frac{360}{360g - v^2}$$
$$t = \int_0^{50} \frac{360}{360g - v^2}\,dv$$
$$= \frac{360}{2\sqrt{360g}}\int_0^{50}\left(\frac{1}{\sqrt{360g} + v} + \frac{1}{\sqrt{360g} - v}\right)dv$$
$$= \frac{3\sqrt{10}}{\sqrt{g}}\left[\ln\left(\sqrt{360g} + v\right) - \ln\left(\sqrt{360g} - v\right)\right]_0^{50}$$
$$= \frac{3\sqrt{10}}{\sqrt{9.8}}\left(\ln\frac{\sqrt{360 \times 9.8} + 50}{\sqrt{360 \times 9.8} - 50}\right)$$
$$= 7.44 \text{ s}$$

79

The height of any point on the paddlewheel is in SHM. The radius gives the amplitude, so $a = 2$. The period in 20 seconds, so $20 = \frac{2\pi}{n} \to n = \frac{\pi}{10}$. The centre of motion is the height above the water surface, so $c = 1.5$.

Let the equation of motion be $x = 2\sin\left(\frac{\pi t}{10}\right) + 1.5$, so the point is initially at the centre of motion and moving upwards.

Let $x = 0$

$$2\sin\left(\frac{\pi t}{10}\right) + 1.5 = 0$$
$$\sin\left(\frac{\pi t}{10}\right) = -\frac{3}{4}$$
$$\frac{\pi t}{10} = \pi + \sin^{-1}\left(\frac{3}{4}\right), 2\pi - \sin^{-1}\left(\frac{3}{4}\right)$$
$$t = 10 + \frac{10}{\pi}\sin^{-1}\left(\frac{3}{4}\right), 20 - \frac{10}{\pi}\sin^{-1}\left(\frac{3}{4}\right)$$

$\therefore$ in each rotation the particle is underwater for:

$$\left(20 - \frac{10}{\pi}\sin^{-1}\left(\frac{3}{4}\right)\right) - \left(10 + \frac{10}{\pi}\sin^{-1}\left(\frac{3}{4}\right)\right)$$
$$= 10 - \frac{20}{\pi}\sin^{-1}\left(\frac{3}{4}\right) \text{ seconds}$$

80 **i**

$$mv\frac{dv}{dx} = mg$$
$$\frac{dv}{dx} = \frac{g}{v}$$
$$= \frac{10}{v}$$

ii

$$\frac{dx}{dv} = \frac{v}{10}$$
$$7.2 = \frac{1}{10}\int_0^v v\,dv$$
$$= \frac{1}{10}\left[\frac{v^2}{2}\right]_0^v$$
$$= \frac{1}{20}(v^2)$$
$$v^2 = 144$$
$$v = 12 \text{ since } v \geq 0$$

80 …

iii

$$mv\frac{dv}{dx} = mg - R$$
$$\frac{dv}{dx} = \frac{mg - R}{mv}$$
$$= \frac{0.2 \times 10 - R}{0.2v}$$
$$= \frac{10 - 5R}{v}$$
$$\frac{dx}{dv} = \frac{v}{10 - 5R}$$
$$0.8 = \frac{1}{10 - 5R}\int_{12}^{6} v\,dv$$
$$0.8(10 - 5R) = \left[\frac{v^2}{2}\right]_{12}^{6}$$
$$8 - 4R = \left(\frac{6^2}{2} - \frac{12^2}{2}\right)$$
$$8 - 4R = -54$$
$$4R = 62$$
$$R = 15.5\text{ N}$$

iv

$$m\frac{dv}{dt} = T - R - mg$$
$$\frac{dv}{dt} = \frac{T - 15.5 - 2}{0.2}$$
$$= 5T - 87.5$$
$$v = \int_0^4 (5T - 87.5)\,dt$$
$$= (5T - 87.5)\Big[t\Big]_0^t$$
$$\frac{dx}{dt} = (5T - 87.5)t$$
$$3.6 = (5T - 87.5)\int_0^4 t\,dt$$
$$3.6 = (5T - 87.5)\left[\frac{t^2}{2}\right]_0^4$$
$$3.6 = 40T - 700$$
$$40T = 703.6$$
$$T = 17.59\text{ N}$$

81

$$\text{Let } \ddot{x} = -n^2(x - c)$$
$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x - c)$$
$$\frac{d}{dx}(v^2) = -2n^2(x - c)$$
$$\Big[v^2\Big]_0^v = -n^2\Big[x^2 - 2cx\Big]_{c-a}^{x}$$
$$v^2 = -n^2\left((x^2 - 2cx) - \left((c - a)^2 - 2c(c\right.\right.$$
$$= -n^2(x^2 - 2cx - c^2 + 2ac - a^2 + 2c^2$$
$$= -n^2(x^2 - 2cx + c^2 - a^2)$$
$$= n^2\left(a^2 - (x^2 - 2cx + c^2)\right)$$
$$= n^2(a^2 - (x - c)^2)$$

82

i

$$m\ddot{x} = mg - \frac{1}{40}mv^2$$
$$\ddot{x} = 10 - \frac{v^2}{40}$$
$$= \frac{1}{40}(400 - v^2)$$

ii

$$\therefore \frac{dv}{dt} = \frac{400 - v^2}{40}$$
$$\frac{dt}{dv} = \frac{40}{400 - v^2}$$
$$t = \int_0^v \frac{40}{400 - v^2}\,dv$$
$$= \int_0^v \left(\frac{1}{20 + v} + \frac{1}{20 - v}\right)dv$$
$$= \Big[\ln(20 + v) - \ln(20 - v)\Big]_0^v$$
$$= \ln\left(\frac{20 + v}{20 - v}\right) - \ln\left(\frac{20}{20}\right)$$
$$= \ln\left(\frac{20 + v}{20 - v}\right)$$

iii

$$\therefore e^t = \frac{20 + v}{20 - v}$$
$$20e^t - ve^t = 20 + v$$
$$v(1 + e^t) = 20(e^t - 1)$$
$$v = \frac{20(e^t - 1)}{e^t + 1}$$
$$= \frac{20(e^t + 1 - 2)}{e^t + 1}$$
$$= 20\left(1 - \frac{2}{1 + e^t}\right)$$

iv

$$\therefore x = \int_0^t 20\left(1 - \frac{2}{1 + e^t}\right)dt$$
$$= 20\int_0^t \left(1 - \frac{2e^{-t}}{e^{-t} + 1}\right)dt$$
$$= 20\Big[t + 2\ln(e^{-t} + 1)\Big]_0^t$$
$$= 20\left(t + 2\,ln(e^{-t} + 1) - 2\ln 2\right)$$
$$= 20\left[t + 2\ln\left(\frac{1 + e^{-t}}{2}\right)\right]$$

83 **i**

$$\ddot{x} = -\frac{k}{x^2}$$
$$-g = -\frac{k}{R^2}$$
$$k = gR^2$$
$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^2}{x^2}$$
$$\frac{1}{2}(v^2 - u^2) = -gR^2\int_R^x x^{-2}\,dx$$
$$v^2 - u^2 = -2gR^2\left[-\frac{1}{x}\right]_R^x$$
$$v^2 - u^2 = 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$
$$\therefore v^2 = \frac{2gR^2}{x} + U^2 - 2gR$$

ii

Let $U^2 = gr, v = 0$

$$\therefore 0 = \frac{2gR^2}{x} + gR - 2gR$$
$$\frac{2gR^2}{x} = gR$$
$$x = \frac{2gR^2}{gR} = 2R$$

The particle reaches a height of $2R$ above the centre of the Earth, so R above the surface of the Earth.

iii

$$v = \sqrt{\frac{2gR^2}{x} + gR - 2gR}$$
$$\frac{dt}{dx} = \frac{1}{\sqrt{\frac{2gR^2}{x} - gR}}$$
$$t = \int_R^{2R}\frac{1}{\sqrt{\frac{2gR^2}{x} - gR}}\,dx$$
$$= \int_R^{2R}\frac{x}{\sqrt{2gR^2x - gRx^2}}\,dx$$
$$= \frac{1}{\sqrt{gR}}\int_R^{2R}\frac{x}{\sqrt{2Rx - x^2}}\,dx$$
$$= \frac{1}{\sqrt{gR}}\int_R^{2R}\frac{x}{\sqrt{R^2 - (x-R)^2}}\,dx$$

$x - R = R\sin\theta$ $x = R(1 + \sin\theta)$ $dx = R\cos\theta\,d\theta$

$$= \frac{1}{\sqrt{gR}}\int_0^{\frac{\pi}{2}}\frac{R(1+\sin\theta)}{\sqrt{R^2 - R^2\sin^2\theta}}(R\cos\theta)\,d\theta$$
$$= \frac{1}{\sqrt{gR}}\int_0^{\frac{\pi}{2}}R(1+\sin\theta)\,d\theta$$
$$= \frac{R}{\sqrt{gR}}\Big[\theta - \cos\theta\Big]_0^{\frac{\pi}{2}}$$
$$= \sqrt{\frac{R}{g}}\left(\left(\frac{\pi}{2} - 0\right) - (0 - 1)\right)$$
$$= \sqrt{\frac{R}{g}}\left(\frac{\pi}{2} + 1\right)\text{ s}$$

84 **i**

$$v\frac{dv}{dx} = -g - kv^2$$
$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$
$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$
$$H = -\int_u^0\frac{v}{g + kv^2}\,dv$$
$$= \frac{1}{2k}\int_0^u\frac{2kv}{g + kv^2}\,dv$$
$$= \frac{1}{2k}\Big[\ln(g + kv^2)\Big]_0^u$$
$$= \frac{1}{2k}(\ln(g + ku^2) - \ln g)$$
$$= \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right)$$
$$= \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right)$$

ii

Let the particles collide at height x

For the first particle:

$$x = -\int_u^{v_1}\frac{v}{g + kv^2}\,dv$$
$$= \frac{1}{2k}\int_{v_1}^u\frac{2kv}{g + kv^2}\,dv$$
$$= \frac{1}{2k}\Big[\ln(g + kv^2)\Big]_{v_1}^u$$
$$= \frac{1}{2k}(\ln(g + ku^2) - \ln(g + k{v_1}^2))$$
$$= \frac{1}{2k}\ln\left(\frac{g + ku^2}{g + kv_1^2}\right)$$

For the second particle:

$$v\frac{dv}{dx} = g - kv^2$$
$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$
$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$
$$H - x = \int_0^{v_2}\frac{v}{g - kv^2}\,dv$$
$$= -\frac{1}{2k}\int_0^{v_2}\frac{-2kv}{g - kv^2}\,dv$$
$$= \frac{1}{2k}\Big[\ln(g - kv^2)\Big]_{v_2}^0$$
$$= \frac{1}{2k}(\ln g - \ln(g - kv_2^2))$$
$$= \frac{1}{2k}\ln\left(\frac{g}{g - kv_2^2}\right)$$
$$\therefore \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right) - \frac{1}{2k}\ln\left(\frac{g + ku^2}{g + kv_1^2}\right) = \frac{1}{2k}\ln\left(\frac{g}{g - kv_2^2}\right)$$
$$\ln\left(\frac{g + ku^2}{g}\right) - \ln\left(\frac{g + ku^2}{g + kv_1^2}\right) = \ln\left(\frac{g}{g - kv_2^2}\right)$$
$$\ln\left(\frac{g + kv_1^2}{g}\right) = \ln\left(\frac{g}{g - kv_2^2}\right)$$
$$(g + kv_1^2)(g - kv_2^2) = g^2$$
$$g^2 - gkv_2^2 + gkv_1^2 - k^2v_1^2v_2^2 = g^2$$
$$k^2v_1^2v_2^2 = gkv_1^2 - gkv_2^2$$
$$\frac{k}{g} = \frac{1}{v_2^2} - \frac{1}{v_1^2}$$
$$\frac{1}{v_2^2} - \frac{1}{v_1^2} = \left(\sqrt{\frac{k}{g}}\right)^2$$
$$= \frac{1}{V^2}$$

85 **i**

$$m\ddot{x} = mg - R$$
$$7\ddot{x} = 7\times 10 - \frac{7v^2}{10}$$
$$\ddot{x} = 10 - \frac{v^2}{10}$$

ii

Let $\ddot{x} = 0, v = v_T$

$$0 = 10 - \frac{v_T^2}{10}$$
$$V_T^2 = 100$$
$$V_T = 10 \text{ ms}^{-1}$$

iii

$$v\frac{dv}{dx} = 10 - \frac{v^2}{10}$$
$$\frac{dv}{dx} = \frac{100 - v^2}{10v}$$
$$\frac{dx}{dv} = \frac{10v}{100 - v^2}$$
$$x = \int_0^v \frac{10v}{100 - v^2}dv$$
$$= -5\Big[\ln(100 - v^2)\Big]_0^v$$
$$= 5\ln\left(\frac{100}{100 - v^2}\right)$$
$$e^{\frac{x}{5}} = \frac{100}{100 - v^2}$$
$$100e^{\frac{x}{5}} - e^{\frac{x}{5}}v^2 = 100$$
$$e^{\frac{x}{5}}v^2 = 100\left(e^{\frac{x}{5}} - 1\right)$$
$$v^2 = 100\left(1 - e^{-\frac{x}{5}}\right)$$

iv

$$\frac{dV}{dt} = -\left(10 + \frac{V^2}{10}\right)$$
$$\frac{dt}{dV} = -\frac{10}{100 + V^2}$$
$$t = -10\int_{\frac{10}{\sqrt{3}}}^{0}\frac{1}{100 + V^2}dV$$
$$= 10\times\frac{1}{10}\left[\tan^{-1}\frac{V}{10}\right]_0^{\frac{10}{\sqrt{3}}}$$
$$= \tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0$$
$$= \frac{\pi}{6}\text{ s}$$

87

$$x = a\cos^2(nt + \alpha) + c$$
$$= a\left(\frac{1}{2}(1 + \cos(2(nt + \alpha)))\right) + c$$
$$= \frac{a}{2}\cos\big(2(nt + \alpha)\big) + \frac{a}{2} + c$$
$$\dot{x} = -\frac{a}{2}\times 2n\sin(2(nt + \alpha))$$
$$= -an\sin\big(2(nt + \alpha)\big)$$
$$\ddot{x} = -2an^2\cos(2(nt + \alpha))$$
$$= -2an^2(2\cos^2(nt + \alpha) - 1)$$
$$= -4n^2\left(a\cos^2(nt + \alpha) - \frac{a}{2}\right)$$
$$= -4n^2\left(a\cos^2(nt + \alpha) + c - \left(\frac{a}{2} + c\right)\right)$$
$$= -(2n)^2\left(x - \left(\frac{a}{2} + c\right)\right)$$

86 i

at A $x = 0, y = h$ and $m = \tan 30° = \frac{1}{\sqrt{3}}$

$$y = mx + b$$
$$y = \frac{x}{\sqrt{3}} + h$$

ii

The horizontal velocities of the plane and the missile must be the same for them to collide, so:

$$\sqrt{8gh}\cos 30° = u\cos\alpha$$
$$\sqrt{8gh}\times\frac{\sqrt{3}}{2} = u\cos\alpha$$
$$\cos\alpha = \frac{\sqrt{6gh}}{u}$$

$$x = ut\cos\alpha \to t = \frac{x}{u\cos\alpha}$$
$$\therefore y = u\left(\frac{x}{u\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$
$$= x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$$
$$= x\tan\alpha - \frac{gx^2}{2u^2}\left(\frac{u^2}{6gh}\right)$$
$$= x\tan\alpha - \frac{x^2}{12h}$$
$$\therefore \frac{x}{\sqrt{3}} + h = x\tan\alpha - \frac{x^2}{12h} \quad \text{from (i)}$$
$$\frac{x^2}{4h} + \left(\frac{1}{\sqrt{3}} - \tan\alpha\right)x + h = 0$$
$$\frac{x^2}{4} + \left(\frac{1}{\sqrt{3}} - \tan\alpha\right)hx + h^2 = 0$$

iii α

$$\Delta = \left(\left(\frac{1}{\sqrt{3}} - \tan\alpha\right)h\right)^2 - 4\left(\frac{1}{12}\right)(h^2)$$
$$= h^2\left(\frac{1}{3} - \frac{2\tan\alpha}{\sqrt{3}} + \tan^2\alpha - \frac{1}{3}\right)$$
$$= h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)$$
$$> 0 \quad \left(\tan\alpha > \frac{2}{\sqrt{3}}\right)$$

∴ there are two points of intersection.

iii β

The time elapsed between the two points of collision is the difference between the x-values divided by the shared horizontal velocity, $\sqrt{6gh}$

From the quadratic formula the difference between the x-values is:

$$R = \frac{2\sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{\sqrt{\Delta}}{a}$$
$$= \frac{\sqrt{h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}}{1/12}$$
$$= \sqrt{144h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}$$
$$\therefore T = \frac{\sqrt{144h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}}{\sqrt{6gh}}$$
$$= \sqrt{\frac{24h}{g}\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}$$
$$= \sqrt{\frac{24h}{g}\tan\alpha\times\frac{1}{3}\times\left(3\tan\alpha - 2\sqrt{3}\right)}$$
$$= \sqrt{\frac{8h\tan\alpha}{g}\left(3\tan\alpha - 2\sqrt{3}\right)}$$

88

$$v\frac{dv}{dx} = -(g + kv)$$

$$\frac{dv}{dx} = -\frac{g + kv}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv}$$

$$H = -\int_V^0 \frac{v}{g + kv}\,dv$$

$$= \int_0^V \frac{\frac{1}{k}(g + kv) - \frac{g}{k}}{g + kv}\,dv$$

$$= \int_0^V \left(\frac{1}{k} - \frac{g}{k^2} \times \frac{k}{g + kv}\right) dv$$

$$= \left[\frac{v}{k} - \frac{g}{k^2}\ln(g + kv)\right]_0^V$$

$$= \frac{V}{k} - \frac{g}{k^2} ln(g + kV) + \frac{g}{k^2}\ln g$$

$$= \frac{V}{k} - \frac{g}{k^2}\ln\left(\frac{g + kV}{g}\right)$$

$$= \frac{V}{k} - \frac{g}{k^2}\ln\left(1 + \frac{kV}{g}\right)$$

89

i

$$v\frac{dv}{dx} = -(v + v^3)$$
$$\frac{dv}{dx} = -(1 + v^2)$$
$$\frac{dx}{dv} = -\frac{1}{1 + v^2}$$
$$x = -\int_1^v \frac{1}{1 + v^2}\,dv$$
$$= \left[\tan^{-1} v\right]_v^1$$
$$= \tan^{-1} 1 - \tan^{-1} v$$
$$= \tan^{-1}(\tan(\tan^{-1} 1 - \tan^{-1} v))$$
$$= \tan^{-1}\left(\frac{\tan(\tan^{-1} 1) - \tan(\tan^{-1} v)}{1 + \tan(\tan^{-1}\sqrt{3}) \times \tan(\tan^{-1} v)}\right)$$
$$= \tan^{-1}\left(\frac{1 - v}{1 + v}\right)$$

ii

$$\frac{dv}{dt} = -(v + v^3)$$
$$\frac{dt}{dv} = -\frac{1}{v + v^3}$$
$$t = -\int_1^V \frac{1}{v + v^3}\,dv$$
$$= \int_V^1 \left(\frac{1}{v} - \frac{v}{1 + v^2}\right) dv$$
$$= \left[\ln v - \frac{1}{2}\ln(1 + v^2)\right]_V^1$$
$$= 0 - \frac{1}{2}\ln 2 - \ln V + \frac{1}{2}\ln(1 + V^2)$$
$$= \frac{1}{2}\ln(1 + V^2) - \frac{1}{2}\ln 2 - \frac{1}{2}\ln V^2$$
$$= \frac{1}{2}\ln\left(\frac{1 + V^2}{2V^2}\right)$$
$$= \ln\sqrt{\frac{1 + V^2}{2V^2}}$$

90

i

$$m\ddot{x} = mg - mkv$$
$$\ddot{x} = g - kv$$
$$\therefore 0 = g - kV_T$$
$$V_T = \frac{g}{k}\ \text{ms}^{-1}$$

ii α

$$\ddot{x} = -(g + kv)$$
$$\frac{dv}{dt} = -(g + kv)$$
$$\frac{dt}{dv} = -\frac{1}{g + kv}$$
$$t = -\int_{V_T}^v \frac{1}{g + kv}\,dv$$
$$= \frac{1}{k}\left[\ln(g + kv)\right]_v^{V_T}$$
$$= \frac{1}{k}\ln\left(\frac{g + kV_T}{g + kv}\right)$$
$$e^{kt} = \frac{g + kV_T}{g + kv}$$
$$ge^{kt} + ke^{kt}v = g + kV_T$$
$$ke^{kt}v = g - ge^{kt} + kV_T$$
$$v = \frac{g}{k}e^{-kt} - \frac{g}{k} + V_T e^{-kt}$$
$$= V_T e^{-kt} - V_T + V_T e^{-kt}$$
$$= V_T(2e^{-kt} - 1)$$

β $\frac{dx}{dt} = V_T(2e^{-kt} - 1)$

$$x = V_T\int_0^t (2e^{-kt} - 1)\,dt$$
$$= V_T\left[-\frac{2}{k}e^{-kt} - t\right]_0^t$$
$$= V_T\left[-\frac{2}{k}e^{-kt} - t + \frac{2}{k}\right]$$
$$= \frac{V_T}{k}(2 - kt - 2e^{-kt})$$

iii

Let $v = 0$ in (ii α)

$$\therefore 2e^{-kt} - 1 = 0$$
$$2 = e^{kt}$$
$$kt = \ln 2$$
$$t = \frac{1}{k}\ln 2$$

substituting in (ii β):

$$\therefore x_{\max} = \frac{V_T}{k}\left(2 - k\left(\frac{1}{k}\ln 2\right) - 2e^{-k\left(\frac{1}{k}\ln 2\right)}\right)$$
$$= \frac{V_T}{k}\left(2 - \ln 2 - \frac{2}{2}\right)$$
$$= \frac{V_T}{k}(1 - \ln 2)$$

91 **i**

$\ddot{x} = -\frac{k}{x^2}$

Let $x = R, \ddot{x} = -g$

$-g = -\frac{k}{R^2}$

$k = gR^2$

ii

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^2}{x^2}$

$\frac{1}{2}(v^2 - u^2) = -gR^2 \int_R^x x^{-2}\,dx$

$v^2 - u^2 = -2gR^2\left[-\frac{1}{x}\right]_R^x$

$= -2gR^2\left(-\frac{1}{x} + \frac{1}{R}\right)$

$\therefore v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$

iii

$v = \sqrt{u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)}$

when $u = \sqrt{2gR}$:

$\therefore \frac{dx}{dt} = \sqrt{\left(\sqrt{2gR}\right)^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)}$

$= \sqrt{\frac{2gR^2}{x}}$

$\frac{dt}{dx} = \sqrt{\frac{x}{2gR^2}}$

$t = \frac{1}{\sqrt{2gR^2}}\int_R^{4R} x^{\frac{1}{2}}\,dx$

$= \frac{1}{\sqrt{2gR^2}}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_R^{4R}$

$= \frac{2}{3\sqrt{2gR^2}}\left((4R)^{\frac{3}{2}} - R^{\frac{3}{2}}\right)$

$= \frac{\sqrt{2}}{3\sqrt{g}}\left(8\sqrt{R} - \sqrt{R}\right)$

$= \frac{7\sqrt{2R}}{3\sqrt{g}}$ s

92 **i**

$\dot{x}_0 = 20\sqrt{2}\cos 45° = 20$

$\dot{y}_0 = 20\sqrt{2}\sin 45° = 20$

$\frac{d\dot{x}}{dt} = -\frac{\dot{x}}{20}$

$\frac{1}{\dot{x}}\,d\dot{x} = -\frac{dt}{20}$

$\int_{20}^{\dot{x}} \frac{1}{\dot{x}}\,d\dot{x} = -\frac{1}{20}\int_0^t dt$

$\Big[\ln|\dot{x}|\Big]_{20}^{\dot{x}} = -\frac{1}{20}t$

$\ln|\dot{x}| - \ln(20) = -\frac{t}{20}$

$\ln|\dot{x}| = \ln(20) - \frac{t}{20}$

$\dot{x} = 20e^{-\frac{t}{20}}$

$\int_0^x \dot{x}\,d\dot{x} = 20\int_0^t e^{-\frac{t}{20}}\,dt$

$\Big[x\Big]_0^x = -400\left[e^{-\frac{t}{20}}\right]_0^t$

$x - 0 = -400\left(e^{-\frac{t}{20}} - 1\right)$

$x = 400\left(1 - e^{-\frac{t}{20}}\right)$

ii

$\frac{d\dot{y}}{dt} = -\frac{\dot{y}}{20} - 10$

$= -\frac{\dot{y} + 200}{20}$

$\frac{d\dot{y}}{\dot{y} + 200} = -\frac{dt}{20}$

$\int_{20}^{\dot{y}} \frac{d\dot{y}}{\dot{y} + 200} = -\frac{1}{20}\int_0^t dt$

$\Big[\ln|\dot{y} + 200|\Big]_{20}^{\dot{y}} = -\frac{1}{20}t$

$\ln|\dot{y} + 200| - \ln(220) = -\frac{t}{20}$

$\ln|\dot{y} + 200| = \ln 220 - \frac{t}{20}$

$\dot{y} + 200 = e^{-\frac{t}{20} + \ln 220}$

$\dot{y} = 220e^{-\frac{t}{20}} - 200$

$y = \int_0^t \left(220e^{-\frac{t}{20}} - 200\right)dt$

$= \left[-4400e^{-\frac{t}{20}} - 200t\right]_0^t$

$= -4400e^{-\frac{t}{20}} - 200t + 4400$

$= 4400\left(1 - e^{-\frac{t}{20}}\right) - 200t$

iii

$\frac{x}{400} = 1 - e^{-\frac{t}{20}}$

$e^{-\frac{t}{20}} = 1 - \frac{x}{400}$ (1)

$= \frac{400 - x}{400}$

$e^{\frac{t}{20}} = \frac{400}{400 - x}$

$t = 20\ln\left(\frac{400}{400 - x}\right)$ (2)

from (1) and (2):

$y = 4400 - 200\left(20\ln\left(\frac{400}{400 - x}\right)\right) - 4400\left(1 - \frac{x}{400}\right)$

$= 4400 - 4000\ln\left(\frac{400}{400 - x}\right) - 4400 + 11x$

$= 11x + 4000\ln\left(\frac{400 - x}{400}\right)$

92 …

iv

$$y = 11x + 4000\ln\left(\frac{400-x}{400}\right)$$
$$= 11x + 4000\ln(400-x) - 4000\ln 400$$
$$\frac{dy}{dx} = 11 - \frac{4000}{400-x}$$
$$= 11 - \frac{4000}{400-50}$$
$$= -0.428571$$
$$\therefore \tan\theta = -0.428571$$
$$\theta = -23°12'$$

93

i

$$\dot{x} = V\cos\alpha$$
$$x = Vt\cos\alpha$$
$$\ddot{y} = -10$$
$$\dot{y} - V\sin\alpha = -10\int_0^t dt$$
$$\dot{y} = V\sin\alpha - 10t$$
$$y = \int_0^t (V\sin\alpha - 10t)\,dt$$
$$= \left[Vt\sin\alpha - 5t^2\right]_0^t$$
$$= Vt\sin\alpha - 5t^2$$

ii

At the first chimney $x = m, y = h$

$$m = Vt\cos\alpha$$
$$\therefore t = \frac{m}{V\cos\alpha}$$
$$h = Vt\sin\alpha - 5t^2$$
$$\therefore h = V\left(\frac{m}{V\cos\alpha}\right)\sin\alpha - 5\left(\frac{m}{V\cos\alpha}\right)^2$$
$$h = m\tan\alpha - \frac{5m^2}{V^2}\sec^2\alpha$$
$$\frac{5m^2}{V^2}(1+\tan^2\alpha) = m\tan\alpha - h$$
$$V^2 = \frac{5m^2(1+\tan^2\alpha)}{m\tan\alpha - h}$$

iii

Similarly at the second chimney

$$V^2 = \frac{5n^2(1+\tan^2\alpha)}{n\tan\alpha - h}$$
$$\therefore \frac{5m^2(1+\tan^2\alpha)}{m\tan\alpha - h} = \frac{5n^2(1+\tan^2\alpha)}{n\tan\alpha - h}$$
$$\frac{m^2}{m\tan\alpha - h} = \frac{n^2}{n\tan\alpha - h}$$
$$m^2 n\tan\alpha - m^2 h = n^2 m\tan\alpha - n^2 h$$
$$\tan\alpha\,(m^2 n - n^2 m) = h(m^2 - n^2)$$
$$\tan\alpha = \frac{h(m+n)(m-n)}{mn(m-n)}$$
$$= \frac{h(m+n)}{mn}$$

94

i

$$\frac{dv}{dt} = 10$$
$$v = \int_0^{10} 10\,dt$$
$$= 10\left[t\right]_0^{10}$$
$$= 100\text{ ms}^{-2}$$

$$v = \int_0^t 10\,dt$$
$$= 10\left[t\right]_0^t$$
$$= 10t$$
$$x = \int_0^{10} 10t\,dt$$
$$= 5\left[t^2\right]_0^{10}$$
$$= 500\text{ ms}^{-2}$$

ii

$$(15+5)\ddot{x} = (15+5)g - 40v$$
$$\ddot{x} = g - 2v$$
$$\frac{dv}{dt} = g - 2v$$
$$\frac{dt}{dv} = \frac{1}{g-2v}$$
$$\int_{10}^t dt = \int_{100}^v \frac{1}{g-2v}\,dv$$
$$\left[t\right]_{10}^t = -\frac{1}{2}\left[\ln(g-2v)\right]_{100}^v$$
$$t - 10 = \frac{1}{2}(\ln(g-200) - \ln(g-2v))$$
$$2(t-10) = \ln\frac{g-200}{g-2v}$$
$$e^{2(t-10)} = \frac{g-200}{g-2v}$$
$$ge^{2(t-10)} - 2e^{2(t-10)}v = g - 200$$
$$2e^{2(t-10)}v = g\left(e^{2(t-10)} - 1\right) + 200$$
$$v = \frac{g}{2} - \frac{g}{2}e^{-2(t-10)} + 100e^{-2(t-10)}$$
$$v = 5 - 5e^{-2(t-10)} + 100e^{-2(t-10)}$$
$$v = 5 + 95e^{-2(t-10)}$$

iii

$$x = 500 + \int_{10}^{120}\left(5 + 95e^{-2(t-10)}\right)dt$$
$$= 500 + \left[5t - \frac{95}{2}e^{-2(t-10)}\right]_{10}^{120}$$
$$= 500 + \left(5(120) - \frac{95}{2}e^{-220}\right) - \left(5(10) - \frac{95}{2}\right)$$
$$= 500 + 600 - \frac{95}{2}e^{-220} - 50 + \frac{95}{2}$$
$$= 1097.5\text{ m}$$

The height above the ground is $2000 - 1097.5 = 902.5$ m.

iv

$$g - 2V_T = 0 \Rightarrow V_T = \frac{g}{2} = \frac{10}{2} = 5$$

The terminal velocity is 5 ms^{-1}.

95

i

amplitude $= \frac{6-3}{2} = 1.5$

period $= 2(11-5) = 12$ hours

ii

b is the centre of motion, so $\frac{3+6}{2} = 4.5$

a is normally the amplitude of the motion, so 1.5, but we make it negative in this case so that at $t = 0$ the cosine curve starts at a minimum. Alternatively we could use $1.5\cos(nt+\pi)$.

$T = \frac{2\pi}{n}$ so $12 = \frac{2\pi}{n} \to n = \frac{\pi}{6}$

iii

$$x = -1.5\cos\left(\frac{\pi t}{6}\right) + 4.5$$

$$\dot{x} = \frac{\pi}{4}\sin\left(\frac{\pi t}{6}\right)$$

$$\ddot{x} = \frac{\pi^2}{24}\cos\left(\frac{\pi t}{6}\right)$$

$$= -\left(\frac{\pi}{6}\right)^2\left(-1.5\cos\left(\frac{\pi t}{6}\right) + 4.5 - 4.5\right)$$

$$= -\left(\frac{\pi}{6}\right)^2(x-4.5)$$

$$= -n^2(x-b)$$

iv

For the first two times the depth is 4 metres:

$$x = 4$$

$$-1.5\cos\left(\frac{\pi t}{6}\right) + 4.5 = 4$$

$$-1.5\cos\left(\frac{\pi t}{6}\right) = -0.5$$

$$\cos\left(\frac{\pi t}{6}\right) = \frac{1}{3}$$

$$\frac{\pi t}{6} = \cos^{-1}\left(\frac{1}{3}\right), 2\pi - \cos^{-1}\left(\frac{1}{3}\right)$$

$$t = \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right), 12 - \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right)$$

$$= 2.35\ldots, 9.64..$$

$$= 2\,h\,21\,m, \quad 9\,h\,39\,m$$

The boat can first enter then leave the harbour from $7{:}21$ am to $2{:}39$ pm.

96

i

$$50\ddot{x} = 50g - 100v$$

$$\therefore \ddot{x} = g - 2v$$

$$\therefore 0 = g - 2V_{TC}$$

$$V_{TC} = \frac{g}{2}\ \text{ms}^{-1}$$

Cynthia's terminal velocity is $\frac{g}{2}$ ms^{-1}

ii

$$60\ddot{x} = 60g - \left(120v + \frac{480}{g}v^2\right)$$

$$\ddot{x} = g - 2v - \frac{8}{g}v^2$$

$$\therefore 0 = g - 2V_{TR} - \frac{8}{g}{V_{TR}}^2$$

$$\frac{8}{g}V_{TR}^2 + 2V_{TR} - g = 0$$

$$V_{TR} = \frac{-k \pm \sqrt{k^2 - 4\left(\frac{2k^2}{g}\right)(-g)}}{2\left(\frac{2k^2}{g}\right)}$$

$$= \frac{-k \pm 3k}{\frac{4k^2}{g}}$$

$$= \frac{g}{4}\ \text{ms}^{-1}\ (\text{since } V_{TR} > 0)$$

Rebel's terminal velocity is $\frac{g}{4}$ ms^{-1}

iii

$$\frac{dv}{dt} = g - 2v$$

$$\frac{dt}{dv} = \frac{1}{g-2v}$$

$$t = \int_0^{\frac{g}{6}} \frac{1}{g-2v}\,dv$$

$$= -\frac{1}{2}\Big[\ln(g-kv)\Big]_0^{\frac{g}{6}}$$

$$= \frac{1}{2}\left(\ln g - \ln\left(g - 2\left(\frac{g}{6}\right)\right)\right)$$

$$= \frac{1}{2}\left(\ln g - \ln\frac{2g}{3}\right)$$

$$= \frac{1}{2}\ln\frac{3}{2}$$

Cynthia has been in free fall for $\frac{1}{k}\ln\frac{3}{2}$ seconds

97 **i**

$$2000\frac{dv}{dt} = 5000 - 10v^2$$
$$\frac{dv}{dt} = \frac{500 - v^2}{200} \text{ ms}^{-2}$$

ii

$$\frac{dt}{dv} = \frac{200}{500 - v^2}$$
$$t = \int_0^v \frac{200}{500 - v^2}\,dv$$
$$= \frac{100}{\sqrt{500}}\int_0^v \left(\frac{1}{\sqrt{500} - v}\right.$$
$$= 2\sqrt{5}\left[-\ln\left(\sqrt{500} - v\right) + \ln\left(\sqrt{500}\right.\right.$$
$$= 2\sqrt{5}\left[\ln\left(\frac{\sqrt{500} + v}{\sqrt{500} - v}\right)\right]_0^v$$
$$= 2\sqrt{5}\left(\ln\left(\frac{\sqrt{500} + v}{\sqrt{500} - v}\right) - 0\right)$$
$$= 2\sqrt{5}\ln\left(\frac{\sqrt{500} + v}{\sqrt{500} - v}\right)$$
$$\frac{t}{2\sqrt{5}} = \ln\left(\frac{\sqrt{500} + v}{\sqrt{500} - v}\right)$$
$$e^{\frac{t}{2\sqrt{5}}} = \frac{\sqrt{500} + v}{\sqrt{500} - v}$$
$$\sqrt{500}e^{\frac{t}{2\sqrt{5}}} - ve^{\frac{t}{2\sqrt{5}}} = \sqrt{500} + v$$
$$v\left(1 + e^{\frac{t}{2\sqrt{5}}}\right) = \sqrt{500}\left(e^{\frac{t}{2\sqrt{5}}} - 1\right)$$
$$v = \sqrt{500}\left(\frac{e^{\frac{t}{2\sqrt{5}}} - 1}{e^{\frac{t}{2\sqrt{5}}} + 1}\right)$$

iii

At the terminal velocity $\frac{dv}{dt} = 0$

$$\frac{500 - v^2}{200} = 0$$
$$v^2 = 500$$
$$v = \sqrt{500}$$
$$= 22.36 \text{ m/s}$$

iv

$$t = 2\sqrt{5}\ln\left(\frac{\sqrt{500} + v}{\sqrt{500} - v}\right) \quad \text{from (ii)}$$
$$= 2\sqrt{5}\ln\left(\frac{\sqrt{500} + 0.99\sqrt{500}}{\sqrt{500} - 0.99\sqrt{500}}\right)$$
$$= 2\sqrt{5}\ln\left(\frac{1.99}{0.01}\right)$$
$$= 2\sqrt{5}\ln(199)$$

98 **i**

$$40\ddot{x} = -\left(400 + \frac{3v^2}{100}\right)$$
$$\ddot{x} = -\left(10 + \frac{3v^2}{4000}\right)$$
$$v\frac{dv}{dx} = -\frac{40000 + 3v^2}{4000}$$
$$\frac{dv}{dx} = -\frac{40000 + 3v^2}{4000v}$$
$$\frac{dx}{dv} = -\frac{4000v}{40000 + 3v^2}$$
$$x = -\int_{500}^v \frac{4000v}{40000 + 3v^2}\,dv$$
$$= \frac{2000}{3}\left[\ln(40000 + 3v^2)\right]_v^{500}$$
$$= \frac{2000}{3}\left(\ln 790000 - \ln(40000 + 3v^2)\right)$$
$$= \frac{2000}{3}\ln\left(\frac{790000}{40000 + 3v^2}\right)$$

ii

$$500 = \frac{2000}{3}\ln\left(\frac{790000}{40000 + 3v^2}\right)$$
$$e^{0.75} = \frac{790000}{40000 + 3v^2}$$
$$40000e^{0.75} + 3e^{0.75}v^2 = 790000$$
$$3e^{0.75}v^2 = 790000 - 40000e^{0.75}$$
$$v^2 = \frac{790000 - 40000e^{0.75}}{3e^{0.75}}$$
$$v = \sqrt{\frac{790000 - 40000e^{0.75}}{3e^{0.75}}}$$
$$= 333.25 \text{ ms}^{-1} \text{ (2 dp)}$$

The velocity of the missile is approximately 333 ms^{-1} when it hits the ship.

99 **i**

$\ddot{x} = -\frac{k}{x^2}$

Let $x = R, \ddot{x} = -g$

$\therefore -g = -\frac{k}{R^2}$

$\therefore k = gR^2$

$\therefore \ddot{x} = -\frac{gR^2}{x^2}$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^2}{x^2}$

$\frac{1}{2}(v^2 - u^2) = -gR^2 \int_R^x \frac{1}{x^2} dx$

$v^2 - u^2 = -2gR^2 \left[-\frac{1}{x}\right]_R^x$

$\therefore v^2 = \frac{2gR^2}{x} - 2gR + u^2$

$= u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

ii

at the highest point above the Earth $x = H + R$, $v = 0$

$\therefore 0 = u^2 + 2gR^2\left(\frac{1}{H+R} - \frac{1}{R}\right)$

$\frac{1}{H+R} - \frac{1}{R} = -\frac{u^2}{2gR^2}$

$\frac{1}{H+R} = \frac{1}{R} - \frac{u^2}{2gR^2}$

$= \frac{2gR - u^2}{2gR^2}$

$\therefore H + R = \frac{2gR^2}{2gR - u^2}$

$H = \frac{2gR^2 - R(2gR - u^2)}{2gR - u^2}$

$= \frac{u^2R}{2gR - u^2}$

iii

$H \to \infty$ as $2gR - u^2 \to 0$

$\therefore u^2 \to 2gR$

$u \to \sqrt{2gR}$

Thus the particle will escape from Earth if $u \geq \sqrt{2gR}$

100 **i**

$m\ddot{x} = -k\dot{x}$

$\frac{d\dot{x}}{dt} = -\frac{k}{m}\dot{x}$

$\frac{d\dot{x}}{\dot{x}} = -\frac{k}{m}\, dt$

$\int_{V\cos\theta}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\frac{k}{m}\int_0^t dt$

$\left[\ln\dot{x}\right]_{V\cos\theta}^{\dot{x}} = -\frac{k}{m}t$

$\ln\dot{x} - \ln(V\cos\theta) = -\frac{k}{m}t$

$\ln\dot{x} = \ln(V\cos\theta) - \frac{k}{m}t$

$\dot{x} = V\cos\theta\, e^{-\frac{k}{m}t}$

ii

$\frac{dx}{dt} = V\cos\theta\, e^{-\frac{k}{m}t}$

$x = V\cos\theta \int_0^t e^{-\frac{k}{m}t}\, dt$

$= V\cos\theta \times \left(-\frac{m}{k}\right)\left[e^{-\frac{k}{m}t}\right]_0^t$

$= \frac{mV\cos\theta}{k}\left(1 - e^{-\frac{k}{m}t}\right)$

iii

$m\ddot{y} = -k\dot{y} - mg$

$\frac{d\dot{y}}{dt} = -\frac{k}{m}\dot{y} - g$

$= -\frac{k\dot{y} + mg}{m}$

$\frac{dt}{d\dot{y}} = -\frac{m}{k\dot{y} + mg}$

$t = -\frac{m}{k}\int_{V\sin\theta}^{\dot{y}} \frac{k}{k\dot{y} + mg}\, d\dot{y}$

$= -\frac{m}{k}\left[\ln|k\dot{y} + mg|\right]_{V\sin\theta}^{\dot{y}}$

$= -\frac{m}{k}(\ln|k\dot{y} + mg| - \ln|kV\sin\theta + mg|)$

$-\frac{k}{m}t = \ln|k\dot{y} + mg| - \ln|kV\sin\theta + mg|$

$\ln|k\dot{y} + mg| = \ln|kV\sin\theta + mg| - \frac{k}{m}t$

$k\dot{y} + mg = (kV\sin\theta + mg)e^{-\frac{k}{m}t}$

$\dot{y} = \left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$

iv

$$\frac{dy}{dt} = \left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$$

$$\begin{aligned} y &= \int_0^t \left(\left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}\right)dt \\ &= \left[-\frac{m}{k}\left(\left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t}\right) - \frac{mgt}{k}\right]_0^t \\ &= -\frac{m}{k}\left(\left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t}\right) - \frac{mgt}{k} + \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right) \\ &= \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right)\left(1 - e^{-\frac{k}{m}t}\right) - \frac{mgt}{k} \end{aligned}$$

v

$$x = \frac{mV\cos\theta}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$\frac{kx}{mV\cos\theta} = 1 - e^{-\frac{k}{m}t} \qquad (1)$$

$$e^{-\frac{k}{m}t} = 1 - \frac{kx}{mV\cos\theta}$$

$$-\frac{k}{m}t = \ln\left(1 - \frac{kx}{mV\cos\theta}\right)$$

$$t = -\frac{m}{k}\ln\left(1 - \frac{kx}{mV\cos\theta}\right) \qquad (2)$$

sub (1), (2) in (iv):

$$\begin{aligned} y &= \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right)\left(\frac{kx}{mV\cos\theta}\right) + \frac{mg}{k}\left(\frac{m}{k}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)\right) \\ &= \frac{m}{k}\left(\frac{mg + kV\sin\theta}{k}\right)\left(\frac{kx}{mV\cos\theta}\right) + \frac{m^2g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right) \\ &= \left(\frac{mg + kV\sin\theta}{kV\cos\theta}\right)x + \frac{m^2g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right) \\ &= \left(\frac{mg}{kV\cos\theta} + \tan\theta\right)x + \frac{m^2g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right) \end{aligned}$$

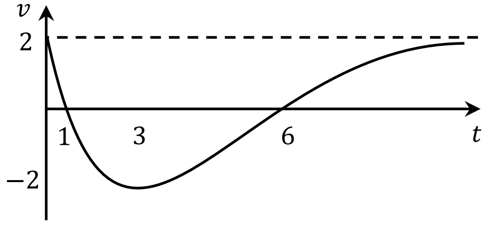
1 Prove that if $a^3 + 5$ is odd, then a is even.

2 **i** Show that $(1-2i)^2 = -3-4i$.
ii Hence solve the equation $x^2 - 5x + (7+i) = 0$

3 Find $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

4 Find the magnitude of $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

5 The graph below shows the velocity of a particle over time.



i At what time is the net force zero?
ii Describe what happens to the net force as $t \to \infty$.

6 Prove $2^{n+4} > 2n + 9$ for $n \geq 1$ by induction.

7 Let $z = 3 - 2i$ and $w = 1 + \sqrt{2}i$
i Find $|z|$
ii Express $\frac{w}{z}$ in the form $a + bi$ where a and b are real numbers

8 Find $\int \frac{x^2}{1+4x^2}\,dx$

9 Simplify $2\begin{pmatrix}1\\1\\4\end{pmatrix} - 3\begin{pmatrix}-1\\-3\\1\end{pmatrix}$

10 A body is projected vertically downwards from a height of $3R$ (from the centre of the Earth) with initial speed u. The acceleration due to gravity at any point on its path is inversely proportional to the square of its distance from the centre of the Earth, $\ddot{x} = -\frac{k}{x^2}$, where x is the distance from the centre of the Earth. Let the radius of the Earth be R, and the acceleration due to gravity at the surface be $-g$.

Prove that the speed at any position x is given by

$$v^2 = u^2 - \frac{2gR}{3} + \frac{2gR^2}{x}$$

11 Given P and Q below, for integral n, can we say $P \Rightarrow Q, P \Leftrightarrow Q,$ or $Q \Rightarrow P$?
P: n is odd
Q: n^2 is odd

12 **i** Find the two square roots of $2i$.
ii Solve $x^2 + 2x + \left(1 - \frac{i}{2}\right) = 0$

13 Use the substitution $u = 1 - \sqrt{x}$ to find $\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx$

14 Show the point $\begin{pmatrix} 5 \\ -3 \\ -5 \end{pmatrix}$ does not lie on $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

15 A particle is in SHM about $x = c$. Is the following statement correct?

The graph for the acceleration of the particle is the graph of displacement reflected about the horizontal axis and stretched vertically by a factor of n^2.

16 Prove $\frac{1}{1\times2} + \frac{1}{2\times3} + \ldots + \frac{1}{(n-1)n} = \frac{n-1}{n}$ for $n \geq 2$ by induction.

17 Given $z = r(\cos\theta + i\sin\theta)$, then $\left|\frac{\bar{z}}{z^2}\right|$ equals

A r

B r^2

C $\frac{1}{r}$

D $-r$

18 Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, find $\int \frac{1}{1-\cos x}\,dx$

19 Prove the following lines are parallel: $\underset{\sim}{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -6 \end{pmatrix}$

20 A body of mass 4 kg is moving in a horizontal straight line experiencing resistance of $2v^2$. If it is propelled by a force of 200 N what is its terminal velocity?

21 Prove $x + \frac{1}{x} \geq 2$

22 **i** Write $\sqrt{3} + i$ in polar form.
ii Hence evaluate $\left(\sqrt{3} + i\right)^6$

23 Find $\int \sin^{-1} x\,dx$

24 Sketch $x = 2, y = 2\cos t, z = 2\sin t$

25 A particle of mass 10 kg is projected at an angle of $\frac{\pi}{3}$ from a tall building, with $V = 100k$. What is the limit, R, of its horizontal displacement? The horizontal displacement is $x = \frac{mV\cos\theta}{k}\left(1 - e^{-\frac{k}{m}t}\right)$

26 Prove the following statement is false: $|2x + 1| \leq 5 \Rightarrow |x| \leq 2$

27 Simplify $e^{-i} \times 2e^{2i}$

28 Find $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

29 Show that $\begin{pmatrix}5\\-1\\1\end{pmatrix}$ lies on $\underset{\sim}{r} = \begin{pmatrix}6\\-3\\-2\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}$

30 A unit mass is released from rest at the surface in which the retardation on the mass is proportional to the square of the distance fallen (x). The net force for this motion is $g - kx^2$ Newtons, with the downward direction as positive. How far will the mass fall before it has constant velocity, for an instant? Express the answer in terms of k and g.

31 Prove $n^3 - 7n + 6$ is divisible by 3 for $n \geq 1$ by induction

32 Which if the following corresponds to the set of points in the complex plane defined by $|z + 2i| = |z|$?

A the point given by $z = -i$

B the line $\text{Im}(z) = -1$

C the circle with centre $-2i$ and radius 1

D the line $\text{Re}(z) = -1$

33 Which expression is equal to $\int x^2 \sin x\,dx$?

A $-x^2\cos x - \int 2x\cos x\,dx$

B $-2x\cos x + \int x^2\cos x\,dx$

C $-x^2\cos x + \int 2x\cos x\,dx$

D $-2x\cos x - \int x^2\cos x\,dx$

34 Sketch $x = \cos t\,, y = 2\sin t$

35 A particle is projected on level ground at an angle to the horizontal and experiences a net resistive force. Which of the following statements could be true?

A The magnitude of the angle of impact is equal to the angle of projection.

B The maximum height occurs after the particle has travelled more than half the horizontal range.

C The magnitude of the velocity of the particle at impact is greater than the velocity at projection

D The range of the particle would be less if it had the same velocity and angle at projection but a larger mass

36 Prove that there are no integers a and b such that $4a + 8b = 34$

37 If $z = 3 - i$ and $w = 2i - 1$ then $\bar{z} - w =$

A $4 - 3i$

B $4 - i$

C $2 - 3i$

D $4 + i$

38 Find $\int \sec^2(2x-1)\tan(2x-1)\ dx$

39 Prove the midsegment theorem, that the interval joining the midpoint of two sides of any triangle is parallel to and half the length of the third side

40 A particle is in Simple Harmonic Motion with equation of motion $x = a\cos(nt+\alpha)+c$. For what values of α will the particle initially be moving to the right? Assume $a, n > 0$ and $0 \le \alpha \le 2\pi$

41 Prove that $2\sqrt{3}-\sqrt{5} < \sqrt{2}$

42 The modulus and argument of $\sqrt{6}-i\sqrt{2}$ are

A $2\sqrt{2}$ and $\frac{\pi}{6}$

B $2\sqrt{2}$ and $-\frac{\pi}{6}$

C $2\sqrt{2}$ and $\frac{\pi}{3}$

D $2\sqrt{2}$ and $-\frac{\pi}{3}$

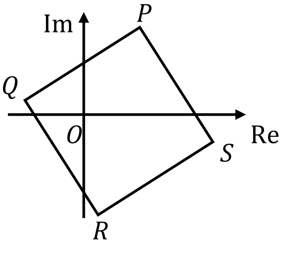
43 Evaluate $\int_{-1}^{1}\frac{1}{5-2t+t^2}dt$

44 Find the scalar product of $\begin{pmatrix}3\\-2\\0\end{pmatrix}$ and $\begin{pmatrix}-2\\1\\3\end{pmatrix}$

45 The velocity of a particle is given by $\dot{x} = x^2-1$ metres per second. What is the acceleration when the particle is at $x=2$?

46 Prove $2n < (n+2)!$ for $n \ge 1$ by induction

47 In the Argand diagram vectors $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}, \overrightarrow{OS}$ represent the complex numbers p, q, r, s respectively where $PQRS$ is a square.



Which of the following statements is correct?

A $q-s = i(p-r)$

B $q-p = i(s-r)$

C $s-q = i(r-p)$

D $q-r = i(p-s)$

48 Evaluate $\int_{-2}^{2}(x+x^3+x^5)\cos x\, dx$

49 A mass exerts a downward force of 1000 N. It is being held in a steady position by three ropes, exerting forces in Newtons of $(10,20,200)$, $(10,-20,300)$ and (a,b,c). Find the value of a, b and c.

50 A stone is thrown vertically upwards at a velocity of 20 ms^{-1}. Assume $g = 10$ m/s^2. Find the maximum height reached by the particle, given the equations of motion

$$y = Vt\sin\alpha - \frac{gt^2}{2} \quad \text{and} \quad x = Vt\cos\alpha$$

51 Find the negation of the following statement, and decide whether the negation is true or not.
$P: \exists m > 0, m^2 - 9 = 0$

52 Solve the quadratic equation $4z^2 + 4z + 5 = 0$.

53 Find $\int \frac{dx}{x^2+4x+5}$

54 Find the unit vector in the direction of $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$

55 Prove $\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

56 Prove $2^{n+4} > (n+4)^2$ for $n \geq 1$ by induction.

57 If $Z_1 = 5 - 2i$ and $Z_2 = 3 + 4i$ then $Z_1\overline{Z_2} =$

A $23 + 14i$

B $7 + 26i$

C $7 - 26i$

D $23 - 26i$

58 Evaluate $\int_0^1 x^3\sqrt{1-x^2}\,dx$

59 Prove the following lines are perpendicular: $\underset{\sim}{r} = \begin{pmatrix}-2\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix}1\\1\end{pmatrix} + \lambda\begin{pmatrix}-2\\-2\end{pmatrix}$

60 A particle of unit mass is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $4kv$ Newtons, its speed is v ms^{-1} and k is a positive constant. At time t seconds the particle has displacement x metres from a fixed point O on the line. If initially it was at the origin with velocity u, where $u > 0$, prove $x = \frac{u-v}{4k}$

61 Give a counterexample to prove the following statement is false.
P: There are no prime numbers divisible by 7

62 Find the conjugate of $z = 2e^{\frac{\pi}{4}i}$

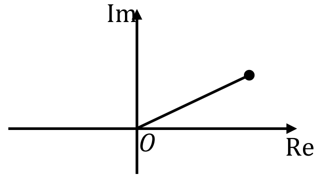
63 Find $\int \tan^3 x \sec x\,dx$

64 Is the triangle is formed by the points $A(1,1,1)$, $B(2,3,3)$ and $C(4,5,4)$ scalene, isosceles or equilateral?

65 A particle is in Simple Harmonic Motion, completing 3 full cycles per second. Prove that $n = 6\pi$.

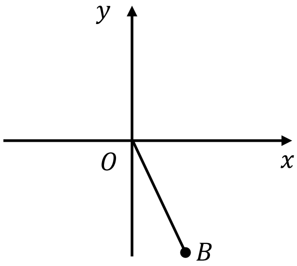
66 Prove $13^n - 4^n$ is divisible by 9 for $n \geq 1$ by induction.

67 If $\overrightarrow{OA} = z$ on the diagram below:

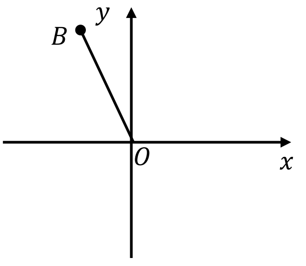


In which of the following diagrams does $\overrightarrow{OB}$ represent iz?

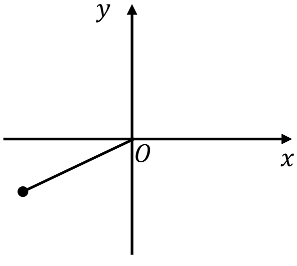
A



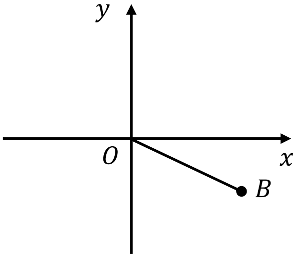
B



C



D



68 **i** Find a and b given that:

$$\frac{1-x}{(x+1)(x+2)} \equiv \frac{a}{x+1} + \frac{b}{x+2}$$

ii Hence find

$$\int \frac{1-x}{(x+1)(x+2)}\,dx$$

69 Prove $2a - b = 1$, given $\underset{\sim}{r} = \begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \underset{\sim}{q} = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ a \\ b \end{pmatrix}$, and that $\underset{\sim}{r}$ and $\underset{\sim}{q}$ are perpendicular.

70 A particle of unit mass is moving in a straight line under the action of a force, $F = x^2 + x + 1$. Find an expression for velocity as a function of its displacement x, if the particle starts from rest at $x = 1$.

71 Given $x > \sin x$ for $x > 0$, prove $\pi x - 2x^2 > \sin 2x$ for $0 < x < \frac{\pi}{2}$

72 **i** Express $1 + i\sqrt{3}$ and $1 + i$ in modulus-argument form.
ii Hence find $(1 + i\sqrt{3})(1 + i)$ in modulus-argument form.
iii Hence find the exact value of $\tan \frac{7\pi}{12}$.

73 Find $\int \tan^3 x\, dx$

74 Find a vector equation of the line through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

75 Prove that a particle where $x = 2\sin\left(2t + \frac{\pi}{3}\right) + 1$ is in Simple Harmonic Motion

76 Prove by contradiction that if the product of any two integers is odd, then both of them must be odd.

77 Given $3x + 2iy - ix + 5y = 7 + 5i$, where x and y are real numbers, then

A $x = -1, y = 2$

B $x = \frac{39}{11}, y = -\frac{8}{11}$

C $x = -\frac{3}{5}, y = \frac{22}{5}$

D $x = -11, y = 8$

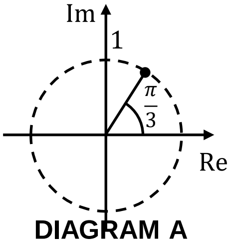
78 Rewrite $\frac{x+1}{x(x-1)^2}$ as the sum or difference of partial fractions

79 Find the angle to the nearest degree between the vectors $-\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$.

80 A particle of unit mass is moving in a straight line under the action of a force, $F = \mu^2 x$, where μ is constant. Given the particle starts from rest at the origin, prove $v = \mu x$.

81 Prove $2^n + 1 \leq 3^n$ for $n \geq 1$ by induction.

82 Diagram shows the complex number z represented in the Argand plane.



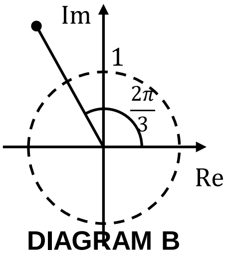


Diagram B shows:

A z^2

B $2iz$

C $-2z$

D $2z^2$

83 Find $\int x \ln x \, dx$

84 Prove that the longest diagonal of a rectangular prism with dimensions a, b and c is given by $\sqrt{a^2 + b^2 + c^2}$.

85 A particle is projected horizontally from a cliff in a medium where air resistance is negligible, travelling R metres horizontally before impact. If gravitational acceleration were to double, what would be the range travelled before impact.

A $\frac{R}{2}$

B $\frac{R}{\sqrt{2}}$

C $2R$

D $\sqrt{2}R$

86 Rewrite the following logic statement as an if-then statement.

$$[P(1) \wedge (\forall k \geq 1)[P(k) \Rightarrow P(k+1)]] \Rightarrow (\forall n \geq 1)\ P(n)$$

What do we know this statement as?

87 If w is a non-real cube root of unity the value $\frac{1}{1+w} + \frac{1}{1+w^2}$ is equal to

A -1 **B** 0

C 1 **D** None of these

88 For $n \geq 0$, let $I_n = \int_0^{\frac{\pi}{4}} \tan^{2n}\theta\, d\theta$

i Show that for $n \geq 1$, $I_n = \frac{1}{2n-1} - I_{n-1}$
ii Hence, or otherwise, calculate I_3

89 Sketch $x = \sin t\,, y = 1 - \cos^2 t$

90 A stone is thrown from the top of a cliff. Its parametric equations of motion are $x = 3t$ and $y = 5 + 2t - 5t^2$. What is its Cartesian equation?

91 Find the contrapositive of the following statement, and decide whether the contrapositive is true or not. P: All birds are ducks

92 If $z = 1 - \sqrt{3}i = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$, then what is the value of z^{21}?

93 For every integer $n \geq 0$ let $I_n = \int_1^{e^2} (\log_e x)^n\, dx$. Show that for $n \geq 1$ $I_n = e^2 2^n - nI_{n-1}$

94 Find a vector equation of the interval from $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

95 The speed v ms^{-1} of a particle moving in simple harmonic motion is given by $v^2 = 6 + 4x - 2x^2$, where displacement from a fixed point O is x m. What is the centre of motion?

A $x = -2$ **B** $x = -1$

C $x = 1$ **D** $x = 2$

96 Prove by contradiction that if a, b are integral and $a + b \geq 9$ then $a \geq 5$ or $b \geq 5$.

97 The square roots of $(3 - 4i)$ in the form $a + bi$ are

A $(-2 - i), (2 + i)$ **B** $(2 - i), (-2 + i)$

C $(1 + 2i), (-1 + 2i)$ **D** $(1 - 2i), (-1 + 2i)$

98 Using the substitution $x = \tan\theta$, or otherwise, find $\int \frac{1}{x\sqrt{x^2+1}}\, dx$

99 Given $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$, and $\overrightarrow{OA} = (1,1,1), \overrightarrow{OB} = (-2,3,2), \overrightarrow{OC} = (3,-1,4)$ and $\overrightarrow{OD} = (0,b,c)$, find the values of b and c.

100 A particle in SHM has speed v ms^{-1} given by $v^2 = -x^2 + 2x + 8$ where x is its displacement from the origin. What is the amplitude?

A 3 m **B** 8 m

C 2π m **D** 9 m

101 Give a counterexample to prove the following statement is false.
P: If $a^2 - b^2 > 0$, where a and b are real, then $a - b > 0$

102 Convert $e^{\frac{\pi}{6}i}$ from exponential into Cartesian form

103 Find $\int \frac{1}{\sqrt{16-4x^2}}\,dx$

104 Given $\underset{\sim}{a} = \overrightarrow{OA} = \begin{pmatrix}1\\1\\4\end{pmatrix}$ and $\underset{\sim}{b} = \overrightarrow{OB} = \begin{pmatrix}-1\\-3\\1\end{pmatrix}$ find $\left(\underset{\sim}{a} - \underset{\sim}{b}\right)\cdot\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

105 A ball of unit mass is projected vertically upwards with speed $\frac{g}{k}$. The acceleration acting against the ball is gravity plus air resistance proportional to its speed. Prove that the time (t) taken to reach the greatest height is $t = \frac{\ln 2}{k}$ seconds.

106 Prove $5^n - 2^n$ is divisible by 3 for $n \geq 1$ by induction.

107 Sketch the region for z in the Argand plane defined by $|z - 1 + i| < 2$ and $-\frac{\pi}{4} \leq \arg(z - 1 + i) \leq \pi$

108 Find $\int \frac{2}{x(x-1)}\,dx$

109 Show the point $\binom{1}{1}$ does not lie on $\underset{\sim}{r} = \binom{0}{4} + \lambda\binom{-2}{3}$

110 A cricket player hits a ball at a velocity of 60 ms^{-1} and the ball just clears a 5 metre high screen which is 80 m away. Find the two possible angles at which the ball could have been hit, to the nearest degree. Assume there is no air resistance and that $g = 10$ ms^{-2}).

The equation of motion is $y = -\frac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$

111 Prove by contradiction that if a^2 is even then a is even.

112 Express $z = \sqrt{3} + i$ in polar form.

A $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ **B** $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

C $2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6}$ **D** $2\cos\frac{\pi}{3} + 2i\sin\frac{\pi}{3}$

113 **i** Find a, b and c given that $\frac{-5x+2}{x^2(x-1)} \equiv \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}$

ii Hence evaluate $\int_2^3 \frac{-5x+2}{x^2(x-1)} dx$

114 Find the equations of the sphere with radius 2 and centre at the origin

115 At time t the displacement, x, of a particle satisfies $t = 1 - e^{-x}$. Find the acceleration of the particle as function of x.

116 Prove $1 - \sin 2x \geq 0$ for all x

117 Find the two square roots of $4e^{\frac{\pi}{3}i}$, expressing them in exponential form.

118 Find $\int \frac{x^2+2x}{x^2+2x+5} dx$

119 Sketch $x = \frac{1}{t}, y = t - 2$

120 A particle is projected upwards from ground level with initial velocity $\sqrt{\frac{g}{k}}$ ms^{-1}, where g is the acceleration due to gravity and k is a positive constant. The particle moves through the air with speed v ms^{-1} and experiences a resistive force. The acceleration of the particle is given by $\ddot{x} = -g - kv^2$. Do NOT prove this. The particle reaches a maximum height, H, before returning to the ground. Using $\ddot{x} = v\frac{dv}{dx}$, or otherwise, show that $H = \frac{\ln 2}{2k}$ metres.

121 Prove $n^2 > n + 1$ for $n \geq 2$ by induction

122 The solutions to the equation $z^3 = 8$ are $z =$

A $2, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

B $2, \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} + \frac{i\sqrt{3}}{2}$

C $2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$

D $2, 1 + i\sqrt{3}, 1 - i\sqrt{3}$

123 Evaluate $\int_0^3 x\sqrt[3]{3-x}\, dx$

124 Sketch $x = \cos t, y = 0, z = \sin t$

125 A particle of mass m kg reaches a terminal velocity of 10 m/s when propelled horizontally by a force F against a resistance of kv^2. If the same particle can reach a terminal velocity of 40 m/s when dropped vertically, against the same resistance of kv^2, find F in terms of m, assuming $g = 10$ m/s^2.

126 If $a > b$ for real a and b, prove that $a^2 + b^2 - 2(a+b) + 2 \geq 0$

127 Prove $2^{2n+1} + 3^{2n+1}$ is divisible by 5 for $n \geq 1$ by induction

128 On the Argand diagram, sketch the locus of the points z such that $|z+1| = |z-i|$

129 Let $t = \tan\frac{\theta}{2}$.

i Show that $\frac{dt}{d\theta} = \frac{1}{2}(1 + t^2)$

ii Show that $\sin\theta = \frac{2t}{1+t^2}$

iii Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \operatorname{cosec}\theta \; d\theta$

130 A cube has three of its edges along the axes. Two opposite vertices are at the origin and $(2,2,2)$. Find the equation of the sphere that just fits inside the cube.

131 Prove $n! > 2n\,,\ n \geq 4$ by induction

132 Let $z = -5 - 12i$ and $\omega = 2 - i$. Find in the form $x + iy$

i $(1 + i)\overline{\omega}$

ii $\frac{z}{2-3i}$

133 **i** Prove $\int_0^a f(a - x)dx = \int_0^a f(x)dx$

ii Hence find $\int_0^{\pi} x \sin x \, dx$

134 Sketch $x = \cos^2 t\,, y = \sin^2 t$

135 A body is moving in a horizontal straight line. At time t seconds, its displacement is x metres from a fixed point O on the line, and its acceleration is $-(v + v^2)$ where $v \geq 0$ is its velocity. The body is initially at O with velocity $u > 0$. Show that $t = \ln\left(\frac{u(1+v)}{v(1+u)}\right)$

136 Given P and Q below, can we say $P \Rightarrow Q, P \Leftrightarrow Q,$ or $Q \Rightarrow P$?

P: Quadrilateral $ABCD$ is a square

Q: Quadrilateral $ABCD$ is a rectangle.

137 Let $z = a + ib$ where a and b are real and non-zero. Which if the following is **not** true?

A $z + \bar{z}$ is real

B $\frac{z}{\bar{z}}$ is non-real

C $z^2 - (\bar{z})^2$ is real

D $z\bar{z}$ is real and positive

138 Find $\int \frac{x}{\sqrt{4x^2-16}} dx$

139 Given $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ find $\overrightarrow{OB}$.

140 A box sits on a slippery ramp which is inclined at an angle of $30°$ to the horizontal. If the box is initially moving down the ramp at 1 ms^{-1}, prove $v = \frac{gt+2}{2}$.

141 A particle falls from the top of a cliff, in a medium where there is no air resistance. Between $t = 1$ and $t = 2$ it will fall:

A as far as it did in the first second

B twice as far as it did in the first second

C three times as far as it did in the first second

D four times as far as it did in the first second

142 **i** Express $z = -3\sqrt{3} + 3i$ in modulus-argument form.
ii Hence find the smallest possible positive integer n so that z^n is real.

143 For every integer $n \geq 0$ $I_n = \int_0^1 x^n (x^2 - 1)^5 \, dx$. Use the recurrence formula $I_n = \frac{n-1}{n+11} I_{n-2}$ for $n \geq 2$ to evaluate $\int_0^1 x^3 (x^2 - 1)^5 \, dx$

144 Show that $\binom{1}{3}$ lies on $\underset{\sim}{r} = \binom{-7}{7} + \lambda \binom{-4}{2}$

145 A particle of unit mass falls from rest from the top of a cliff in a medium where the resistive force is kv . How far has it fallen when it reaches a speed half its terminal velocity, in terms of g and k?

146 Prove $1 + x \geq 2\sqrt{x}$ for $x \geq 0$

147 Find the cube roots of -1 in polar form.

148 Find $\int \sin^3 x \; dx$

149 Sketch $x = t^2, y = t - 1$

150 A particle in SHM has speed v ms^{-1} given by $v^2 = 64 - 16x^2$where x is its displacement from the origin. What is the period of the motion?

A $\frac{\pi}{8}$ s **B** $\frac{\pi}{4}$ s

C $\frac{\pi}{2}$ s **D** π s

1 Suppose a is odd
Let $a = 2n + 1$
$\therefore a^3 + 5 = (2n+1)^3 + 5$
$= 8n^3 + 12n^2 + 6n + 1 + 5$
$= 2(4n^3 + 6n^2 + 3)$
$= 2p$ for integral p since n is integral
$\therefore$ if a is odd then $a^3 + 5$ is even
$\therefore$ if $a^3 + 5$ is odd then a is even by contrapositive □

2 i
$(1-2i)^2 = 1 - 4i - 4 = -3 - 4i$

ii
$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(7+i)}}{2(1)} = \frac{5 \pm \sqrt{25 - 28 - 4i}}{2} = \frac{5 \pm \sqrt{-3-4i}}{2} = \frac{5 \pm (1-2i)}{2} = \frac{4+2i}{2}, \frac{6-2i}{2} = 2 + i, 3 - i$$

3
$$\int \frac{1}{\sqrt{9-4x^2}}dx = \frac{1}{2}\int \frac{2}{\sqrt{3^2-(2x)^2}}dx = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$$

4
$$\left|\underset{\sim}{u}\right| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$$

5 i
The net force is zero when the velocity is constant, which only occurs at the turning point at $t = 3$

ii
As $t \to \infty, v \to 2$, so the acceleration is approaching zero from the positive side, but never reaches zero. This means the net force also approaches zero from the positive side but never reaches it.

6 Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= 2^{1+4} = 32;$ RHS $= 2(1) + 9 = 11$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $2^{k+4} > 2k + 9$

RTP $P(k+1)$ $\quad 2^{k+5} > 2k + 11$

LHS $= 2^{k+5}$
$= 2(2^{k+4})$
$> 2(2k+9)$ from $P(k)$
$> 4k + 18$
$> 2k + 11 + 2k + 7$
$> 2k + 11$ since $k \geq 0$
$=$ RHS
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

7 i
$$|z| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$$

ii
$$\frac{w}{z} = \frac{1+\sqrt{2}i}{3-2i} \times \frac{3+2i}{3+2i} = \frac{3 + 2i + 3\sqrt{2}i - 2\sqrt{2}}{3^2 + 2^2} = \frac{3-2\sqrt{2}}{13} + \frac{2+3\sqrt{2}}{13}i$$

8
$$\int \frac{x^2}{1+4x^2}dx = \int \frac{\frac{1}{4}(1+4x^2) - \frac{1}{4}}{1+4x^2}dx = \frac{1}{4}\int\left(1 - \frac{1}{2}\times\frac{2}{1+4x^2}\right)dx = \frac{x}{4} - \frac{1}{8}\tan^{-1}(2x) + c$$

9
$$2\begin{pmatrix}1\\1\\4\end{pmatrix} - 3\begin{pmatrix}-1\\-3\\1\end{pmatrix} = \begin{pmatrix}2(1)-3(-1)\\2(1)-3(-3)\\2(4)-3(1)\end{pmatrix} = \begin{pmatrix}5\\11\\5\end{pmatrix}$$

10
$$\ddot{x} = -\frac{k}{x^2}$$
Let $\ddot{x} = -g, x = R$
$$-g = -\frac{k}{R^2}$$
$$k = gR^2$$
$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -gR^2x^{-2}$$
$$\frac{1}{2}(v^2 - u^2) = -gR^2\int_{3R}^{x} x^{-2}\,dx$$
$$v^2 - u^2 = 2gR^2\left[-\frac{1}{x}\right]_x^{3R}$$
$$v^2 = u^2 + 2gR^2\left(-\frac{1}{3R} + \frac{1}{x}\right)$$
$$v^2 = u^2 - \frac{2gR}{3} + \frac{2gR^2}{x}$$

11 Since the square of an odd number is also odd, and the square root of an odd perfect square is odd then both statements are equivalent, so $P \Leftrightarrow Q$

12 i
$2i = 2e^{\frac{\pi}{2}i}$
The square roots are $\pm\left(\sqrt{2}e^{\frac{\pi}{4}i}\right) = \pm(1+i)$

ii
$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)\left(1 - \frac{i}{2}\right)}}{2(1)} = \frac{-2 \pm \sqrt{4 - 4 + 2i}}{2} = \frac{-2 \pm (1+i)}{2} = \frac{-3-i}{2}, \frac{-1+i}{2}$$

13 $\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx$

$$u = 1 - x^{\frac{1}{2}}$$
$$du = -\frac{1}{2}x^{-\frac{1}{2}}dx$$
$$dx = -2\sqrt{x}du = -2(1-u)du$$

$$= \int \frac{1-u}{u} \times \left(-2(1-u)\right) du$$
$$= -2\int \frac{(1-u)^2}{u} du$$
$$= -2\int \left(\frac{1}{u} - 2 + u\right) du$$
$$= -2\ln|u| + 4u - u^2 + c$$
$$= -2\ln\left|1-\sqrt{x}\right| + 4(1-\sqrt{x}) - (1-\sqrt{x})^2 + c$$

14 $\begin{pmatrix} 5 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ -1-\lambda \\ 1+3\lambda \end{pmatrix}$

$$5 = 1 + 2\lambda \ \rightarrow \lambda = 2$$
$$-3 = -1 - \lambda \ \rightarrow \lambda = 2$$
$$-5\& = 1 + 3\lambda \ \rightarrow \lambda = -2$$

No value of λ corresponds $\begin{pmatrix} 5 \\ -3 \\ -5 \end{pmatrix}$, so it is not on the line

15 False. The graph of displacement needs to be moved vertically by $-c$ before the other transformations, so that the graph of acceleration is centred about $\ddot{x} = 0$.

16 Let $P(n)$ represent the proposition.

$P(2)$ is true since LHS $= \frac{1}{1\times 2} = \frac{1}{2}$; RHS $= \frac{2-1}{2} = \frac{1}{2}$

If $P(k)$ is true for some arbitrary $k \geq 2$ then

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{(k-1)k} = \frac{k-1}{k}$$

RTP $P(k+1)$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{(k-1)k} + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

$$\text{LHS} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{(k-1)k} + \frac{1}{k(k+1)}$$
$$= \frac{k-1}{k} + \frac{1}{k(k+1)} \quad \text{from } P(k)$$
$$= \frac{(k-1)(k+1)+1}{k(k+1)}$$
$$= \frac{k^2-1+1}{k(k+1)}$$
$$= \frac{k}{k+1}$$
$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction □

17 $\left|\frac{\bar{z}}{z^2}\right| = \frac{|\bar{z}|}{|z^2|} = \frac{|z|}{|z|^2} = \frac{r}{r^2} = \frac{1}{r}$

ANSWER (C)

18 $\int \frac{1}{1-\cos x}\,dx$

$$t = \tan\frac{x}{2}$$
$$dx = \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{1 - \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int \frac{1}{1+t^2-1+t^2}dt$$
$$= 2\int \frac{dt}{2t^2}$$
$$= \int t^{-2}\,dt$$
$$= -\frac{1}{t} + c$$
$$= -\cot\frac{x}{2} + c$$

19 $\begin{pmatrix} 4 \\ -6 \end{pmatrix} = -2\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ so the lines are parallel

20 $m\ddot{x} = 200 - 2v^2$

Let $\ddot{x} = 0, m = 4, v = v_T$

$$0 = 200 - 2v_T^2$$
$$v_T^2 = \frac{200}{2}$$
$$v_T = 10 \quad (v_T > 0)$$

21 $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0$

$$x - 2 + \frac{1}{x} \geq 0$$
$$x + \frac{1}{x} \geq 2$$

22 **i**

$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ by inspection (exact triangles)

ii

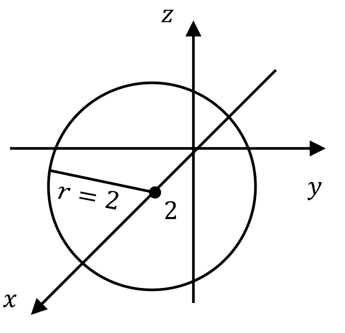
$$\left(\sqrt{3}+i\right)^6 = 2^6\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6$$
$$= 64(\cos\pi + i\sin\pi)$$
$$= 64(-1)$$
$$= -64$$

23 $\int \sin^{-1}x\,dx$

$$u = \sin^{-1}x \qquad \frac{dv}{dx} = 1$$
$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad v = x$$

$$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$$
$$= x\sin^{-1}x + \frac{1}{2}\int (-2x)(1-x^2)^{-\frac{1}{2}}dx$$
$$= x\sin^{-1}x + \frac{1}{2} \times 2(1-x^2)^{\frac{1}{2}} + c$$
$$= x\sin^{-1}x + \sqrt{1-x^2} + c$$

24 A circle of radius 2 parallel to the yz plane, centred at $(2,0,0)$.



25

$$R = \lim_{t\to\infty} \frac{10 \times 100k \times \cos\frac{\pi}{3}}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$
$$= \lim_{t\to\infty} 500\left(1 - e^{-\frac{k}{m}t}\right)$$
$$= 500 \text{ m}$$

26 Let $x = -3$

$|2(-3)+1| = 5 \leq 5$ yet $|-3| > 2$, so the statement is false

27 $e^{-i} \times 2e^{2i} = 2e^{i}$

28 $\int \frac{x^3}{\sqrt{1+x^2}}dx$

$= \int \frac{x^3}{u} \times \frac{u\,du}{x}$

$= \int x^2\,du$

$= \int (u^2-1)\,du$

$= \left(\frac{u^3}{3} - u\right) + c$

$= \frac{\sqrt{(1+x^2)^3}}{3} - \sqrt{1+x^2} + c$

$u^2 = 1 + x^2$ $2u\,du = 2x dx$ $dx = \frac{u\,du}{x}$

29 Let $\lambda = 1$

$\underset{\sim}{r} = \begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$ lies on $\underset{\sim}{r}$

30 $g - kx^2 = 0$

$kx^2 = g$

$x^2 = \frac{g}{k}$

$x = \sqrt{\frac{g}{k}}$

31 Let $P(n)$ represent the proposition.

$P(1)$ is true since $1^3 - 7(1) + 6 = 0$

If $P(k)$ is true for some arbitrary odd $k \geq 1$ then $k^3 - 7k + 6 = 3m$ for integral m

RTP $P(k+1)$ $(k+1)^3 - 7(k+1) + 6 = 3p$ for integral p

LHS $= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 6$
$= k^3 - 7k + 6 + 3k^2 + 3k - 6$
$= 3m + 3(k^2 + k - 2)$
$= 3p$ since m, k integral
= RHS
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

32 The perpendicular bisector of $(0, -2)$ and the origin, so $\mathrm{Im}(z) = -1$
ANSWER (B)

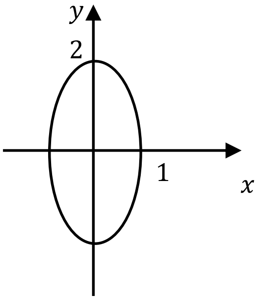
33 $\int x^2 \sin x\,dx$

$= -x^2 \cos x + 2\int x \cos x\,dx$

$u = x^2$	$\frac{dv}{dx} = \sin x$
$\frac{du}{dx} = 2x$	$v = -\cos x$

ANSWER (C)

34 An ellipse centred at the origin with a vertical semimajor axis of 2 and a horizontal semiminor axis of 1.



35 A False - the particle impacts at a steeper angle
B True – the horizontal velocity is higher during the ascent, so the maximum height occurs closer to impact than projection.
C False – the particle will be slower at impact on level ground
D False – if the particle had greater mass then air resistance would have less affect, so the range would increase for the same velocity and angle of projection.
ANSWER (B)

36 $4(a + 2b) = 34$
There are no integral solutions, as the LHS would then be a multiple of 4 while the RHS is not □

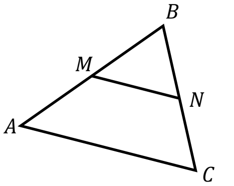
37 $\bar{z} - w = (3 + i) - (2i - 1) = 4 - i$
ANSWER (B)

38 $\int \sec^2(2x-1)\tan(2x-1)\;dx$

$= \frac{1}{2}\int 2\sec^2(2x-1) \times (\tan(2x-1))^1\,dx$

$= \frac{\tan^2(2x-1)}{4} + c$

39 Let M and N be the midpoints of AB and BC respectively in ΔABC.



$\therefore \overrightarrow{BM} = \frac{1}{2}\overrightarrow{AB}, \overrightarrow{BN} = \frac{1}{2}\overrightarrow{BC}$

$\overrightarrow{MN} = \overrightarrow{BM} + \overrightarrow{BN}$

$= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC})$

$= \frac{1}{2}\overrightarrow{AC}$

$\therefore$ the interval joining the midpoint of two sides of any triangle is parallel to and half the length of the third side

40 $\dot{x} = -an\sin(nt + \alpha)$
Let $t = 0, \dot{x} > 0$
$-an\sin\alpha > 0$
$\sin\alpha < 0$ since $a, n > 0$
$\pi < \alpha < 2\pi$

41 Suppose by contradiction that $2\sqrt{3} - \sqrt{5} \geq \sqrt{2}$
$\therefore \left(2\sqrt{3} - \sqrt{5}\right)^2 \geq \left(\sqrt{2}\right)^2$
$12 - 4\sqrt{15} + 5 \geq 2$
$15 \geq 4\sqrt{15}$
$225 \geq 240$ #
Which is a contradiction, so $2\sqrt{3} - \sqrt{5} < \sqrt{2}$ □

42 $|z| = \sqrt{(\sqrt{6})^2 + (-\sqrt{2})^2} = \sqrt{8} = 2\sqrt{2}$

$\arg z = \tan^{-1}\left(-\frac{\sqrt{2}}{\sqrt{6}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

ANSWER (B)

43 $\int_{-1}^{1} \frac{1}{5 - 2t + t^2}\,dt$

$= \int_{-1}^{1} \frac{dt}{(t-1)^2 + 2^2}$

$= \frac{1}{2}\left[\tan^{-1}\left(\frac{t-1}{2}\right)\right]_{-1}^{1}$

$= \frac{1}{2}(0 - \left(-\frac{\pi}{4}\right)$

$= \frac{\pi}{8}$

44 $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = 3(-2) - 2(1) + 0(3) = -8$

45 $\ddot{x} = v\frac{dv}{dx}$

$= (x^2 - 1)(2x)$

Let $x = 2$

$\ddot{x} = (2^2 - 1)(2(2))$

$= 12$ ms^{-2}

46 Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= 2(1) = 2$; RHS $= (1+2)! = 6$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $2k < (k+2)!$

RTP $P(k+1)$ $2(k+1) < (k+3)!$

LHS $= 2k + 2$

$< (k+2)! + 2$ from $P(k)$

$< (k+3)(k+2)!$ since $k > 1$

$< (k+3)!$

= RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

47 Diagonals of a square are of equal length and at right angles, so $\overrightarrow{QS} = i\overrightarrow{PR} \rightarrow q - s = i(p - r)$. This is always true.

ANSWER (A)

48 $x + x^3 + x^5$ is odd

$\cos x$ is even

$\therefore (x + x^3 + x^5)\cos x$ is odd

$\int_{-a}^{a} f(x)\,dx = 0$ when $f(x)$ is odd

$\therefore \int_{-2}^{2} (x + x^3 + x^5)\cos x\,dx = 0$

49 $\begin{pmatrix} 10 + 10 + a \\ 20 + (-20) + b \\ 200 + 300 + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1000 \end{pmatrix}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -20 \\ 0 \\ 500 \end{pmatrix}$

$a = -20, b = 0$ and $c = 500$

50 The formula for vertical displacement is a parabola, so use the axis of symmetry then substitute back in. The formula for x is added as a distraction.

Axis of symmetry is

$$-\frac{V\sin\alpha}{2\left(-\frac{g}{2}\right)} = \frac{V\sin\alpha}{g}$$

Maximum height is

$y = V\left(\frac{V\sin\alpha}{g}\right)\sin\alpha - \frac{g}{2}\left(\frac{V\sin\alpha}{g}\right)^2$

$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$

$= \frac{V^2\sin^2\alpha}{2g}$

When $V = 20, \alpha = 90°, g = 10$

$y_{\text{max}} = \frac{20^2}{20} = 20$ m

51 $\neg P: \forall m \leq 0, m^2 - 9 \neq 0$

The negation is false, since for $m = -3, m^2 - 9 = 0$

52

$$4z^2 + 4z + 5 = 0$$
$$(2z+1)^2 + 2^2 = 0$$
$$(2z+1)^2 - (2i)^2 = 0$$
$$(2z + 1 + 2i)(2z + 1 - 2i) = 0$$
$$2z = -1 \pm 2i$$
$$z = -\frac{1}{2} \pm i$$

53 $\int \frac{dx}{x^2 + 4x + 5}$

$= \int \frac{dx}{(x+2)^2 + 1^2}$

$= \tan^{-1}(x+2) + c$

54 $|\underset{\sim}{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

$\underset{\sim}{\hat{v}} = \frac{1}{\sqrt{3}}\underset{\sim}{i} - \frac{1}{\sqrt{3}}\underset{\sim}{j} + \frac{1}{\sqrt{3}}\underset{\sim}{k}$

55

$\ddot{x} = \frac{d^2x}{dt^2}$

$= \frac{d}{dt}\left(\frac{dx}{dt}\right)$

$= \frac{d}{dt}(v)$

$= \frac{dv}{dt}$ (1)

$= \frac{dv}{dx} \times \frac{dx}{dt}$

$= \frac{dv}{dx} \times v \rightarrow = v\frac{dv}{dx}$ (2)

$= \frac{dv}{dx} \times \frac{d}{dv}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ (3)

$\therefore \ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ from (1), (2), (3)

56 Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= 2^{1+4} = 32$; RHS $= (1+4)^2 = 25$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $2^{k+4} > (k+4)^2$

RTP $P(k+1)$ $\quad 2^{k+5} > (k+5)^2$

LHS $= 2^{k+5}$
$= 2(2^{k+4})$
$> 2(k+4)^2$ from $P(k)$
$> 2k^2 + 16k + 32$
$> (k^2 + 10k + 25) + (k^2 + 6k + 7)$
$> (k+5)^2$ since $k \geq 0$
$=$ RHS
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

57 $Z_1\overline{Z_2} = (5-2i)(3-4i)$
$= 15 - 20i - 6i - 8$
$= 7 - 26i$
ANSWER (C)

58

$$\int_0^1 x^3\sqrt{1-x^2}\,dx$$
$$= \int_1^0 x^3 \cdot u \cdot \left(-\frac{u\,du}{x}\right)$$
$$= \int_0^1 (1-u^2)u^2\,du$$
$$= \int_0^1 (u^2 - u^4)\,du$$
$$= \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1$$
$$= \left(\left(\frac{1}{3} - \frac{1}{5}\right) - (0-0)\right)$$
$$= \frac{2}{15}$$

$u^2 = 1 - x^2$ $2u\,du = -2x dx$ $dx = -\dfrac{u\,du}{x}$

59 $\begin{pmatrix}-1\\1\end{pmatrix}\cdot\begin{pmatrix}-2\\-2\end{pmatrix} = (-1)(-2) + (1)(-2) = 0$so the lines are perpendicular.

60

$$v\frac{dv}{dx} = -4kv$$
$$\frac{dv}{dx} = -4k$$
$$\frac{dx}{dv} = -\frac{1}{4k}$$
$$x = -\frac{1}{4k}\int_u^v dv$$
$$= \frac{1}{4k}\left[v\right]_v^u$$
$$= \frac{1}{4k}(u-v)$$
$$= \frac{u-v}{4k}$$

61 7 is prime and is divisible by 7.

62 $\bar{z} = 2e^{-\frac{\pi}{4}i}$

63

$$\int \tan^3 x \sec x\,dx$$
$$= \int \tan^2 x \times \tan x \sec x\,dx$$
$$= \int (\sec^2 x - 1)\tan x \sec x\,dx$$
$$= \int (\tan x \sec x\,(\sec x)^2 - \tan x \sec x)\,dx$$
$$= \frac{\sec^3 x}{3} - \sec x + c$$

64 $|\overrightarrow{AB}| = \sqrt{(2-1)^2 + (3-1)^2 + (3-1)^2} = 3$
$|\overrightarrow{AC}| = \sqrt{(4-1)^2 + (5-1)^2 + (4-1)^2} = \sqrt{34}$
$|\overrightarrow{BC}| = \sqrt{(4-2)^2 + (5-3)^2 + (4-3)^2} = 3$
$\therefore \Delta ABC$ is isosceles.

65

$$T = \frac{1}{3}$$
$$\therefore \frac{2\pi}{n} = \frac{1}{3}$$
$$n = 6\pi$$

66 Let $P(n)$ represent the proposition.

$P(1)$ is true since $13^1 - 4^1 = 9$

If $P(k)$ is true for some arbitrary odd $k \geq 1$ then $13^k - 4^k = 9m$ for integral m

RTP $P(k+2)$ $\quad 13^{k+1} - 4^{k+1} = 9p$ for integral p

LHS $= 13(13^k) - 4(4^k)$
$= 13(13^k - 4^k) + 9(4^k)$
$= 13(9m) + 9(4^k)$
$= 9(13m + 4^k)$
$= 9p$ since m, k integral
$=$ RHS
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

67 iz is OA rotated anticlockwise 90°.
ANSWER (B)

68 i

$$a = \frac{1-(-1)}{(-1)+2} = 2$$
$$b = \frac{1-(-2)}{(-2)+1} = -3$$

ii

$$\int \frac{1-x}{(x+1)(x+2)}\,dx$$
$$= \int \left(\frac{2}{x+1} - \frac{3}{x+2}\right)dx$$
$$= 2\ln|x+1| - 3\ln|x+2| + c$$

69

$$\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\a\\b\end{pmatrix} = 0$$
$$-1 + 2a - b = 0$$
$$2a - b = 1$$

70 $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = x^2 + x + 1$

$$\frac{1}{2}v^2 = \int_1^x (x^2 + x + 1)\,dx$$
$$v^2 = 2\left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_1^x$$
$$v^2 = 2\left(\left(\frac{x^3}{3} + \frac{x^2}{2} + x\right) - \left(\frac{1}{3} + \frac{1}{2} + 1\right)\right)$$
$$v^2 = \frac{2x^3}{3} + x^2 + 2x - \frac{11}{3}$$
$$v = \sqrt{\frac{2x^3 + 3x^2 + 6x - 11}{3}}$$

($v > 0$ given initial conditions)

71
$$x > \sin x \qquad (1)$$
$$\frac{\pi}{2} - x > \sin\left(\frac{\pi}{2} - x\right) \qquad (2)$$

$(1) \times (2)$:
$$\frac{\pi}{2}x - x^2 > \sin x \sin\left(\frac{\pi}{2} - x\right)$$
$$\frac{\pi}{2}x - x^2 > \sin x \cos x$$
$$\pi x - 2x^2 > 2\sin x \cos x$$
$$\therefore \pi x - 2x^2 > \sin 2x$$

72
i
by inspection (exact triangles):
$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

ii
$$(1 + i\sqrt{3})(1 + i)$$
$$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$= 2\sqrt{2}\left(\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right)$$
$$= 2\sqrt{2}\left(\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right)$$

iii
$$(1 + i\sqrt{3})(1 + i) = 1 + i + \sqrt{3}i - \sqrt{3}$$
$$= (1 - \sqrt{3}) + (1 + \sqrt{3})i$$
$$\tan\left(\frac{7\pi}{12}\right) = \frac{2\sqrt{2}\sin\left(\frac{7\pi}{12}\right)}{2\sqrt{2}\cos\left(\frac{7\pi}{12}\right)}$$
$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$$

73
$$\int \tan^3 x\,dx$$
$$= \int \tan^2 x \tan x\,dx$$
$$= \int (\sec^2 x - 1)\tan x\,dx$$
$$= \int (\sec^2 x \tan x - \tan x)\,dx$$
$$= \frac{\tan^2 x}{2} + \ln|\cos x| + c$$

74 $\underset{\sim}{r} = \begin{pmatrix}1\\1\end{pmatrix} + \lambda\begin{pmatrix}2-1\\-3-1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-4\end{pmatrix}$ is one possibility.

75
$$x = 2\sin\left(2t + \frac{\pi}{3}\right) + 1$$
$$\dot{x} = 4\cos\left(2t + \frac{\pi}{3}\right)$$
$$\ddot{x} = -8\sin\left(2t + \frac{\pi}{3}\right)$$
$$= -2^2\left(2\sin\left(2t + \frac{\pi}{3}\right) + 1 - 1\right)$$
$$= -2^2(x - 1)$$

The particle is in SHM.

76 Suppose by contradiction that at least one of the integers is even and the product is odd (*)
Let the integers be $2m + j$ and $2n$, where m, n are integral and $j = 0,1$.
$$p = (2m + j)(2n)$$
$$= 4mn + 2nj$$
$$= 2(2mn + nj)$$
$= 2p$ for integral p since m, n integral #
Which contradicts $(*)$ since the product cannot be odd and even.
∴ if the product of any two integers is odd, then both of them must be odd □

77
$$3x + 2iy - ix + 5y = (3x + 5y) + (2y - x)i$$
$\therefore 3x + 5y = 7$ (1)
$2y - x = 5$ (2)
$(1) + 3(2)$: $11y = 22 \rightarrow y = 2$
sub in (2): $2(2) - x = 5 \rightarrow x = -1$
ANSWER (A)

78
$$\frac{x + 1}{x(x - 1)^2} \equiv \frac{a}{x} + \frac{b}{x - 1} + \frac{c}{(x - 1)^2}$$
$$a = \frac{(0) + 1}{((0) - 1)^2} = 1$$
$$c = \frac{(1) + 1}{1} = 2$$
coefficients of x^2: $a + b = 0 \rightarrow b = -1$
$$\therefore \frac{x + 1}{x(x - 1)^2} \equiv \frac{1}{x} - \frac{1}{x - 1} + \frac{2}{(x - 1)^2}$$

79
$$\cos\theta = \frac{(-1,2,3)\cdot(1,-1,4)}{\sqrt{(-1)^2 + 2^2 + 3^2} \times \sqrt{1^2 + (-1)^2 + 4^2}}$$
$$= \frac{-1 - 2 + 12}{\sqrt{14} \times \sqrt{18}}$$
$$\theta = 55°$$

80
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \mu^2 x$$
$$\frac{1}{2}v^2 = \mu^2\int_0^x x\,dx$$
$$v^2 = 2\mu^2\left[\frac{x^2}{2}\right]_0^x$$
$$v^2 = \mu^2 x^2$$

$v = \mu x$ $\quad$ $v > 0$ given initial conditions

81 Let $P(n)$ represent the proposition.

$P(1)$ is true since LHS $= 2^1 + 1 = 3$; RHS $= 3^1 = 3$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $2^k + 1 \leq 3^k$

RTP $P(k+1)$ $\quad 2^{k+1} + 1 \leq 3^{k+1}$

$$\begin{aligned} \text{LHS} &= 2^{k+1} + 1 \\ &= 2\left(2^k + 1\right) - 1 \\ &\leq 2\left(3^k\right) - 1 && \text{from } P(k) \\ &\leq 3\left(3^k\right) - 3^k - 1 \\ &\leq 3^{k+1} - \left(3^k + 1\right) \\ &\leq 3^{k+1} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

82 The argument in B has doubled, so the answer involves z^2, A or D.
$|z| = 1$, so $|z^2| = 1$, so we need to use $2z^2$ to get a modulus greater than 1.
ANSWER (D)

83

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx = \frac{x^2 \ln x}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right) + c = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

$u = \ln x$	$\frac{dv}{dx} = x$
$\frac{du}{dx} = \frac{1}{x}$	$v = \frac{x^2}{2}$

84 Let the opposite corners of the rectangular prism be A and B. $\overrightarrow{AB} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$, as it equals the sum of the vectors along the length, width and height.
$\therefore \left|\overrightarrow{AB}\right| = \sqrt{a^2 + b^2 + c^2}$

85 Taking the point of projection as the origin, and downwards as positive, the trajectory is of the form $y = \frac{gx^2}{2}$, and at impact $h = \frac{gR^2}{2} \to R = \sqrt{\frac{2h}{g}}$. Doubling gravity has the same effect as stretching the parabolic path vertically by a factor of 2, so $y = gx^2$, and at impact $h = gx^2 \to x = \sqrt{\frac{h}{g}} = \frac{R}{\sqrt{2}}$.
ANSWER (B)

86 If $P(1)$ is true, and for all $k \geq 1$ if $P(k)$ is true then $P(k+1)$ is true, then for all $n \geq 1$ $P(n)$ is true.

We know this statement as mathematical induction, specifically for a statement that is true for all positive integers.

87 Note: $1 + w + w^2 = 0$

$$\begin{aligned} \frac{1}{1+w} + \frac{1}{1+w^2} &= \frac{1 + w^2 + 1 + w}{(1+w)(1+w^2)} \\ &= \frac{0 + 1}{1 + w^2 + w + w^3} \\ &= \frac{1}{0 + 1} \\ &= 1 \end{aligned}$$

ANSWER (C)

88 **i**

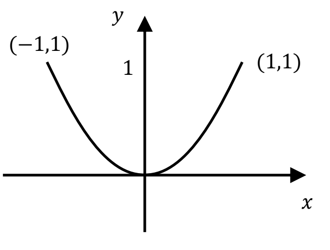
$$\begin{aligned} &I_n + I_{n-1} \\ &= \int_0^{\frac{\pi}{4}} \tan^{2n}\theta \, d\theta + \int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta \, d\theta \\ &= \int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta \left(1 + \tan^2\theta\right) d\theta \\ &= \int_0^{\frac{\pi}{4}} \sec^2\theta \, (\tan\theta)^{2n-2} \, d\theta \\ &= \left[\frac{\tan^{2n-1}\theta}{2n-1}\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2n-1} - 0 \\ &\therefore I_n = \frac{1}{2n-1} - I_{n-1} \end{aligned}$$

ii

$$\begin{aligned} I_3 &= \frac{1}{2(3)-1} - I_2 \\ &= \frac{1}{5} - \left(\frac{1}{2(2)-1} - I_1\right) \\ &= \frac{1}{5} - \frac{1}{3} + I_1 \\ &= -\frac{2}{15} + \int_0^{\frac{\pi}{4}} \tan^2\theta \, d\theta \\ &= -\frac{2}{15} + \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - 1\right) d\theta \\ &= -\frac{2}{15} + \Big[\tan\theta - \theta\Big]_0^{\frac{\pi}{4}} \\ &= -\frac{2}{15} + \left(\left(1 - \frac{\pi}{4}\right) - (0 - 0)\right) \\ &= \frac{13}{5} - \frac{\pi}{4} \end{aligned}$$

89 $-1 \leq \sin t \leq 1 \ \rightarrow -1 \leq x \leq 1$
$1 - \cos^2 t = \sin^2 t \ \rightarrow \ 0 \leq y \leq 1$

$$\begin{aligned} y &= 1 - \cos^2 t \\ &= \sin^2 t \\ &= x^2 \quad \text{for} \quad -1 \leq x \leq 1 \end{aligned}$$



90

$$\begin{aligned} t &= \frac{x}{3} \\ y &= 5 + 2\left(\frac{x}{3}\right) - 5\left(\frac{x}{3}\right)^2 \\ &= 5 + \frac{2x}{3} - \frac{5x^2}{9} \end{aligned}$$

91 $\neg Q \Rightarrow \neg P$: If it is not a duck then it is not a bird.
The negation is false, since some things that are not ducks are birds, as a counterexample a magpie is not a duck but it is a bird.

92

$$\begin{aligned} z^{21} &= 2^{21}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)^{21} \\ &= 2^{21}(\cos(-7\pi) + i\sin(-7\pi)) \\ &= 2^{21}(\cos\pi + i\sin\pi) \\ &= 2^{21}(-1) \\ &= -2^{21} \end{aligned}$$

ANSWER (B)

93 $I_n = \int_1^{e^2} (\log_e x)^n \, dx$

$u = (\log_e x)^n$	$\frac{dv}{dx} = 1$
$\frac{du}{dx} = n(\log_e x)^{n-1} \times \frac{1}{x}$	$v = x$

$= \left[x(\log_e x)^n\right]_1^{e^2} - n\int_1^{e^2} (\log_e x)^{n-1} \, dx$

$\therefore I_n = e^2 \times (2)^n - 0 - nI_{n-1}$

$I_n = e^2 2^n - nI_{n-1}$

94 $\underset{\sim}{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -4-(-2) \\ -3-3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -6 \end{pmatrix} \quad 0 \le \lambda \le 1$

is one possibility.

95 $v^2 = 6 + 4x - 2x^2$

$= -2(x^2 - 2x - 3)$

$= -2\left((x-1)^2 - 2^2\right)$

$= \left(\sqrt{2}\right)^2 (2^2 - (x-1)^2)$

The centre is $x = 1$

ANSWER (C)

96 Suppose by contradiction that $2\sqrt{3} - \sqrt{5} \ge \sqrt{2}$

$\therefore$ Suppose $a + b \ge 9$ and $a < 5$ and $b < 5$ *

$\therefore a + b \le 4 + 4$ since a, b integral

≤ 8 #

Which contradicts $(*)$ since $a + b$ cannot be ≥ 9 and ≤ 8, hence $a \ge 5$ or $b \ge 5$.

97 Use the formula since it is multiple choice

$\pm\left(\sqrt{\frac{|z| + \text{Re}(z)}{2}} + \text{sgn}\left(\text{Im}(z)\right)\sqrt{\frac{|z| + Re(z)}{2}}\, i\right)$

$= \pm\left(\sqrt{\frac{\sqrt{(3)^2 + (-4)^2} + 3}{2}} - \sqrt{\frac{\sqrt{(3)^2 + (-4)^2} - 3}{2}}\, i\right)$

$= \pm(2 - i)$

ANSWER (B)

98 $\int \frac{1}{x\sqrt{x^2+1}} dx$

$x = \tan\theta$
$dx = \sec^2\theta d\theta$

$= \int \frac{1}{\tan\theta\sqrt{\tan^2\theta + 1}} \times \sec^2\theta \, d\theta$

$= \int \frac{1}{\tan\theta \sec\theta} \times \sec^2\theta \, d\theta$

$= \int \frac{\sec\theta}{\tan\theta} d\theta$

$= \int \frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta} d\theta$

$= \int \text{cosec}\,\theta \, d\theta$

$= -\ln|\text{cosec}\,\theta + \cot\theta| + c$

99 $\overrightarrow{AB} = k\overrightarrow{CD}$

$\begin{pmatrix} -2-1 \\ 3-1 \\ 2-1 \end{pmatrix} = k\begin{pmatrix} 0-3 \\ b+1 \\ c-4 \end{pmatrix}$

$\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = k\begin{pmatrix} -3 \\ b+1 \\ c-4 \end{pmatrix}$

$-3 = -3k \quad \to \quad k = 1$

$\therefore 2 = b + 1 \quad \to \quad b = 1$

$1 = c - 4 \quad \to \quad c = 5$

100 $v^2 = -x^2 + 2x + 8$

$= -(x^2 - 2x - 8)$

$= -\left((x-1)^2 - 3^2\right)$

$= 1^2(3^2 - (x-1)^2)$

The amplitude is 3 m

ANSWER (A)

101 Let $a = -2, b = 1$

$(-2)^2 - (1)^2 = 3 > 0, -2 - 1 = -3 < 0$

102 $e^{\frac{\pi}{6}i} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i \times \frac{1}{2}$

103 $\int \frac{1}{\sqrt{16 - 4x^2}} \, dx$

$= \frac{1}{2}\int \frac{2}{\sqrt{4^2 - (2x)^2}} dx$

$= \frac{1}{2}\sin^{-1}\frac{2x}{4} + c$

$= \frac{1}{2}\sin^{-1}\frac{x}{2} + c$

104 $\left(\underset{\sim}{a} - \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

$= \left|\underset{\sim}{a} - \underset{\sim}{b}\right|^2$

$= \left|\begin{pmatrix} 1+1 \\ 1+3 \\ 4-1 \end{pmatrix}\right|^2$

$= 2^2 + 4^2 + 3^2$

$= 29$

105 $\ddot{x} = -(g + kv)$

$\frac{dv}{dt} = -(g + kv)$

$\frac{dt}{dv} = -\frac{1}{g + kv}$

$t = -\int_{\frac{g}{k}}^{0} \frac{1}{g + kv} dv$

$= \frac{1}{k}\left[\ln(g + kv)\right]_0^{\frac{g}{k}}$

$= \frac{1}{k}\left(\ln(g + g) - \ln g\right)$

$= \frac{\ln 2}{k}$

106 Let $P(n)$ represent the proposition.

$P(1)$ is true since $5^1 - 2^1 = 3$

If $P(k)$ is true for some arbitrary odd $k \ge 1$ then $5^k - 2^k = 3m$ for integral m

RTP $P(k+1) \quad 5^{k+1} - 2^{k+1} = 3p$ for integral p

LHS $= 5(5^k) - 2(2^k)$

$= 5(5^k - 2^k) + 3(2^k)$

$= 5(3m) + 3(2^k) from P(k)$

$= 3(5m + 2^k)$

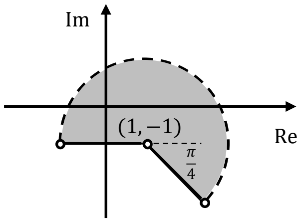
$= 3p \quad$ since m, k integral

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \ge 1$ by induction

107 This is intersection of the inside of the circle centred at $(1,-1)$ with radius 2, with the sector with the same centre, with argument clockwise from $-\frac{\pi}{4}$ to π.



108
$$\int \frac{2}{x(x-1)}dx$$
$$= 2\int\left(\frac{1}{x-1}-\frac{1}{x}\right)dx$$
$$= 2\Big(\ln|x-1| - \ln|x|\Big) + c$$
$$= 2\ln\left|\frac{x-1}{x}\right| + c$$

109 $\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-2\lambda\\4+3\lambda\end{pmatrix}$

$1 = -2\lambda \to \lambda = -\frac{1}{2}$

$1 = 4 + 3\lambda \to \lambda = -1$

No value of λ corresponds $\begin{pmatrix}1\\1\end{pmatrix}$, so it is not on the line.

110
$$5 = -\frac{10\times 80^2}{2\times 60^2}(1+\tan^2\theta) + 80\tan\theta$$
$$5 = -\frac{80}{9}(1+\tan^2\theta) + 80\tan\theta$$
$$45 = -80 - 80\tan^2\theta + 720\tan\theta$$
$$80\tan^2\theta - 720\tan\theta + 125 = 0$$
$$\tan\theta = \frac{720\pm\sqrt{720^2-4(80)(125)}}{2(80)}$$
$$= 0.17709\ldots, 08.82290\ldots$$
$$\theta = 10°, 84°$$

111 Suppose a^2 is even and a is odd (*)
Let $a = 2k+1$ for integral k.
$\therefore a^2 = (2k+1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
$= 2p + 1$ for integral p since k is integral #
Which contradicts (*) since a^2 cannot be even and odd,
$\therefore$ if a^2 is even then a is even □

112
$$|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$$
$$\arg z = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$
$$\therefore z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
ANSWER (C)

113 **i**
$$b = \frac{-5(0)+2}{(0)-1} = -2$$
$$c = \frac{-5(1)+2}{(1)^2} = -3$$
coefficients of x^2: $a + c = 0 \to a = 3$

ii
$$\int_2^3 \qquad dx$$
$$= \int_2^3\left(\qquad \right)dx$$
$$= \left[3\ln|x| + \frac{2}{x} - 3\ln|x-1|\right]_2^3$$
$$= \left(3\ln 3 + \frac{2}{3} - 3\ln 2\right) - (3\ln 2 + 1 + 0)$$
$$= 3\ln\frac{3}{4} - \frac{1}{3}$$

114 $x^2 + y^2 + z^2 = 4$ or $\left|\begin{pmatrix}x\\y\\z\end{pmatrix}\right| = 4$

115
$$\frac{dt}{dx} = e^{-x}$$
$$v = e^x$$
$$\ddot{x} = v\frac{dv}{dx}$$
$$= e^x(e^x)$$
$$= e^{2x}$$

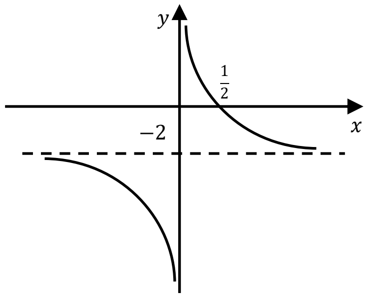
116 LHS $= 1 - \sin 2x$
$= \sin^2 x + \cos^2 x - 2\sin x\cos x$
$= (\sin x - \cos x)^2$
≥ 0 since $\mathbb{R}^2 \geq 0$

117 The square roots are
$$\pm\left(\sqrt{4}e^{\left(\frac{\pi}{3}\div 2\right)i}\right)$$
$$= \pm\left(2e^{\frac{\pi}{6}i}\right)$$
$$= 2e^{\frac{\pi}{6}i}, 2e^{\left(\frac{\pi}{6}-\pi\right)i}$$
$$= 2e^{\frac{\pi}{6}i}, 2e^{-\frac{5\pi}{6}i}$$

118
$$\int\frac{x^2+2x}{x^2+2x+5}dx$$
$$= \int\frac{x^2+2x+5-5}{x^2+2x+5}dx$$
$$= \int\left(1 - \frac{5}{(x+1)^2+2^2}\right)dx$$
$$= x - \frac{5}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$$

119 $t = \frac{1}{x} \rightarrow \ y = \frac{1}{x} - 2$

A standard hyperbola moved down two units



120 $v\frac{dv}{dx} = -(g + kv^2)$

$\frac{dv}{dx} = -\frac{g + kv^2}{v}$

$\frac{dx}{dv} = -\frac{v}{g + kv^2}$

$x = -\frac{1}{2k}\int_{\sqrt{\frac{g}{k}}}^{0} \frac{2kv}{g + kv^2}\,dv$

$= \frac{1}{2k}\left[\ln(g + kv^2)\right]_0^{\sqrt{\frac{g}{k}}}$

$= \frac{1}{2k}\left(\ln\left(g + k\left(\frac{g}{k}\right)\right) - \ln g\right)$

$= \frac{1}{2k}\ln 2$

$= \frac{\ln 2}{2k}$ metres

121 Let $P(n)$ represent the proposition.

$P(2)$ is true since LHS $= 2^2 = 4$; RHS $= 2 + 1 = 3$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $k^2 > k + 1$

RTP $P(k+1)$ $(k+1)^2 > k + 2$

LHS $= k^2 + 2k + 1$

$> k + 1 + 2k + 1$ from $P(k)$

$> 3k + 2$

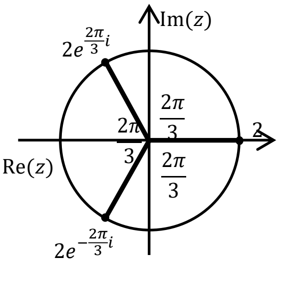
$> k + 2$ since $k > 2$

= RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction □

122 Sketching the roots we see:



Using exact triangles and reflection we can see the roots are $2, -1 + \sqrt{3}i, -1 - \sqrt{3}i$

ANSWER (C)

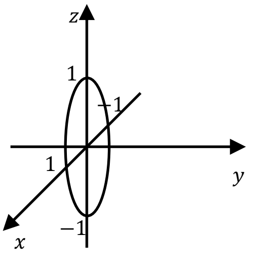
123 $\int_0^3 x\sqrt[3]{3 - x}\,dx = \int_0^3 (3 - x)\sqrt[3]{x}\,dx$

$= \int_0^3 \left(3x^{\frac{1}{3}} - x^{\frac{4}{3}}\right)dx$

$= \left[3 \times \frac{3}{4}x^{\frac{4}{3}} - \frac{3}{7}x^{\frac{7}{3}}\right]_0^3$

$= \frac{9\sqrt[3]{81}}{4} - \frac{3\sqrt[3]{2187}}{7}$

124 A unit circle on the xz plane, centred at the origin.



125 Horizontally:

$F - k(10)^2 = 0$

$F = 100k$

Vertically:

$m(10) - k(40)^2 = 0$

$10m - 1600k = 0$

$m = 160k$

$k = \frac{m}{160}$

$\therefore F = 100\left(\frac{m}{160}\right)$

$= \frac{5m}{8}$

126 LHS $= a^2 + b^2 - 2a - 2b + 2$

$= a^2 - 2a + 1 + b^2 - 2b + 1$

$= (a - 1)^2 + (b - 1)^2$

$\geq 0 + 0$

$= 0$

$\therefore a^2 + b^2 - 2(a + b) + 2 \geq 0$

127 Let $P(n)$ represent the proposition.

$P(1)$ is true since $2^{2(1)+1} + 3^{2(1)+1} = 2^3 + 3^3 = 35 = 5(7)$ which is divisible by 5.

If $P(k)$ is true for some arbitrary $k \geq 1$ then $2^{2k+1} + 3^{2k+1} = 5m$ for integral m

RTP $P(k+1)$ $2^{2k+3} + 3^{2k+3} = 5p$ for integral p

LHS $= 4(2^{2k+1}) + 9(3^{2k+1})$

$= 4(2^{2k+1} + 3^{2k+1}) + 5(3^{2k+1})$

$= 4(5m) + 5(3^{2k+1})$ from $P(k)$

$= 5(4m + 3^{2k+1})$

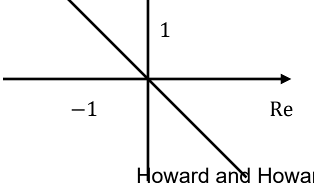
$= 5p$ since m, k integral

= RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

128 This is the perpendicular bisector of $(-1, 0)$ and $(0, 1)$

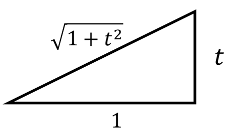


129 **i**

$$t = \tan\frac{\theta}{2}$$
$$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$$
$$= \frac{1}{2}\left(1 + \tan^2\frac{\theta}{2}\right)$$
$$= \frac{1}{2}(1 + t^2)$$

ii

$$\tan\frac{\theta}{2} = \frac{t}{1}$$
$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$$
$$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$$



$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2 \times \frac{t}{\sqrt{1+t^2}} \times \frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

iii

$$\int \operatorname{cosec}\theta \; d\theta$$
$$= \int \frac{1+t^2}{2t} \times \frac{2dt}{1+t^2} \quad \text{from (i)}$$
$$= \int \frac{dt}{t}$$
$$= \ln|t| + c$$
$$= \ln\left|\tan\frac{\theta}{2}\right| + c$$

130 The sphere is centred at $(1,1,1)$ with radius 1, so

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$$

131 Let $P(n)$ represent the proposition.

$P(4)$ is true since LHS $= 4! = 24$; RHS $= 2(4) = 8$

If $P(k)$ is true for some arbitrary $k \geq 4$ then $k! > 2k$

RTP $P(k+1)$ $\qquad (k+1)! > 2k + 2$

LHS $= (k+1) \times k!$
$\geq (k+1) \times 2k$ $\qquad$ from $P(k)$
$\geq k(2k+2)$
$\geq 2k + 2$ for $k \geq 4$
$=$ RHS
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 4$ by induction $\square$

132 **i**

$$(1+i)\bar{\omega} = (1+i)(2+i)$$
$$= 2 + i + 2i - 1$$
$$= 1 + 3i$$

ii

$$\frac{z}{2-3i} = -\frac{5+12i}{2-3i} \times \frac{2+3i}{2+3i}$$
$$= -\frac{10 + 15i + 24i - 36}{2^2 + 3^2}$$
$$= -\frac{-26 + 39i}{13}$$
$$= 2 - 3i$$

133 **i**

$$\int_0^a f(a-x)\,dx$$
$$= \int_a^0 f(u)\,(-du)$$
$$= \int_0^a f(u)\,du$$
$$= \int_0^a f(x)\,dx$$

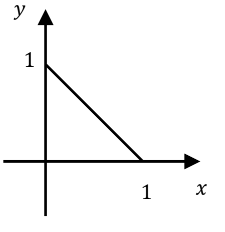
$u = a - x$
$du = -dx$
$dx = -du$

ii

$$\int_0^\pi x\sin x\,dx = \int_0^\pi (\pi - x)\sin(\pi - x)\,dx$$
$$= \int_0^\pi (\pi - x)\sin x\,dx \quad \text{since } \sin(\pi - x)$$
$$= \pi\int_0^\pi \sin x\,dx - \int_0^\pi x\sin x\,dx$$
$$\therefore 2\int_0^\pi x\sin x\,dx = -\pi\Big[\cos x\Big]_0^\pi$$
$$\int_0^\pi x\sin x\,dx = -\frac{\pi}{2}(-1-1)$$
$$= \pi$$

134 $-1 \leq \cos t \leq 1 \;\rightarrow 0 \leq \cos^2 t \leq 1 \;\rightarrow 0 \leq x \leq 1$
Similarly $0 \leq y \leq 1$

$\sin^2 t + \cos^2 t = 1 \;\rightarrow x + y = 1$ for $0 \leq x, y \leq 1$



135

$$\frac{dv}{dt} = -(v + v^2)$$
$$\frac{dt}{dv} = -\frac{1}{v(1+v)}$$
$$t = -\int_u^v \frac{1}{v(1+v)}\,dv$$
$$= -\int_u^v \left(\frac{1}{v} - \frac{1}{1+v}\right)dv$$
$$= -\Big[\ln v - \ln(1+v)\Big]_u^v$$
$$= -\Big(\ln v - \ln(1+v)\Big) - \Big(\ln u - \ln(1+u)\Big)$$
$$= \ln\left(\frac{u(1+v)}{v(1+u)}\right)$$

136 A square is a rectangle, but a rectangle is not necessarily a square, so $P \Rightarrow Q$

137 $z + \bar{z} = a + ib + a - ib = 2a$ which is real
$\frac{z}{\bar{z}} = \frac{a+ib}{a-ib} \times \frac{a+ib}{a+ib} = \frac{a^2 - b^2 + 2abi}{a^2 + b^2}$ which is non-real
$z^2 - (\bar{z})^2 = (a+ib)^2 - (a-ib)^2$
$= (a^2 - b^2 + 2abi) - (a^2 - b^2 - 2abi) = 4abi$ which is non−real
$z\bar{z} = (a+ib)(a-ib) = a^2 + b^2$ which is real and positive
ANSWER (C)

138 $\int \frac{x}{\sqrt{4x^2-16}}dx$

$= \frac{1}{8}\int (8x)(4x^2-16)^{-\frac{1}{2}}$

$= \frac{1}{8}\times 2(4x^2-16)^{\frac{1}{2}} + c$

$= \frac{\sqrt{4x^2-16}}{4} + c$

$= \frac{\sqrt{x^2-4}}{2} + c$

139 $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}-2\\1\\3\end{pmatrix}$

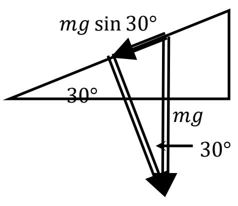
$= \begin{pmatrix}-1\\3\\4\end{pmatrix}$

140 $m\ddot{x} = mg\sin 30°$

$\frac{dv}{dt} = \frac{g}{2}$

$v - 1 = \frac{gt}{2}$

$v = \frac{gt+2}{2}$ ms^{-1}



141 $y = 5t^2$ assuming $g = 10$

$y_1 = 5, y_2 = 20$

$\frac{20-5}{5} = 3$

ANSWER (C)

142 **i**

$|z| = \sqrt{\left(-3\sqrt{3}\right)^2 + 3^2} = 6$

$\arg z = \pi - \tan^{-1}\left(\frac{-3}{3\sqrt{3}}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$

$\therefore z = 6\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$

ii

$z^n = 6^n\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)^n$

$= 6^n\left(\cos\left(\frac{5n\pi}{6}\right) + i\sin\left(\frac{5n\pi}{6}\right)\right)$

For z^n to be real $\frac{5n\pi}{6} = k\pi$ for integral k

$\therefore \frac{5n}{6}$ must be integral. $n = 6$ is the smallest positive value for which this occurs.

143

$\int_0^1 x^3(x^2-1)^5\,dx$

$= I_3$

$= \frac{3-1}{3+11}I_1$

$= \frac{1}{7}\int_0^1 x(x^2-1)^5\,dx$

$= \frac{1}{14}\int_0^1 2x(x^2-1)^5\,dx$

$= \frac{1}{14}\left[\frac{x^2-1}{6}\right]_0^1$

$= \frac{1}{14}\left(0 + \frac{1}{6}\right)$

$= \frac{1}{84}$

144 Let $\lambda = -2$

$\underset{\sim}{r} = \begin{pmatrix}-7\\7\end{pmatrix} - 2\begin{pmatrix}-4\\2\end{pmatrix} = \begin{pmatrix}-7+8\\7-4\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$

$\therefore \begin{pmatrix}1\\3\end{pmatrix}$ lies on $\underset{\sim}{r}$

145 $\ddot{x} = g - kv$

$0 = g - kV_T \Rightarrow V_T = \frac{g}{k}$

$v\frac{dv}{dx} = g - kv$

$\frac{dv}{dx} = \frac{g-kv}{v}$

$\frac{dx}{dv} = \frac{v}{g-kv}$

$x = \int_0^{\frac{g}{2k}} \frac{v}{g-kv}dv$

$= \int_0^{\frac{g}{2k}} \frac{-\frac{1}{k}(g-kv) + \frac{g}{k}}{g-kv}dv$

$= \int_0^{\frac{g}{2k}} \left(-\frac{1}{k} - \frac{g}{k^2}\times\frac{-k}{g-kv}\right)dv$

$= \left[\frac{v}{k} + \frac{g}{k^2}\ln|g-kv|\right]_{\frac{g}{2k}}^{0}$

$= \left(0 + \frac{g}{k^2}\ln g\right) - \left(\frac{g}{2k^2} + \frac{g}{k^2}\ln\frac{g}{2}\right)$

$= \frac{g}{k^2}\left(\ln 2 - \frac{1}{2}\right)$

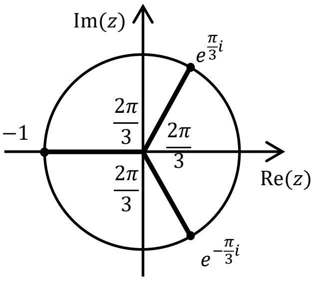
146 $(1-\sqrt{x})^2 \geq 0$

$1 - 2\sqrt{x} + x \geq 0$

$1 + x \geq 2\sqrt{x}$

147 -1 is a cube root of -1, and the other roots must be spaced by $\frac{2\pi}{3}$ as shown. The cube roots of -1 are $-1, e^{\frac{\pi}{3}i}$ and $e^{-\frac{\pi}{3}i}$.

The third root could be written as $e^{\frac{5\pi}{3}i}$ instead.



148 $\int \sin^3 x\ dx$

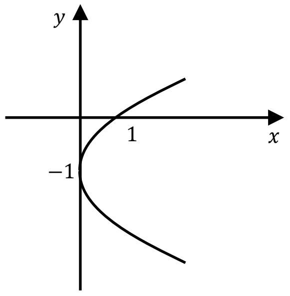
$= \int (1-\cos^2 x)\sin x\,dx$

$= \int (\sin x - \sin x(\cos x)^2)\,dx$

$= -\cos x + \frac{\cos^3 x}{3} + c$

149 $t = \pm\sqrt{x} \rightarrow y = \pm\sqrt{x} - 1$

A concave right parabola moved down one unit



150 $v^2 = 64 - 16x^2$

$= -16(x^2 - 4)$

$= 4^2(2^2 - x^2)$

$n = 4 \rightarrow T = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

ANSWER (C)

REVISION EXERCISE 8 – MIXED TOPICS – MEDIUM

1 Consider the following sequence, with terms in the form $(k+1)^2-1$ where k is a positive integer

$$3, 8, 15, 24, 35, 48$$

Prove that the product of any two consecutive terms of the above sequence can be written as the product of 4 consecutive numbers.

2 Which expression is a correct factorisation of $z^3 - i$?

A $(z-i)(z^2+iz+1)$ **B** $(z+i)(z^2-iz-1)$

C $(z+i)(z-i)^2$ **D** $(z+i)^3$

3 Find $\int \frac{1}{\sqrt{25-9x^2}}dx$

4 Show that $\underset{\sim}{u} = (2,3,6), \underset{\sim}{v} = (3,-6,2)$ and $\underset{\sim}{w} = (6,2,-3)$ are mutually perpendicular.

5 The position vector, $\underset{\sim}{r}(t)$, of a particle at time t is given by $\underset{\sim}{r}(t) = (e^{-t}\sin t)\underset{\sim}{i} + (e^{-t}\cos t)\underset{\sim}{j}$. Show that the speed of the particle at time t is ke^{-t}, where k is a constant that should be determined.

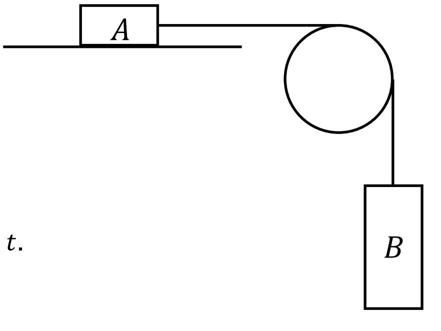
6 Given $x > \ln(1+x)$ for all $x > 0$, prove that $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} > \ln(1+n)$ for $n \geq 1$ by induction.

7 Find the real values of x and y for which $\frac{x}{i} - \frac{y}{1+i} = -1-3i$

8 Find $\int e^x e^{e^x}\,dx$

9 Prove that the point that divides $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m{:}n$ is $P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

10 Particle A of mass m kg and Particle B of mass $2m$ kg are connected by a light inextensible string passing over a frictionless pulley. Particle A is on a slippery horizontal surface. Initially the particles are at rest.
After Particle A has travelled x metres to the right it is travelling at v metres per second. Find the x in terms of g and t.



11 Prove by contrapositive that if m is an integer and m^2 is not divisible by 4 then m is odd.

12 **i** Express $-2+2i$ in modulus-argument form.
ii Simplify $(-2+2i)^{8k}$, where k is an integer.

13 Find $\int \frac{x^6+3x^2-1}{x^3+1}dx$

14 Relative to a fixed origin O, the point A has position vector $10\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, and the point B has position vector $8\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$. The line ℓ passes through the points A and B.

i Find the vector $\overrightarrow{AB}$.
ii Find a vector equation for the line ℓ.
iii The point C has position vector $3\underset{\sim}{i} + 12\underset{\sim}{j} + 3\underset{\sim}{k}$. The point P lies on ℓ. Given that the vector $\overrightarrow{CP}$ is perpendicular to ℓ, find the position vector of the point P.

15 A particle is oscillating between $x = 1$ and $x = 5$ in Simple Harmonic Motion. The particle is initially at $x = 3$, the equilibrium position, moving to the left. The particle takes 3 seconds to first reach the leftmost extremity. Find the simplest expression for its displacement as a function of time.

16 Each of n famous mathematics students (where $n \geq 2$) shakes hands with all the other mathematics students once each.
i How many handshakes will there be?
ii Prove that your answer to (i) is correct using induction.

17 Simplify $-(4e^{3i})$, leaving your answer in exponential form

18 Find $\int_0^1 \frac{x+2}{x^2+7x+12} dx$

19 Find the Cartesian equation of the following curve, leaving your answer with y as the subject.

$$x = \frac{1}{\operatorname{cosec}\theta}, y = \cot\theta$$

20 A particle moves in a straight line with its position x metres at time t seconds given by $x = 2 + \sin 4t + \sqrt{3}\cos 4t$.
i Prove that the particle is in simple harmonic motion.
ii By expressing $\sin 4t + \sqrt{3}\cos 4t$ in the form $R\sin(4t + \alpha)$, find the amplitude and equilibrium position of the particle.
iii Find the maximum speed of the particle.

21 Prove by contradiction that if n is an integer and $n^3 - 1$ is odd then n is even.

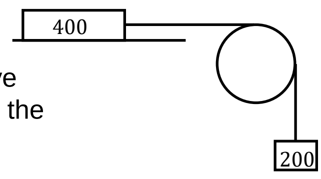
22 The cube roots of unity are $z_1 = 1, z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

i Prove that each non-real cube root of unity is the square root of the other.
ii Let ω be the second square root of z_3, Find ω, leaving the answer in Cartesian form.
iii Find the smallest positive value of n that solves $\omega^n = 1$

23 Find $\int \frac{e^{2x}}{\sqrt{e^x+1}} dx$ using the substitution $u^2 = e^x + 1$ or otherwise.

24 The points in the three dimensional space that satisfy $x^2 + y^2 = 1$ describe what 3d solid?

25 A 400 gram package lying on a rough horizontal surface is attached to a horizontal string which passes over a smooth pulley. The movement of the package is opposed by a frictional force of μR, where R is the normal reactive force of the table on the package. When a mass of 200 grams is attached to the other end of the string, the package is on the point of moving.
Find μ, the coefficient of friction.



26 Prove $(3n+1)\times 7^n - 1$ is divisible by 9 for $n \geq 1$, given $7^{n+1}+2$ is divisible by 3 for $n \geq 1$, by induction.

27 Let $z = 3(\cos\theta + i\sin\theta)$

i Find $\overline{1-z}$

ii Express the imaginary part of $\frac{1}{1-z}$ in terms of θ

28 Find $\int(\tan^4 x - \sec^4 x)\,dx$

29 If $\underset{\sim}{u} = 5\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k}, \underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k}$ then show that $\underset{\sim}{u} + \underset{\sim}{v}$ is perpendicular to $\underset{\sim}{u} - \underset{\sim}{v}$.

30 A particle of mass 1 kg is projected vertically upwards from ground level with a velocity of u ms^{-1}. The particle is subject to a constant gravitational force and a resistance which is proportional to twice the square of its velocity v ms^{-1}, (with k being the constant of proportionality). Let x be the displacement in metres from the ground after t seconds and let g be the acceleration due to gravity. Which of the following expressions gives the maximum height reached by the particle?

A $\displaystyle\int_u^0 \frac{v}{g+2kv^2}\,dv$

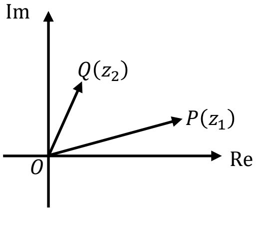
B $\displaystyle\int_u^0 \frac{v}{g-2kv^2}\,dv$

C $\displaystyle\int_0^u \frac{v}{g+2kv^2}\,dv$

D $\displaystyle\int_0^u \frac{v}{g-2kv^2}\,dv$

31 Prove that the product of two integers is even if and only if one or both are even.

32 The points P and Q in the first quadrant represent the complex numbers z_1 and z_2 respectively, as shown in the diagram above. Which statement about the complex number $z_2 - z_1$ is true?



A It is represented by the vector QP

B Its principal argument lies between $\frac{\pi}{2}$ and π

C Its real part is positive

D Its modulus is greater than $|z_1 + z_2|$

33 Prove $\int \operatorname{cosec} x\,dx = \ln\left|\tan\frac{x}{2}\right| + c$ using the substitution $t = \tan\frac{x}{2}$.

34 If $\underset{\sim}{u} = (2,2,3), \underset{\sim}{v} = (-1,2,1)$ and $\underset{\sim}{w} = (3,1,0)$ and $\underset{\sim}{u} + \lambda\underset{\sim}{v}$ is perpendicular to $\underset{\sim}{w}$ find λ

35 One of the Egyptian pyramids is 130 m high and the length of each side of its square base is 250 m. Is it possible to throw a stone with initial speed 25 ms^{-1} from the top of the pyramid so that it strikes the ground beyond the base? Assume $g = 10$ ms^{-2} and ignore air resistance. You may assume $y = \tan\theta\, x - \frac{gx^2}{2V^2}(1+\tan^2\theta) + h$ is the trajectory of a particle projected from a height of h at an angle to the horizontal θ with initial velocity V.

36 Prove that $\sin(n\pi + x) = (-1)^n \sin x$ for $n \geq 1$ by induction

37 **i** On an Argand diagram, sketch the locus of the point P representing the complex number z which moves so that $|z-2| = 1$.
ii Find the range of possible values of $|z|$ and $\arg(z)$
iii The points P_1 and P_2 such that OP_1 and OP_2 are tangents to the locus, (O is the origin) represent the complex numbers z_1 and z_2 respectively. Express z_1 and z_2 in modulus-argument form.
iv Evaluate ${z_1}^2 + {z_2}^2$, give your answer in its simplest form.

38 Find $\int 2^x \sin x \, dx$ using Integration by parts

39 A vector in $\mathbb{R}^3$ makes angles of α, β and γ with the x, y and z axes respectively. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

40 The velocity of a particle projected vertically into the air where air resistance is proportional to velocity is $\dot{y} = \left(\frac{mg}{k} + V\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$. Given $\frac{m}{k} = 0.1$ and $V = g = 10$, find the time to maximum height.

41 Prove $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$ for $x, y > 0$.

42 Which if the following cannot be the argument of a complex number z such that $z^9 = -1 + i$?

A $\dfrac{11\pi}{36}$

B $\dfrac{\pi}{12}$

C $\dfrac{29\pi}{36}$

D $\dfrac{19\pi}{36}$

43 **i** If $I_{m,n} = \int x^m \ln^n x \, dx$ prove $I_{m,n} = \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1} I_{m,n-1}$
ii Hence find $\int x^2 \ln^2 x \, dx$

44 Consider the two lines

$$\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \qquad \ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix} -5 \\ 15 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

i Find the point of intersection, A, of ℓ_1 and ℓ_2.
ii Find the angle between ℓ_1 and ℓ_2, to the nearest degree.
iii Show that the point $B\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$ lies on ℓ_1.
iv Find the shortest distance from B to ℓ_2, to 2 dp.

45 The velocity of a particle is given by $\underset{\sim}{v} = -2\cos(2t)\,\underset{\sim}{i} + \underset{\sim}{j}$ m/s, $t \geq 0$ seconds. Find the acceleration when the particle first has a speed of $\sqrt{3}$ m/s.

46 Prove by induction that for $n \geq 1$

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

47 **i** By using de Moivre's theorem, show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$
ii Hence show that $\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}$, where $t = \tan\theta$
iii Hence find the general solutions of the equation $3\tan\theta - \tan^3\theta = 0$

48 For what values of a is the following integral defined?

$$\int_0^2 \frac{dx}{1-\sqrt{x-a}}$$

49 A curve is given by $\underset{\sim}{r}(t) = \sec(t)\,\underset{\sim}{i} + \frac{1}{\sqrt{2}}\tan(t)\,\underset{\sim}{j}$. Show that its Cartesian equation is $x^2 - 2y^2 = 1$.

50 A mathematician stands on a set of scales in a stationary elevator, which reads a mass of 50 kg. They press the button for a higher floor, and on the journey the reading on the scales varies between 45 kg and 55 kg. If $g = 10\ \text{ms}^{-2}$, what was the acceleration of the elevator when it started to ascend?

51 Prove $x^4 + x^2y + 4y^2 \geq 5x^2y$

52 Given the cube roots of unity are $1, \omega$ and ω^2, find the sum of the reciprocals of the roots.

53 Find $\int \frac{2}{25x^2-10x+10}\,dx$

54 The position of a particle at any time t is given by:

$$\underset{\sim}{r}(t) = (2\cos t + 1)\underset{\sim}{i} + (\sin 2t + \sin t)\underset{\sim}{j}, \qquad t \geq 0$$

i Find the velocity of the particle at $t = \frac{\pi}{6}$.
ii Find the value of k for which $3k\underset{\sim}{i} + 4\underset{\sim}{j}$ is perpendicular to $\frac{d\underset{\sim}{r}}{dt}$ at $t = \frac{\pi}{6}$.
iii Show that if $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$ then $\sin(t) = \frac{y}{x}$ and $\cos(t) = \frac{1}{2}(x-1)$.
iv Hence show that the Cartesian equation of the path is $4y^2 = 3x^2 + 2x^3 - x^4$.

55 A particle is oscillating between A and B, 4 m apart, in Simple Harmonic Motion. The time for a particle to travel from B to A and back is 10 seconds. Find the velocity and acceleration at M, the midpoint of OB where O is the centre of AB. Assume that the positive direction is measured from A towards B.

56 Prove $\frac{d^n}{dx^n}(xe^{2x}) = 2^{n-1}(2x+n)e^{2x}$ for $n \geq 1$ by induction.

57 **i** Factorise $z^5 + 1 = 0$ over the real field.
ii List the roots of $z^5 + 1 = 0$ in $r\text{cis}\,\theta$ form.
iii Deduce that $2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} - 1 = 0$

58 Find $\int x\cos(x^2+4)\,dx$

59 Points $A(2,2,5), B(1,-1,-4), C(3,3,10)$ and $D(8,6,3)$ are the vertices of a pyramid with a triangular base.
i Calculate the lengths AB and AC, and the angle BAC, to the nearest minute.
ii Show that $\overrightarrow{AD}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

60 A particle is moving in a straight line and its position x, in metres from the origin O at time t seconds is given by $x = 3\cos 2t + 4\sin 2t + 2$
i Express $3\cos 2t + 4\sin 2t$ in the form $R\cos(2t - \alpha)$ where $0 \leq \alpha \leq \frac{\pi}{2}$ and $R > 0$.
ii Prove that the particle is undergoing simple harmonic motion.
iii Find the maximum speed of the particle. When does the particle first reach this maximum speed? Provide your answer to 2 decimal places.

61 Prove $3n^2+3n+1<2^n,\ n\geq 8$ by induction.

62 For the complex number $w=1-i\sqrt{3}$:
i Find $|w|$ and $\arg(w)$
ii Express $\overline{w}, w^2$ and $\frac{1}{w}$ in the form $a+ib$.

63 Find $\int \frac{e^{2x}-e^x}{e^x+1}dx$

64 Prove that if $\left(\underset{\sim}{c}-\underset{\sim}{b}\right)\cdot\underset{\sim}{a}=0$ and $\left(\underset{\sim}{c}-\underset{\sim}{a}\right)\cdot\underset{\sim}{b}=0$ then $\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\cdot\underset{\sim}{c}=0$

65 Find the constant acceleration produced by a force of $1000\ \mathrm{N}$ acting horizontally on a mass of $50\ \mathrm{kg}$ if it is opposed by a constant frictional force of $100\ \mathrm{N}$.

66 Prove that if 1 is added to the product of any 4 consecutive positive integers, the resulting number will always be a square number.

67 Let $z=1-3i$ and $w=2+i.$
i Express zw in the form $a+ib.$
ii Express zw in modulus-argument form.
iii Hence find x if $\frac{\sqrt{2}(\cos x-i\sin x)}{2+i}=\frac{1-3i}{5}$

68 Find $\int \frac{e^x+1}{e^{2x}-5e^x+6}dx$ using partial fractions.

69 Find a vector equation for the line through $(4,3)$ with gradient $m=-1$

70 A particle of unit mass experiences a resistive force, in Newtons, of 5% of the square of its velocity when it moves through the air. The particle is projected vertically upwards from a point A with velocity u metres per second. The highest point reached is B, directly above A. Assume that $g=10\ \mathrm{ms}^{-2}$ and take upwards as the positive direction.

i Show that the acceleration of the particle as it rises is given by
$$\ddot{x}=-\frac{v^2+200}{20}$$
ii Show that the distance metres of the particle from A as it rises is given by
$$x=10\log_e\left(\frac{200+u^2}{200+v^2}\right)$$

71 The nth Fermat number, F_n, is defined by $F_n=2^{2^n}+1$ for $n=0,1,2,3$. Prove by mathematical induction, that for all positive integers: $F_0\times F_1\times F_2\times\ldots\times F_{n-1}=F_n-2$

72 Find r,θ such that $e^{i\pi}-e^{\frac{\pi}{2}i}=re^{i\theta}$

73 Find $\int x^3\sqrt{x^2+1}\,dx$ using the substitution $u^2=x^2+1$ or otherwise.

74 Let $\underset{\sim}{r} = 2\sin(2t)\,\underset{\sim}{i} + 3\cos(2t)\,\underset{\sim}{j}$ be a time dependent position vector of an object. The path of the object takes is:

A A straight line **B** A parabola

C A circle **D** An ellipse

75 When stowed (pointing vertically) the cross-section of the Parkes radio telescope can be modelled by $y = \frac{x^2}{100}$, where $(0,0)$ represents the vertex of the parabola. The dish has a diameter of 64 metres. A ball is thrown from the centre of the dish at a speed of 15 ms^{-1} at an angle to the horizontal of $45°$. What horizontal distance will it travel before impact? Assume $g = 10$ ms^{-2} and ignore air resistance. You may assume $y = \tan\theta\, x - \frac{gx^2}{2V^2}(1 + \tan^2\theta)$ is the trajectory of a particle projected at an angle to the horizontal θ with initial velocity V.

76 Prove by contrapositive for m, n positive integers that if mn is divisible by 3 then m or n must be divisible by 3.

77 Solve $z^2 = 21 - 20i$, leaving your answer in the form $a + ib$ for a, b real.

78 Find $\int x\cos 4x^2 \cos 2x^2\, dx$

79 The displacement of two fireworks are given by $x = \cos t\,, y = \sin t\,, z = 100t$ and $x = -\cos 2t\,, y = -\sin 2t\,, z = 100t$ respectively, all measurements in metres.
i Describe the trajectory of both fireworks, commenting on their starting positions and any similarities or differences.
ii Find the coordinates of the point of collision of the two fireworks.

80 A particle of mass m is projected from the top of a very tall cliff, with an initial velocity V ms^{-1} at an angle of θ to the horizontal. The particle experiences the effect of gravity and a resistance proportional to its velocity in both the horizontal and vertical directions. As $t \to \infty$ the particle approaches vertical flight at terminal velocity.
i Given $\ddot{y} = -g - k\dot{y}$, find the terminal velocity of the particle in terms of g and k.
ii Normally when we calculate a terminal velocity we find a positive answer. Why is the answer negative in this case?

81 Prove $x^n - y^n$ is divisible by $x + y$ for even integers $n \geq 2$, for integral $x, y,\ x > y$ by induction.

82 The complex number $z = a + ib$ for a, b real solves $4z - 3\bar{z} = \frac{1-18i}{2-i}$. Find the values of a, b.

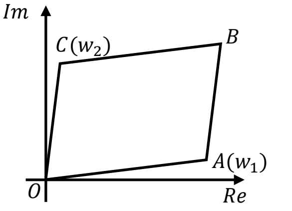
83 Find $\int \sqrt{\frac{1-x}{1+x}}\, dx$

84 The point $A(3, -2, 6)$ lies on $\underset{\sim}{r} = \begin{pmatrix} 13 \\ 8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$. The point P has position vector $(-p\underset{\sim}{i} + 2p\underset{\sim}{k})$ relative to O, where p is a constant.
i Given that $\overrightarrow{PA}$ is perpendicular to $\underset{\sim}{r}$, find the value of p
ii Given also that B is a point on the line $\underset{\sim}{r}$ and $\angle BPA = 45°$, find the two possible coordinates of B.

85 The acceleration of a particle is given by $\ddot{x} = v^4 + v^2$. If initially the particle has a velocity of 1 ms^{-1} at $x = 2$, prove $x = \ln\sqrt{\frac{2v^2}{v^2+1}} + 2$

86 Prove by contradiction that there are infinite number of odd integers.

87 In the Argand diagram above, $OABC$ is a rhombus with $\angle COA = \frac{\pi}{3}$. The points A and C represent the complex numbers ω_1 and ω_2 respectively.



i Explain why $\omega_2 = \omega_1 \text{cis}\frac{\pi}{3}$
ii Write down, in terms of ω_1 only, the complex numbers represented by the vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$.
iii By considering $i(\omega_1 + \omega_2)$, show that the diagonals OB and AC of the rhombus are perpendicular.

88 Find $\int \ln|\cos x| \operatorname{cosec}^2 x \, dx$ using Integration by parts

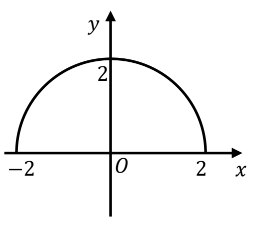
89 Prove that the diagonals of a parallelogram bisect each other.

90 A ball is rolled up a frictionless $30°$ ramp, with an initial velocity of 5 m/s. Find how long it will take to reach its highest point on the ramp. Assume $g = 10 \text{ ms}^{-2}$.

91 Prove that if m, n are integers that $m^2 - n^2$ is odd iff the sum and difference of m and n are both odd.

92 The locus of z if $\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$ is best shown as:

A



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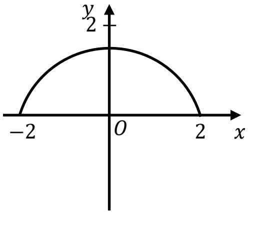
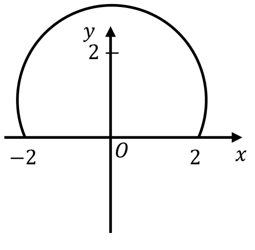
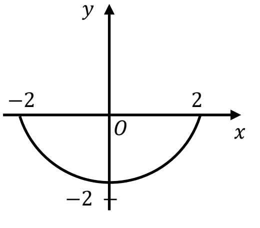


C



D



93 If $I_n = \int x^{-n} e^{2x} \, dx$ prove $I_n = -\frac{e^{2x}}{(n-1)x^{n-1}} + \frac{2}{n-1} I_{n-1}$ for $n \geq 1$

94 The motion of two particles is described by $\underset{\sim}{r_1} = t\underset{\sim}{i} + t^2\underset{\sim}{j} + t^3\underset{\sim}{k}$ and $\underset{\sim}{r_2} = (1+2t)\underset{\sim}{i} + (1+6t)\underset{\sim}{j} + (1+14t)\underset{\sim}{k}$ where $t \geq 0$.
i Show that the two particles do not collide
ii Find any points at which the two paths intersect.

95 At time t a particle in simple harmonic motion has $\ddot{x} = \dot{x} = k$ for $k < 0$.
i Describe the particle's displacement and velocity at t.
ii Write an inequality for the numerical value of acceleration and velocity an instant after time t.

96 Prove by induction that $N \times S_N - \sum_{R=1}^{N-1} S_R = \sum_{R=1}^{N} R^3$ for integral $N > 1$, given $S_R = \sum_{K=1}^{R} K^2$.

97 **i** Find the five roots of the equation $z^5 = 1$ in polar form
ii Show that $z^5 - 1 = (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$
iii Hence show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

98 Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2\left(x - \frac{\pi}{4}\right)}$

99 Given $\underset{\sim}{r} = \lambda \sin t\, \underset{\sim}{i} + \left(1 + \frac{2}{\lambda}\right)\cos t\, \underset{\sim}{j}$ represents the circle centred at the origin with radius 2, find λ.

100 A particle moves along the x-axis such that its velocity $v\ \mathrm{ms}^{-1}$ is given by $v^2 = -4x^2 + 8x + 32$
i By expressing the acceleration as a function in terms of x, prove that the particle is undergoing simple harmonic motion.
ii Find the amplitude.
iii Find the maximum acceleration.

101 Given k is a positive integer, prove $\frac{2k+2}{2k+3} > \frac{2k}{2k+1}$

102 **i** If $|z| = 1$, show that $z^n + z^{-n} = 2\cos n\theta$
ii Hence solve $5z^4 - z^3 + 6z^2 - z + 5 = 0$, leaving answers in the form $z = a + ib$.

103 Find $-\int \frac{2x}{\sqrt{3 - 2x^2 - x^4}}\,dx$

104 At time t, the position vector of a particle is given by $\underset{\sim}{r}(t) = \sqrt{t}\,\underset{\sim}{i} + \frac{1}{t+1}\underset{\sim}{j}$. This particle moves along the path with equation:

A $y(x^2 + 1) = 1$

B $x^2(y - 1) = 0$

C $y = x^2 - 1$

D $y = \dfrac{x^2 - 1}{x^2}$

105 A 10 kg box sits on a horizontal table, where the coefficient of friction is $0.2R$, where R is the normal reactive force of the table on the box. If a force of 12 N is applied to the side of the box will it move? Assume $g = 10$.

106 Prove $\int_0^{\infty} x^n e^{-x}\,dx = n!$ for $n \geq 0$ by induction.

107 $1, \omega$ and ω^2 are the cube roots of unity and each are represented by the points A_1, A_2 and A_3 respectively on an Argand diagram. Find the value of $A_1A_2 \times A_1A_3$, where 'A_1A_2' represents the length A_1A_2.

108 Find $\int \frac{\ln^3 x}{x}\,dx$

109 Relative to a fixed origin O, the point A has position vector $2\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}$, the point B has position vector $5\underset{\sim}{i} + 2\underset{\sim}{j} + 10\underset{\sim}{k}$, and the point D has position vector $-\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$. The line ℓ passes through the points A and B.

i Find the vector $\overrightarrow{AB}$.
ii Find a vector equation for the line ℓ.
iii Show that the size of the angle BAD is $109°$, to the nearest degree.
iv The points A, B and D, together with a point C, are the vertices of the parallelogram $ABCD$, where $AB = DC$. Find the position vector of C.

110 A particle of unit mass falls from rest under gravity in a medium for which the resistance to the motion is proportional to the square of the velocity, kv^2.

i Write an equation for the acceleration of the particle.
ii Show that the terminal velocity, V_T, is given by $V_T = \sqrt{\frac{g}{k}}$.
iii Show that the position, x, of the particle in terms of its velocity, v, is given by $x = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$.

111 Prove $\frac{x^2+y^2}{2} \geq \left(\frac{x+y}{2}\right)^2$ for $x, y > 0$

112 Express $1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{99}$ in the form $x + iy$, x, y real.

113 Prove

$$\int \frac{\tan^2 x + \tan x + \sec^2 x - 2}{\tan x - 1}\,dx = \ln\left|\frac{\tan x - 1}{\cos x}\right| + 2x + c$$

114 Prove that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its sides.

115 A projectile is launched from the point $\underset{\sim}{r}_0 = 10\underset{\sim}{j}$ such that the velocity vector is given by $\underset{\sim}{v}(t) = 20\underset{\sim}{i} + (30 - 9.8t)\underset{\sim}{j}\ \mathrm{ms}^{-1}$. Find the position vector of the projectile. Air resistance is negligible.

116 Prove $16 \times 3^{2n-1} + 21 \times 2^{2n-1}$ is divisible by 30 for $n \geq 1$ by induction.

117 Given $z = \frac{a+3i}{2+ai}$ for real a is a complex number in the first quadrant.

i If $a = 4$ find $|z|$
ii Prove that if $\arg z = \frac{\pi}{4}$ that a can only equal 1

118 Express $\frac{2x+3}{(x+1)(x^2+4)}$ in the form $\frac{a}{x+1} + \frac{bx+c}{x^2+4}$ and hence evaluate $\int_0^2 \frac{2x+3}{(x+1)(x^2+4)}\,dx$

119 The motion of two different particles is described by
$\underset{\sim}{r}_1 = \underset{\sim}{i} + (5t + 4p)\underset{\sim}{j} + 3\underset{\sim}{k}$ and
$\underset{\sim}{r}_2 = (4t + p)\underset{\sim}{i} + (7t + q)\underset{\sim}{j} + (t - q)\underset{\sim}{k}$, where $t \geq 0$ in hours and p, q are real constants. Unit vectors are in metres. Find a pair of values for p and q such that the particles collide, and find the time and place of collision, to the nearest second and metre respectively.

120 A particle is projected $20\ \mathrm{ms}^{-1}$ at an angle of $\frac{\pi}{3}$ to the horizontal with air resistance proportional to velocity. If $\frac{m}{k} = \sqrt{3}$ and $g = 10$, find the time taken until the particle is travelling upwards at an angle of $\frac{\pi}{4}$, to 2 dp. It is given that $\dot{x} = V\cos\theta\, e^{-\frac{k}{m}t}$ and $\dot{y} = \left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$.

121 Prove that if a rational number is divided by another rational number the result is rational.

122 **i** Express $z = \sqrt{3} + i$ in modulus-argument form.
ii Show that $z^7 + 64z = 0$

123 Find $\int \frac{2x^3}{(x^2-1)^2} dx$ using the substitution $u = x^2 - 1$ or otherwise.

124 An ant spirals down and around the outside of a sphere of radius 1 unit, starting at the top and ending at the bottom. Which set of parametric equations could represent its motion, and how many seconds does it take to reach the bottom?

A
$x = \sin t \cos 2t$
$y = \sin t \sin 2t$
$z = \cos t$
Time = π seconds

B
$x = \sin t \cos 2t$
$y = \sin t \sin 2t$
$z = \cos t$
Time = 1 second

C
$x = \cos t \cos 2t$
$y = \sin t \sin 2t$
$z = \cos t$
Time = π seconds

D
$x = \cos t \cos 2t$
$y = \sin t \sin 2t$
$z = \cos t$
Time = 1 second

125 The acceleration of a particle along a straight line is given by $\ddot{x} = t \sin t$ at time t seconds. Initially $x = 0$ and $\dot{x} = u$.
i Find an expression for velocity as a function of time.
ii Find an expression for displacement as a function of time.

126 Prove by contrapositive that if $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$ then $x \geq 0$.

127 Simplify $e^{-1+\frac{\pi}{2}i}$

128 Find $\int \sec^6 x \, dx$

129 Sketch $x = \sin^2 t$, $y = \sin t$

130 A particle of unit mass is initially at the origin with velocity u ms^{-1}, and undergoes uniform acceleration of $\ddot{x} = a$ ms^{-2}. Prove that when the particle is s units to the right of the origin that $v^2 = u^2 + 2as$.

131 Prove that $\sqrt{5}$ is irrational by contradiction.

132 Determine the possible values of the real constant λ if
$$\left|\frac{3+i}{1+2i} + \lambda\right| = \sqrt{\lambda + 2}$$

133 Find $\int \frac{dx}{x^2\sqrt{x^2+1}}$ using the substitution $x = \tan\theta$ or otherwise.

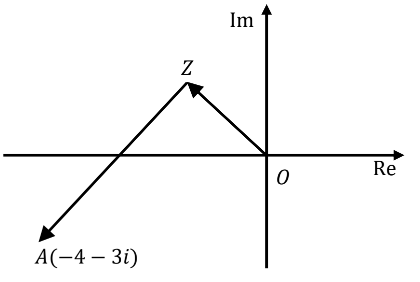
134 Relative to a fixed origin O, the point A has position vector $\begin{pmatrix}-2\\4\\7\end{pmatrix}$ and the point B has position vector $\begin{pmatrix}-1\\3\\8\end{pmatrix}$. The line ℓ_1 passes through the points A and B.

i Find the vector $\overrightarrow{AB}$.
ii Hence find a vector equation for the line ℓ_1
iii The point P has position vector $\begin{pmatrix}0\\2\\3\end{pmatrix}$. Given that $\angle PBA = \theta$, show that $\cos\theta = \frac{1}{3}$
iv The line ℓ_2 passes through the point P and is parallel to the line ℓ_1. Find a vector equation for the line ℓ_2
v The points C and D both lie on the line ℓ_2. Given that $AB = PC = DP$ and the x coordinate of C is positive, find the coordinates of C and the coordinates of D.

135 The tide in a certain harbour can be modelled by simple harmonic motion, with $x = 3\cos\frac{\pi t}{6} + 2$ giving the height of the water t hours after high tide. If a type of oyster can only live when they are under water $\frac{5}{6}$ of the time, what is the maximum height where they can survive? Answer to 2 dp.

136 Consider the statement: ‘There are no prime numbers divisible by 6’. Prove that it is true, or find a counterexample to prove that it is false.

137 The point A represents the complex number $-4 - 3i$. $\angle OZA = 90°$ and $|ZA| = 2|z|$. Find the complex number represented by the point Z.



A $-1 + \sqrt{2}i$ **B** $-1 + 2i$

C $-2 + i$ **D** $-\sqrt{2} + i$

138 Evaluate $\int_4^{e+3} \ln|x - 3|\, dx$ using Integration by parts

139 In the irregular pentagon $ABCDE$, the midpoints of AB and CD, and of BC and DE are joined by two intervals, whose midpoints are also connected by an interval. Prove that this interval is parallel to AE and one quarter its length.

140 A rock is thrown vertically from the bottom of a mine shaft at $20\ \mathrm{ms^{-1}}$, and just reaches the surface before falling back down the shaft. Find the depth of the mine shaft, assuming $g = 10\ \mathrm{ms^{-2}}$.

141 Given $u_{n+1} = \sqrt{3u_n}$ and $u_1 = 1$:
i Show that $u_2 = 3^{\frac{1}{2}}, u_3 = 3^{\frac{1}{2}+\frac{1}{4}}$ and $u_4 = 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$
ii Hence prove by mathematical induction that $u_n = 3^{1-2^{1-n}}$ for $n \geq 1$.
iii Find the limiting value of u_n as $n \to \infty$.

142 Show all complex numbers represented by $z = 1 + 2e^{\theta i}$ for $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$.

143 If $I_n = \int \frac{\sin 3x}{x^n}dx$ and $J_n = \int \frac{\cos 3x}{x^n}dx$ prove $I_n = -\frac{\sin 3x}{(n-1)x^{n-1}} + \frac{3}{n-1}J_{n-1}$ for $n \geq 1$

144 The lines ℓ_1: $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$ meet at C.

i Find the coordinates of C.

ii The point A is on ℓ_1 where $\lambda = 0$ and point B is on ℓ_2 where $\mu = -1$. Find the size of $\angle ACB$ to the nearest minute.

iii Hence or otherwise, find the area of ΔABC, to 2 decimal places.

145 Prove that a particle with $x = 3\cos^2 t$ is in simple harmonic motion. Clearly state its centre of motion and amplitude.

146 Prove by contradiction that the product of a rational and irrational number is irrational.

147 If ω is a complex root of unity, prove that: $(a+b)(a+\omega b)(a+\omega^2 b) = a^3 + b^3$

148 For what values of a and b is the following statement always true?

$$\int_a^b (x^3 - x^2)\,dx > 0$$

149 The sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = 1$ lies entirely within the sphere $x^2 + y^2 + z^2 = 9$. Find the maximum value of $a^2 + b^2 + c^2$.

150 The velocity $v\ \text{ms}^{-1}$ of a particle moving in simple harmonic motion along the x-axis is given by $v^2 = 24 + 2x - x^2$.

i By letting $v^2 = 0$, find the two points between which the particle is oscillating.

ii What is the amplitude of the motion?

iii Find acceleration of the particle in terms of x.

iv Find the period of oscillation.

v Find the maximum speed of the particle.

151 Prove for real x, that $|x^2 - x| + |x - 1| \geq |(x+1)(x-1)|$

152 Solve the equation $z^2 + (z+1)^2 = 0$, where z is a complex number.

153 Find $\int \frac{dx}{(x^2+1)\tan^{-1}x}$

154 Find the Cartesian equation of the curve $x = e^t + e^{-t}, y = e^t - e^{-t}$ and sketch the curve.

155 A car is travelling along a straight horizontal road at a speed of $32\ \text{ms}^{-1}$ when it brakes sharply then skids. Friction brings the car to rest. If the coefficient of friction between the tyres and road is $0.8R$, where R is the normal reactive force between the car and the road, calculate

i the deceleration (assume $g = 10\ \text{ms}^{-2}$)

ii the distance travelled by the car before it comes to rest.

156 Prove by induction that for $n \geq 1$

$$1^2 \times 2 + 2^2 \times 3 + \ldots + n^2(n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

157 $z_1 = 1 + 2i$ and $z_2 = 3 - i$. The value of ${z_1}^2 \div \overline{z_2}$.

A $\frac{-1-3i}{2}$

B $\frac{1+3i}{2}$

C $\frac{1-3i}{2}$

D $\frac{-1+3i}{2}$

158 Find $\int \frac{e^{4x}}{e^{2x}+1} dx$

159 The parallelogram $OABC$ has $\overrightarrow{OA} = (1,3,1)$ and $\overrightarrow{OC} = (-1,1,2)$. Find the area of the parallelogram to 2 dp.

160 A particle is moving in a straight line and performing Simple Harmonic Motion. At time t seconds it has displacement x metres from a fixed point O on the line, given by $x = 2\cos\left(2t - \frac{\pi}{4}\right)$.

i Show that $v^2 - x\ddot{x} = 16$

ii Sketch the graph of x as a function of time for $0 \leq t \leq \pi$ showing clearly the coordinates of the end points.

iii Show that the particle first returns to its starting point after one quarter of its period.

iv Find the time taken by the particle to travel the first 100 metres of its motion.

161 ΔABC has sides a, b, c. If $a^2 + b^2 + c^2 = ab + bc + ca$, show that ΔABC is isosceles.

162 Given $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ and $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ prove $e^{ix} = \cos x + i\sin x$

163 It can be shown that $\frac{3x^3+x^2+12x-12}{(x^2+4)(x^2+2x+4)} = \frac{x-3}{x^2+2x+4} + \frac{2x}{x^2+4}$ (Do NOT prove this.)

Use this result to show $\int_0^1 \frac{3x^3+x^2+12x-12}{(x^2+4)(x^2+2x+4)} dx = \ln\frac{5\sqrt{7}}{8} + \frac{4}{\sqrt{3}}\left(\frac{\pi}{6} - \tan^{-1}\frac{2}{\sqrt{3}}\right)$

164 Prove that a point P lies on the line AB if $\overrightarrow{OP} = \lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OB}$.

165 A particle of unit mass is allowed to fall under gravity from rest in a medium which exerts a resistance proportional to the speed (v) of the particle.

i Show that the particle reaches a terminal velocity, T, given by $T = \frac{g}{k}$ (where k is a positive constant).

ii Show that the distance fallen to reach half its terminal velocity, $\frac{T}{2}$, is given by $x = \frac{T^2}{g}\left(\ln 2 - \frac{1}{2}\right)$

iii Determine an expression for the time taken to reach a speed of $\frac{T}{2}$.

166 If a and b are integers, prove that $(a+b)^2 - (a-b)^2$ is divisible by 4

167 The complex number $z = a + ib$ for a, b real solves $z + 2\bar{z} = |z + 2|$. Find the values of a, b.

168 Find $\int \frac{\sin 2x + \cos 2x}{\cos x} dx$

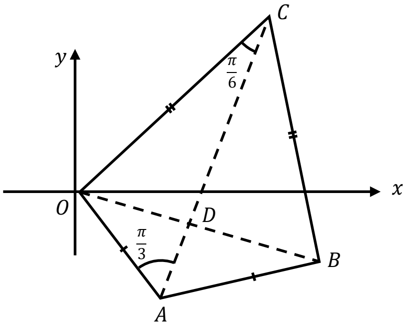
169 Prove that for any triangle ABC and any point P that $\overrightarrow{PA} \cdot \overrightarrow{BC} + \overrightarrow{PB} \cdot \overrightarrow{CA} + \overrightarrow{PC} \cdot \overrightarrow{AB} = 0$

170 A particle of unit mass is projected at an angle of $\frac{\pi}{6}$ below the horizontal at a velocity of $\frac{2g}{k}$ ms^{-1}. Air resistance is proportional to the speed of the particle. Find an expression for the vertical displacement as a function of time.

171 Prove by induction that for $n \geq 1$

$$\sum_{r=1}^{n} (-1)^r r^2 = \frac{(-1)^n n(n+1)}{2}$$

172 In the Argand diagram, $OABC$ is a kite. $OA = AB, OC = CB$, $\overrightarrow{OA} = w, \angle OAC = \frac{\pi}{3}$ and $\angle OCA = \frac{\pi}{6}$.
Find the complex numbers $\overrightarrow{OB}$ and $\overrightarrow{OC}$ in terms of w.



173 Find $\int \frac{1}{\sin x + \cos x} dx$ using the substitution $t = \tan\frac{x}{2}$.

174 The lines ℓ_1: $\underset{\sim}{r} = \begin{pmatrix} 11 \\ 2 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix} -5 \\ 11 \\ p \end{pmatrix} + \mu \begin{pmatrix} q \\ 2 \\ 2 \end{pmatrix}$ are perpendicular.

i Show that $q = -3$.
ii Given that the lines intersect, find the value of p.
iii Find the point of intersection.

175 A particle of unit mass moves in a straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity u, where $u > 0$.
i Show that v is related to the displacement x by the formula

$$x = \arctan\left(\frac{u - v}{1 + uv}\right)$$

ii Show that the time t which has elapsed when the particle is travelling with velocity v is given by

$$t = \log_e \sqrt{\frac{u^2(1 + v^2)}{v^2(1 + u^2)}}$$

176 Prove that all composite integers n have a prime divisor k where $k \leq \sqrt{n}$.

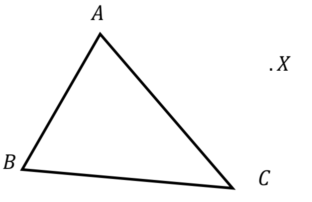
177 The points A and B represent the complex numbers $z_1 = 2 - i$ and $z_2 = 8 + i$ respectively. Find all possible complex numbers z_3, represented by C, such that ΔABC is isosceles and right angled at C.

178 Find $\int \tan^{-1} x \, dx$ using Integration by parts

179 Three spheres centred at A, B and C of radii $r_A = 1, r_B = 2$ and $r_C = 3$ units respectively are, and are arranged so that each sphere touches each of the others. Prove $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0$.

180 A particle of unit mass is dropped in a medium where the resistance to motion is proportional to the speed, so $R = -kv$.
i Find the equation for acceleration as a function of g and the constant of proportionality, k.
ii Show that the velocity of the particle after t seconds is given by
$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$
iii Show that the displacement of the particle is given by
$$x = \frac{g}{k^2}\left[\ln\left(\frac{g}{g - kv}\right) - \frac{kv}{g}\right]$$

181 The diagram shows a point X outside a triangle ABC.
Show that
$$AX + BX + CX > \frac{AB + BC + CA}{2}$$



182 If ω is one of the complex cube roots of unity prove that if n is a positive integer, then $1 + \omega^n + \omega^{2n} = 3$ or 0 depending on whether n is or is not a multiple of 3.

183 If $I_n = \int x^n\sqrt{2x+1}\,dx$ prove $I_n = \frac{x^n\sqrt{(2x+1)^3}}{2n+3} - \frac{n}{2n+3}I_{n-1}$ for $n \geq 1$

184 If four points A, B, C and D in the same plane are such that $AC \perp BD$, then $AB^2 + CD^2 = BC^2 + DA^2$.

185 The launch speed of a projectile is five times the speed it has at its maximum height. Air resistance is negligible. Calculate the angle of projection. to the nearest minute.

186 If a, b, c, d are real numbers, prove $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$

187 Given $z_1 = i\sqrt{2}$ and $z_2 = \frac{2}{1-i}$
i Express z_1 and z_2 in modulus-argument form.
ii If $z_1 = wz_2$, find w in modulus-argument form.
iii On an Argand diagram, plot the points P, Q and R, where P represents z_1, Q represents z_2 and R represents $(z_1 + z_2)$.
iv Show that $\arg(z_1 + z_2) = \frac{3\pi}{8}$ and hence find the exact value of $\tan\frac{3\pi}{8}$.

188 Prove $\int_a^{3a} f(x-a)\,dx = \int_{-a}^{a} f(a-x)\,dx$

189 A particle moves so that its vector at time $t, t \geq 1$, is given by $\underset{\sim}{r}(t) = \left(t + \frac{1}{t}\right)\underset{\sim}{i} + \left(t - \frac{1}{t}\right)\underset{\sim}{j}$
i Find an expression for the distance from the origin of the particle at time t.
ii Find the speed of the particle at time $t = 5$.
iii Find the Cartesian equation of the path of the particle and state the domain and range
iv A second particle has a velocity vector given by
$$\underset{\sim}{v_B}(t) = \left(2 - \frac{2}{t^2}\right)\underset{\sim}{i} + \left(2 + \frac{2}{t^2}\right)\underset{\sim}{j}, \qquad t \geq 0$$
When $t = 1$, the position vector is given by $\underset{\sim}{r_B}(t) = 4\underset{\sim}{i}$. Show that the position vector of this second particle and of the first particle at time t are parallel.

190 A particle of mass m kg is fired vertically in a medium where resistance is proportional to the square of speed, kv. If positive motion is measured up from the point of projection for the upward flight U, and down from the maximum height for the downward flight D, the equations of motion for the two parts of its flight are:

A $\ddot{x_U} = -\frac{mg+kv}{m}, \ddot{x_D} = \frac{mg-kv}{m}$

B $\ddot{x_U} = -\frac{mg+kv}{m}, \ddot{x_D} = -\frac{mg-kv}{m}$

C $\ddot{x_U} = \frac{mg+kv}{m}, \ddot{x_D} = \frac{mg-kv}{m}$

D $\ddot{x_U} = \frac{mg+kv}{m}, \ddot{x_D} = -\frac{mg-kv}{m}$

191 Consider the statement: '$3^n + 2$ is prime for all positive integers n'. Prove that it is true, or find a counterexample to prove that it is false.

192 If $|z| = 1$, prove $\bar{z} = \frac{1}{z}$

193 Find $\int \frac{\cos x}{1+\sin^2 x} dx$

194 An equilateral triangle is inscribed into a circle of radius r. Prove that from any point P on the circle $PA^2 + PB^2 + PC^2 = 6r^2$

195 The acceleration of a particle in the $x - y$ plane is $-g\underset{\sim}{j}$. At time $t = 0$ the particle leaves the point with position vector $\underset{\sim}{r}(0) = h\underset{\sim}{j}$ with velocity $\underset{\sim}{v}(0) = V\cos(\theta)\,\underset{\sim}{i} + V\sin(\theta)\,\underset{\sim}{j}$.

i Show that the particle's position vector at time t is given by

$$\underset{\sim}{r}(t) = V\cos(\theta)\,t\,\underset{\sim}{i} + \left(V\sin(\theta)\,t - \frac{1}{2}gt^2 + h\right)\underset{\sim}{j}$$

ii Show that the particle's path is given by $y = h + \tan(\theta)\,x - \frac{g\sec^2(\theta)}{2V^2}x^2$.

196 **i** Show that: $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$.

Ii Hence, use the principle of mathematical induction to establish the result:

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$

197 If $z = \frac{\lambda+4i}{1+\lambda i}$ for the real constant λ, find the possible values of λ if z is purely real.

198 Find $\int \frac{x^2-4x+2}{(x-2)^3} dx$

199 The line ℓ_1 has vector equation $\underset{\sim}{r} = \begin{pmatrix}-6\\4\\-1\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\3\end{pmatrix}$ and the line ℓ_2 has vector equation $\underset{\sim}{r} = \begin{pmatrix}-6\\4\\-1\end{pmatrix} + \mu\begin{pmatrix}3\\-4\\1\end{pmatrix}$.

The lines ℓ_1 and ℓ_2 intersect at the point A and the acute angle between ℓ_1 and ℓ_2 is θ.

i Write down the coordinates of A.

ii Find the value of $\cos\theta$.

iii The point X lies on ℓ_1 where $\lambda = 4$. Find the coordinates of X.

iv Find the vector $\overrightarrow{AX}$

v Hence, or otherwise, show that $|\overrightarrow{AX}| = 4\sqrt{26}$.

vi The point Y lies on ℓ_2. Given that $\overrightarrow{YX}$ is perpendicular to ℓ_1, find the length of $\overrightarrow{AY}$, giving your answer to 3 significant figures.

200 A circular wheel has masses of m and $2m$ attached to the rim, separated by angles of $\frac{\pi}{2}$. The wheel starts with the large mass directly on top, and the smaller mass to the left as shown in Figure 1. In Figure 1 the wheel is unbalanced and would start turning anticlockwise.

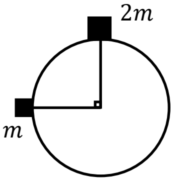


Figure 1

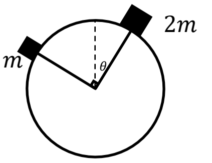


Figure 2

How many degrees, θ, does it need to be rotated to be in balance, so that the two masses are above the centre as shown in Figure 2?

1
$$\left((k+1)^2-1\right)\cdot\left((k+2)^2-1\right)$$
$$=(k+1+1)(k+1-1)(k^2+4k+4-1)$$
$$=(k+2)(k)(k^2+4k+3)$$
$$=(k+2)(k)(k+3)(k+1)$$
$$=k(k+1)(k+2)(k+3)$$

2
$$z^3-i=z^3+i^3=(z+i)(z^2-zi+i^2)$$
$$=(z+i)(z^2-iz-1)$$
ANSWER (B)

3
$$\int\frac{1}{\sqrt{25-9x^2}}dx$$
$$=\frac{1}{3}\int\frac{3}{\sqrt{5^2-(3x)^2}}dx$$
$$=\frac{1}{3}\sin^{-1}\left(\frac{3x}{5}\right)+c$$

4
$$\underset{\sim}{u}\cdot\underset{\sim}{v}=(2)(3)+(3)(-6)+(6)(2)=0$$
$$\underset{\sim}{u}\cdot\underset{\sim}{w}=(2)(6)+(3)(2)+(6)(-3)=0$$
$$\underset{\sim}{v}\cdot\underset{\sim}{w}=(3)(6)+(-6)(2)+(2)(-3)=0$$
$\therefore\ \underset{\sim}{u}, \underset{\sim}{v}$, and $\underset{\sim}{w}$ are mutually perpendicular.

5
$$\dot{\underset{\sim}{r}}=(\sin t\times(-e^{-t})+e^{-t}\cos t)\underset{\sim}{i}$$
$$+\left(\cos t\times(-e^{-t})+e^{-t}(-\sin t)\right)\underset{\sim}{j}$$
$$=\left(e^{-t}(\cos t-\sin t)\right)\underset{\sim}{i}-\left(e^{-t}(\cos t+\sin t)\right)\underset{\sim}{j}$$
$$\left|\dot{\underset{\sim}{r}}\right|=\sqrt{\left(e^{-t}(\cos t-\sin t)\right)^2+\left(-\left(e^{-t}(\cos t+\sin t)\right)\right)^2}$$
$$=e^{-t}\sqrt{\cos^2 t-2\cos t\sin t+\sin^2 t+\cos^2 t+2\cos t\sin t+}$$
$$=e^{-t}\sqrt{2(\cos^2 t+\sin^2 t)}$$
$$=\sqrt{2}e^{-t}$$

6
Let $P(n)$ represent the proposition.
$P(1)$ is true since $\text{LHS}=\frac{1}{1}=1;\ \text{RHS}=\ln(1+1)=\ln 2\approx 0.693$
If $P(k)$ is true for some arbitrary $k\geq 1$ then
$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}>\ln(1+k)$$
RTP $P(k+1)$
$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\frac{1}{k+1}>\ln(2+k)$$
$$\text{LHS}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\frac{1}{k+1}$$
$$>\ln(1+k)+\frac{1}{k+1}\qquad\text{from } P(k)$$
$$>\ln(1+k)+\ln\left(1+\frac{1}{k+1}\right)$$
$$>\ln(1+k)+\ln\left(\frac{k+2}{k+1}\right)$$
$$>\ln(1+k)+\ln(k+2)-\ln(k+1)$$
$$>\ln(k+2)$$
$$=\text{RHS}$$
$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\geq 1$ by induction

7
$$\frac{x}{i}-\frac{y}{1+i}$$
$$=\frac{x(1+i)-iy}{i(i+1)}$$
$$=\frac{x+i(x-y)}{-1+i}\times\frac{-1-i}{-1-i}$$
$$=\frac{-x-xi-xi+yi+x-y}{1+1}$$
$$=\frac{-y+i(y-2x)}{2}$$
$$\therefore -\frac{y}{2}=-1\rightarrow y=2$$
$$\frac{y-2x}{2}=-3$$
$$2-2x=-6$$
$$2x=8$$
$$x=4$$

8
$$\int e^x e^{e^x}dx=e^{e^x}+c$$

9
Let $\overrightarrow{OP}=\underset{\sim}{p}, \overrightarrow{OA}=\underset{\sim}{a}$ and $\overrightarrow{OB}=\underset{\sim}{b}$
$$\overrightarrow{AP}=\frac{m}{m+n}\overrightarrow{AB}$$
$$\underset{\sim}{p}-\underset{\sim}{a}=\frac{m}{m+n}\left(\underset{\sim}{b}-\underset{\sim}{a}\right)$$
$$\underset{\sim}{p}=\frac{m}{m+n}\left(\underset{\sim}{b}-\underset{\sim}{a}\right)+\underset{\sim}{a}$$
$$=\frac{m\underset{\sim}{b}-m\underset{\sim}{a}+(m+n)\underset{\sim}{a}}{m+n}$$
$$=\frac{m\underset{\sim}{b}+n\underset{\sim}{a}}{m+n}$$
$$=\frac{m\left(x_2\underset{\sim}{i}+y_2\underset{\sim}{j}\right)+n\left(x_1\underset{\sim}{i}+y_1\underset{\sim}{j}\right)}{m+n}$$
$$=\frac{mx_2+nx_1}{m+n}\underset{\sim}{i}+\frac{my_2+ny_1}{m+n}\underset{\sim}{j}$$
$\therefore P$ is $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$

10
$$(m+2m)\ddot{x}=(2mg-T)-(0-T)$$
$$3m\ddot{x}=2mg$$
$$\ddot{x}=\frac{2g}{3}$$
$$\therefore\frac{dv}{dt}=\frac{2g}{3}$$
$$v=\frac{2gt}{3}$$
$$\therefore\frac{dx}{dt}=\frac{2gt}{3}$$
$$x=\frac{gt^2}{3}$$

11
Suppose m is even.
Let $m=2k$ for integral k
$$m^2=(2k)^2$$
$$=4k^2$$
$$=4p\text{ for integral }p\text{ since }k\text{ is integral}$$
$\therefore$ if m is even then m^2 is divisible by 4.
$\therefore$ if m^2 is not divisible by 4 then m must be odd by contrapositive.

12 **i**

$|-2+2i| = \sqrt{2^2+2^2} = 2\sqrt{2}$

$\arg(-2+2i) = \pi - \tan^{-1}\left(\frac{2}{2}\right) = \frac{3\pi}{4}$

ii

$$(-2+2i)^{8k} = \left(2\sqrt{2}\operatorname{cis}\frac{3\pi}{4}\right)^{8k}$$
$$= 2^{12k}\operatorname{cis}6k\pi$$
$$= 2^{12k} \text{ for integral } k$$

13

$$\int \frac{x^6+3x^2-1}{x^3+1}dx$$
$$= \int \frac{x^3(x^3+1)-(x^3+1)+3x^2}{x^3+1}dx$$
$$= \int \left(x^3-1+\frac{3x^2}{x^3+1}\right)dx$$
$$= \frac{x^4}{4} - x + \ln|x^3+1| + c$$

14 **i**

$\overrightarrow{AB} = (8-10)\underset{\sim}{i} + (3-2)\underset{\sim}{j} + (4-3)\underset{\sim}{k} = -2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

ii

$$\ell\text{: } \underset{\sim}{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} = \begin{pmatrix}10\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

iii

Let P be $\begin{pmatrix}10-2\lambda\\2+\lambda\\3+\lambda\end{pmatrix}$

$$\overrightarrow{CP} = \begin{pmatrix}10-2\lambda-3\\2+\lambda-12\\3+\lambda-3\end{pmatrix} = \begin{pmatrix}7-2\lambda\\-10+\lambda\\\lambda\end{pmatrix}$$
$$\begin{pmatrix}7-2\lambda\\-10+\lambda\\\lambda\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix} = 0$$
$$-14+4\lambda-10+\lambda+\lambda = 0$$
$$6\lambda = 24$$
$$\lambda = 4$$
$$P\begin{pmatrix}10-2(4)\\2+(4)\\3+(4)\end{pmatrix} = \begin{pmatrix}2\\6\\7\end{pmatrix}$$

15

$a = \frac{5-1}{2} = 2$

$T = 4\times 3 = 12$

$\frac{2\pi}{n} = 12 \rightarrow n = \frac{\pi}{6}$

The particle starts at the centre and is initially moving left, which can most simply be modelled by the negative sine curve.

$x = 3 - 2\sin\left(\frac{\pi t}{6}\right)$

16 **i**

If there are n students then there will be $\binom{n}{2}$ handshakes.

ii

Let $P(n)$ represent the proposition.

$P(2)$ is true since 2 people involves 1 handshake, and $\binom{2}{2} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then k students will involve $\binom{k}{2}$ handshakes

RTP $P(k+1)$ $k+1$ students will involve $\binom{k+1}{2}$ handshakes

Consider a group of k students, who have shaken each others hands, which involves $\binom{k}{2}$ handshakes from $P(k)$.

The $k+1^{th}$ person joins them and shakes hands with each of the k students already there.

The number of handshakes is

$$\binom{k}{2} + k = \frac{k!}{2!\,(k-2)!} + k$$
$$= \frac{k!+2k(k-2)!}{2!\,(k-2)!}$$
$$= \frac{(k-1)k!+2k(k-1)(k-2)!}{2!\,(k-1)(k-2)!}$$
$$= \frac{(k-1)k!+2k!}{2!\,(k-1)!}$$
$$= \frac{k!\,(k-1+2)}{2!\,(k-1)!}$$
$$= \frac{(k+1)!}{2!\,(k-1)!}$$
$$= \binom{k+1}{2}$$

= RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction

17 $-\left(4e^{3i}\right) = e^{-\pi i}\times 4e^{3i} = 4e^{(3-\pi)i}$

18

$$\frac{x+2}{x^2+7x+12} = \frac{x+2}{(x+3)(x+4)} = \frac{a}{x+3} + \frac{b}{x+4}$$
$$a = \frac{(-3)+2}{(-3)+4} = -1$$
$$b = \frac{(-4)+2}{(-4)+3} = 2$$
$$\int_0^1 \frac{x+2}{x^2+7x+12}dx$$
$$= \int_0^1 \left(-\frac{1}{x+3} + \frac{2}{x+4}\right)dx$$
$$= \Big[-\ln|x+3| + 2\ln|x+4|\Big]_0^1$$
$$= (-\ln 4 + 2\ln 5) - (-\ln 3 + 2\ln 4)$$
$$= -\ln 4 + \ln 25 + \ln 3 - \ln 16$$
$$= \ln\frac{75}{64}$$

19

$$x = \frac{1}{\operatorname{cosec}\theta} \rightarrow \operatorname{cosec}\theta = \frac{1}{x}$$
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$
$$1 + y^2 = \frac{1}{x^2}$$
$$y^2 = \frac{1-x^2}{x^2}$$
$$y = \pm\frac{\sqrt{1-x^2}}{x}$$

20 **i**

$x = 2 + \sin 4t + \sqrt{3}\cos 4t$
$\dot{x} = 4\cos 4t - 4\sqrt{3}\sin 4t$
$\ddot{x} = -16\sin 4t - 4\sqrt{3}\cos 4t$
$= -4^2\left(2 + \sin 4t + \sqrt{3}\cos 4t - 2\right)$
$= -4^2(x-2)$
∴ the particle is in SHM.

ii

$R = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2, \alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$
$\therefore x = 2\sin\left(4t + \frac{\pi}{3}\right) + 2$
$a = 2, c = 2$

iii

$v^2 = n^2(a^2 - x^2)$
$v_{\max}^2 = n^2(a^2 - 0)$
$v_{\max} = 4 \times 2 = 8\ \text{ms}^{-1}$

21 Suppose $n^3 - 1$ is odd and n is odd. *
Let $n = 2k + 1$ for integral k
$\therefore n^3 - 1 = (2k+1)^3 - 1$
$= 8k^3 + 12k^2 + 6k + 1 - 1$
$= 2(4k^3 + 6k^2 + 3k)$
$= 2p$ for integral p since k is integral
$\therefore n^3 - 1$ is even #

Which contradicts (*) since $n^3 - 1$ cannot be both odd and even, hence if $n^3 - 1$ is odd then n is even.

22 **i**

In polar form the roots are $z_1 = 1, z_2 = \text{cis}\frac{2\pi}{3}$ and $z_3 = \text{cis}\frac{4\pi}{3}$.
Since $\left(\text{cis}\frac{2\pi}{3}\right)^2 = \text{cis}\frac{4\pi}{3}$, z_2 is a square root of z_3.

ii

The second square root is the first multiplied by negative one, so $\omega = -z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

iii

ω is the square root of a cube root of unity, so is a sixth root of unity, $\therefore n = 6$.

23

$\int \frac{e^{2x}}{\sqrt{e^x + 1}}dx$
$= \int \frac{e^{2x}}{u} \times \frac{2u\,du}{e^x}$
$= 2\int e^x\,du$
$= 2\int (u^2 - 1)\,du$
$= \frac{2u^3}{3} - 2u + c$
$= \frac{2\sqrt{(e^x+1)^3}}{3} - 2\sqrt{e^x + 1} + c$

$u^2 = e^x + 1$
$2u\,du = e^x dx$
$dx = \frac{2u\,du}{e^x}$

24 This is the unit circle extended vertically parallel to the z-axis, which gives a cylinder.

25 $(0.4 + 0.2)\ddot{x} = (0.2g - T) - (T - \mu \times 0.4g)$
$0.6\ddot{x} = 0.2g(1 - 2\mu)$
$\ddot{x} = \frac{g(1-2\mu)}{3}$
If the package is on the point of moving $\ddot{x} = 0$
$\therefore 1 - 2\mu = 0$
$\mu = 0.5$

26 Let $P(n)$ represent the proposition.

$P(1)$ is true since $(3(1) + 1) \times 7^1 - 1 = 27 = 9(3)$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then $(3k+1) \times 7^k - 1 = 9m$ for integral m

RTP $P(k+1)$ $(3k+4) \times 7^{k+1} - 1 = 9p$ for integral p

$\text{LHS} = (3k + 1 + 3) \times 7\left(7^k\right) - 1$
$= 7[(3k+1) \times 7^k] + 21\left(7^k\right) - 1$
$= 7[(3k+1) \times 7^k - 1] + 21\left(7^k\right) + 6$
$= 7[9m] + 3[7^{k+1} + 2]$ from $P(k)$
$= 9(7m) + 3(3q)$ for integral q since $7^{n+1} + 2$ is divisible by 3
$= 9(7m + q)$
$= 9p$ since m, q integral
$= \text{RHS}$
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

27 **i**

$\overline{1 - z} = \overline{1 - 3\cos\theta - 3i\sin\theta}$
$= 1 - 3\cos\theta + 3i\sin\theta$

ii

$\frac{1}{1-z} = \frac{1}{1 - 3(\cos\theta + i\sin\theta)}$
$= \frac{1}{(1 - 3\cos\theta) - 3i\sin\theta}$
$= \frac{(1 - 3\cos\theta) + 3i\sin\theta}{(1 - 3\cos\theta)^2 + (3\sin\theta)^2}$
$= \frac{(1 - 3\cos\theta) + 3i\sin\theta}{1 - 6\cos\theta + 9\cos^2\theta + 9\sin^2\theta}$
$= \frac{(1 - 3\cos\theta) + 3i\sin\theta}{10 - 6\cos\theta}$
$\therefore \text{Im}\left(\frac{1}{1-z}\right) = \frac{3\sin\theta}{10 - 6\cos\theta}$

28

$\int (\tan^4 x - \sec^4 x)\,dx$
$= \int (\tan^2 x - \sec^2 x)(\tan^2 x + \sec^2 x)\,dx$
$= \int (-1)\,(2\sec^2 x - 1)dx$
$= -2\tan x + x + c$

29 $\underset{\sim}{u} + \underset{\sim}{v} = (5+1)\underset{\sim}{i} + (-1+3)\underset{\sim}{j} + (-3-5)\underset{\sim}{k}$
$= 6\underset{\sim}{i} + 2\underset{\sim}{j} - 8\underset{\sim}{k}$
$\underset{\sim}{u} - \underset{\sim}{v} = (5-1)\underset{\sim}{i} + (-1-3)\underset{\sim}{j} + (-3+5)\underset{\sim}{k}$
$= 4\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$
$\left(\underset{\sim}{u} + \underset{\sim}{v}\right) \cdot \left(\underset{\sim}{u} - \underset{\sim}{v}\right) = (6)(4) + (2)(-4) + (-8)(2) = 0$
$\therefore$ $\underset{\sim}{u} + \underset{\sim}{v}$ is perpendicular to $\underset{\sim}{u} - \underset{\sim}{v}$.

30 $v\frac{dv}{dx} = -(g + 2kv^2)$

$\frac{dv}{dx} = -\frac{(g + 2kv^2)}{v}$

$\frac{dx}{dv} = -\frac{v}{g + 2kv^2}$

$H = -\int_u^0 \frac{v}{g + 2kv^2}\,dv$

$= \int_0^u \frac{v}{g + 2kv^2}\,dv$

ANSWER (C)

Note: the 2 is actually irrelevant in the question as it would be absorbed by k.

31 If both integers are odd, let the integers be $2m + 1, 2n + 1$ for integral m, n.

$p = (2m + 1)(2m + 1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn + m + n) + 1$

$= 2p + 1$ for integral p since m, n integral #

∴ if both integers are odd then the product is odd.
By contrapositive, if the product is even then at least one of the integers must be even

Conversely, one or both of the integers is even, let the integers be $2m + j$ and $2n$, where m, n are integral and $j = 0,1$.

$p = (2m + j)(2n)$

$= 4mn + 2nj$

$= 2(2mn + nj)$

$= 2p$ for integral p since m, n integral

∴ if at least one of the integers is even then the product is even

∴ the product of two integers is even if and only if one or both are even

32 $z_2 - z_1$ is the vector from P to Q, so $\overrightarrow{PQ}$.

ANSWER (B)

33 $\int \operatorname{cosec} x\,dx$

$t = \tan\frac{x}{2}$ $dx = \frac{2dt}{1 + t^2}$

$= \int \frac{1 + t^2}{2t} \times \frac{2dt}{1 + t^2}$

$= \int \frac{dt}{t}$

$= \ln|t| + c$

$= \ln\left|\tan\frac{x}{2}\right| + c$

34 $\underset{\sim}{u} + \lambda\underset{\sim}{v} = \begin{pmatrix} 2 - \lambda \\ 2 + 2\lambda \\ 3 + \lambda \end{pmatrix}$

$\begin{pmatrix} 2 - \lambda \\ 2 + 2\lambda \\ 3 + \lambda k \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = 0$

$6 - 3\lambda + 2 + 2\lambda = 0$

$\lambda = 8$

35 Let $y = 0, x = 125, V = 25, g = 10$ and $h = 130$.

$0 = \tan\theta\,(125) - \frac{10 \times 125^2}{2 \times 25^2}(1 + \tan^2\theta) + 130$

$125\tan\theta - 125 - 125\tan^2\theta + 130 = 0$

$125\tan^2\theta - 125\tan\theta - 5 = 0$

$\Delta = (-125)^2$

$= 18125$

∴ there are two solutions for $\tan\theta$, so it is possible to clear the base of the pyramid.

36 Let $P(n)$ represent the proposition.

$P(1)$ is true since
LHS $= \sin(\pi + x) = \sin\pi\cos x + \cos\pi\sin x$
$= 0 + (-1)\sin x$;
RHS $= (-1)^1\sin x = (-1)\sin x$

If $P(k)$ is true for some arbitrary $k \geq 1$ then
$\sin(k\pi + x) = (-1)^k\sin x$

RTP $P(k+1)$ $\sin((k + 1)\pi + x) = (-1)^{k+1}\sin x$

LHS $= \sin\big(k\pi + (\pi + x)\big)$

$= \sin(k\pi)\cos(\pi + x) + \cos(k\pi)\sin(\pi + x)$

$= 0 \times \cos(\pi + x) + (-1)^k \times (-1)^k\sin x$ from $P(k)$

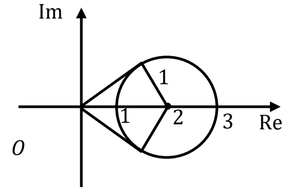
$= (-1)^{k+1}\sin x$

$=$ RHS

∴ $P(k) \Rightarrow P(k + 1)$

∴ $P(n)$ is true for $n \geq 1$ by induction

37 **i**



ii

$1 \leq |z| \leq 3$ (closest and furthest points on the circle are the real intercepts.

$-\sin^{-1}\left(\frac{1}{2}\right) \leq \arg z \leq \sin^{-1}\left(\frac{1}{2}\right)$ tangents from $z = 0$

$-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{6}$

iii

$|P_1| = |P_2| = \sqrt{3}$ exact triangles

∴ $z_1 = \sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = \sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

iv

${z_1}^2 + {z_2}^2$

$= \left(\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^2$

$+ \left(\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right)^2$

$= 3\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} + \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$

$= 3\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} + \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]$

$= 3\left(2\left(\frac{1}{2}\right)\right)$

$= 3$

38 $I = \int 2^x\sin x\,dx$

$u = \sin x$ $\frac{dv}{dx} = 2^x$ $\frac{du}{dx} = \cos x$ $v = \frac{2^x}{\ln 2}$
$u = \cos x$ $\frac{dv}{dx} = 2^x$ $\frac{du}{dx} = -\sin x$ $v = \frac{2^x}{\ln 2}$

$= \frac{2^x\sin x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\cos x\,dx$

$= \frac{2^x\sin x}{\ln 2} - \frac{1}{\ln 2}\left[\frac{2^x\cos x}{\ln 2} + \frac{1}{\ln 2}\int 2^x\sin x\,dx\right]$

$= \frac{2^x\sin x}{\ln 2} - \frac{2^x\cos x}{\ln^2 2} - \frac{1}{\ln^2 2}\int 2^x\sin x\,dx$

∴ $\left(1 + \frac{1}{\ln^2 2}\right)I = \frac{2^x\sin x}{\ln 2} - \frac{2^x\cos x}{\ln^2 2}$

$I = \frac{\ln^2 2}{\ln^2 2 + 1}\left(\frac{2^x\sin x}{\ln 2} - \frac{2^x\cos x}{\ln^2 2}\right)$

$= \frac{2^x(\ln 2\sin x - \cos x)}{\ln^2 2 + 1}$

39 Let the vector be $\underset{\sim}{u} = (x, y, z)$

$\therefore x = \left|\underset{\sim}{u}\right| \cos\alpha, y = \left|\underset{\sim}{u}\right| \cos\beta, z = \left|\underset{\sim}{u}\right| \cos\gamma$

$$\left|\underset{\sim}{u}\right|^2 = x^2 + y^2 + z^2$$
$$= \left|\underset{\sim}{u}\right|^2 \cos^2\alpha + \left|\underset{\sim}{u}\right|^2 \cos^2\beta + \left|\underset{\sim}{u}\right|^2 \cos^2\gamma$$
$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

40

$$\dot{y} = (0.1 \times 10 + 10)e^{-10t} - 0.1 \times 10$$
$$= 11e^{-10t} - 1$$
Let $\dot{y} = 0$
$$11e^{-10t} = 1$$
$$e^{10t} = 11$$
$$10t = \ln 11$$
$$t = \ln 11 \div 10$$
$$= 0.24 \text{ seconds}$$

41

$$\text{LHS} - \text{RHS} = \frac{1}{x} + \frac{1}{y} - \frac{4}{x+y}$$
$$= \frac{y(x+y) + x(x+y) - 4xy}{xy(x+y)}$$
$$= \frac{xy + y^2 + x^2 + xy - 4xy}{xy(x+y)}$$
$$= \frac{x^2 - 2xy + y^2}{xy(x+y)}$$
$$= \frac{(x-y)^2}{xy(x+y)}$$
$$\geq 0$$

$$\therefore \frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$$

42

$$-1 + i = \sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$$
$$\therefore \arg(-1+i) = \frac{3\pi}{4}$$
$\arg(z^9) = 9 \arg z$

A $9 \times \frac{11\pi}{36} = \frac{11\pi}{4} = 2\pi + \frac{3\pi}{4} = \frac{3\pi}{4}$

B $9 \times \frac{\pi}{12} = \frac{3\pi}{4}$

C $9 \times \frac{29\pi}{36} = \frac{29\pi}{4} = 6\pi + \frac{5\pi}{4} = \frac{5\pi}{4}$

D $9 \times \frac{19\pi}{36} = \frac{19\pi}{4} = 4\pi + \frac{3\pi}{4} = \frac{3\pi}{4}$

ANSWER (C)

43 **i**

$$u = \ln^n x \qquad \frac{dv}{dx} = x^m$$
$$\frac{du}{dx} = n \ln^{n-1} x \times \frac{1}{x} \qquad v = \frac{x^{m+1}}{m+1}$$

$$I_{m,n} = \int x^m \ln^n x \, dx$$
$$= \frac{x^{m+1} \ln^n x}{m+1} - \frac{n}{m+1} \int x^m \ln^{n-1} x \, dx$$
$$\therefore I_{m,n} = \frac{x^{m+1} \ln^n x}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

ii

$$\int x^2 \ln^2 x$$
$$= I_{2,2}$$
$$= \frac{x^3 \ln^2 x}{3} - \frac{2}{3} I_{2,1}$$
$$= \frac{x^3 \ln^2 x}{3} - \frac{2}{3}\left(\frac{x^3 \ln x}{3} - \frac{1}{3} I_{2,0}\right)$$
$$= \frac{x^3 \ln^2 x}{3} - \frac{2x^3 \ln x}{9} + \frac{2}{9} \int x^2 \, dx$$
$$= \frac{x^3 \ln^2 x}{3} - \frac{2x^3 \ln x}{9} + \frac{2}{9}\left(\frac{x^3}{3}\right) + c$$
$$= \frac{x^3 \ln^2 x}{3} - \frac{2x^3 \ln x}{9} + \frac{2x^3}{27} + c$$

44 **i**

$$\begin{pmatrix} 6 - \lambda \\ -3 + 2\lambda \\ -2 + 3\lambda \end{pmatrix} = \begin{pmatrix} -5 + 2\mu \\ 15 - 3\mu \\ 3 + \mu \end{pmatrix}$$
$$-2 + 3\lambda = 3 + \mu \rightarrow \mu = 3\lambda - 5$$
sub in x:
$$6 - \lambda = -5 + 2(3\lambda - 5)$$
$$6 - \lambda = 6\lambda - 15$$
$$7\lambda = 21$$
$$\lambda = 3$$
on ℓ_1: $\begin{pmatrix} 6 - (3) \\ -3 + 2(3) \\ -2 + 3(3) \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}$

$\mu = 3(3) - 5 = 4$

on ℓ_2: $\begin{pmatrix} -5 + 2(4) \\ 15 - 3(4) \\ 3 + (4) \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}$

$\therefore A \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}$

ii

$$\left|\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}\right| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$$
$$\left|\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$$
$$\cos\theta = \frac{\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}{\sqrt{14} \times \sqrt{14}}$$
$$= \frac{(-1)(2) + (2)(-3) + (3)(1)}{14}$$
$$= -\frac{5}{14}$$
$$\theta = 111°$$

iii

Let $\lambda = 1$

$$\underset{\sim}{r} = \begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$$

iv

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix}$$

The projection of $\overrightarrow{AB}$ onto ℓ_2 is

$$\frac{\begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}{2^2 + (-3)^2 + 1^2} \times \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$
$$= \frac{4 + 12 - 6}{14} \times \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$
$$= \frac{5}{7} \times \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{10}{7} \\ -\frac{15}{7} \\ \frac{5}{7} \end{pmatrix}$$

The vector from B to ℓ_2

$$= \begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix} - \begin{pmatrix} \frac{10}{7} \\ -\frac{15}{7} \\ \frac{5}{7} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{4}{7} \\ -\frac{13}{7} \\ -\frac{47}{7} \end{pmatrix}$$

$\therefore$ the shortest distance

$$= \sqrt{\left(\frac{4}{7}\right)^2 + \left(-\frac{13}{7}\right)^2 + \left(-\frac{47}{7}\right)^2}$$
$$= 6.99 \text{ units}$$

45 $\left|\underset{\sim}{v}\right| = \sqrt{(-2\cos 2t)^2 + 1^2}$
$= \sqrt{4\cos^2 2t + 1}$
Let $\left|\underset{\sim}{v}\right| = \sqrt{3}$
$\therefore 4\cos^2 2t + 1 = 3$
$\cos^2 2t = \frac{1}{2}$
$\cos 2t = \pm\frac{1}{\sqrt{2}}$
$2t = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$
$t = \frac{\pi}{8}, \frac{3\pi}{8}, \ldots$
$\underset{\sim}{\dot{v}} = 4\sin(2t)\, \underset{\sim}{i}$
Let $t = \frac{\pi}{8}$
$\underset{\sim}{\dot{v}} = 4\sin\frac{\pi}{4}\underset{\sim}{i}$
$= 2\sqrt{2}\underset{\sim}{i}\ \text{ms}^{-2}$

46 $P(1)$ is true since LHS $= \frac{1}{1\times 3} = \frac{1}{3}$
RHS $= \frac{1}{2}\left(1 - \frac{1}{2(1)+1}\right) = \frac{1}{3}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then
$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\left(1 - \frac{1}{2k+1}\right)$
RTP $P(k+1)$
$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$
$= \frac{1}{2}\left(1 - \frac{1}{2k+3}\right)$

LHS $= \frac{1}{2}\left(1 - \frac{1}{2k+1}\right) + \frac{1}{(2k+1)(2k+3)}$ from $P(k)$
$= \frac{1}{2}\left(1 - \frac{1}{2k+1} + \frac{2}{(2k+1)(2k+3)}\right)$
$= \frac{1}{2}\left(1 - \frac{2k+3}{(2k+1)(2k+3)} + \frac{2}{(2k+1)(2k+3)}\right)$
$= \frac{1}{2}\left(1 - \frac{2k+1}{(2k+1)(2k+3)}\right)$
$= \frac{1}{2}\left(1 - \frac{1}{2k+3}\right)$
$=$ RHS
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

47 **i**
$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ (1)
$= \cos^3\theta + 3\cos^2\theta\sin\theta\, i$
$= (\cos^3\theta - 3\cos\theta\sin^2\theta)$
$+ i(3\cos^2\theta\sin\theta - \sin^3\theta)$ (2)
Equating real and imaginary parts of (1) and (2):
$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

ii
$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$
$= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta} \div \frac{\cos^3\theta}{\cos^3\theta}$
$= \frac{\left(\frac{3\sin\theta}{\cos\theta} - \left(\frac{\sin\theta}{\cos\theta}\right)^3\right)}{1 - 3\left(\frac{\sin\theta}{\cos\theta}\right)^2}$
$= \frac{3t - t^3}{1 - 3t^2}$

iii
$3\tan\theta - \tan^3\theta = 0$
$\therefore \tan 3\theta = 0$ from (ii)
$3\theta = k\pi$ for integral k
$\theta = \frac{k\pi}{3}$

48 $x - a \geq 0$ for $[0,2]$
$\therefore a \leq 0$

$1 - \sqrt{x-a} \neq 0$
$\sqrt{x-a} \neq 1$
$x - a \neq 1$
$\therefore a < -1, a > 1$
The integral is defined only if $a < -1$.

49 $x = \sec(t)$
$y = \frac{1}{\sqrt{2}}\tan(t) \rightarrow \tan(t) = \sqrt{2}y$
$\tan^2(t) + 1 = \sec^2(t)$
$\left(\sqrt{2}y\right)^2 + 1 = x^2$
$x^2 - 2y^2 = 1$

50 The scales are calibrated for $g = 10$, so they measure the force in Newtons and divide by 10 to give the mass. As the lift accelerates the net acceleration exceeds g, so the higher reading of 55 kg is seen, representing 550 N.
$50(g + a) = 550$
$10 + a = 11$
$a = 1\text{ ms}^{-2}$
The elevator initially accelerates at 1 ms^{-2}.

51 $(x^2 - 2y)^2 \geq 0$
$x^4 - 4x^2y + 4y^2 \geq 0$
$x^4 + x^2y + 4y^2 \geq 5x^2y$

52 $\frac{1}{\omega} + \frac{1}{\omega^2} + 1$
$= \frac{\omega^2 + \omega + \omega^3}{\omega^3}$
$= \frac{\omega^2 + \omega + 1}{1}$
$= 0$

53 $\int \frac{2}{25x^2 - 10x + 10}dx$
$= \int \frac{2}{25x^2 - 10x + 1 + 9}dx$
$= \frac{2}{5}\int \frac{5}{(5x-1)^2 + 3^2}dx$
$= \frac{2}{5} \times \frac{1}{3}\tan^{-1}\left(\frac{5x-1}{3}\right) + c$
$= \frac{2}{15}\tan^{-1}\left(\frac{5x-1}{3}\right) + c$

54 **i**

$$\frac{d\underset{\sim}{r}}{dt} = (-2\sin t)\underset{\sim}{i} + (2\cos 2t + \cos t)\underset{\sim}{j}$$

when $t = \frac{\pi}{6}$

$$\frac{d\underset{\sim}{r}}{dt} = \left(-2\sin\left(\frac{\pi}{6}\right)\right)\underset{\sim}{i} + \left(2\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)\right)\underset{\sim}{j}$$

$$= -\underset{\sim}{i} + \left(1 + \frac{\sqrt{3}}{2}\right)\underset{\sim}{j}$$

ii

$$\begin{pmatrix} -1 \\ 1 + \frac{\sqrt{3}}{2} \end{pmatrix} \cdot \begin{pmatrix} 3k \\ 4 \end{pmatrix} = 0$$

$$-3k + 4 + 2\sqrt{3} = 0$$

$$k = \frac{4 + 2\sqrt{3}}{3}$$

iii

$$x = 2\cos t + 1 \rightarrow \cos t = \frac{1}{2}(x - 1)$$

$$\begin{aligned} y &= \sin 2t + \sin t \\ &= 2\sin t\cos t + \sin t \\ &= \sin t\,(2\cos t + 1) \\ &= \sin t \times x \end{aligned}$$

$$\therefore \sin t = \frac{y}{x}$$

iv

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{y}{x}\right)^2 + \left(\frac{1}{2}(x - 1)\right)^2 = 1$$

$$\frac{y^2}{x^2} + \frac{x^2 - 2x + 1}{4} = 1$$

$$4y^2 + x^4 - 2x^3 + x^2 = 4x^2$$

$$4y^2 = 3x^2 + 2x^3 - x^4$$

55

$$\frac{2\pi}{n} = 10 \rightarrow n = \frac{\pi}{5}$$

$$a = \frac{4}{2} = 2$$

At M, $x = 1$

$$v^2 = n^2(a^2 - x^2) = \left(\frac{\pi}{5}\right)^2 (2^2 - 1^2) = \frac{3\pi^2}{25}$$

$$\therefore v = \pm\frac{\sqrt{3}\pi}{5}$$

$$a = -n^2 x = -\left(\frac{\pi}{5}\right)^2 \times 1 = -\frac{\pi^2}{25}$$

56 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{d}{dx}(xe^{2x}) = e^{2x} + x \cdot 2e^{2x} = (2x+1)e^{2x} = 2^{1-1}(2x+1)e^{2x}$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{d^k}{dx^k}(xe^{2x}) = 2^{k-1}(2x+k)e^{2x}$

RTP $P(k+1)$ $\frac{d^{k+1}}{dx^{k+1}}(xe^{2x}) = 2^k(2x + k + 1)e^{2x}$

$$\begin{aligned} \text{LHS} &= \frac{d^{k+1}}{dx^{k+1}}(xe^{2x}) \\ &= \frac{d}{dx}\left(\frac{d^k}{dx^k}\right) \\ &= \frac{d}{dx}\left(2^{k-1}(2x+k)e^{2x}\right) from P(k) \\ &= 2^{k-1} \times [e^{2x}(2) + (2x+k) \times 2e^{2x}] \\ &= 2^{k-1} \times 2e^{2x}[1 + 2x + k] \\ &= 2^k(2x + k + 1)e^{2x} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

57 **i**

$$z^5 + 1 = 0$$

$$\therefore (z+1)(z^4 - z^3 + z^2 - z + 1) = 0$$

ii

$z = -1 = \cos\pi + i\sin\pi$ is one root, and the other roots are spaced by $\frac{2\pi}{5}$, so the roots are:

$$-1, \cos\left(\pm\frac{\pi}{5}\right) + i\sin\left(\pm\frac{\pi}{5}\right), \cos\left(\pm\frac{3\pi}{5}\right) + i\sin\left(\pm\frac{3\pi}{5}\right)$$

iii

$$\therefore \sum\alpha = -\frac{b}{a}$$

$$-1 + \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) + \cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)$$

$$+ \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) + \cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right) = 0$$

$$-1 + \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) - i\sin\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right)$$

$$+ i\sin\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) - i\sin\left(\frac{3\pi}{5}\right) = 0$$

$$2\cos\left(\frac{\pi}{5}\right) + 2\cos\left(\frac{3\pi}{5}\right) - 1 = 0$$

58

$$\int x\cos(x^2 + 4)\,dx$$

$$= \frac{1}{2}\int (2x)\cos(x^2 + 4)\,dx$$

$$= \frac{1}{2}\sin(x^2 + 4) + c$$

59 **i**

$$\overrightarrow{AB} = (1 - 2, -1 - 2, -4 - 5) = (-1, -3, -9)$$

$$\overrightarrow{AC} = (3 - 2, 3 - 2, 10 - 5) = (1,1,5)$$

$$|AB| = \sqrt{(-1)^2 + (-3)^2 + (-9)^2} = \sqrt{91}$$

$$|AC| = \sqrt{(1)^2 + (1)^2 + (5)^2} = \sqrt{27} = 3\sqrt{3}$$

$$\cos\angle BAC = \frac{(-1,-3,-9)\cdot(1,1,5)}{\sqrt{91} \times 3\sqrt{3}}$$

$$= \frac{(-1)(1) + (-3)(1) + (-9)(5)}{3\sqrt{273}}$$

$$\angle BAC = 171°19'$$

ii

$$\overrightarrow{AD} = (8 - 2, 6 - 2, 3 - 5) = (6, 4, -2)$$

$$\overrightarrow{AD} \cdot \overrightarrow{AB} = (6)(-1) + (4)(-3) + (-2)(-9) = 0$$

$$\overrightarrow{AD} \cdot \overrightarrow{AC} = (6)(1) + (4)(1) + (-2)(5) = 0$$

$\therefore \overrightarrow{AD}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$

60 **i**

$$R = \sqrt{3^2 + 4^2} = 5, \alpha = \tan^{-1}\frac{4}{3} = 0.927$$

ii

$$x = 5\cos(2t - 0.927) + 2$$

$$\dot{x} = -10\sin(2t - 0.927)$$

$$\begin{aligned} \ddot{x} &= -20\cos 0.927 \\ &= -2^2(5\cos(2t - 0.927) + 2 - 2) \\ &= -2^2(x - 2) \end{aligned}$$

The particle is in SHM.

iii

$\dot{x} = -10\sin(2t - 0.927)$, so the maximum speed is 10 ms^{-1}.

This occurs at the centre of motion, which will occur for the first time when $\cos(2t - 0.927) = 0$.

$$2t - 0.927 = \frac{\pi}{2}$$

$$t = \frac{\frac{\pi}{2} + 0.927}{2}$$

$$= 1.25 \text{ seconds (2 dp)}$$

61 Let $P(n)$ represent the proposition.

$P(8)$ is true since $\text{LHS} = 3(8)^2+3(8)+1 = 217;\ \text{RHS} = 3^1 = 256$

If $P(k)$ is true for some arbitrary $k \geq 8$ then $3k^2+3k+1 < 2^k$

RTP $P(k+1)$ $\qquad 3(k+1)^2+3(k+1)+1 < 2^{k+1}$

$$\begin{aligned}\text{LHS} &= 3(k+1)^2+3(k+1)+1\\ &= 3k^2+6k+3+3k+3+1\\ &= 3k^2+3k+1+6k+6\\ &< 2^k+6k+6 \qquad \text{from } P(k)\\ &< 2^k+2^k \ \text{ for } k \geq 8\\ &< 2^{k+1}\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 8$ by induction

62 **i**

$|w| = 2, \arg(w) = -\frac{\pi}{3}$ by inspection

ii

$$\bar{w} = 1 + i\sqrt{3}$$
$$w^2 = \left(1 - i\sqrt{3}\right)^2 = 1 - 2\sqrt{3}i - 3 = -2 - 2\sqrt{3}i$$
$$\frac{1}{w} = \frac{1}{1-\sqrt{3}i} \times \frac{1+\sqrt{3}i}{1+\sqrt{3}i} = \frac{1+\sqrt{3}i}{1+3} = \frac{1}{4} + \frac{\sqrt{3}}{4}i$$

63

$$\begin{aligned}&\int \frac{e^{2x}-e^x}{e^x+1}\,dx\\ &= \int \frac{(e^x(e^x+1)-2e^x)}{e^x+1}\,dx\\ &= \int \left(e^x - \frac{2e^x}{e^x+1}\right)dx\\ &= e^x - 2\ln|e^x+1| + c\end{aligned}$$

64

$$\left(\underset{\sim}{c} - \underset{\sim}{b}\right)\cdot \underset{\sim}{a} = 0 \ \rightarrow \ \underset{\sim}{a}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \quad (1)$$
$$\left(\underset{\sim}{c} - \underset{\sim}{a}\right)\cdot \underset{\sim}{b} = 0 \ \rightarrow \ \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \quad (2)$$
$$(2)-(1): \quad \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{c} = 0$$
$$\therefore \left(\underset{\sim}{b} - \underset{\sim}{a}\right)\cdot\underset{\sim}{c} = 0$$

65

$$\begin{aligned}50\ddot{x} &= 1000 - 100\\ \ddot{x} &= \frac{900}{50}\\ &= 18\ \text{ms}^{-2}\end{aligned}$$

66

$$\begin{aligned}&k(k+1)(k+2)(k+3)+1\\ &= (k^2+k)(k^2+5k+6)+1\\ &= k^4+5k^3+6k^2+k^3+5k^2+6k+1\\ &= k^4+6k^3+11k^2+6k+1\\ &= k^4+3k^3+k^2+3k^3+9k^2+3k+k^2+3k+1\\ &= (k^2+3k+1)^2\end{aligned}$$

67 **i**

$$zw = (1-3i)(2+i) = 2+i-6i+3 = 5-5i$$

ii

$$|zw| = \sqrt{5^2+5^2} = 5\sqrt{2}$$
$$\arg(zw) = -\tan^{-1}\left(\frac{5}{5}\right) = -\frac{\pi}{4}$$
$$\therefore zw = 5\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

iii

$$\frac{\sqrt{2}(\cos x - i\sin x)}{2+i} = \frac{1-3i}{5}$$
$$5\sqrt{2}(\cos x - i\sin x) = (1-3i)(2+i)$$
$$\therefore \cos x - i\sin x = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) \ \text{ from (i), (ii)}$$
$$= \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)$$
$$\therefore x = \frac{\pi}{4}$$

68

$$\frac{e^x+1}{e^{2x}-5e^x+6} = \frac{e^x+1}{(e^x-3)(e^x-2)} = \frac{a}{e^x-3} + \frac{b}{e^x-2}$$
$$a = \frac{e^{\ln 3}+1}{e^{\ln 3}-2} = \frac{3+1}{3-2} = 4$$
$$b = \frac{e^{\ln 2}+1}{e^{\ln 2}-3} = \frac{2+1}{2-3} = -3$$
$$\begin{aligned}&\int \frac{e^x+1}{e^{2x}-5e^x+6}\,dx\\ &= \int \left(\frac{4}{e^x-3} - \frac{3}{e^x-2}\right)dx\\ &= \int \left(\frac{4e^{-x}}{1-3e^{-x}} - \frac{3e^{-x}}{1-2e^{-x}}\right)dx\\ &= \frac{4}{3}\ln|1-3e^{-x}| - \frac{3}{2}\ln|1-2e^{-x}| + c\end{aligned}$$

69 Let $m = -1$ be represented by the vector $\begin{pmatrix}-1\\1\end{pmatrix}$

One possible vector is $\underset{\sim}{r} = \begin{pmatrix}4\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\end{pmatrix}$

70 **i**

$$\begin{aligned}\ddot{x} &= -10 - 0.05v^2\\ &= -\frac{v^2+200}{20}\end{aligned}$$

ii

$$\begin{aligned}v\frac{dv}{dx} &= -\frac{v^2+200}{20}\\ \frac{dv}{dx} &= -\frac{v^2+200}{20v}\\ \frac{dx}{dv} &= -\frac{20v}{v^2+200}\\ x &= -\int_u^v \frac{20v}{v^2+200}\,dv\\ &= 10\Big[\ln(v^2+200)\Big]_v^u\\ &= 10(\ln(u^2+200) - \ln(v^2+200))\\ &= 10\log_e\left(\frac{200+u^2}{200+v^2}\right)\end{aligned}$$

71 Let $P(n)$ represent the proposition.

$P(0)$ is true since $F_0 = 2^{2^0} + 1 = 3$; $F_1 = 2^{2^1} + 1 = 5$ and $F_0 = F_1 - 2 \rightarrow 3 = 5 - 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $F_0 \times F_1 \times F_2 \times \ldots \times F_{k-1} = F_k - 2$

RTP $P(k+1)$ $F_0 \times F_1 \times F_2 \times \ldots \times F_{k-1} \times F_k = F_{k+1} - 2$

$$\begin{aligned}\text{LHS} &= F_0 \times F_1 \times F_2 \times \ldots \times F_{k-1} \times F_k \\ &= (F_k - 2) \times F_k \text{ from } P(k) \\ &= \left(2^{2^k} + 1 - 2\right)\left(2^{2^k} + 1\right) \\ &= \left(2^{2^k} - 1\right)\left(2^{2^k} + 1\right) \\ &= 2^{2^{k+1}} - 1 \\ &= F_{k+1} - 2 \\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 0$ by induction

72 $e^{i\pi} - e^{\frac{\pi}{2}i} = -1 - i = \sqrt{2}e^{-\frac{3\pi}{4}i}$

$\therefore r = \sqrt{2}, \theta = -\dfrac{3\pi}{4}$

73

$$\begin{aligned}&\int x^3\sqrt{x^2+1}\,dx \\ &= \int x^3 \cdot u \cdot \frac{u\,du}{x} \\ &= \int x^2 u^2\,du \\ &= \int (u^2-1)u^2\,du \\ &= \int (u^4 - u^2)\,du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + c \\ &= \frac{\sqrt{(x^2+1)^5}}{5} - \frac{\sqrt{(x^2+1)^3}}{3} + c\end{aligned}$$

$$\begin{aligned}u^2 &= x^2 \\ 2u\,du &= 2x\,dx \\ dx &= \frac{u\,du}{x}\end{aligned}$$

74

$$\begin{aligned}x = 2\sin 2t &\rightarrow \sin 2t = \frac{x}{2} \\ y = 3\cos 2t &\rightarrow \cos 2t = \frac{y}{3}\end{aligned}$$

$$\sin^2 2t + \cos^2 2t = 1$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Which is an ellipse
ANSWER (D)

75 Let $\theta = 45°, g = 10, V = 5$

$$\begin{aligned}y &= \tan 45°\, x - \frac{10x^2}{2 \times 15^2}(1 + \tan^2 45°) \\ &= x - \frac{2}{45}x^2\end{aligned}$$

at impact:

$$\begin{aligned}x - \frac{2}{45}x^2 &= \frac{x^2}{100} \\ \frac{49x^2}{900} - x &= 0 \\ x\left(\frac{49}{900}x - 1\right) &= 0 \\ x &= 0, \frac{900}{49} \\ &= 0, 18.37\end{aligned}$$

The ball will travel 18.37 metres horizontally before impact.

76 Suppose neither m nor n are divisible by 3.

Let $m = 3j + p$ and $n = 3k + q$ for integral j, k and $p, q =$ 1 or 2.

$$\begin{aligned}mn &= (3j+p)(3k+q) \\ &= 9jk + 3jq + 3kp + pq \\ &= 3(3jk + jq + kp) + pq \\ &= 3c + pq \text{ for integral } c \text{ since } 3, j, k, p, q \text{ are integral}\end{aligned}$$

Now $pq = 1, 2$ or 4, so not a multiple of 3, so mn is not a multiple of 3.

$\therefore$ if m and n are not divisible by 3 then mn is not divisible by 3.

$\therefore$ if mn is divisible by 3 then m or n must be divisible by 3 by contrapositive.

77

$$\begin{aligned}(a+ib)^2 &= 21 - 20i \\ &= 25 - 20i - 4 \\ &= 25 - 20i + 4i^2 \\ &= (\pm(5-2i))^2\end{aligned}$$

$$\therefore z = \pm(5 - 2i)$$

78

$$\begin{aligned}&\int x\cos 4x^2 \cos 2x^2\,dx \\ &= \int x \times \frac{1}{2}(\cos 2x^2 + \cos 6x^2)\,dx \\ &= \frac{1}{2}\int x\cos 2x^2\,dx + \frac{1}{2}\int x\cos 6x^2\,dx \\ &= \frac{1}{8}\int 4x\cos 2x^2\,dx + \frac{1}{24}\int 12x\cos 6x^2\,dx \\ &= \frac{\sin 2x^2}{8} + \frac{\sin 6x^2}{24} + c\end{aligned}$$

79 **i** The first firework starts at $(1,0,0)$, rising in a right hand spiral, completing one turn every 2π seconds and rising one hundred metres per second. The second firework starts at $(-1,0,0)$, rising in a right hand spiral, completing one turn every π seconds and rising one hundred metres per second. They rise at the same rate, with the second firework spinning more quickly.
ii After π seconds the first firework will have completed half a turn while the second firework has completed a full turn, so they collide at $(\cos\pi, \sin\pi, 100\pi) = (-1,0,314)$.

80 **i**

Let $\ddot{y} = 0, \dot{y} = V_T$

$$\begin{aligned}0 &= -g - kV_T \\ V_T &= -\frac{g}{k}\text{ ms}^{-1}\end{aligned}$$

ii
Normally we are dealing with horizontal or downward flight and the particle is moving in the positive direction. In this case we have defined the upward direction as positive, so we get a negative terminal velocity as it is moving in the opposite direction.

81 Let $P(n)$ represent the proposition.

$P(2)$ is true since $x^2 - y^2 = (x-y)(x+y) = m(x+y)$ for integral m since x, y are integral.

If $P(k)$ is true for some arbitrary even $k \geq 2$ then $x^k - y^k = m(x+y)$ for integral m

RTP $P(k+2)$ $x^{k+2} - y^{k+2} = p(x+y)$ for integral p

$$\begin{aligned}\text{LHS} &= x^2(x^k) - y^2(y^k)\\ &= x^2(x^k - y^k) + (x^2 - y^2)(y^k)\\ &= x^2(m(x+y)) + (x+y)(x-y)(y^k) \quad \text{from } P(k)\\ &= (x+y)(mx^2 + (x-y)y^k)\\ &= (x+y)p \quad \text{since } m, x, y, k \text{ are integral}\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for even $n \geq 2$ by induction

82

$$\begin{aligned}4(a+ib) - 3(a-ib) &= \frac{1-18i}{2-i}\\ a + 7bi &= \frac{1-18i}{2-i} \times \frac{2+i}{2+i}\\ &= \frac{2+i-36i+18}{2^2+1^2}\\ &= 4 - 7i\end{aligned}$$

$\therefore a = 4, b = -1$

83

$$\begin{aligned}&\int \sqrt{\frac{1-x}{1+x}}\,dx\\ &= \int \sqrt{\frac{1-x}{1+x}} \times \sqrt{\frac{1-x}{1-x}}\,dx\\ &= \int \frac{1-x}{\sqrt{1-x^2}}\,dx\\ &= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{2} \times (-2x)(1-x^2)^{-\frac{1}{2}}\right)dx\\ &= \sin^{-1} x + \frac{1}{2} \times 2(1-x^2)^{\frac{1}{2}} + c\\ &= \sin^{-1} x + \sqrt{1-x^2} + c\end{aligned}$$

84 **i**

$$\overrightarrow{PA} = (3-(-p), -2-0, 6-2p) = \begin{pmatrix} 3+p \\ -2 \\ 6-2p \end{pmatrix}$$

$$\begin{pmatrix} 3+p \\ -2 \\ 6-2p \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 0$$

$$\begin{aligned}6 + 2p - 4 - 6 + 2p &= 0\\ 4p &= 4\\ p &= 1\end{aligned}$$

ii

ΔAPB is right angled isosceles with $AP = AB$

$$\therefore \overrightarrow{PA} = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

$$|\overrightarrow{PA}| = \sqrt{4^2 + (-2)^2 + 4^2} = 6$$

$$\left|\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}\right| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\begin{aligned}\therefore \overrightarrow{OB} &= \overrightarrow{OA} \pm 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}\\ &= (3, -2, 6) \pm 2(2, 2, -1)\\ &= (-1, -6, 8), (7, 2, 4)\end{aligned}$$

85

$$\begin{aligned}v\frac{dv}{dx} &= v^4 + v^2\\ \frac{dv}{dx} &= v^3 + v\\ \frac{dx}{dv} &= \frac{1}{v^3+v}\\ x - 2 &= \int_1^v \frac{dv}{v^3+v}\\ x - 2 &= \int_1^v \left(\frac{1}{v} - \frac{v}{v^2+1}\right)dv\\ &= \left[\ln v - \frac{1}{2}\ln(v^2+1)\right]_1^v\\ &= \ln v - \frac{1}{2}\ln(v^2+1) - 0 + \frac{1}{2}\ln 2\\ &= \frac{1}{2}\ln\left(\frac{2v^2}{v^2+1}\right)\\ x &= \ln\sqrt{\frac{2v^2}{v^2+1}} + 2\end{aligned}$$

86 Suppose there is a finite number of odd integers (*)
Let the largest odd integer be k
$k+2$ is also odd, so we have another odd integer which contradicts (*)
$\therefore$ there are infinite number of odd integers

87 **i**

$\overrightarrow{OC}$ is $\overrightarrow{OA}$ rotated $\frac{\pi}{3}$ anticlockwise, since the sides of a rhombus are equal, and $\angle COA = \frac{\pi}{3}$

$\therefore \overrightarrow{OC} = \overrightarrow{OA} \times \text{cis}\frac{\pi}{3}$

$\therefore \omega_2 = \omega_1 \text{cis}\frac{\pi}{3}$

ii

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OC} + \overrightarrow{OA} = \omega_2 + \omega_1$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \omega_2 - \omega_1$

iii

$$\begin{aligned}&i(\omega_1 + \omega_2)\\ &= i\left(\omega_1 + \omega_1 \text{cis}\frac{\pi}{3}\right)\\ &= \omega_1\left(\text{cis}\frac{\pi}{2}\right)\left(1 + \text{cis}\frac{\pi}{3}\right)\\ &= \omega_1\left(\text{cis}\frac{\pi}{2} + \text{cis}\frac{5\pi}{6}\right)\\ &= \omega_1\left(i + \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\\ &= \omega_1\left(i - \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\\ &= \omega_1\left(-\frac{\sqrt{3}}{2} + \frac{3}{2}i\right)\\ &= \sqrt{3}\omega_1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\\ &= \sqrt{3}\omega_1\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i - 1\right)\\ &= \sqrt{3}\omega_1\left(\text{cis}\frac{\pi}{3} - 1\right)\\ &= \sqrt{3}\left(\omega_1 \text{cis}\frac{\pi}{3} - \omega_1\right)\\ &= \sqrt{3}(\omega_2 - \omega_1)\end{aligned}$$

$\therefore \overrightarrow{AC}$ is at right angles to $\overrightarrow{OB}$ (and $\sqrt{3}$ times its magnitude).

88

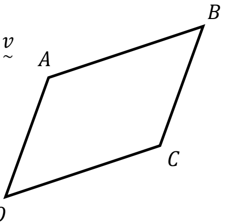
$$\int \ln|\cos x| \csc^2 x\,dx$$

$u = \ln\lvert\cos x\rvert$	$\frac{dv}{dx} = \csc^2 x$
$\frac{du}{dx} = -\tan x$	$v = -\cot x$

$$\begin{aligned}&= -\ln|\cos x|\cot x - \int \tan x \cot x\,dx\\ &= -\ln|\cos x|\cot x - \int dx\\ &= -\ln|\cos x|\cot x - x + c\end{aligned}$$

89 In the parallelogram $OABC$

Let $\overrightarrow{OA} = \overrightarrow{CB} = \underset{\sim}{u}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \underset{\sim}{v}$



The diagonals are

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \underset{\sim}{v} - \underset{\sim}{u}$

and $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \underset{\sim}{u} + \underset{\sim}{v}$.

The midpoint of $\overrightarrow{OB} = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$

and the midpoint of $\overrightarrow{AC}$ is

$\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \underset{\sim}{u} + \frac{1}{2}\left(\underset{\sim}{v} - \underset{\sim}{u}\right) = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$

Since both diagonals share the same midpoint they bisect each other.

90 The ball is subject to gravity, part of which is balanced by the normal force from the ramp, while $g \sin\alpha$ acts against the motion of the ball.

$$\ddot{x} = -10\sin 30^\circ$$
$$= -5$$
$$\frac{dv}{dt} = -5$$

The velocity drops by $5\ \text{m/s/s}$ so it will be stationary after 1 second, which is the time it takes to reach its highest point.

91 If $m^2 - n^2$ is odd then $(m+n)(m-n)$ is odd, using the difference of two squares.

$\therefore$ $m+n$ is odd and $m-n$ is odd, since only two odd numbers have an odd product, $\therefore$ the sum and difference of m and n are both odd.

Conversely, if $m+n$ and $m-n$ are both odd then $(m+n)(m-n)$ is odd, since the product of two odd numbers is odd.

$\therefore m^2 - n^2$ is odd, using the difference of two squares.

$\therefore m^2 - n^2$ is odd iff the sum and difference of m and n are both odd

92 The interval joining $(-2,0)$ and $(2,0)$ subtends an angle at the circumference is $\frac{\pi}{4}$, so the angle subtended at the centre is $\frac{\pi}{2}$, so we want three quarters of the circumference.

ANSWER (C)

93

$$I_{n-1} = \int x^{-(n-1)} e^{2x}\,dx$$

$$\boxed{\begin{array}{ll} u = x^{-(n-1)} & \dfrac{dv}{dx} = e^{2x} \\ \dfrac{du}{dx} = -(n-1)x^{-n} & v = \dfrac{1}{2}e^{2x} \end{array}}$$

$$= \frac{e^{2x}}{2x^{n-1}} + \frac{n-1}{2}\int x^{-n}e^{2x}\,dx$$

$$I_{n-1} = \frac{e^{2x}}{2x^{n-1}} + \frac{n-1}{2}I_n$$

$$\frac{n-1}{2}I_n = -\frac{e^{2x}}{2x^{n-1}} + I_{n-1}$$

$$I_n = -\frac{e^{2x}}{(n-1)x^{n-1}} + \frac{2}{n-1}I_{n-1}$$

94 **i**

$t = 1 + 2t \ \rightarrow \ t = -1$

The two particles do not share the same x-value for any positive t, so they do not collide.

ii

$t_1 = 1 + 2t_2 \quad (1)$

$t_1^2 = 1 + 6t_2 \quad (2)$

$t_1^3 = 1 + 14t_2 \quad (3)$

$(2) - 3(1)$: $t_1^2 - 3t_1 = -2$

$t_1^2 - 3t_1 + 2 = 0$

$(t_1 - 2)(t_1 - 1) = 0$

$t_1 = 1,2$

sub in (1): $1 = 1 + 2t_2 \rightarrow t_2 = 0$

$2 = 1 + 2t_2 \rightarrow t_2 = \frac{1}{2}$

check (3): $1^3 = 1 + 14(0)$

$2^3 = 1 + 14\left(\frac{1}{2}\right)$

The paths intersect at $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{r} = 2\underset{\sim}{i} + 4\underset{\sim}{j} + 8\underset{\sim}{k}$

95 **i**

If the velocity and acceleration are both negative then the particle is to the right of the centre moving towards the centre.

ii

The particle is moving towards the centre so the magnitude of the acceleration is decreasing and the magnitude of the velocity is increasing. Since they are both negative $\ddot{x} > \dot{x}$.

96 $P(2)$ is true since $\text{LHS} = 2 \times (1^2 + 2^2) - (1^2) = 9$;

$\text{RHS} = 1^3 + 2^3 = 9$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$k \times S_k - \sum_{R=1}^{k-1} S_R = \sum_{R=1}^{k} R^3$$

RTP $P(k+1)$

$$(k+1) \times S_{k+1} - \sum_{R=1}^{k} S_R = \sum_{R=1}^{k+1} R^3$$

$$\text{LHS} = (k+1)(S_k + (k+1)^2) - \sum_{R=1}^{k-1} S_R - S_k$$

$$= k \times S_k - \sum_{R=1}^{k-1} S_R + (k+1)^3$$

$$= \sum_{R=1}^{k} R^3 + (k+1)^3 \quad \text{from } P(k)$$

$$= \sum_{R=1}^{k+1} R^3$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

97 **i**

$$\left(re^{i\theta}\right)^5 = e^{2\pi k i} \text{ for } 4k = -2,-1,0,1,2$$
$$r^5 e^{5\theta i} = e^{2\pi k i}$$
$$r = 1, 5\theta = 2\pi k$$
$$\theta = \frac{2\pi k}{5}$$
$$= -\frac{4\pi}{5}, -\frac{2\pi}{5}, 0, \frac{2\pi}{5}, \frac{4\pi}{5}$$
$$\therefore z = 1, \cos\left(\pm\frac{4\pi}{5}\right) + i\sin\left(\pm\frac{4\pi}{5}\right), \cos\left(\pm\frac{2\pi}{5}\right) + i\sin\left(\pm\frac{2\pi}{5}\right)$$

ii

$$z^5 - 1 = (z-\alpha)(z-\beta)(z-\gamma)(z-\delta)(z-\epsilon)$$
$$= (z-1)\left(z - \left(\cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)\right)\right)\left(z - \left(\cos\left(\frac{2}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)\right)\left(z - \left(\cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)\right)\right)\left(z - \left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right)\right)$$
$$= (z-1)\left(z - \left(\cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)\right)\right)\left(z - \left(\cos\left(\frac{2}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)\right)\left(z - \left(\cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)\right)\right)\left(z - \left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right)\right)$$
$$= (z-1)\left(z - \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)\left(z - \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)\right)\left(z - \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right)\left(z - \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)\right)$$
$$= (z-1)\left(z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right) + \sin^2\left(\frac{2\pi}{5}\right)\right)\left(z^2 - 2z\cos\left(\frac{4\pi}{5}\right) + \cos^2\left(\frac{4\pi}{5}\right) + \sin^2\left(\frac{4\pi}{5}\right)\right)$$
$$= (z-1)\left(z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{4\pi}{5}\right) + 1\right)$$

iii

$$\alpha + \beta + \gamma + \delta + \epsilon = -\frac{b}{a}$$
$$1 + \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) = 0$$
$$1 + \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) = 0$$
$$2\left[\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right)\right] = -1$$
$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$$

98

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2\left(x - \frac{\pi}{4}\right)}$$
$$= \int_0^{\frac{\pi}{2}} \sec^2\left(x - \frac{\pi}{4}\right) dx$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x \, dx$$
$$= \Big[\tan x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= 1 - (-1)$$
$$= 2$$

99

$$x = \lambda \sin t$$
$$y = \left(1 + \frac{2}{\lambda}\right)\cos t$$
$$x^2 + y^2 = \lambda^2 \sin^2 t + \left(1 + \frac{2}{\lambda}\right)^2 \cos^2 t$$

Let $\lambda = 2$

$$= 4\sin^2 t + \left(1 + \frac{2}{2}\right)^2 \cos^2 t$$
$$= 4(\sin^2 t + \cos^2 t)$$
$$= 4$$

$\therefore$ if $\lambda = 2$ then $\underset{\sim}{r}$ is the circle centred at the origin with radius 2.

100 **i**

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{d}{dx}(-2x^2 + 4x + 16)$$
$$= -4x + 4$$
$$= -4(x-1)$$

$\therefore$ the particle is in SHM.

ii

$$v^2 = n^2(a^2 - (x-c)^2) \quad (1)$$
$$-4x^2 + 8x + 32 = 4(8 + 2x - x^2)$$
$$= 2^2(3^2 - (x-1)^2)$$

The amplitude is 3

iii

The maximum acceleration occurs at the leftmost extremity of motion, so $x = 1 - 3 = -2$

$\ddot{x}_{\max} = -4(-2-1) = 12 \text{ ms}^{-2}$

101

$$\text{LHS} - \text{RHS} = \frac{2k+2}{2k+3} - \frac{2k}{2k+1}$$
$$= \frac{(2k+2)(2k+1) - 2k(2k+3)}{(2k+1)(2k+3)}$$
$$= \frac{4k^2 + 6k + 2 - 4k^2 - 6k}{(2k+1)(2k+3)}$$
$$= \frac{2}{(2k+1)(2k+3)}$$
$$\geq 0$$
$$\therefore \frac{2k+2}{2k+3} > \frac{2k}{2k+1}$$

102 **i**

Let $z = \cos\theta + i\sin\theta$

$$\begin{aligned} z^n + z^{-n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\ &= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta) \\ &= 2\cos(n\theta) \end{aligned}$$

ii

$$\begin{aligned} 5z^4 - z^3 + 6z^2 - z + 5 &= 0 \\ 5z^2 - z + 6 - z^{-1} + 5z^{-2} &= 0 \\ 5(z^2 + z^{-2}) - (z^1 + z^{-1}) + 6 &= 0 \\ 5(2\cos 2\theta) - (2\cos\theta) + 6 &= 0 \\ 10(2\cos^2\theta - 1) - 2\cos\theta + 6 &= 0 \\ 20\cos^2\theta - 2\cos\theta - 4 &= 0 \\ 10\cos^2\theta - \cos\theta - 2 &= 0 \end{aligned}$$

$$\begin{aligned} \cos\theta &= \frac{1 \pm \sqrt{(-1)^2 - 4(10)(-2)}}{2(10)} \\ &= \frac{1 \pm 9}{20} \\ &= -\frac{2}{5}, \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \sin\theta &= \pm\sqrt{1 - \cos^2\theta} \\ &= \pm\frac{\sqrt{21}}{5}, \pm\frac{\sqrt{3}}{2} \quad \text{respectively} \end{aligned}$$

$$\therefore z = -\frac{2}{5} \pm \frac{\sqrt{21}}{5}i, \qquad \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

103

$$\begin{aligned} &-\int \frac{2x}{\sqrt{3 - 2x^2 - x^4}}\,dx \\ &= -2\int \frac{x}{\sqrt{4 - (1 + x^2)^2}}\,dx \\ &= \cos^{-1}\left(\frac{x^2 + 1}{2}\right) + c \end{aligned}$$

104

$x = \sqrt{t} \to t = x^2$

$y = \dfrac{1}{t + 1}$

$y = \dfrac{1}{x^2 + 1}$

$y(x^2 + 1) = 1$

ANSWER (A)

105

$R = mg = 5 \times 10 = 50\text{ N}$

$m\ddot{x} = F - 0.2R$

$5\ddot{x} = 12 - 10$

$\ddot{x} = 0.4$

The box will move, with an acceleration of 0.4 ms^{-2}.

106 Let $P(n)$ represent the proposition.

$P(0)$ is true since $\int_0^\infty x^0 e^{-x}\,dx = \int_0^\infty e^{-x}\,dx = -\left[e^{-x}\right]_0^\infty = -(0 - 1) = 1 = 1!$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\int_0^\infty x^k e^{-x}\,dx = k!$

RTP $P(k+1)$ $\int_0^\infty x^{k+1}e^{-x}\,dx = (k+1)!$

$\text{LHS} = \int_0^\infty x^{k+1}e^{-x}\,dx$

$u = x^{k+1}$	$\dfrac{dv}{dx} = e^{-x}$
$\dfrac{du}{dx} = (k+1)x^k$	$v = -e^{-x}$

$$\begin{aligned} &= -\left[x^{k+1}e^{-x}\right]_0^\infty + (k+1)\int_0^\infty x^k e^{-x}\,dx \\ &= -(0 - 0) + (k+1) \times k! \quad \text{from } P(k) \\ &= (k+1)! \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

107 ΔOA_1A_2 is isosceles, with $\angle OA_1A_2 = \frac{2\pi}{3}$ and $|OA_1| = |OA_2| = 1$

ΔOA_1A_3 is congruent to ΔOA_1A_2, so $A_1A_2 \times A_1A_3 = (A_1A_2)^2$

Using the Cosine Rule in ΔOA_1A_2:

$$\begin{aligned} &(A_1A_2)^2 \\ &= |OA_1|^2 + |OA_2|^2 - 2|OA_1| \times |OA_2|\cos\angle OA_1A_2 \\ &= 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\left(\frac{2\pi}{3}\right) \\ &= 2 - 2\left(-\cos\frac{\pi}{3}\right) \\ &= 2 + 2\left(\frac{1}{2}\right) \\ &= 3 \end{aligned}$$

108

$$\begin{aligned} &\int \frac{\ln^3 x}{x}\,dx \\ &= \int \frac{1}{x}(\ln x)^3\,dx \\ &= \frac{\ln^4 x}{4} + c \end{aligned}$$

109 **i**

$$\begin{aligned} \overrightarrow{AB} &= (5 - 2)\underset{\sim}{i} + \big(2 - (-1)\big)\underset{\sim}{j} + (10 - 5)\underset{\sim}{k} \\ &= 3\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k} \end{aligned}$$

ii

$$\underset{\sim}{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}$$

iii

$$\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 3^2 + 5^2} = \sqrt{43}$$

$$\begin{aligned} \overrightarrow{AD} &= (-1 - 2)\underset{\sim}{i} + \big(1 - (-1)\big)\underset{\sim}{j} + (4 - 5)\underset{\sim}{k} \\ &= -3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} \end{aligned}$$

$$\left|\overrightarrow{AD}\right| = \sqrt{(-3)^2 + 2^2 + (-1)^2} = \sqrt{14}$$

$$\begin{aligned} \cos\angle BAD &= \frac{(3)(-3) + (3)(2) + (5)(-1)}{\sqrt{43} \times \sqrt{14}} \\ \angle BAD &= \cos^{-1}\left(\frac{-8}{\sqrt{43 \times 14}}\right) \\ &= 109° \end{aligned}$$

iv

$$\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DC} = \overrightarrow{OD} + \overrightarrow{AB} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix}$$

110 **i**

$\ddot{x} = g - kv^2$

ii

Let $\ddot{x} = 0, v = V_T$

$$\begin{aligned} 0 &= g - kV_T^2 \\ kV_T^2 &= g \\ V_T &= \sqrt{\frac{g}{k}} \end{aligned}$$

iii

$$\begin{aligned} v\frac{dv}{dx} &= g - kv^2 \\ \frac{dv}{dx} &= \frac{g - kv^2}{v} \\ \frac{dx}{dv} &= \frac{v}{g - kv^2} \\ x &= \int_0^v \frac{v}{g - kv^2}\,dv \\ &= \frac{1}{2k}\Big[\ln(g - kv^2)\Big]_v^0 \\ &= \frac{1}{2k}\Big(\ln g - \ln(g - kv^2)\Big) \\ &= \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right) \end{aligned}$$

111 LHS – RHS

$$= \frac{x^2+y^2}{2} - \frac{x^2+2xy+y^2}{4}$$
$$= \frac{2x^2+2y^2-x^2-2xy-y^2}{4}$$
$$= \frac{x^2-2xy+y^2}{4}$$
$$= \left(\frac{x}{2}-\frac{y}{2}\right)^2$$
$$\geq 0$$
$$\therefore \frac{x^2+y^2}{2} \geq \left(\frac{x+y}{2}\right)^2$$

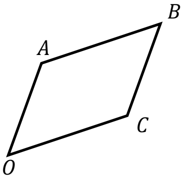
112 $1+(1+i)+(1+i)^2+\ldots+(1+i)^{99}$

$$= \frac{1\left((1+i)^{100}-1\right)}{1+i-1}$$
$$= \frac{(1+i)^{100}-1}{i}$$
$$= \frac{\left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^{100}-1}{i}$$
$$= \frac{2^{50}\operatorname{cis}25\pi - 1}{i}$$
$$= -\frac{2^{50}+1}{i}\times\frac{i}{i}$$
$$= (2^{50}+i)i$$

113 $\int \frac{\tan^2 x+\tan x+\sec^2 x-2}{\tan x - 1}\,dx$

$$= \int \frac{\tan x\,(\tan x-1)+2(\tan x-1)+\sec^2 x}{\tan x-1}\,dx$$
$$= \int \left(\tan x + 2 + \frac{\sec^2 x}{\tan x - 1}\right)dx$$
$$= -\ln|\cos x| + 2x + \ln|\tan x - 1| + c$$
$$= \ln\left|\frac{\tan x - 1}{\cos x}\right| + 2x + c$$

114 In the parallelogram $OABC$



Let $\overrightarrow{OA}=\overrightarrow{CB}=\underset{\sim}{u}$ and $\overrightarrow{OC}=\overrightarrow{AB}=\underset{\sim}{v}$

The diagonals are

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\underset{\sim}{v}-\underset{\sim}{u}$

and $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\underset{\sim}{u}+\underset{\sim}{v}$.

$$\left|\overrightarrow{OB}\right|^2+\left|\overrightarrow{AC}\right|^2 = \left|\underset{\sim}{u}+\underset{\sim}{v}\right|^2+\left|\underset{\sim}{v}-\underset{\sim}{u}\right|^2$$
$$= \left(\underset{\sim}{u}+\underset{\sim}{v}\right)\cdot\left(\underset{\sim}{u}+\underset{\sim}{v}\right)+\left(\underset{\sim}{v}-\underset{\sim}{u}\right)\cdot\left(\underset{\sim}{v}-\underset{\sim}{u}\right)$$
$$= \underset{\sim}{u}\cdot\underset{\sim}{u}+2\underset{\sim}{u}\cdot\underset{\sim}{v}+\underset{\sim}{v}\cdot\underset{\sim}{v}+\underset{\sim}{u}\cdot\underset{\sim}{u}-2\underset{\sim}{u}\cdot\underset{\sim}{v}+\underset{\sim}{v}$$
$$= \left|\underset{\sim}{u}\right|^2+\left|\underset{\sim}{v}\right|^2+\left|\underset{\sim}{u}\right|^2+\left|\underset{\sim}{v}\right|^2$$
$$= \left|\overrightarrow{OA}\right|^2+\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{CB}\right|^2+\left|\overrightarrow{OC}\right|^2$$

∴ the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its sides.

115 $\underset{\sim}{r}(t) = (20t+c_1)\underset{\sim}{i} + (30t-4.9t^2+c_2)\underset{\sim}{j}$

Let $t=0, x=0, y=10$

$20(0)+c_1=0 \rightarrow\ c_1=0$

$30(0)-4.9(0)^2+c_2=10 \rightarrow c_2=10$

$\therefore \underset{\sim}{r}(t)=20t\underset{\sim}{i}+(30t-4.9t^2+10)\underset{\sim}{j}$

116 Let $P(n)$ represent the proposition.

$P(1)$ is true since $16\times 3^{2(1)-1}+21\times 2^{2(1)-1}=90=30(3)$.

If $P(k)$ is true for some arbitrary $k\geq 1$ then $16\times 3^{2k-1}+21\times 2^{2k-1}=30m$ for integral m

RTP $P(k+1)$ $16\times 3^{2k+1}+21\times 2^{2k+1}=30p$ for integral p

$$\text{LHS} = 9\left(16\times 3^{2k-1}\right)+4\left(21\times 2^{2k-1}\right)$$
$$= 9\left(16\times 3^{2k-1}+21\times 2^{2k-1}\right)-5\left(21\times 2^{2k-1}\right)$$
$$= 9(30m)-5\times 3\times 2\left(7\times 2^{2k-2}\right) \quad \text{from } P(k)$$
$$= 30(9m-7\times 2^{2k-2})$$
$$= 30p \quad \text{since } m,k \text{ are integral}$$
$$= \text{RHS}$$

$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\geq 1$ by induction

117 **i**

$$z = \frac{4+3i}{2+4i}\times\frac{2-4i}{2-4i}$$
$$= \frac{8-16i+6i+12}{2^2+4^2}$$
$$= \frac{20-10i}{20}$$
$$= 1-\frac{1}{2}i$$
$$\therefore |z| = \sqrt{1^2+\left(-\frac{1}{2}\right)^2}$$
$$= \frac{\sqrt{5}}{2}$$

ii

$$z = \frac{a+3i}{2+ai}\times\frac{2-ai}{2-ai}$$
$$= \frac{2a-a^2i+6i+3a}{2^2+a^2}$$
$$= \frac{5a+(6-a^2)i}{a^2+4}$$
$$= \frac{5a}{a^2+4}+\frac{6-a^2}{a^2+4}i$$

$\arg z = \tan^{-1}\frac{6-a^2}{5a}$

$\therefore \frac{6-a^2}{5a} = \tan\frac{\pi}{4}$

$6-a^2=5a$

$a^2+5a-6=0$

$(a+6)(a-1)=0$

$\therefore a=-6,1$

$\therefore a=1$ is the only solution, since z is in the first quadrant.

118 $a = \frac{2(-1)+3}{(-1)^2+4} = \frac{1}{5}$

equating coefficients of x^2: $a+b=0 \ \rightarrow\ b=-\frac{1}{5}$

equating constants: $4a+c=3\ \rightarrow \frac{4}{5}+c=3\ \rightarrow c=\frac{11}{5}$

$$\int_0^2 \frac{2x+3}{(x+1)(x^2+4)}\,dx$$
$$= \frac{1}{5}\int_0^2\left(\frac{1}{x+1}-\frac{x-11}{x^2+4}\right)dx$$
$$= \frac{1}{5}\int_0^2\left(\frac{1}{x+1}-\frac{1}{2}\times\frac{2x}{x^2+4}+\frac{11}{x^2+4}\right)dx$$
$$= \frac{1}{5}\left[\ln|x+1|-\frac{1}{2}\ln|x^2+4|+\frac{11}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$
$$= \frac{1}{5}\left(\left(\ln 3-\frac{1}{2}\ln 8+\frac{11}{2}\times\left(\frac{\pi}{4}\right)\right)-\left(0-\frac{1}{2}\ln 4-0\right)\right)$$
$$= \frac{1}{5}\left(\ln 3+\frac{1}{2}\ln\frac{4}{8}+\frac{11\pi}{8}\right)$$
$$= \frac{1}{5}\left(\ln\frac{3}{\sqrt{2}}+\frac{11\pi}{8}\right)$$
$$= \frac{1}{5}\left(\ln\frac{3\sqrt{2}}{2}+\frac{11\pi}{8}\right)$$

119

$$\begin{pmatrix}1\\5t+4p\\3\end{pmatrix}=\begin{pmatrix}4t+p\\7t+q\\t-q\end{pmatrix}$$

$3=t-q \rightarrow \ t=q+3 \quad (1)$

$1=4t+p \rightarrow t=\dfrac{1-p}{4} \quad (2)$

$\therefore$ from $(1),(2)$: $q+3=\dfrac{1-p}{4} \rightarrow 4q+12=1-p \quad (3)$

$5t+4p=7t+q \rightarrow t=\dfrac{4p-q}{2} \quad (4)$

$\therefore$ from $(2),(4)$: $\dfrac{1-p}{4}=\dfrac{4p-q}{2}$

$1-p=8p-2q$

$2q+1=9p \quad (5)$

$(3)-2(5)$: $10=1-19p \rightarrow 19p=-9 \rightarrow p=-\dfrac{9}{19}$

sub in (5): $2q+1=9\left(-\dfrac{9}{19}\right) \rightarrow 2q=-\dfrac{100}{19} \rightarrow q=-\dfrac{50}{19}$

sub in (1):

$t=-\dfrac{50}{19}+3=\dfrac{7}{19}\text{h}$

$=22$ min 6 seconds (nearest second)

$$\begin{pmatrix}1\\5t+4p\\3\end{pmatrix}=\begin{pmatrix}1\\5\left(\frac{7}{19}\right)+4\left(-\frac{9}{19}\right)\\3\end{pmatrix}=\begin{pmatrix}1\\-\frac{1}{19}\\3\end{pmatrix}$$

120

$\dot{x}=10e^{-\frac{t}{\sqrt{3}}}$

$\dot{y}=(\sqrt{3}\times 10+10\sqrt{3})e^{-\frac{t}{\sqrt{3}}}-\sqrt{3}\times 10$

$=10\sqrt{3}\left(2e^{-\frac{t}{\sqrt{3}}}-1\right)$

$\tan\dfrac{\pi}{4}=\dfrac{\dot{y}}{\dot{x}}$

$1=\dfrac{10\sqrt{3}\left(2e^{-\frac{t}{\sqrt{3}}}-1\right)}{10e^{-\frac{t}{\sqrt{3}}}}$

$10e^{-\frac{t}{\sqrt{3}}}=10\sqrt{3}\left(2e^{-\frac{t}{\sqrt{3}}}-1\right)$

$e^{-\frac{t}{\sqrt{3}}}(1-2\sqrt{3})=-\sqrt{3}$

$e^{\frac{t}{\sqrt{3}}}=\dfrac{2\sqrt{3}-1}{\sqrt{3}}$

$\dfrac{t}{\sqrt{3}}=\ln\left(\dfrac{2\sqrt{3}-1}{\sqrt{3}}\right)$

$t=\sqrt{3}\ln\left(\dfrac{2\sqrt{3}-1}{\sqrt{3}}\right)$

$=0.61$ seconds

121

Let the numbers be $\frac{p}{q}$ and $\frac{m}{n}$, where p,q,m,n are integral

$\dfrac{p}{q}\div\dfrac{m}{n}=\dfrac{p}{q}\times\dfrac{n}{m}$

$=\dfrac{pn}{qm}$

$=\dfrac{a}{b}$ for integral a,b since p,q,m,n are integral

$\therefore$ the product of two rational numbers is rational

122

i

$z=\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ by inspection

ii

z^7+64z

$=\left(2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)\right)^7+64\left(2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)\right)$

$=128\left(\cos\dfrac{7\pi}{6}+i\sin\dfrac{7\pi}{6}\right)+128\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)$

$=128\left[\left(-\cos\dfrac{\pi}{6}-i\sin\dfrac{\pi}{6}\right)+\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)\right]$

$=0$

123

$\displaystyle\int\frac{2x^3}{(x^2-1)^2}dx$

$=2\displaystyle\int\frac{x^3}{u^2}\times\frac{du}{2x}$

$=\displaystyle\int\frac{x^2}{u^2}du$

$=\displaystyle\int\frac{u+1}{u^2}du$

$=\displaystyle\int\left(\frac{1}{u}+u^{-2}\right)du$

$=\ln|u|-\dfrac{1}{u}+c$

$=\ln|x^2-1|-\dfrac{1}{x^2-1}+c$

$u=x^2-1$
$du=2x\,dx$
$dx=\dfrac{du}{2x}$

124

The height, $z=\cos t$, goes from 1 to -1, so t goes from 0 to π.

The x and y components are a circle that starts with a radius of 0, expands to a radius of 1 at $t=\frac{\pi}{2}$ (halfway), then shrinks back to a radius of 1. Multiplying the standard unit circle parametric equations by $\sin t$ achieves this.

ANSWER (A)

125

i

$\dfrac{dv}{dt}=t\sin t$

$v-u=\displaystyle\int_0^t t\sin t\,dt$

$=-\Big[t\cos t\Big]_0^t+\displaystyle\int_0^t\cos t\,dt$

$=-t\cos t+\Big[\sin t\Big]_0^t$

$v=-t\cos t+\sin t+u$

$u=t$	$\dfrac{dv}{dx}=\sin t$
$\dfrac{du}{dt}=1$	$v=-\cos t$

ii

$\dfrac{dx}{dt}=-t\cos t+\sin t+u$

$u=-t$	$\dfrac{dv}{dx}=\cos t$
$\dfrac{du}{dt}=-1$	$v=\sin t$

$x=-\Big[t\sin t\Big]_0^t+\displaystyle\int_0^t\sin t\,dt+\int_0^t(\sin t+u)\,dt$

$=-t\sin t-\Big[2\cos t+ut\Big]_0^t$

$=-t\sin t-2\cos t+ut+2$

126

Suppose $x<0$.

$x^5-4x^4+3x^3-x^2+3x-4$

$=(x^5+3x^3+3x)-(4x^4+x^2+4)$

<0 since $x^5+3x^3+3x<0, 4x^4+x^2+4>0$ for $x<0$

$\therefore$ if $x<0$ then $x^5-4x^4+3x^3-x^2+3x-4<0$.

$\therefore$ if $x^5-4x^4+3x^3-x^2+3x-4\geq 0$ then $x\geq 0$ by contrapositive.

127

$e^{-1+\frac{\pi}{2}i}=e^{-1}\cdot e^{\frac{\pi}{2}i}=\dfrac{1}{e}\cdot i=\dfrac{i}{e}$

128 $\int \sec^6 x\,dx$

$= \int \sec^2 x \sec^4 x\,dx$

$= \int \sec^2 x\,(\tan^2 x + 1)^2\,dx$

$= \int \sec^2 x\,(\tan^4 x + 2\tan^2 x + 1)\,dx$

$= \int \sec^2 x\,(\tan x)^4\,dx + 2\int \sec^2 x\,(\tan x)^2\,dx$

$+ \int \sec^2 x\,dx$

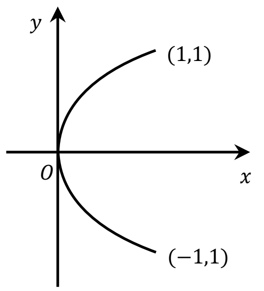
$= \frac{\tan^5 x}{5} + \frac{2\tan^3 x}{3} + \tan x + c$

129 $0 \le \sin^2 t \le 1 \rightarrow 0 \le x \le 1$

$-1 \le \sin t \le 1 \rightarrow -1 \le y \le 1$

$\therefore x = y^2$ for $-1 \le y \le 1$

This is part of a concave right parabola with vertex at the origin.



130 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = a$

$\frac{1}{2}(v^2 - u^2) = a\int_0^s dx$

$v^2 - u^2 = 2a\Big[x\Big]_0^s$

$v^2 - u^2 = 2as$

$v^2 = u^2 + 2as$

131 Suppose $\sqrt{5} = \frac{p}{q}$ for integral p, q is a solution and p and q have no common factor except 1 (*)

$\therefore 5 = \frac{p^2}{q^2}$

$5q^2 = p^2$

Now the LHS is a multiple of 5

$\therefore p^2$ is a multiple of 5

$\therefore p$ is a multiple of 5

Let $p = 5m$ for integral m

$\therefore 5q^2 = 25m^2$

$q^2 = 5m^2$

Now $5m^2$ is a multiple of 5

$\therefore q^2$ is a multiple of 5

$\therefore q$ is a multiple of 5 #

Now p and q have a common factor of 5 which contradicts (*), hence $\sqrt{5}$ is irrational

132 $\frac{3+i}{1+2i} + \lambda$

$= \frac{3+i}{1+2i} \times \frac{1-2i}{1-2i} + \lambda$

$= \frac{3-6i+i+2}{1^2+2^2} + \lambda$

$= \frac{5-5i}{5} + \lambda$

$= (1+\lambda) - i$

$\therefore |(1+\lambda) - i| = \sqrt{\lambda + 2}$

$\sqrt{(1+\lambda)^2 + (-1)^2} = \sqrt{\lambda+2}$

$1 + 2\lambda + \lambda^2 + 1 = \lambda + 2$

$\lambda^2 + \lambda = 0$

$\lambda(\lambda + 1) = 0$

$\lambda = 0, -1$

133 $\int \frac{dx}{x^2\sqrt{x^2+1}}$

$x = \tan\theta$ $dx = \sec^2\theta\,d\theta$

$= \int \frac{1}{\tan^2\theta\sqrt{\tan^2\theta + 1}}\sec^2\theta\,d\theta$

$= \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}d\theta$

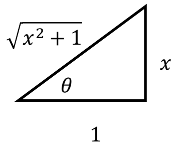
$= \int \frac{\sec\theta}{\tan^2\theta}d\theta$

$= \int \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta}d\theta$

$= \int \cos\theta\,(\sin\theta)^{-2}\,d\theta$

$= -\frac{1}{\sin\theta} + c$

$= -\frac{\sqrt{x^2+1}}{x} + c$



134 **i**

$\overrightarrow{AB} = \begin{pmatrix} -1-(-2) \\ 3-4 \\ 8-7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

ii

$\ell_1\!: \ \underset{\sim}{r} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

iii

$\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, |\overrightarrow{BA}| = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}$

$\overrightarrow{BP} = \begin{pmatrix} 0-(-1) \\ 2-3 \\ 3-8 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}, |\overrightarrow{BP}|$

$= \sqrt{(1)^2 + (-1)^2 + (-5)^2} = \sqrt{27} = 3\sqrt{3}$

$\cos\theta = \frac{\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}}{\sqrt{3} \times 3\sqrt{3}}$

$= \frac{(-1)(1) + (1)(-1) + (-1)(-5)}{9}$

$= \frac{3}{9}$

$= \frac{1}{3}$

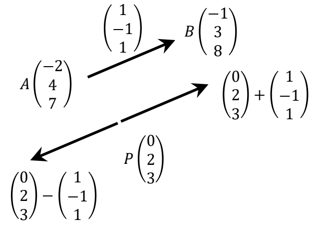
iv

$\ell_2\!: \ \underset{\sim}{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

v

$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \pm \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

$\therefore C\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $D\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$



135 $n = \frac{\pi}{6} \rightarrow T = \frac{2\pi}{\frac{\pi}{6}} = 12$ hours

$\therefore$ the oysters need to be underwater for 10 hours during each cycle, so above water for two hours only, so 1 hour either side of high tide.

$x_1 = 3\cos\frac{\pi}{6} + 2$

$= 4.60$ m (2 dp)

The maximum height of any living oyster is 4.60 metres.

136 True. Any number divisible by 6 is also divisible by 2 and 3, so must be composite

137

$$z + 2iz = -4 - 3i$$
$$x + iy + 2ix - 2y = -4 - 3i$$
$$x - 2y = -4 \quad (1)$$
$$2x + y = -3 \quad (2)$$
$$(1) + 2(2):\ 5x = -10 \ \rightarrow\ x = -2$$

ANSWER (C)

138

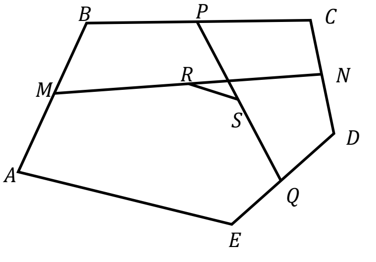
$$\int_4^{e+3} \ln|x-3|\,dx$$

$u = \ln\lvert x-3\rvert$	$\frac{dv}{dx} = 1$
$\frac{du}{dx} = \frac{1}{x-3}$	$v = x$

$$= \Big[x\ln|x-3|\Big]_4^{e+3} - \int_4^{e+3} \frac{x}{x-3}\,dx$$
$$= \big((e+3) - 0\big) - \int_4^{e+3} \frac{x-3+3}{x-3}\,dx$$
$$= e + 3 - \int_4^{e+3}\left(1 + \frac{3}{x-3}\right)dx$$
$$= e + 3 - \Big[x + 3\ln|x-3|\Big]_4^{e+3}$$
$$= e + 3 - \Big(\big((e+3)+3\big) - (4+0)\Big)$$
$$= 1$$

139

Let the midpoints of AB, CD, BC, DE, MN and PQ be M, N, P, Q, R and S respectively.
Let $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$ etc



$$\underset{\sim}{r} = \frac{1}{2}\left(\underset{\sim}{m} + \underset{\sim}{n}\right) = \frac{1}{2}\left(\frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + \frac{1}{2}\left(\underset{\sim}{c} + \underset{\sim}{d}\right)\right)$$
$$= \frac{1}{4}\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}\right)$$
$$\underset{\sim}{s} = \frac{1}{2}\left(\underset{\sim}{p} + \underset{\sim}{q}\right) = \frac{1}{2}\left(\frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right) + \frac{1}{2}\left(\underset{\sim}{d} + \underset{\sim}{e}\right)\right)$$
$$= \frac{1}{4}\left(\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d} + \underset{\sim}{e}\right)$$
$$\overrightarrow{RS} = \underset{\sim}{s} - \underset{\sim}{r} = \frac{1}{4}\left(\underset{\sim}{e} - \underset{\sim}{a}\right) = \frac{1}{4}\overrightarrow{AE}$$

The interval is parallel to AE and one quarter its length.

140

$$\ddot{x} = -10$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -10$$
$$\frac{1}{2}v^2 = -10x + c$$

Let $v = 20, x = 0$

$$\frac{1}{2}(20)^2 = -10(0) + c \ \rightarrow\ c = 200$$
$$\frac{1}{2}v^2 = -10x + 200$$

Let $v = 0$

$$0 = -10x + 200$$
$$x = 20$$

The mine shaft is 20 metres deep.

141

i

$$u_2 = \sqrt{3u_1} = \sqrt{3(1)} = 3^{\frac{1}{2}}$$
$$u_3 = \sqrt{3u_2} = \sqrt{3\left(3^{\frac{1}{2}}\right)} = \left(3^{\frac{3}{2}}\right)^{\frac{1}{2}} = 3^{\frac{3}{4}} = 3^{\frac{1}{2}+\frac{1}{4}}$$
$$u_4 = \sqrt{3u_3} = \sqrt{3\left(3^{\frac{3}{4}}\right)} = \left(3^{\frac{7}{4}}\right)^{\frac{1}{2}} = 3^{\frac{7}{8}} = 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$$

ii

Let $P(1)$ represent the proposition.

$P(1)$ is true since $u_1 = 3^{1-2^{1-1}} = 3^{1-1} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $u_k = 3^{1-2^{1-k}}$

RTP $P(k+1)$ $u_{k+1} = 3^{1-2^{-k}}$

$$\text{LHS} = u_{k+1}$$
$$= \sqrt{3u_k}$$
$$= \sqrt{3 \times 3^{1-2^{1-k}}} \text{ from } P(k)$$
$$= \sqrt{3^{1+1-2^{1-k}}}$$
$$= \sqrt{3^{2-2^{1-k}}}$$
$$= \sqrt{3^{2(1-2^{-k})}}$$
$$= 3^{1-2^{-k}}$$
$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

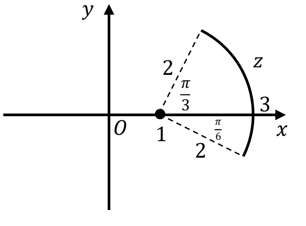
$\therefore P(n)$ is true for $n \geq 1$ by induction

iii

$u_n = 3^{1-2^{1-n}}$
as $n \to \infty, 2^{1-n} \to 0, u_n \to 3^{1-0} \to 3$

142

$z = 2e^{\theta i}$ the circle centred at $z = 0$ with radius 2.
The restriction $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$ gives the arc between arguments $-\frac{\pi}{6}$ and $\frac{\pi}{3}$.
Adding 1 moves the arc 1 unit to the right.



143

$u = \sin 3x$	$\frac{dv}{dx} = x^{-n}$
$\frac{du}{dx} = 3\cos 3x$	$v = \frac{x^{-n+1}}{-n+1} = -\frac{1}{(n-1)x^{n-1}}$

$$I_n = -\frac{\sin 3x}{(n-1)x^{n-1}} + \frac{3}{n-1}\int \frac{\cos 3x}{x^{n-1}}\,dx$$
$$= -\frac{\sin 3x}{(n-1)x^{n-1}} + \frac{3}{n-1}J_{n-1}$$

144 **i**

$$\begin{pmatrix}2+\lambda\\3+2\lambda\\-4+\lambda\end{pmatrix}=\begin{pmatrix}5\mu\\9\\-3+2\mu\end{pmatrix}$$

$3+2\lambda=9\ \to \lambda=3$

$$C \text{ is}\begin{pmatrix}2+(3)\\3+2(3)\\-4+(3)\end{pmatrix}=\begin{pmatrix}5\\9\\-1\end{pmatrix}$$

ii

from (i) $\mu=1$

$$A \text{ is}\begin{pmatrix}2+(0)\\3+2(0)\\-4+(0)\end{pmatrix}=\begin{pmatrix}2\\3\\-4\end{pmatrix}$$

$$B \text{ is}\begin{pmatrix}5(-1)\\9\\-3+2(-1)\end{pmatrix}=\begin{pmatrix}-5\\9\\-5\end{pmatrix}$$

$$\overrightarrow{CA}=\begin{pmatrix}2-5\\3-9\\-4-(-1)\end{pmatrix}=\begin{pmatrix}-3\\-6\\-3\end{pmatrix}$$

$$|\overrightarrow{CA}|=\sqrt{(-3)^2+(-6)^2+(-3)^2}=\sqrt{54}=3\sqrt{6}$$

$$\overrightarrow{CB}=\begin{pmatrix}-5-5\\9-9\\-5-(-1)\end{pmatrix}=\begin{pmatrix}-10\\0\\-4\end{pmatrix}$$

$$|\overrightarrow{CB}|=\sqrt{10^2+0^2+4^2}=\sqrt{116}=2\sqrt{29}$$

$$\overrightarrow{CA}\cdot\overrightarrow{CB}=(-3)(-10)+(-6)(0)+(-3)(-4)=42$$

$$\cos\angle ACB=\frac{42}{3\sqrt{6}\times 2\sqrt{29}}$$

$$\angle ACB=57°57'$$

iii

$$\text{Area}=\frac{1}{2}|\overrightarrow{CA}|\times|\overrightarrow{CB}|\times\sin\angle ACB$$

$$=\frac{1}{2}\times 3\sqrt{6}\times 2\sqrt{29}\times\sin 57°57'$$

$$=33.54\text{ unit}^2$$

145

$$x=3\cos^2 t$$

$$=3\left(\frac{1}{2}(1+\cos 2t)\right)$$

$$=\frac{3}{2}+\frac{3}{2}\cos 2t$$

$$\dot{x}=-3\sin 2t$$

$$\ddot{x}=-6\cos 2t$$

$$=-4\left(\frac{3}{2}\cos 2t\right)$$

$$=-2^2\left(\frac{3}{2}+\frac{3}{2}\cos 2t-\frac{3}{2}\right)$$

$$=-2^2\left(x-\frac{3}{2}\right)$$

The particle is in SHM with $c=\frac{3}{2}$ and $a=\frac{3}{2}$.

146 Suppose the product of a rational and irrational number is rational

Let $a=\frac{p}{r}$ and $ab=\frac{m}{n}$, where p,q,m,n are integral and b is irrational (*)

$$\therefore \frac{p}{r}\times b=\frac{m}{n}$$

$$b=\frac{mr}{np}$$

$$=\frac{c}{d}\ \text{ for integral } c,d \text{ since } p,q,m,n \text{ are integral}$$

This contradicts (*) since b cannot be rational and irrational.

∴ the product of a rational and irrational number is irrational

147

$$(a+b)(a+\omega b)(a+\omega^2 b)$$

$$=(a^2+ab\omega+ab+b^2\omega)(a+\omega^2 b)$$

$$=a^3+a^2b\omega^2+a^2b\omega+ab^2\omega^3+a^2b+ab^2\omega^2+ab^2\omega+b^3\omega^3$$

$$=a^3+a^2b(\omega^2+\omega+1)+ab^2(\omega^3+\omega^2+\omega)+b^3$$

$$=a^3+a^2b(0)+ab^2(1+\omega^2+\omega)+b^3$$

$$=a^3+ab^2(0)+b^3$$

$$=a^3+b^3$$

148 $x^3>x^2$ for $x>1$

so $1\le b<a$

149 The spheres have respective radii of 1 and 3, so the centre of the smaller sphere must lie within 2 units of the centre of the larger sphere to lie entirely within the first sphere. This sphere is $x^2+y^2+z^2=4$, so if the centre (a,b,c) is on or inside this sphere we have $a^2+b^2+c^2\le 4$.

150 **i**

$$24+2x-x^2=0$$

$$x^2-2x-24=0$$

$$(x-6)(x+4)=0$$

$$x=-4,6$$

The particle is oscillating between $x=-4$ and $x=6$.

ii

$$a=\frac{6-(-4)}{2}=5$$

iii

$$v^2=24+2x-x^2$$

$$=1^2(5^2-(x-1)^2)$$

$n=1, c=1$

$\therefore \ddot{x}=-(x-1)=1-x$

iv

$$T=\frac{2\pi}{n}=2\pi$$

v

$v^2=1^2(5^2-(x-1)^2)$ from (iii)

The maximum velocity occurs at $x=c$

$\therefore v_{max}^2=5^2$

∴ the maximum speed is 5 ms^{-1}

151

$$|x^2-x|+|x-1|\ge|(x^2-x)+(x-1)|$$

$$\ge|x^2-1|$$

$$\ge|(x+1)(x-1)|$$

152

$$z^2+(z+1)^2=0$$

$$2z^2+2z+1=0$$

$$z=\frac{-2\pm\sqrt{2^2-4(2)(1)}}{2(2)}$$

$$=\frac{-2\pm\sqrt{-4}}{4}$$

$$=\frac{-2\pm 2i}{4}$$

$$=\frac{-1\pm i}{2}$$

153

$$\int\frac{dx}{(x^2+1)\tan^{-1}x}$$

$$=\int\frac{\frac{1}{x^2+1}}{\tan^{-1}x}dx$$

$$=\ln|\tan^{-1}x|+c$$

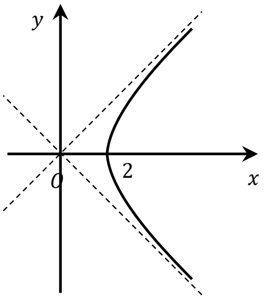
154 $e^t + e^{-t} > 0$, so $x > 0$

$x^2 = e^{2t} + 2 + e^{-t}$ (1)

$y^2 = e^{2t} - 2 + e^{-t}$ (2)

(1) − (2):

$x^2 - y^2 = 4$, for $x > 0$, which t=is the right branch of the hyperbola cutting the x-axis at 2, with $y = \pm x$ as the asymptotes.



155 **i**

$$m\ddot{x} = -0.8mg$$
$$\ddot{x} = -0.8 \times 10$$
$$= -8 \text{ ms}^{-2}$$

The car decelerates at 8 ms^{-2}

ii

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -8$$
$$\frac{1}{2}(v^2 - 32^2) = -8x$$
$$v^2 - 32^2 = -16x$$
$$x = \frac{32^2 - v^2}{16}$$

Let $v = 0$

$$x = \frac{32^2}{16} = 64 \text{ m}$$

156 $P(1)$ is true since $\text{LHS} = 1^2 \times 2 = 2$; $\text{RHS} = \frac{(1)(1+1)(1+2)(3(1)+1)}{12} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$1^2 \times 2 + 2^2 \times 3 + \ldots + k^2(k+1) = \frac{k(k+1)(k+2)(3k+1)}{12}$$

RTP $P(k+1)$

$$1^2 \times 2 + 2^2 \times 3 + \ldots + k^2(k+1) + (k+1)^2(k+2) = \frac{(k+1)(k+2)(k+3)(3k+4)}{12}$$

$$\text{LHS} = \frac{k(k+1)(k+2)(3k+1)}{12} + (k+1)^2(k+2)$$
$$= \frac{(k+1)(k+2)}{12}\big(k(3k+1) + 12(k+1)\big)$$
$$= \frac{(k+1)(k+2)}{12}(3k^2 + 13k + 12)$$
$$= \frac{(k+1)(k+2)}{12}(k+3)(3k+4)$$
$$= \frac{(k+1)(k+2)(k+3)(3k+4)}{12}$$
$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

157

$$z_1^2 \div \overline{z_2} = \frac{(1+2i)^2}{3+i} = \frac{1+4i-4}{3+i} \times \frac{3-i}{3-i}$$
$$= \frac{(-3+4i)(3-i)}{9+1} = \frac{-9+3i+12i+4}{10}$$
$$= \frac{-5+15i}{10} = \frac{-1+3i}{2}$$

ANSWER (D)

158

$$\int \frac{e^{4x}}{e^{2x}+1}dx$$
$$= \int \frac{e^{2x}(e^{2x}+1) - (e^{2x}+1) + 1}{e^{2x}+1}dx$$
$$= \int \left(e^{2x} - 1 + \frac{1}{e^{2x}+1}\right)dx$$
$$= \int \left(e^{2x} - 1 + \frac{e^{-2x}}{1+e^{-2x}}\right)dx$$
$$= \frac{1}{2}e^{2x} - x - \frac{1}{2}\ln|1 + e^{-2x}| + c$$

159

$$\angle AOC = \cos^{-1}\left(\frac{(1,3,1)\cdot(-1,1,2)}{\sqrt{1^2+3^2+1^2} \times \sqrt{(-1)^2+1^2+2^2}}\right)$$
$$= \cos^{-1}\left(\frac{(1)(-1)+(3)(1)+(1)(2)}{\sqrt{11} \times \sqrt{6}}\right)$$
$$= \cos^{-1}\frac{4}{\sqrt{66}}$$
$$= 60°30'$$
$$\text{Area}_{AOC} = \frac{1}{2} \times |\overrightarrow{OA}| \times |\overrightarrow{OC}| \times \sin\angle AOC$$
$$= \frac{1}{2} \times \sqrt{11} \times \sqrt{6} \times \sin 60°30'$$
$$= 3.5355\ldots$$
$$\therefore \text{Area}_{OACB} = 2(3.5355\ldots)$$
$$= 7.07 \text{ units}^2$$

160 **i**

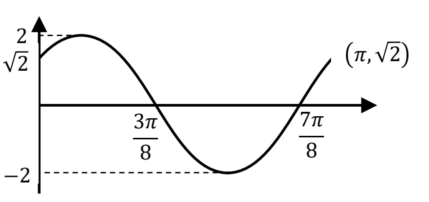
$$x = 2\cos\left(2t - \frac{\pi}{4}\right)$$
$$v = -4\sin\left(2t - \frac{\pi}{4}\right)$$
$$\ddot{x} = -8\cos\left(2t - \frac{\pi}{4}\right)$$

$$v^2 - x\ddot{x}$$
$$= \left(-4\sin\left(2t - \frac{\pi}{4}\right)\right)^2 - \left(2\cos\left(2t - \frac{\pi}{4}\right)\right)\left(-8\cos\left(2t - \frac{\pi}{4}\right)\right)$$
$$= 16\sin^2\left(2t - \frac{\pi}{4}\right) + 16\cos^2\left(2t - \frac{\pi}{4}\right)$$
$$= 16\left(\sin^2\left(2t - \frac{\pi}{4}\right) + \cos^2\left(2t - \frac{\pi}{4}\right)\right)$$
$$= 16$$

ii

Let $t = 0 \rightarrow x = 2\cos\left(-\frac{\pi}{4}\right) = 2\cos\frac{\pi}{4} = \sqrt{2}$

Let $t = \pi \rightarrow x = 2\cos\left(2\pi - \frac{\pi}{4}\right) = 2\cos\frac{7\pi}{4} = 2\cos\frac{\pi}{4} = \sqrt{2}$



iii

$$\sqrt{2} = 2\cos\left(2t - \frac{\pi}{4}\right)$$
$$\cos\left(2t - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$2t - \frac{\pi}{4} = \frac{\pi}{4}$$
$$2t = \frac{\pi}{2}$$
$$t = \frac{\pi}{4}$$
$$T = \frac{2\pi}{2} = \pi$$

$\therefore$ the particle first returns to its starting point after one quarter of its period.

iv

Each period the particle travels 4 times its amplitude, so 8 metres.

$\frac{100}{8} = 12.5$, so the particle takes 12.5 times its period, or $\frac{25\pi}{2}$ seconds.

161
$$(a-b)^2 \geq 0$$
$$a^2-2ab+b^2 \geq 0$$
$$a^2+b^2 \geq 2ab \quad (1)$$
Similarly
$b^2+c^2>0 \quad (2)$
$a^2+c^2 \geq 0$
$(1)+(2)+(3)$:
$$2(a^2+b^2+c^2) \geq 2(ab+bc+ac)$$
$$a^2+b^2+c^2 \geq ab+bc+ac$$
Equality occurs when $a=b=c$, so if $a^2+b^2+c^2=ab+bc+ac$ then $a=b=c$ so ΔABC is isosceles.

162
$$e^{ix}=1+\frac{ix}{1!}+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\ldots$$
$$=1+xi-\frac{x^2}{2!}+\frac{x^3}{3!}i-\ldots$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)i$$
$$=\cos x+i\sin x$$

163
$$\int_0^1 \frac{3x^3+x^2+12x-12}{(x^2+4)(x^2+2x+4)}dx$$
$$=\int_0^1\left(\frac{x-3}{x^2+2x+4}+\frac{2x}{x^2+4}\right)dx$$
$$=\int_0^1\left(\frac{\frac{1}{2}(2x+2)-4}{x^2+2x+4}+\frac{2x}{x^2+4}\right)dx$$
$$=\int_0^1\left(\frac{1}{2}\times\frac{2x+2}{x^2+2x+4}-\frac{4}{(x+1)^2+\left(\sqrt{3}\right)^2}+\frac{2x}{x^2+4}\right)dx$$
$$=\left[\frac{1}{2}\ln|x^2+2x+4|-\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+\ln|x^2+4|\right]_0^1$$
$$=\left(\frac{1}{2}\ln 7-\frac{4}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}+\ln 5\right)$$
$$-\left(\frac{1}{2}\ln 4-\frac{4}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}+\ln 4\right)$$
$$=\ln\sqrt{7}-\frac{4}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}+\ln 5-\ln 2-\ln 4+\frac{4}{\sqrt{3}}\left(\frac{\pi}{6}\right)$$
$$=\ln\frac{5\sqrt{7}}{8}+\frac{4}{\sqrt{3}}\left(\frac{\pi}{6}-\tan^{-1}\frac{2}{\sqrt{3}}\right)$$

164
P lies on AB then $\overrightarrow{AP}=k\overrightarrow{AB}$.
$$\overrightarrow{OP}-\overrightarrow{OA}=k\left(\overrightarrow{OB}-\overrightarrow{OA}\right)$$
$$\overrightarrow{OP}=(1-k)\overrightarrow{OA}+k\overrightarrow{OB}$$
$$=\lambda\overrightarrow{OA}+(1-\lambda)\overrightarrow{OB} \text{ for } \lambda+k=1$$

165
i
$\ddot{x}=g-kv$
Let $\ddot{x}=0, v=T$
$0=g-kT$
$T=\frac{g}{k}$

165 …
ii
$$v\frac{dv}{dx}=g-kv$$
$$\frac{dv}{dx}=\frac{g-kv}{v}$$
$$\frac{dx}{dv}=\frac{v}{g-kv}$$
$$x=\int_0^{\frac{T}{2}}\frac{v}{g-kv}dv$$
$$=\int_0^{\frac{T}{2}}\frac{-\frac{1}{k}(g-kv)+\frac{g}{k}}{g-kv}dv$$
$$=\int_0^{\frac{T}{2}}\left(-\frac{1}{k}-\frac{g}{k^2}\times\frac{-k}{g-kv}\right)dv$$
$$=\int_{\frac{T}{2}}^{0}\left(\frac{1}{k}+\frac{g}{k^2}\times\frac{-k}{g-kv}\right)dv$$
$$=\left[\frac{v}{k}+\frac{g}{k^2}\ln(g-kv)\right]_{\frac{T}{2}}^{0}$$
$$=\left(0+\frac{g}{k^2}\ln g\right)-\left(\frac{T}{2k}+\frac{g}{k^2}\ln\left(g-\frac{kT}{2}\right)\right)$$
$$=\frac{g}{k^2}\ln g-\frac{g}{k^2}\ln\left(g-\frac{g}{2}\right)-\frac{T}{2k}$$
$$=\frac{T^2}{g}\ln\left(\frac{g}{g/2}\right)-\frac{T^2}{2g}$$
$$=\frac{T^2}{g}\left(\ln 2-\frac{1}{2}\right)$$

iii
$$\frac{dv}{dt}=g-kv$$
$$\frac{dt}{dv}=\frac{1}{g-kv}$$
$$t=\int_0^{\frac{T}{2}}\frac{dv}{g-kv}$$
$$=\frac{1}{k}\int_{\frac{T}{2}}^{0}\frac{-k}{g-kv}dv$$
$$=\frac{1}{k}\left[\ln(g-kv)\right]_{\frac{T}{2}}^{0}$$
$$=\frac{1}{k}\left(\ln g-\ln\left(g-\frac{kT}{2}\right)\right)$$
$$=\frac{1}{k}\left(\ln g-\ln\left(g-\frac{g}{2}\right)\right)$$
$$=\frac{1}{k}\ln 2$$

166
$(a+b)^2-(a-b)^2$
$=\big((a+b)+(a-b)\big)\big((a+b)-(a-b)\big)$
$=(2a)(2b)$
$=4ab$
$=4p$ for integral p since a, b integral

∴ if a and b are integers, then $(a+b)^2-(a-b)^2$ is divisible by 4

167
$$(a+ib)+2(a-ib)=|a+ib+2|$$
$$3a-ib=|(a+2)+ib|$$
equating real and imaginary parts
$$3a=|(a+2)+ib| \qquad b=0$$
$$9a^2=a^2+4a+4$$
$$8a^2-4a-4=0$$
$$2a^2-a-1=0$$
$$(2a+1)(a-1)=0$$
$$a=-\frac{1}{2},1$$
Since $z=-\frac{1}{2}+0i=-\frac{1}{2}$ does not solve the original equation, $z=1+0i=1$ is the only solution

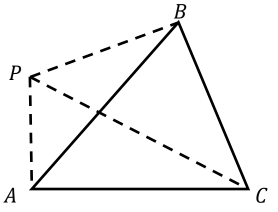
168 $\int \frac{\sin 2x + \cos 2x}{\cos x} dx$

$= \int \frac{2\sin x \cos x + 2\cos^2 x - 1}{\cos x} dx$

$= \int (2\sin x + 2\cos x - \sec x)\, dx$

$= -2\cos x + 2\sin x - \ln|\tan x + \sec x| + c$

169 Let $\overrightarrow{OP} = \underset{\sim}{p}, \overrightarrow{OA} = \underset{\sim}{a}$ etc



$\overrightarrow{PA} \cdot \overrightarrow{BC} + \overrightarrow{PB} \cdot \overrightarrow{CA} + \overrightarrow{PC} \cdot \overrightarrow{AB}$

$= \left(\underset{\sim}{a} - \underset{\sim}{p}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{b}\right) + \left(\underset{\sim}{b} - \underset{\sim}{p}\right) \cdot \left(\underset{\sim}{a} - \underset{\sim}{c}\right) + \left(\underset{\sim}{c} - \underset{\sim}{p}\right)$

$\cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

$= \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{p} \cdot \underset{\sim}{c} + \underset{\sim}{p} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{p} \cdot \underset{\sim}{a} + \underset{\sim}{p} \cdot \underset{\sim}{c}$

$+ \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{p} \cdot \underset{\sim}{b} + \underset{\sim}{p} \cdot \underset{\sim}{a}$

$= 0$

170 $\ddot{y} = g - k\dot{y}$

initially $\dot{y} = \frac{2g}{k} \sin\frac{\pi}{6} = \frac{g}{k}$

$\ddot{y} = g - k\left(\frac{g}{k}\right) = 0$

Since acceleration is zero the particle maintains a constant vertical velocity,

$\therefore \dot{y} = \frac{g}{k}$

$y = \frac{gt}{k}$

171 $P(1)$ is true since $\text{LHS} = (-1)^1 \times 1^2 = -1$; $\text{RHS} = \frac{(-1)^1(1)(1+1)}{2} = -1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$\sum_{r=1}^{k} (-1)^r r^2 = \frac{(-1)^k k(k+1)}{2}$$

RTP $P(k+1)$

$$\sum_{r=1}^{k+1} (-1)^r r^2 = \frac{(-1)^{k+1}(k+1)(k+2)}{2}$$

$\text{LHS} = \sum_{r=1}^{k} (-1)^r r^2 + (-1)^{k+1}(k+1)^2$

$= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2$ from $P(k)$

$= \frac{(-1)^k (k+1)}{2}\left(k + 2(-1)(k+1)\right)$

$= \frac{(-1)^k (k+1)}{2}(-k-2)$

$= \frac{(-1)^{k+1}(k+1)(k+2)}{2}$

$= \text{RHS}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

172 ΔODA is right angled (diagonals of a kite are perpendicular)

$\therefore OD = \frac{\sqrt{3}}{2}|w|$ (exact triangles)

$OB = 2 \times OD = \sqrt{3}|w|$

$\angle DCB = \angle DCO = \frac{\pi}{6}$ (angles of a kite are bisected by the diagonals)

$\therefore \Delta OBC$ is equilateral (isosceles triangle with apex angle $\frac{\pi}{3}$)

$\therefore \angle COB = \frac{\pi}{3}$

$\angle AOC = \frac{\pi}{2}$ (angle sum ΔAOC)

$\angle AOB = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

$\therefore \overrightarrow{OB} = \sqrt{3}w \operatorname{cis}\frac{\pi}{6}$

$= \sqrt{3}w\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= \frac{w(3 + \sqrt{3}i)}{2}$

$\overrightarrow{OC} = \overrightarrow{OB} \operatorname{cis}\frac{\pi}{3}$

$= \sqrt{3}w \operatorname{cis}\frac{\pi}{6} \operatorname{cis}\frac{\pi}{3}$

$= \sqrt{3}w \operatorname{cis}\frac{\pi}{2}$

$= \sqrt{3}wi$

173 $\int \frac{1}{\sin x + \cos x} dx$

$= \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$

$$t = \tan\frac{x}{2}$$
$$dx = \frac{2dt}{1+t^2}$$

$= 2\int \frac{dt}{2t + 1 - t^2}$

$= 2\int \frac{dt}{2 - (t-1)^2}$

$= 2\int \frac{dt}{(\sqrt{2} + t - 1)(\sqrt{2} - t + 1)}$

$= \frac{1}{\sqrt{2}} \int \left(\frac{1}{t - 1 + \sqrt{2}} + \frac{1}{1 + \sqrt{2} - t}\right) dt$

$= \frac{1}{\sqrt{2}}\left(\ln\left|t - 1 + \sqrt{2}\right| - \ln\left|1 + \sqrt{2} - t\right|\right) + c$

$= \frac{1}{\sqrt{2}} \ln\left|\frac{t - 1 + \sqrt{2}}{1 + \sqrt{2} - t}\right| + c$

$= \frac{1}{\sqrt{2}} \ln\left|\frac{\tan\frac{x}{2} - 1 + \sqrt{2}}{1 + \sqrt{2} - \tan\frac{x}{2}}\right| + c$

174 **i**

$\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} q \\ 2 \\ 2 \end{pmatrix} = 0$

$-2q + 2 - 8 = 0$

$2q = -6$

$q = -3$

ii

$\begin{pmatrix} 11 - 2\lambda \\ 2 + \lambda \\ 17 - 4\lambda \end{pmatrix} = \begin{pmatrix} -5 - 3\mu \\ 11 + 2\mu \\ p + 2\mu \end{pmatrix}$

$11 - 2\lambda = -5 - 3\mu \rightarrow 3\mu - 2\lambda = -16$ (1)

$2 + \lambda = 11 + 2\mu \rightarrow 2\mu - \lambda = -9$ (2)

$2(2) - (1)$: $\mu = -2$

sub in (2): $2(-2) - \lambda = -9 \rightarrow \lambda = 5$

$\therefore 17 - 4(5) = p + 2(-2) \rightarrow p - 4 = -3 \rightarrow p = 1$

iii

$\begin{pmatrix} 11 - 2(5) \\ 2 + (5) \\ 17 - 4(5) \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$

175 **i**

$$\ddot{x} = -(v+v^3)$$
$$v\frac{dv}{dx} = -(v+v^3)$$
$$\frac{dv}{dx} = -(1+v^2)$$
$$\frac{dx}{dv} = -\frac{1}{1+v^2}$$
$$x = -\int_u^v \frac{dv}{1+v^2}$$
$$= \Big[\arctan(v)\Big]_v^u$$
$$= \arctan u - \arctan v$$
$$= \arctan(\tan(\arctan u - \arctan v))$$
$$= \arctan\left(\frac{\tan(\arctan u) - \tan(\arctan v)}{1+\tan(\arctan u)\tan(\arctan v)}\right)$$
$$= \arctan\left(\frac{u-v}{1+uv}\right)$$

ii

$$\frac{dv}{dt} = -(v+v^3)$$
$$\frac{dt}{dv} = -\frac{1}{v+v^3}$$
$$t = -\int_u^v \frac{dv}{v+v^3}$$
$$= \int_v^u \left(\frac{1}{v} - \frac{v}{1+v^2}\right)dv$$
$$= \left[\ln v - \frac{1}{2}\ln(1+v^2)\right]_v^u$$
$$= \left(\ln u - \frac{1}{2}\ln(1+u^2)\right) - \left(\ln v - \frac{1}{2}\ln(1+v^2)\right)$$
$$= \frac{1}{2}\left(\ln u^2 + \ln(1+v^2) - \frac{1}{2}\ln(1+u^2) - \ln v^2\right)$$
$$= \frac{1}{2}\ln\left(\frac{u^2(1+v^2)}{v^2(1+u^2)}\right)$$
$$= \log_e\sqrt{\frac{u^2(1+v^2)}{v^2(1+u^2)}}$$

176 Suppose n is a composite integer and has no prime divisors less than or equal to $\sqrt{n}$ (*)

If k is a divisor then $\frac{n}{k}$ is also a divisor

$$\frac{n}{k} \le \frac{n}{\sqrt{n}}$$
$$\le \sqrt{n} \quad \#$$

Now if $\frac{n}{k}$ is prime then this is a contradiction of (*) since here we have a prime divisor $\le \sqrt{n}$, and if $\frac{n}{k}$ is composite it can be broken down into yet smaller prime divisors which still contradict (*).

$\therefore$ all composite integers n have a prime divisor k where $k \le \sqrt{n}$

177 C can be in two positions, C_1 and C_2, forming a square with diagonal AB.
The diagonals of a square are equal and bisect each other at right angles.
Let $M = \frac{1}{2}(2-i+8+i) = 5$ be the midpoint of AB.
$\overrightarrow{MB} = 8+i-5 = 3+i$
$\overrightarrow{MC_1} = i(3+i) = 3i-1 \;\therefore C_1 = 5+3i-1 = 4+3i$
$\overrightarrow{MC_2} = -i(3+i) = -3i+1 \;\therefore C_2 = 5-3i+1 = 6-3i$

178

$$\int \tan^{-1} x\,dx$$
$$= x\tan^{-1} x - \int \frac{x}{1+x^2}dx$$
$$= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$$

$u = \tan^{-1} x$	$\frac{dv}{dx} = 1$
$\frac{du}{dx} = \frac{1}{1+x^2}$	$v = x$

179 Since the spheres are touching, the distance between their centres are $1+2=3, 1+3=4$ and $2+3=5$, so a $3:4:5$ triangle which is right angled at A.
$\therefore AB \perp AC$ so $\overrightarrow{AB}\cdot\overrightarrow{AC} = 0$.

180 **i**

$$\frac{dv}{dt} = g - kv$$
$$\frac{dt}{dv} = \frac{1}{g-kv}$$
$$t = \int_0^v \frac{dv}{g-kv}$$
$$= \frac{1}{k}\Big[\ln(g-kv)\Big]_v^0$$
$$= \frac{1}{k}(\ln g - \ln(g-kv))$$
$$kt = \ln g - \ln(g-kv)$$
$$\ln(g-kv) = \ln g - kt$$
$$g - kv = ge^{-kt}$$
$$kv = g - ge^{-kt}$$
$$v = \frac{g}{k}(1-e^{-kt})$$

ii

$$v\frac{dv}{dx} = g - kv$$
$$\frac{dv}{dx} = \frac{g-kv}{v}$$
$$\frac{dx}{dv} = \frac{v}{g-kv}$$
$$x = \int_0^v \frac{v}{g-kv}dv$$
$$= \int_0^v \frac{-\frac{1}{k}(g-kv)+\frac{g}{k}}{g-kv}dv$$
$$= \int_0^v \left(-\frac{1}{k} - \frac{g}{k^2}\times\frac{-k}{g-kv}\right)dv$$
$$= \left[\frac{v}{k} + \frac{g}{k^2}\ln(g-kv)\right]_v^0$$
$$= \left(0 + \frac{g}{k^2}\ln g\right) - \left(\frac{v}{k} + \frac{g}{k^2}\ln(g-kv)\right)$$
$$= \frac{g}{k^2}\left[\ln\left(\frac{g}{g-kv}\right) - \frac{kv}{g}\right]$$

181 In ΔAXB, $AX + BX > AB$ (1) (triangle inequality)
Similarly in ΔBXC and ΔAXC:
$BX + CX > BC$ (2) and $AX + CX > AC$ (3)
(1) + (2) + (3):
$2(AX+BX+CX) > AB + BC + AC$
$AX + BX + CX > \frac{AB+AC+BC}{2}$

182 Let $n = 3k + j$ for integral k and $j = 0,1,2$
$1 + \omega^n + \omega^{2n}$
$= 1 + \omega^{3k+j} + \omega^{6k+2j}$
$= 1 + (\omega^3)^k\cdot\omega^j + (\omega^3)^{2k}\cdot\omega^{2j}$
$= 1 + \omega^j + \omega^{2j}$
If $j = 0$ then $1 + \omega^n + \omega^{2n} = 1 + 1 + 1 = 3$
$\therefore$ if n is a multiple of 3 then $1 + \omega^n + \omega^{2n} = 3$
If $j = 1$ then $1 + \omega^n + \omega^{2n} = 1 + \omega + \omega^2 = 0$
If $j = 2$ then $1 + \omega^n + \omega^{2n} = 1 + \omega^2 + \omega^4 = 1 + \omega^2 + \omega = 0$
$\therefore$ if n is not a multiple of 3 then $1 + \omega^n + \omega^{2n} = 0$

183 $I_n = \int x^n\sqrt{2x+1}\,dx$

$u = x^n \quad \frac{dv}{dx} = (2x+1)^{\frac{1}{2}}$

$\frac{du}{dx} = nx^{n-1} \quad v = \frac{1}{2}\times\frac{2}{3}(2x+1)^{\frac{3}{2}} = \frac{\sqrt{(2x+1)^3}}{3}$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}\int x^{n-1}(2x+1)\sqrt{2x+1}\,dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{2n}{3}\int x^n\sqrt{2x+1}\,dx$$

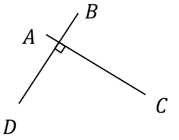
$$-\frac{n}{3}\int x^{n-1}\sqrt{2x+1}\,dx$$

$$I_n = \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{2n}{3}I_n - \frac{n}{3}I_{n-1}$$

$$\frac{2n+3}{3}I_n = \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}I_{n-1}$$

$$I_n\& = \frac{x^n\sqrt{(2x+1)^3}}{2n+3} - \frac{n}{2n+3}I_{n-1}$$

184 Let $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$ etc



$$\left(\underset{\sim}{c} - \underset{\sim}{a}\right)\cdot\left(\underset{\sim}{d} - \underset{\sim}{b}\right) = 0$$

$$\underset{\sim}{c}\cdot\underset{\sim}{d} - \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{d} + \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$$

$$\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{d} = \underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{d} \quad (1)$$

$AB^2 + CD^2$

$= \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CD}\right|^2$

$= \left(\underset{\sim}{b} - \underset{\sim}{a}\right)\cdot\left(\underset{\sim}{b} - \underset{\sim}{a}\right) + \left(\underset{\sim}{d} - \underset{\sim}{c}\right)\cdot\left(\underset{\sim}{d} - \underset{\sim}{c}\right)$

$= \underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{d}\cdot\underset{\sim}{d} - 2\underset{\sim}{c}\cdot\underset{\sim}{d} + \underset{\sim}{c}\cdot\underset{\sim}{c}$

$= \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{d}\cdot\underset{\sim}{d} - 2\left(\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{d}\right)$

$= \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{d}\cdot\underset{\sim}{d} - 2\left(\underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{d}\right)$ from (1)

$= \underset{\sim}{c}\cdot\underset{\sim}{c} - 2\underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{a} - 2\underset{\sim}{a}\cdot\underset{\sim}{d} + \underset{\sim}{d}\cdot\underset{\sim}{d}$

$= \left(\underset{\sim}{c} - \underset{\sim}{b}\right)\cdot\left(\underset{\sim}{c} - \underset{\sim}{b}\right) + \left(\underset{\sim}{a} - \underset{\sim}{d}\right)\cdot\left(\underset{\sim}{a} - \underset{\sim}{d}\right)$

$= \left|\underset{\sim}{c} - \underset{\sim}{b}\right|^2 + \left|\underset{\sim}{a} - \underset{\sim}{d}\right|^2$

$= \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{DA}\right|^2$

$= BC^2 + DA^2$

185 The horizontal speed is constant, and matches the speed at the maximum height.
Let the initial velocity be $5v$ and the initial horizontal velocity be v.

$\cos\theta = \frac{v}{5v}$

$\theta = \cos^{-1}\left(\frac{1}{5}\right)$

$= 78°28'$

186

$$(ad - bc)^2 \geq 0$$

$$a^2d^2 - 2abcd + b^2c^2 \geq 0$$

$$a^2d^2 + b^2c^2 \geq 2abcd$$

$$a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \geq a^2c^2 + 2abcd + b^2d^2$$

$$(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$$

187 **i**

$z_1 = i\sqrt{2} = \sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$z_2 = \frac{2}{1-i} = \frac{2}{\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

ii

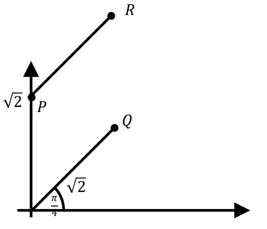
$z_1 = wz_2$

$\therefore w = \frac{z_1}{z_2}$

$= \sqrt{2}\,\text{cis}\left(\frac{\pi}{2}\right) \div \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

iii



iv

$|z_1| = |z_2| = \sqrt{2}$ so $OPRQ$ is a rhombus.

$\therefore OR$ bisects $\angle POQ$

$\therefore \arg(z_1 + z_2) = \arg z_2 + \frac{1}{2}(\arg z_1 - \arg z_2)$

$= \frac{1}{2}(\arg z_1 + \arg z_2)$

$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$

$= \frac{3\pi}{8}$

$z_1 + z_2 = i\sqrt{2} + \frac{2}{1-i}\times\frac{1+i}{1+i}$

$= i\sqrt{2} + \frac{2(1+i)}{1^2 + 1^2}$

$= 1 + (1 + \sqrt{2})i$

$\therefore \tan\left(\frac{3\pi}{8}\right) = \frac{1+\sqrt{2}}{1} = 1 + \sqrt{2}$

188

$\int_a^{3a} f(x-a)\,dx$

$= \int_0^{2a} f(u)\,du$

$= \int_a^{-a} f(a-v)\,(-dv)$

$= \int_{-a}^{a} f(a-v)dv$

$= \int_{-a}^{a} f(a-x)\,dx$

$u = x - a$

$du = dx$

$v = a - u$

$dv = -du$

$du = -dv$

189 **i**

$$\left|\underset{\sim}{r}(t)\right| = \sqrt{\left(t+\frac{1}{t}\right)^2+\left(t-\frac{1}{t}\right)^2}$$
$$= \sqrt{t^2+2+\frac{1}{t^2}+t^2-2+\frac{1}{t^2}}$$
$$= \sqrt{2t^2+\frac{2}{t^2}}$$
$$= \sqrt{2(t^2+\frac{1}{t^2}}$$

ii

$$\underset{\sim}{v_A} = \frac{d}{dt}\left(\left(t+\frac{1}{t}\right)\underset{\sim}{i}+\left(t-\frac{1}{t}\right)\underset{\sim}{j}\right)$$
$$= \left(1-\frac{1}{t^2}\right)\underset{\sim}{i}+\left(1+\frac{1}{t^2}\right)\underset{\sim}{j}$$
$$\left|\underset{\sim}{v_A}\right| = \sqrt{\left(1-\frac{1}{t^2}\right)^2+\left(1+\frac{1}{t^2}\right)^2}$$
$$= \sqrt{1-\frac{2}{t^2}+\frac{1}{t^4}+1+\frac{2}{t^2}+\frac{1}{t^4}}$$
$$= \sqrt{2+\frac{2}{t^4}}$$

iii

$x^2 = t^2+2+\frac{1}{t^2}, y^2 = t^2-2+\frac{1}{t^2}$
$\therefore x^2-y^2 = 4$
This is a hyperbola with branches to the right and left, with vertices at $(-2,0)$ and $(2,0)$
domain $(-\infty,-2] \cup [2,\infty)$
range $(-\infty,\infty)$

iv

$$\underset{\sim}{v_B}(t) = \left(2-\frac{2}{t^2}\right)\underset{\sim}{i}+\left(2+\frac{2}{t^2}\right)\underset{\sim}{j}$$
$$\underset{\sim}{r_B}(t) = \left(2t+\frac{2}{t}+c_1\right)\underset{\sim}{i}+\left(2t-\frac{2}{t}+c_2\right)\underset{\sim}{j}$$
Let $t = 1, \underset{\sim}{r_B}(t) = 4\underset{\sim}{i}$
$\therefore 2+2+c_1 = 4 \to c_1 = 0$
$2-2+c_2 = 0 \to c_2 = 0$
$$\therefore \underset{\sim}{r_B}(t) = \left(2t+\frac{2}{t}\right)\underset{\sim}{i}+\left(2t-\frac{2}{t}\right)\underset{\sim}{j}$$
$$\underset{\sim}{r_A}(t) = \begin{pmatrix} t+\frac{1}{t} \\ t-\frac{1}{t} \end{pmatrix}$$
$$\underset{\sim}{r_B}(t) = \begin{pmatrix} 2t+\frac{2}{t} \\ 2t-\frac{2}{t} \end{pmatrix} = 2\begin{pmatrix} t+\frac{1}{t} \\ t-\frac{1}{t} \end{pmatrix} = 2\underset{\sim}{r_A}(t)$$
$\therefore O, A$ and B are collinear, so the position vectors are parallel.

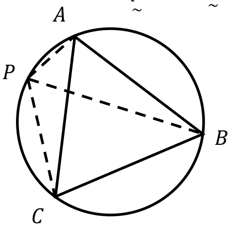
190 For the upward flight:
$$m\ddot{x_U} = -mg-kv \to \ddot{x_U} = -\frac{mg+kv}{m}$$
For the downward flight:
$$m\ddot{x_D} = mg-kv \to \ddot{x_D} = \frac{mg-kv}{m}$$
ANSWER (A)

191 False. $3^5+2 = 245$ which is divisible by 5.

192 Let $z = e^{\theta i}$
$\therefore \bar{z} = e^{-\theta i}$
$$= \frac{1}{e^{\theta i}}$$
$$= \frac{1}{z}$$

193
$$\int \frac{\cos x}{1+\sin^2 x}dx$$
$$= \tan^{-1}(\sin x)+c$$

194 Let O be the centre of the circle, and $\overrightarrow{OP} = \underset{\sim}{p}, \overrightarrow{OA} = \underset{\sim}{a}$ etc.



$PA^2+PB^2+PC^2$
$= \left|\overrightarrow{PA}\right|^2+\left|\overrightarrow{PB}\right|^2+\left|\overrightarrow{PC}\right|^2$
$= \left(\underset{\sim}{a}-\underset{\sim}{p}\right)\cdot\left(\underset{\sim}{a}-\underset{\sim}{p}\right)+\left(\underset{\sim}{b}-\underset{\sim}{p}\right)\cdot\left(\underset{\sim}{b}-\underset{\sim}{p}\right)+\left(\underset{\sim}{c}-\underset{\sim}{p}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{p}\right)$
$= \underset{\sim}{a}\cdot\underset{\sim}{a}-2\underset{\sim}{a}\cdot\underset{\sim}{p}+\underset{\sim}{p}\cdot\underset{\sim}{p}+\underset{\sim}{b}\cdot\underset{\sim}{b}-2\underset{\sim}{b}\cdot\underset{\sim}{p}+\underset{\sim}{p}\cdot\underset{\sim}{p}+\underset{\sim}{c}\cdot\underset{\sim}{c}-2\underset{\sim}{c}\cdot\underset{\sim}{p}+\underset{\sim}{p}\cdot\underset{\sim}{p}$
$= \left|\underset{\sim}{a}\right|^2+\left|\underset{\sim}{b}\right|^2+\left|\underset{\sim}{c}\right|^2+3\left|\underset{\sim}{p}\right|^2-2\underset{\sim}{p}\cdot\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)$
$= r^2+r^2+r^2+3r^3-2p\cdot(0)$
$= 6r^2$

195 **i**

$\underset{\sim}{a}(t) = -g\underset{\sim}{j}$
$\underset{\sim}{v}(t) = (c_1)\underset{\sim}{i}+(-gt+c_2)\underset{\sim}{j}$
Let $t = 0, \underset{\sim}{v}(0) = V\cos(\theta)\,\underset{\sim}{i}+V\sin(\theta)\,\underset{\sim}{j}$
$c_1 = V\cos(\theta), c_2 = V\sin(\theta)$
$\therefore \underset{\sim}{v}(t) = V\cos(\theta)\,\underset{\sim}{i}+(-gt+V\sin(\theta))\underset{\sim}{j}$
$$\underset{\sim}{r}(t) = (V\cos(\theta)\,t+c_3)\underset{\sim}{i}+\left(V\sin(\theta)\,t-\frac{1}{2}gt^2+c_4\right)\underset{\sim}{j}$$
Let $t = 0, \underset{\sim}{r}(0) = h\underset{\sim}{j}$
$c_3 = 0, c_4 = h$
$$\therefore \underset{\sim}{r}(t) = V\cos(\theta)\,t\underset{\sim}{i}+\left(V\sin(\theta)\,t-\frac{1}{2}gt^2+h\right)\underset{\sim}{j}$$

ii

$$x = V\cos(\theta)\,t \to t = \frac{x}{V\cos(\theta)}$$
$$y = V\sin(\theta)\left(\frac{x}{V\cos(\theta)}\right)-\frac{1}{2}g\left(\frac{x}{V\cos(\theta)}\right)^2+h$$
$$= h+\tan(\theta)\,x-\frac{g\sec^2(\theta)}{2V^2}x^2$$

196 **i**

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
$= \cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2$
$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$
$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

ii

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1 + i\sin\theta_1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then
$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_k + i\sin\theta_k) = \cos(\theta_1+\theta_2+\ldots+\theta_k) + i\sin(\theta_1+\theta_2+\ldots+\theta_k)$

RTP $P(k+1)$
$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_k + i\sin\theta_k)(\cos\theta_{k+1} + i\sin\theta_{k+1})$
$= \cos(\theta_1+\theta_2+\ldots+\theta_k+\theta_{k+1}) + i\sin(\theta_1+\theta_2+\ldots+\theta_k+\theta_{k+1})$

$\text{LHS} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_k + i\sin\theta_k)(\cos\theta_{k+1} + i\sin\theta_{k+1})$
$= (\cos(\theta_1+\theta_2+\ldots+\theta_k) + i\sin(\theta_1+\theta_2+\ldots+\theta_k))(\cos\theta_{k+1} + i\sin\theta_{k+1})$
$= \cos(\theta_1+\theta_2+\ldots+\theta_k)\cos\theta_k + i\cos(\theta_1+\theta_2+\ldots+\theta_k)\sin\theta_k + i\sin(\theta_1+\theta_2+\ldots+\theta_k)\cos\theta_k + i^2\sin(\theta_1$
$= \cos(\theta_1+\theta_2+\ldots+\theta_k)\cos\theta_k - \sin(\theta_1+\theta_2+\ldots+\theta_k)\sin\theta_k + i(\cos(\theta_1+\theta_2+\ldots+\theta_k)\sin\theta_k + \sin(\theta_1+\theta_2+\ldots+\theta_k)\cos\theta_k)$
$= \cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)$
$= \text{RHS}$
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

197

$\dfrac{\lambda+4i}{1+\lambda i}\times\dfrac{1-\lambda i}{1-\lambda i}$
$= \dfrac{\lambda-\lambda^2 i+4i+4\lambda}{1^2+\lambda^2}$
$= \dfrac{5\lambda+(4-\lambda^2)i}{1+\lambda^2}$
$\therefore 4-\lambda^2 = 0 \rightarrow \lambda = \pm 2$

198

$\displaystyle\int\frac{x^2-4x+2}{(x-2)^3}dx$
$\displaystyle= \int\frac{(x-2)^2-2}{(x-2)^3}dx$
$\displaystyle= \int\left(\frac{1}{x-2} - 2(x-2)^{-3}\right)du$
$\displaystyle= \ln|x-2| - 2\times\left(-\frac{1}{2}\right)(x+1)^{-2} + c$
$\displaystyle= \ln|x-2| + \frac{1}{(x-2)^2} + c$

199 **i**

$\begin{pmatrix}-6\\4\\-1\end{pmatrix}$ by inspection

ii

$\left|\begin{pmatrix}4\\-1\\3\end{pmatrix}\right| = \sqrt{4^2+(-1)^2+3^2} = \sqrt{26}$
$\left|\begin{pmatrix}3\\-4\\1\end{pmatrix}\right| = \sqrt{3^2+(-4)^2+1^2} = \sqrt{26}$
$\begin{pmatrix}4\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\1\end{pmatrix} = 12+4+3 = 19$
$\cos\theta = \dfrac{19}{\sqrt{26}\times\sqrt{26}} = \dfrac{19}{26}$

iii

$\overrightarrow{OX} = \begin{pmatrix}-6\\4\\-1\end{pmatrix} + 4\begin{pmatrix}4\\-1\\3\end{pmatrix} = \begin{pmatrix}10\\0\\11\end{pmatrix}$

iv

$\overrightarrow{AX} = \begin{pmatrix}10-(-6)\\0-4\\11-(-1)\end{pmatrix} = \begin{pmatrix}16\\-4\\12\end{pmatrix}$

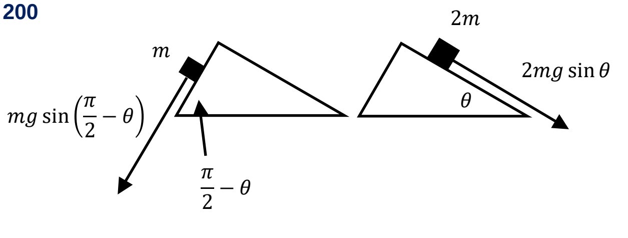
199 ... **v**

$|\overrightarrow{AX}| = \sqrt{16^2+(-4)^2+12^2} = \sqrt{416} = \sqrt{16}\times\sqrt{26}$
$= 4\sqrt{26}$

vi

Y is on $\underset{\sim}{r} = \begin{pmatrix}-6+3\mu\\4-4\mu\\-1-\mu\end{pmatrix}$
$\therefore \overrightarrow{YX} = \begin{pmatrix}10+6-3\mu\\0-4+4\mu\\11+1+\mu\end{pmatrix} = \begin{pmatrix}16-3\mu\\-4+4\mu\\12+\mu\end{pmatrix}$
$\begin{pmatrix}16-3\mu\\-4+4\mu\\12+\mu\end{pmatrix}\cdot\begin{pmatrix}4\\-1\\3\end{pmatrix} = 0$
$64-12\mu+4-4\mu+36+3\mu = 0$
$104-13\mu = 0$
$\mu = 8$
$\therefore \overrightarrow{YX} = \begin{pmatrix}16-3(8)\\-4+4(8)\\12+(8)\end{pmatrix} = \begin{pmatrix}-8\\28\\20\end{pmatrix}$
$|\overrightarrow{YX}| = \sqrt{(-8)^2+28^2+20^2} = \sqrt{1248}$
$|\overrightarrow{YX}|^2 + |\overrightarrow{AX}|^2 = |\overrightarrow{AY}|^2$
$|\overrightarrow{AY}| = \sqrt{\left(4\sqrt{26}\right)^2 + \left(\sqrt{1248}\right)^2}$
$= 40.8$

200



$mg\sin\left(\dfrac{\pi}{2}-\theta\right) = 2mg\sin\theta$
$\cos\theta = 2\sin\theta$
$\tan\theta = \dfrac{1}{2}$
$\theta = 26°34'$

1 Let $a, b, x, y > 0$. Prove that $(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2$

2 Solve for p, q, r over the complex numbers, given:

$$p + q + r = 1 \qquad pq + pr + qr = 9 \qquad pqr = 9$$

3 Find $\int \frac{\sin(\tan\theta)}{\cos^2\theta} d\theta$

4 The lines between A, B and C are the vertices of a cube and form the edge, diagonal of a side and diagonal of the cube itself. Find the volume of the cube, given:
$\overrightarrow{OA} = (0,5,2), \overrightarrow{OB} = (8,2,7)$ and $\overrightarrow{OC} = (11,0,1)$.

5 A particle is moving in simple harmonic motion, with amplitude of 10 metres and a period of 10 seconds.

i Prove that it would take the particle $\frac{5}{\pi}\cos^{-1}\left(\frac{3}{5}\right)$ seconds to travel from one of the extremities of its path to a point 4 metres away.

ii At what speed, to the nearest whole integer, would it pass through this position?

6 Prove $n(n+1)(n+2)(3n+5)$ is divisible by 24 for $n \geq 1$ by induction

7 Solve the following pair of simultaneous equations for the complex numbers z and w:

$$2z + 3iw = 0 \qquad (1)$$
$$(1 - i)z + 2w = i - 7 \qquad (2)$$

8 Given that $xy = x + 1$, evaluate $\int_3^5 x\, dy$

9 The points $A(1,3,1), B(0,-2,3), C(2,4,3)$ and D form the parallelogram $ABCD$, where the vertices are in that order around the parallelogram.

i Find the coordinates of D.

ii E, B and D are collinear, with B the midpoint of ED. Determine the coordinates of E.

iii The point F is such that $ABEF$ is also a parallelogram. Find the coordinates of F.

iv Show that B is the midpoint of FC.

v Prove that $ADBF$ is another parallelogram.

10 A particle is projected from a point O. After 5 seconds its horizontal and vertical displacements are 60 m and 57.5 m respectively. Assume $g = 10$ ms^{-2} and ignore air resistance. Find its initial velocity. The equations of motion are

$$x = vt\cos\theta \quad \text{and} \quad y = -\frac{1}{2}gt^2 + vt\sin\theta$$

11 For real x, y prove that if $y^3 + yx^2 \leq x^3 + xy^2$ then $y \leq x$.

12 Show that $(3+i)^n + (3-i)^n$ is purely real.

13 Find $\int \frac{\sec^2 x}{1-\tan^2 x} dx$

14 A rectangular prism with sides of length 2, 8 and 16 units has both ends of one of its longest diagonals along the x-axis, and its centre is at the origin. Prove for any point P on the surface of the prism $|OP| \leq 9$.

15 The acceleration of a particle moving in a straight line is given by $\ddot{x} = 1 + \ln x$. Given that the particle starts at rest 1 cm to the right of the origin, find the velocity when $x = e^2$.

16 Prove $x^n + x^{n-2} + x^{n-4} + \ldots + \frac{1}{x^{n-4}} + \frac{1}{x^{n-2}} + \frac{1}{x^n} \geq n + 1,$ for $x > 0, n > 0$ by induction.
[Hint: separate base cases are required to prove the result for n even or odd].

17 The complex roots of $iz^2 + \sqrt{3}z - 1 = 0$ are α and β.

i Find α and β in Cartesian form

ii Show that $\alpha^2\beta^2 + 1 = 0$

18 Evaluate

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{2e^{\frac{2}{x^2}}}{x^3}\, dx$$

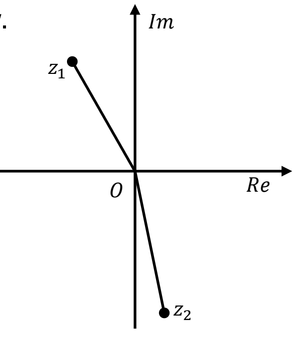
19 Find the cartesian equation of the path of a particle whose position vector is

$\underset{\sim}{r}(t) = (\sec(t) - 1)\underset{\sim}{i} + (\tan(t) - 2)\underset{\sim}{j}$ for $0 \leq t < \frac{\pi}{2}$.

20 A projectile is fired horizontally with an initial velocity of V ms^{-1} from a position H metres above ground level. The acceleration due to gravity is g ms^{-2}, and air resistance is negligible. Show that the horizontal range is $V\sqrt{\frac{2H}{g}}$.

21 Given the arithmetic mean is greater than or equal to the geometric mean, if $a, b, c > 0$ then prove $a^7 + b^7 + c^7 \geq a^4b^3 + b^4c^3 + c^4a^3$

22 z_1, z_2 and z_3 are three complex numbers which satisfy $z_1 + z_2 + z_3 = 0$
z_1 and z_2 are indicated in the Argand diagram as shown in the figure below.



i Copy the diagram and sketch on the same diagram a possible location for z_3. Explain your decision.

ii Given that the arguments of z_1, z_2 and z_3 are α, β and γ respectively, and their moduli are $1, k$ and $2 - k$ respectively, where $0 < k < 2$. Express z_1, z_2 and z_3 in modulus-argument form.

iii Prove that

$$\begin{cases} \cos\alpha + k\cos\beta + (2-k)\cos\gamma = 0 \\ \sin\alpha + k\sin\beta + (2-k)\sin\gamma = 0 \end{cases}$$

iv From (iii), by eliminating α or otherwise, prove that

$$k^2 + (2-k)^2 + 2k(2-k)\cos(\beta - \gamma) = 1$$

v By considering $|\cos(\beta - \gamma)| \leq 1$, find the range of values of k.

23 Find $\int(2x \operatorname{cosec} x - x^2 \cot x \operatorname{cosec} x)\, dx$

24 Prove that $\left|\underset{\sim}{a}\right|\underset{\sim}{b}+\left|\underset{\sim}{b}\right|\underset{\sim}{a}$ is perpendicular to $\left|\underset{\sim}{a}\right|\underset{\sim}{b}-\left|\underset{\sim}{b}\right|\underset{\sim}{a}$ for any two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

25 A particle moves in a straight line with displacement in centimetres from the point $x = 0$ at time t seconds given by $x = C\cos 2t + D\sin 2t$, where C and D are constants.

i Show that the motion is simple harmonic by showing that the acceleration has the form $\ddot{x} = -n^2x$, where n is a constant.

ii It is known that when $t = \frac{\pi}{3}, x = \frac{\sqrt{3}}{2}$ and $\dot{x} = -5$.

$\boldsymbol{\alpha}$ Find C and D

$\boldsymbol{\beta}$ Find the amplitude of the motion.

26 Let $w, x, y, z > 0$.

i Prove that $\frac{x}{y} + \frac{y}{x} \geq 2$

ii Deduce that $\frac{x+y+z}{w} + \frac{w+y+z}{x} + \frac{w+x+z}{y} + \frac{w+x+y}{z} \geq 12$

iii Hence prove that if $w + x + y + z = 1$, then $\frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 16$

27 The points $A(\omega)$ and $B(\phi)$ are on the Argand diagram such that $\angle AOB = \frac{2\pi}{3}$ and ΔAOB is isosceles with $|OA| = |OB|$. Prove that $(\omega + \phi)^2 = \omega\phi$

28 Find $\int \frac{e^{\log_2 x}}{x}\,dx$

29 **i** Find the vector equation of the line passing through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

ii Hence prove that the cartesian form of the line through A and B is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

iii Rewrite the following line in vector form, and hence find two points through which the line passes.

$$\frac{x-3}{-2} = \frac{y+2}{3} = z - 1$$

30 A particle moves in a straight line with acceleration given by $\ddot{x} = (4x - 2)\text{ms}^{-2}$, where x is the displacement. Initially the particle is at the origin with velocity $1\ \text{ms}^{-1}$.

i If the velocity at time t seconds is $v\ \text{ms}^{-1}$, show that $v^2 = (1 - 2x)^2$.

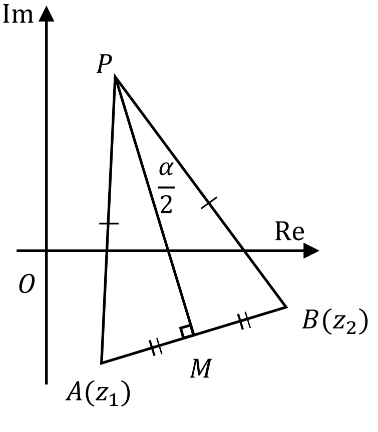
ii Hence show that $\ddot{x} = -2v$

iii Find expressions for x and v in terms of t.

iv Show that the particle approaches, but never reaches, $x = \frac{1}{2}$.

31 Prove $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ is an integer for $n \geq 1$ by induction.

32 The diagram shows an isosceles triangle ABP in the Argand diagram, with base AB and $\angle APB = \alpha$. PM is the perpendicular bisector of AB and so bisects $\angle APB$. Suppose that A and B represent the complex numbers z_1 and z_2 respectively.



i In terms of z_1, z_2 and α, find the complex numbers represented by :

$\boldsymbol{\alpha}$ the vector AM

$\boldsymbol{\beta}$ the vector MP

ii Hence show that P represents the complex number

$$\frac{1}{2}\left(1 - i\cot\frac{\alpha}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)z_2$$

33 Find $\int \frac{1+\sin 2x}{\cos x+\sin x} dx$

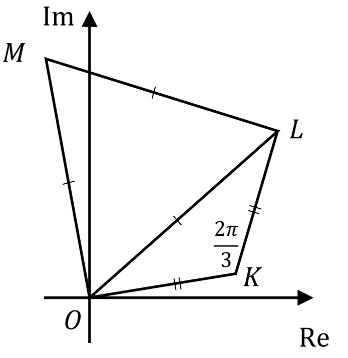
34 The point $P(x, y, z)$ is twice as far from $(0,0,0)$ as it is from $(3,0,0)$. Prove that P lies on a sphere, and find its centre and radius.

35 A particle is in simple harmonic motion between $x = 2$ and $x = 6$, taking 8 seconds to move from one extremity of its motion to the other. Sketch a graph of its acceleration as a function of displacement.

36 Prove that there are no rational solutions to $x^3 + 3x + 5 = 0$ using contradiction

37 The points K and M in a complex plane represent the complex numbers α and β respectively. The triangle OKL is isosceles and $\angle OKL = \frac{2\pi}{3}$. The triangle OLM is equilateral.

Show that $3\alpha^2 + \beta^2 = 0$.



38 Find $\int \frac{xe^{\sqrt{2x^2-1}} \sin\left(e^{\sqrt{2x^2-1}}\right)}{\sqrt{2x^2-1}} dx$

39 What is the difference between the parametric curves $f(t) = (t, t, t^2), g(t) = (t^2, t^2, t^4)$ and $h(t) = (\sin t, \sin t, \sin^2 t)$?

40 A particle of unit mass moves in a straight line. It is placed at the origin on the x-axis and is then released from rest. When at position x, its acceleration is given by $\ddot{x} = -9x + \frac{5}{(2-x)^2}$. Prove that the particle moves between two points on the x-axis and find these points.

41 Prove $\sqrt{n} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1,\ n \geq 1$ by induction.

42 Prove $e^{ix} = \cos x + i \sin x$ by substituting the functions $f(x) = e^{ix}, g(x) = \cos x$ and $h(x) = \sin x$ into the Maclaurin series:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

43 Find $\int \frac{\sin^3 x}{\cos x - 1} dx$

44 If $\underset{\sim}{a} = \underset{\sim}{i} + m\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = m\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, and the angle between the vectors is $\frac{\pi}{3}$ find possible values for m.

45 For $0 \leq t \leq \frac{1}{2}$ the velocity in metres per second of a particle is given by $v = \frac{10}{\sqrt{1-t^2}} + \frac{1}{(1-t)^2}$. During the given time interval find:
i The distance travelled by the particle.
ii The maximum velocity attained.

46 Prove $a^n - b^n < na^{n-1}(a-b)$ for $0 < b < a$

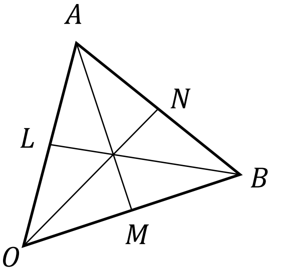
47 Given $ae^{i\alpha} + be^{i\beta} = re^{i\theta}$.

i Prove $\cos\theta = \frac{a\cos\alpha + b\cos\beta}{r}$ and $\sin\theta = \frac{a\sin\alpha + b\cos\beta}{r}$

ii Prove $r = \sqrt{a^2 + b^2 + 2ab\cos(\alpha - \beta)}$

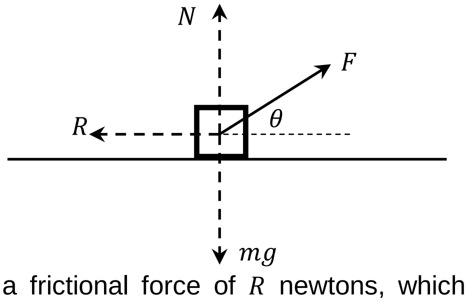
iii Hence prove $|z_1| + |z_2| \geq |z_1 + z_2|$

48 Use integration to prove $\int \frac{2}{x\ln 7}dx = \log_7 x^2 + c$



49 Prove that the medians of an equilateral triangle are also the altitudes.

50 A block of mass 5 kg is to be moved along a rough horizontal surface by a force of F newtons, inclined at an angle of θ with the direction of motion where $0 \leq \theta \leq \frac{\pi}{2}$.



The motion is resisted by a frictional force of R newtons, which is proportional to the normal reaction force of N newtons exerted on the block by the surface, such that $R = 0.2N$.
Assume $g = 10$ ms^{-2}.

i Show that $F = \frac{50}{5\cos\theta + \sin\theta}$ newtons, when the block is about to move.

ii Calculate the minimum value of F needed to overcome the frictional resistance between the block and the surface.

51 Prove $\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$ by contradiction for real $x, y \neq 0$.

52 The points $z = x + iy$ on the curve $x^2 - 2x + y^2 - 2y - 2 = 0,\ y \neq 1$ can also be represented by $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\theta$. Find the complex numbers z_1 and z_2, and the angle θ.

53 Find $\int \frac{dx}{\sec x - 1}$

54 Two model airplanes race around a circular course, with the second airplane taking off T seconds after the first plane. Their position vectors are $\underset{\sim}{r_1}(t) = \sin t\ \underset{\sim}{i} + \cos t\ \underset{\sim}{j} + \sin t\ \underset{\sim}{k}$ and $\underset{\sim}{r_2}(t) = \sin(2t - \alpha)\ \underset{\sim}{i} + \cos(2t - \alpha)\ \underset{\sim}{j} + \sin(2t - \alpha)\ \underset{\sim}{k}$, where time is measured in seconds from when the first airplane took off. They collide when they have both completed one and a half laps. Find T given the first plane takes 20 seconds to complete one lap.

55 A body of unit mass is projected vertically upwards, under gravity, from the ground in a medium that produces a resistance force of kv^2, where v is the velocity and k is a positive constant. The acceleration due to gravity is g.

i If the initial velocity of the body is v_0, prove that the maximum height, H, of the body above the ground is given by

$$H = \frac{1}{2k}\log_e\left(1 + \frac{kv_0^2}{g}\right)$$

ii In a second projection vertically upwards of the body, it is noticed that the maximum height reached is $2H$. Show that the initial velocity was $\left(e^{2kH}+1\right)^{\frac{1}{2}}v_0$.

56 **i** Show that

$$\frac{(n+9)^2(n+10)^2 - n^2(n-1)^2}{4} = 5(2n+9)(n^2+9n+45)$$

ii Prove by mathematical induction that for positive integers, $n \geq 1$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2}{4}(n+1)^2$$

iii Hence prove

$$n^3 + (n+1)^3 + (n+2)^3 + \ldots + (n+9)^3 = 5(2n+9)(n^2+9n+45)$$

57 If $z_1 = 2i$ and $z_2 = 1 + 3i$ are two complex numbers, describe the loci of z such that $z = z_1 + k(z_2 - z_1)$, when

i $k = 1$

ii $0 < k < 1$

iii k is any real number

iv When $0 < k < 1$ the locus can also be written in the form $\arg\left(\frac{z-\alpha}{z-\beta}\right) = \theta$. Find α, β and θ.

58 Find $\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$

59 The parametric equations $x = \cos t, y = \sin t$ gives a unit circle, and as t increases from zero the point moves anticlockwise from $(1,0)$. Find the parametric equations of a circle where as t increases from zero the point moves clockwise from $\left(\sqrt{3}, 1\right)$, on a circle centred about the origin.

60 The only force acting on a particle moving in a straight line is a resistance $\lambda(c+v)$ acting in the same line. The particle is of unit mass, its velocity is v, and λ and c are positive constants. The particle starts to move with velocity $u\ (>0)$ and comes to rest in time T. At time $\frac{1}{2}T$ its velocity is $\frac{1}{4}u$. Show that

i $c = \frac{1}{8}u$

ii at time t, $8\frac{v}{u} = 9e^{-\lambda t} - 1$

61 Prove by contrapositive that if $2^n - 1$ is prime then n is prime, for $n > 1$.

You are given $a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + a^{m-3}b^3 + \ldots + b^{m-1})$

62 ω is a non-real cube root of unity.

i Find the value of $\frac{1}{\omega^2} + \frac{1}{\omega}$

ii Show that

$$\frac{1+2\omega+3\omega^2}{2+3\omega+\omega^2} + \frac{1+2\omega+3\omega^2}{3+\omega+2\omega^2} = -1$$

63 **i** Given that $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2}\, dx$ where n is a positive integer, find a recurrence relationship for I_{2n+1} in terms of I_{2n-1}.
ii Hence evaluate I_5.

64 Given $\left|\underset{\sim}{u}\right| = 2, \left|\underset{\sim}{v}\right| = 3, \left|\underset{\sim}{w}\right| = 1, \underset{\sim}{u} \cdot \underset{\sim}{v} = 2$ and $\underset{\sim}{w} \cdot \left(\underset{\sim}{u} + \underset{\sim}{v}\right) = -1$, prove $\left|\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}\right| = 4$.

65 A particle of unit mass is projected vertically upward under gravity with a speed v in a medium where resistance is k times the speed, where k is a positive constant. If the particle reaches its greatest height H, in time T, show that $v = gT + kH$.

66 Prove $1 + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$ for $n \geq 1$ by induction.

67 Let α, β and γ be the roots of the cubic equation $x^3 + Ax^2 + Bx + 8 = 0$, where A, B are real. Furthermore $\alpha^2 + \beta^2 = 0$ and $\beta^2 + \gamma^2 = 0$.
i Explain why β is real and α and γ are not real.
ii Show that α and γ are purely imaginary.
iii Find A and B.

68 Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

69 It is given that $\underset{\sim}{a} = \underset{\sim}{b} + \underset{\sim}{c}$ where $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$. Furthermore $\underset{\sim}{b}$ is in the direction $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, and $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are perpendicular. Find $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

70 A particle of unit mass is projected vertically upwards under gravity, the air resistance to the motion being $\frac{gv^2}{k^2}$ where the speed is v, and k is a constant.
i Show that during the upward motion of the ball $\ddot{x} = -\frac{g}{k^2}(k^2 + v^2)$, where x is the upward displacement.
ii Hence, show that the greatest height reached is $\frac{k^2}{2g} \ln\left(1 + \frac{u^2}{k^2}\right)$, where u is the speed of projection.

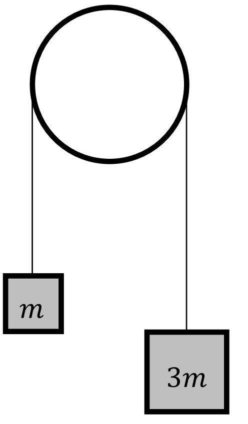
71 **i** For $a, b > 0$ prove that $\frac{a+b}{2} \geq \sqrt{ab}$
ii The generalisation of the result in (i) is $\frac{a_1+a_2+a_3+\ldots+a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$. Use this result to prove that $n! \leq \left(\frac{n+1}{2}\right)^n$ for $n \geq 1$. Do not use mathematical induction.

72 **i** Given z is a root of $az^3 + bz^2 + cz + d$ where a, b, c, d are real, prove $\bar{z}$ is also a root.
ii Find all three roots of $z^3 - 6z^2 + 13z - 20$ given $1 + 2i$ is one of the roots.

73 Given α, β are the roots of $ax^2 + bx + c = 0$, prove:
$$\int \frac{\cos x\,(2a \sin x + b)}{(\sin x - \alpha)(\sin x - \beta)} dx = a \ln|a \sin^2 x + b \sin x + c| + c_1$$

74 Find a relation linking a and b if $\underset{\sim}{r} = (1 - \sqrt{a} \sin t)\underset{\sim}{i} + \left(1 - \frac{1}{b}\cos t\right)\underset{\sim}{j}$ is the vector equation of a circle.

75 Particles of mass m and $3m$ kilograms are connected by a light inextensible string which passes over a smooth fixed pulley. The string hangs vertically on each side, as shown in the diagram.



The particles are released from rest and move under the influence of gravity. The air resistance on each particle is kv Newtons, when the speed of the particles is v ms^{-1} and the acceleration due to gravity is g ms^{-2} and is taken as positive throughout the question and is assumed to be constant. k is a positive constant.

i Show that the equation of motion is:

$$\ddot{x} = \frac{mg - kv}{2m}$$

ii Find the terminal velocity V or maximum speed of the system stating your answer in terms of m, g and k.

iii Prove that the time elapsed since the beginning of the motion is given by:

$$t = \frac{2m}{k}\ln\left(\frac{mg}{mg - kv}\right)$$

iv If the bodies attain a velocity equal to half of the terminal speed, show that the time is equal to $\frac{V}{g}\ln 4$.

76 **i** Given the product of two consecutive positive integers is divisible by 2, prove by induction that the product of 3 consecutive positive integers is divisible by 6.

ii Hence prove that the sum of the cubes of any 3 consecutive positive integers is divisible by 9.

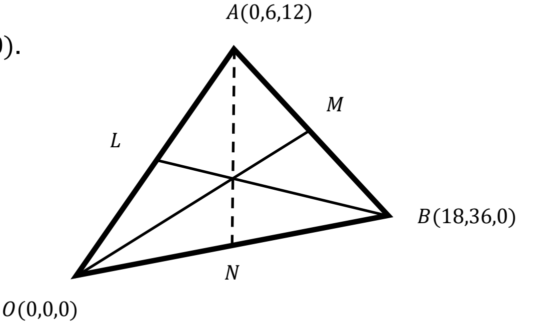
77 Z and W represent the complex numbers z and w respectively. If $|z| = 2$ and $w = \frac{z+3}{z}$, find the locus of W.

78 The length of an arc joining $P(a, f(a))$ to $Q(b, f(b))$ on a smooth continuous curve is given by

$$\ell = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Use the formula to prove that the circumference of a circle of radius r is $C = 2\pi r$.

79 A triangle has vertices $O(0,0,0), A(0,6,12)$ and $B(18,36,0)$. Find the equations of the three medians and show that they are concurrent.



80 A stone is projected from a point on the ground and it just clears a fence d metres away. The height of the fence is h metres. The angle of projection to the horizontal is θ and the speed of projection is v m/s, and air resistance is negligible. The displacement equations are

$$x = vt\cos\theta \text{ and } y = -\frac{1}{2}gt^2 + vt\sin\theta$$

i Show that

$$v^2 = \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$$

ii Show that the maximum height reached by the stone is

$$\frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$$

iii Show that the stone will just clear the fence at its highest point if $\tan\theta = \frac{2h}{d}$

81 Prove $\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^n\alpha = \frac{\sin(2^{n+1}\alpha)}{2^{n+1}\sin\alpha}$ for $n \geq 0$ by induction.

82 If $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then find the value of $z_1^2 + z_2^2 + z_3^2$

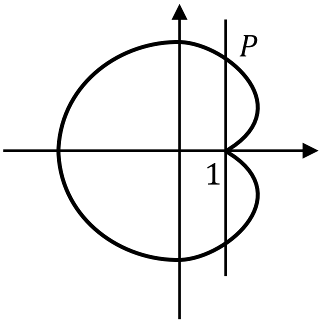
83 Evaluate

$$\int_{-1}^{1} \frac{\tan^{-1} x}{1 + \sin^2 x}\,dx$$

84 A cardioid is defined by the following pair of parametric equations

$$x = 2\cos\theta - \cos 2\theta$$
$$y = 2\sin\theta - \sin 2\theta$$

for $0 \leq \theta \leq 2\pi$, as shown below.



The cardioid curve and the vertical line $x = 1$ intersect in the 1st quadrant at a point P. Find the coordinates of the point P.

85 A particle of unit mass is dropped from rest in a medium which causes a resistance of kv, where v ms^{-1} is the particle's velocity and k is a constant.

i Show that the terminal velocity, V_T is given by $V_T = \frac{g}{k}$

ii Find the time taken to reach a velocity of $\frac{1}{2}V_T$.

iii Find the distance travelled in this time.

86 Given a, b, c are positive real numbers with $a > b$ and $c^2 > ab$, prove by contradiction that

$$\frac{a+c}{\sqrt{a^2+c^2}} - \frac{b+c}{\sqrt{b^2+c^2}} > 0$$

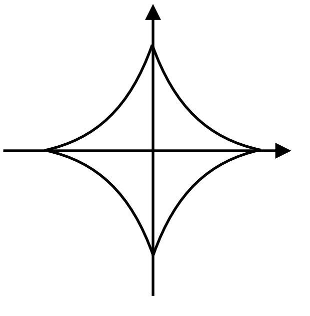
87 Solve the equation

$$\frac{2\bar{z}(1-2i)}{5z} + \frac{i}{1+2i} = \frac{2-3i}{z}$$

88 Find the smallest value of $\int_0^1 (x^2 - a)^2\,dx$ as a varies.

89 The parametric equation of the Astroid curve is given by $x = a\cos^3 t\,, y = a\sin^3 t$. Prove the cartesian equation of the curve is

$$\sqrt[3]{\frac{x^2}{4}} + \sqrt[3]{\frac{y^2}{4}} = 1$$



90 A particle is projected from the origin at an angle of α with an initial velocity of V, and it passes through a point (m, n). Air resistance is negligible.

i Prove that $gm^2\tan^2\alpha - 2mV^2\tan\alpha + gm^2 + 2nV^2 = 0$, where g is the acceleration due to gravity.

ii Prove that there are two possible trajectories if

$$(V^2 - gn)^2 > g^2(m^2 + n^2)$$

91 Given the sequence $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ for $n \geq 1$ and $a_1 = 1$:

i Prove $\frac{a_n - \sqrt{2}}{a_n + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}}$ by induction

ii Hence find the limiting value of a_n as $n \to \infty$.

92 Show that the equation $2z^3 - z = \bar{z}$ is solved by any complex number whose cube is equal to its real component, and thus that $z = 0, \pm 1$ are all solutions.

93 Which expression must be equal to $\int_0^a [f(a-x) + f(a+x)]dx$?

A $\displaystyle\int_0^a f(x)\,dx$ **B** $\displaystyle\int_0^{2a} f(x)\,dx$

C $\displaystyle 2\int_0^a f(x)\,dx$ **D** $\displaystyle\int_{-a}^a f(x)\,dx$

94 Find the two ends of the curve with position vector $\underset{\sim}{r}(t) = \begin{pmatrix} \arccos(t) \\ 1 + \sin(\pi t) \\ \sqrt{-t} \end{pmatrix}$.

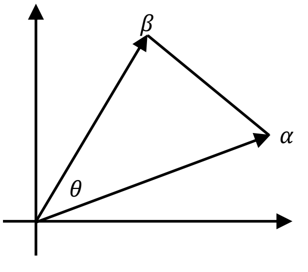
95 A particle of mass 1 kg is projected from a point O with velocity u m/s along a smooth horizontal table in a medium whose resistance is RV^2 Newtons when the particle has velocity V m/s, R being constant.
i Find its velocity as a function of t.
ii An equal particle is projected from O simultaneously with the first particle but vertically upwards under gravity with velocity u in the same medium. Show that the velocity V of the first particle when the second is momentarily at rest is given $\frac{1}{V} = \frac{1}{u} + \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$ where $Ra^2 = g$.

96 **i** If $x > 0, y > 0$ show that $x + y \geq 2\sqrt{xy}$
ii Hence show that if $x > 0, y > 0, z > 0$ then

$$(x+y)(y+z)(z+x) \geq 8xyz$$

iii If a, b, c are the sides of a triangle with semi-perimeter $S = \frac{1}{2}(a+b+c)$ then Heron's formula states that the area of the triangle is given by $A = \sqrt{S(S-a)(S-b)(S-c)}$.
By choosing suitable values for x, y, z show that $A^2 \leq \frac{(a+b+c)abc}{16}$

97 **i** Use the results $z + \bar{z} = 2\text{Re}(z)$ and $|z|^2 = z\bar{z}$ for the complex number z to show that $|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 = 2\text{Re}(\alpha\bar{\beta})$.
ii The diagram shows the angle θ between the complex numbers α and β. Prove that $|\alpha||\beta|\cos\theta = \text{Re}(\alpha\bar{\beta})$



98 **i** Prove

$$\int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left(f(x) + f\left(\frac{1}{x}\right)\right)}dx = \int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x}\right)}{x\left(f(x) + f\left(\frac{1}{x}\right)\right)}dx$$

ii Hence or otherwise evaluate

$$I = \int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x + \sin\frac{1}{x}\right)}dx$$

99 The vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are defined as $\underset{\sim}{u} = \underset{\sim}{a} + 2\underset{\sim}{b}$ and $\underset{\sim}{v} = 5\underset{\sim}{a} - 4\underset{\sim}{b}$, where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are unit vectors. Given that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular, determine the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

100 A projectile fired with velocity V and at an angle $45°$ to the horizontal, just clears the tops of two vertical posts of height $8a^2$ and the posts are $12a^2$ apart. There is no air resistance, and the acceleration due to gravity is g.

i If the projectile is at the point (x, y) at time t, derive expressions for x and y in terms of t.

ii Hence show that the equation of the path of the projectile is $y = x - \frac{gx^2}{V^2}$.

iii Show that the range of the projectile is $\frac{V^2}{g}$.

iv If the first post is b units from the origin, show:

$\boldsymbol{\alpha}$ $\frac{V^2}{g} = 2b + 12a^2$

$\boldsymbol{\beta}$ $8a^2 = b - \frac{gb^2}{V^2}$

v Hence or otherwise prove that $V = 6a\sqrt{g}$

101 A_n and B_n are two series given by:

$$A_n = 1^2 + 5^2 + 9^2 + \ldots + (4n-3)^2$$
$$B_n = 3^2 + 7^2 + 11^2 + \ldots$$

i Find the nth term of B_n

ii If $S_{2n} = A_n - B_n$, prove that $S_{2n} = -8n^2$

iii Hence, or otherwise, evaluate:

$$101^2 - 103^2 + 105^2 - 107^2 + \ldots + 1993^2 - 1995^2$$

102 **i** By considering the expansion of $(\cos\theta + i\sin\theta)^3$ and de Moivre's theorem, show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

ii Deduce that $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$, where $\cos 3\theta = \frac{1}{2}$

iii Find the roots of $8x^3 - 6x - 1 = 0$ in the form $x = \cos\alpha$, for $0 \le \alpha \le 2\pi$.

iv Hence evaluate $\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)$.

103 **i** Prove the following recursion formula holds:

$$\int x^m \log_e^n x\, dx = \frac{x^{m+1}\log_e^n x}{m+1} - \frac{n}{m+1}\int x^m \log_e^{n-1} x\, dx$$

ii Evaluate $\int x^3 \log_e^3 x\, dx$

104 The shortest distance, d, from the point B to the line AC satisfies:

$$d^2 = \left(\underset{\sim}{b} - \underset{\sim}{a}\right)\cdot\left(\underset{\sim}{b} - \underset{\sim}{a}\right) - \left[\frac{\left(\underset{\sim}{b} - \underset{\sim}{a}\right)\cdot\left(\underset{\sim}{c} - \underset{\sim}{a}\right)}{\left|\underset{\sim}{c} - \underset{\sim}{a}\right|}\right]^2$$

Do NOT prove this.

The point E lies on the interval AB, and is 2 units from the line AC. Find E using the points $A(2,1,4), B(3,1,1)$ and $C(2,3,2)$.

105 A particle of mass m is attracted towards the origin by a force of magnitude $\frac{\mu m}{x^2}$ for $x \neq 0$, where the distance from the origin is x and μ is a positive constant.

i Prove that $\frac{d}{dx}\left[\sqrt{bx - x^2} + \frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right)\right] = -\sqrt{\frac{x}{b-x}}$ for $x \ge 0$

ii If the particle starts at rest at a distance b to the right of the origin, show that its velocity v is given by $v^2 = 2\mu\left(\frac{b-x}{bx}\right)$.

iii Find the time required for the particle to reach a point halfway towards the origin.

106 Given the sequence $a_{n+1} = a_n(a_n + 1)$ for $n \geq 1$ and $a_1 = 1$:

i Prove by mathematical induction that

$$a_{n+1} = 1 + \sum_{r=1}^{n} a_r^2$$

ii Show that

$$(2a_{n+1} + 1)^2 = (2a_n + 1)^2 + (2a_{n+1})^2$$

iii Hence deduce that

$$(2a_{n+1} + 1)^2 = (2a_1 + 1)^2 + \sum_{r=2}^{n+1} (2a_r)^2$$

iv Find a_2, a_3 and a_4 and use them to evaluate a_5.

v Express a_5 as the sum of 5 positive integers.

vi Hence prove that $3^2 + 4^2 + 12^2 + 84^2 + 3612^2 = 3613^2$

107 **i** Use De Moivre's Theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

ii Show that the equation $16x^4 - 16x^2 + 1 = 0$ has roots $x = \cos\frac{\pi}{12}, x = -\cos\frac{\pi}{12}, x = \cos\frac{5\pi}{12}$ and $x = -\cos\frac{5\pi}{12}$.

iii By considering this equation as a quadratic equation in x^2, prove $\cos\frac{5\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$.

108 Let $I_n = \int_0^1 (1 - x^2)^n \, dx$ and $J_n = \int_0^1 x^2(1 - x^2)^n \, dx$.

i Apply integration by parts to I_n to show that $I_n = 2nJ_{n-1}$.

ii Hence show that $I_n = \frac{2n}{2n+1} I_{n-1}$.

iii Show that $J_n = I_n - I_{n-1}$, and hence deduce that $J_n = \frac{1}{2n+3} I_n$.

iv Hence write down a reduction formula for J_n in terms of J_{n-1}

109 Prove that if three points $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are collinear that they must satisfy $\lambda\underset{\sim}{a} + \mu\underset{\sim}{b} + \nu\underset{\sim}{c} = 0$ where $\lambda + \mu + \nu = 0$

110 A plane of mass M kg on landing, experiences a variable resistive force due to air resistance of magnitude Bv^2 Newtons, where v is the speed of the plane. That is $M\ddot{x} = -Bv^2$.

i Show that the distance (D_1) travelled in slowing the plane from speed V to speed U under the effect of air resistance only, is given by:

$$D_1 = \frac{M}{B}\ln\left(\frac{V}{U}\right)$$

After the brakes are applied, the plane experiences a constant resistive force of A Newtons (due to brakes) as well as a variable resistive force, Bv^2. That is, $M\ddot{x} = -(A + Bv^2)$.

ii After the brakes are applied when the plane is at U, show that the distance D_2, required to come to rest is given by:

$$D_2 = \frac{M}{2B}\ln\left[1 + \frac{B}{A}U^2\right]$$

iii Use the above information to estimate the total stopping distance after landing, for a 100 tonne plane if it slows from $90\ \mathrm{m}/s$ to 60 m/s under a resistive force of $125v^2$ Newtons and is finally to rest with the assistance of a constant braking force of magnitude $75\,000$ Newtons. (Note: 1 Newton (N) = 1 kg.m/s^2.

111 **i** Given $e^u \geq 1 + u$ prove that $\log_e x \leq x - 1$ for $x > 0$ by making a suitable substitution.

ii Show that $\log_e\left(\frac{c_1c_2c_3\ldots c_n}{\mu^n}\right) \leq \frac{c_1+c_2+c_3+\ldots+c_n}{\mu} - n$, where $c_1, c_2, c_3, \ldots, c_n > 0$ and $\mu > 0$

iii Hence if $\mu = \frac{c_1+c_2+c_3+\ldots+c_n}{n}$, show that $\sqrt[n]{c_1c_2c_3\ldots c_n} \leq \frac{c_1+c_2+c_3+\ldots+c_n}{n}$

iv Hence use part (iii) to find a lower bound for $\frac{101}{103} + \frac{103}{105} + \frac{105}{107} + \ldots + \frac{197}{199} + \frac{199}{101}$

112 Let $z = \cos\theta + i\sin\theta$ and suppose that n is a positive integer.

i Use the identities $z^n + z^{-n} = 2\cos n\theta$ and $2\cos A\sin B = \sin(A+B) - \sin(A-B)$ to show that

$$(z^{2n} + z^{2n-2} + z^{2n-4} + \ldots + z^{-2n})\sin\theta = \sin(2n+1)\theta$$

ii Use part (i) and the identity $\cos 3A = 4\cos^3 A - 3\cos A$ to deduce that

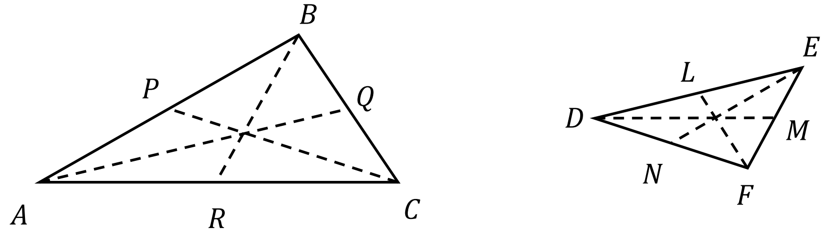
$$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \frac{\sin 7\theta}{\sin\theta}$$

iii Hence show that $\cos\frac{2\pi}{7}$ is a root of the equation

$$8x^3 + 4x^2 - 4x - 1 = 0$$

113 Consider the integral $I_n = \int_0^1 \sqrt{x}(1-x)^n\,dx, \quad n = 0, 1, 2, 3, \ldots$ By finding a recurrence relationship for I_n in terms of I_{n-1}, evaluate I_3.

114 The sides of triangle DEF are parallel to the medians of triangle ABC as shown, and the triangles are similar. Let P, Q, R, L, M, N be the midpoints of AB, BC, CA, DE, EF, FD as shown.



Prove that the medians of triangle DEF are parallel to the sides of triangle ABC.

115 An object is to undergo vertical motion on a bungee cord in a vacuum so that air resistance can be neglected. The only forces it experiences are gravity, mg and an elastic force, $-kmx$, where x is the particle's displacement from the origin. Initially, the object is at its lowest point given by $x = -a$. All constants are positive.

i Using a force diagram show that $\ddot{x} = -g - kx$.

ii By integration show that $v^2 = k\left(\left(a - \frac{g}{k}\right)^2 - \left(x + \frac{g}{k}\right)^2\right)$

iii Show that the motion is described by $x = \left(\frac{g}{k} - a\right)\cos\left(\sqrt{k}t\right) - \frac{g}{k}$

116 A positive integer is said to be blue if no two adjacent digits are the same. Examples include 9, 23, 143 and 2452, but not 22, 344 or 1132. Let $B(n)$ represent the total number of n-digit blue integers, $O(n)$ represent the number of odd n-digit blue integers, and $E(n)$ represent the number of even n-digit blue integers. Blue integers cannot start with 0.

i Explain why $B(n) = 9^n$

ii Explain why $O(k+1) = 4 \times O(k) + 5 \times E(k)$

iii Using induction prove that $O(n) = \frac{9^n + (-1)^{n-1}}{2}$

iv Hence, or otherwise, find an expression for $E(n)$, the number of even n-digit blue integers.

117 The equation $z^5 = 1$ has roots $1, \omega, \omega^2, \omega^3, \omega^4$, where $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

i Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

ii Show that $\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$

iii Hence, show that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$.

118

i Show that $\int \frac{dx}{5 - 4\cos x} = \frac{2}{3}\tan^{-1}\left(3\tan\frac{x}{2}\right) + c$

ii Given that $\int_0^{\pi} \frac{dx}{5 - 4\cos x} = \frac{\pi}{3}$ show that $\int_0^{\pi} \frac{\cos x}{5 - 4\cos x}\,dx = \frac{\pi}{6}$

iii If $u_n = \int_0^{\pi} \frac{\cos nx}{5 - 4\cos x}\,dx$ show that $u_{n+1} + u_{n-1} - \frac{5}{2}u_n = 0$

119 Given P is the point on the line through A and B closest to the origin O, and $\overrightarrow{OP} = \underset{\sim}{p}, \overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$, prove

$$\underset{\sim}{p} = \underset{\sim}{a} - \frac{\left[\underset{\sim}{a} \cdot \left(\underset{\sim}{a} - \underset{\sim}{b}\right)\right]\left(\underset{\sim}{a} - \underset{\sim}{b}\right)}{\left|\underset{\sim}{a} - \underset{\sim}{b}\right|^2}$$

120 A particle of mass m kg moves in a straight line with velocity v metres per second, under a constant force P Newtons, and a resistance R Newtons. Initially the particle has a speed v_0 metres per second. If $R = 5 + 3v$ and $P = 10$:

i Show that $v = \frac{5}{3}\left(1 - e^{-\frac{3t}{m}}\right) + v_0 e^{-\frac{3t}{m}}$.

ii Find the terminal velocity of the particle.

iii When the particle accelerates from v_0 to v_1 show that the distance travelled, x metres, is given by:

$$x = \frac{m}{9}\left[3(v_0 - v_1) + 5\ln\left(\frac{5 - 3v_0}{5 - 3v_1}\right)\right]$$

121 The integers $a, b\ d$ are connected by the relation $a = b + d$.

i Use the binomial expansion of $(b + d)^n$, where n is a positive integer, to show that $a^n - b^{n-1}(b + nd)$ is divisible by d^2.

ii In the result of part (i) replace b by $a - d$. Hence show that if a is the first term, d the common difference and l the nth term of an arithmetic progression, then $a^n - l(a - d)^{n-1}$ is divisible by d^2.

iii Deduce that $5^{682} - 2^{692}$ is divisible by 9.

122 **i** For $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos(n\theta)$.

ii If $z + \frac{1}{z} = u$, find an expression for $z^3 + \frac{1}{z^3}$ solely in terms of u.

iii It can be shown that $z^5 + \frac{1}{z^5} = u^5 - 5u^3 + 5u$. (Do NOT prove this).

Show that

$$1 + \cos 10\theta = 2(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)^2$$

123 **i** Show

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \frac{1}{2}\sin x} = \frac{2\pi}{3\sqrt{3}}$$

ii Show

$$\int_0^{2a} f(x)\,dx = \int_0^a [f(x) + f(2a - x)]\,dx$$

iii Hence evaluate

$$\int_0^{\pi} \frac{x\,dx}{1 + \frac{1}{2}\sin x}$$

124 A series of vectors $\underset{\sim}{v}_k$ are such that $\left|\underset{\sim}{v}_1\right| = 3, \left|\underset{\sim}{v}_2\right| = 4$ and $\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 10$, and $\underset{\sim}{v}_k = \text{proj}_{\underset{\sim}{v}_{k-2}}\underset{\sim}{v}_{k-1}$, where $\text{proj}_{\underset{\sim}{a}}\underset{\sim}{b}$ is the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$. Find

$$S = \sum_{n=1}^{\infty} \left|\underset{\sim}{v}_n\right|$$

125 **i** A particle P of unit mass starts from rest at a point O and falls under gravity in a medium where the resistance to its motion has magnitude kv, where v is the speed of the particle and k is a positive constant.

$\boldsymbol{\alpha}$ Find the equation of motion of P.

$\boldsymbol{\beta}$ Show that the expression for its velocity v at any time t is given by

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

$\boldsymbol{\gamma}$ Find an expression for its terminal velocity V_T

ii A second particle Q, also of unit mass, is fired vertically upwards from O with initial speed u, so that P and Q leave O simultaneously.

$\boldsymbol{\alpha}$ Find the equation of motion of Q.

$\boldsymbol{\beta}$ Find an expression for t when Q comes to rest.

iii Show that, at the instant Q comes to rest, the velocity of P is given by:

$$v = \frac{V_T u}{V_T + u}$$

126 If $0 < b \le a$, prove

$$\frac{(a-b)^2}{8a} \le \frac{a+b}{2} - \sqrt{ab} \le \frac{(a-b)^2}{8b}$$

127 The roots of $z^n = 1,\ n$ a positive integer, are

$$z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \qquad k = 1, 2, \dots, n$$

If z_k is such that $z_k, z_k^2, z_k^3, \dots, z_k^n$ generates all the roots of $z^n = 1$, then z_k is called a primitive root of $z^n = 1$.

i Show that z_1 is a primitive root of $z^n = 1$

ii Show that z_5 is a primitive root of $z^6 = 1$

iii Suppose that the highest common factor of n and k is h, ie. $n = ph$ and $k = qh$ for p, q integers. Show that for z_k to be a primitive root of $z^n = 1$, then $h = 1$.

128 Let $I_n = \int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^n}$, where n is a positive integer.

i Find the value of I_1

ii Using integration by parts, show that

$$I_n = \frac{2nI_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)}$$

iii Hence evaluate

$$I_3 = \int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^3}$$

129 Prove that the perpendicular bisectors of a triangle are concurrent.

130 A particle of unit mass is projected upwards in a medium where it experiences a resistance of magnitude kv^2 where k is a positive constant and v is the velocity of the particle. During the downward motion the terminal velocity of the particle is V. It's initial velocity of projection is $\frac{1}{3}$ of this terminal velocity.

i By considering the forces on the particle during its downward motion, show that $kV^2 = g$.

ii Show that during the upward motion the acceleration of the particle $\ddot{x}$ is given by

$$\ddot{x} = -g\left(1 + \frac{v^2}{V^2}\right)$$

iii If the distance travelled by the particle in its upward motion is x when its velocity is v, show that the maximum height H reached is given by

$$H = \frac{V^2}{2g}\ln\left(\frac{10}{9}\right)$$

iv The velocity of the particle is v when it has fallen a distance y from its maximum height. Show that

$$y = \frac{V^2}{2g}\ln\left[\frac{V^2}{V^2 - v^2}\right]$$

v The velocity of the particle is U when it returns to its point of projection. Show that

$$\frac{V}{U} = \sqrt{10}$$

131 Let $T(m,y) = \frac{{}^mC_0}{y} - \frac{{}^mC_1}{y+1} + \frac{{}^mC_2}{y+2} - \ldots + (-1)^m \frac{{}^mC_m}{y+m}$

i If it is given that $T(k,x) = \frac{k!}{x(x+1)(x+2)\ldots(x+k)}$ for a particular value of k, show that

$$T(k,x) - T(k,x+1) = T(k+1,x)$$

ii Hence prove by induction that for for $n \geq 1$

$$T(n,x) = \frac{{}^nC_0}{x} - \frac{{}^nC_1}{x+1} + \frac{{}^nC_2}{x+2} - \ldots + (-1)^n \frac{{}^nC_n}{x+n} = \frac{n!}{x(x+1)(x+2)\ldots(x+n)}$$

You may assume that ${}^{m+1}C_r = {}^mC_r + {}^mC_{r-1}$.

iii Hence prove that

$$\frac{{}^nC_0}{1} - \frac{{}^nC_1}{3} + \frac{{}^nC_2}{5} - \ldots + (-1)^n \frac{{}^nC_n}{2n+1} = \frac{2^n n!}{1 \times 3 \times 5 \times \ldots \times (2n+1)}$$

132 Solve the equation $8x^3 - 6x - \sqrt{2} = 0$ using De Moivre's Theorem, leaving answers to 3 decimal places.

133 Double factorials are defined as:

$$n!! = \begin{cases} n(n-2)(n-4)\ldots 2 & \text{for even integers } n \\ n(n-2)(n-4)\ldots 1 & \text{for odd integers } n \end{cases}$$

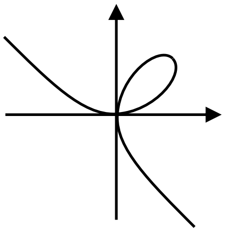
i Prove that for even integers $n = 2k$ that $(2k)!! = 2^k k!$

ii Show $(2k+1)!!\,2^k k! = (2k+1)!$ and thus for odd integers $n = 2k+1$ that $(2k+1)!! = \frac{(2k+1)!}{2^k k!}$

iii Prove

$$I_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$$

134 The Folium of Descartes has parametric equations $x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}$, for constant a. Find its cartesian equation.



135 A particle is projected vertically upwards under gravity in a medium where the resistance to motion is proportional to the square of its speed. The speed of projection is equal to the terminal velocity V ms^{-1}, of the particle when it falls in the same medium.

i Let x be the height of the particle above its point of projection and v ms^{-1} is its velocity at time t. Show that $\ddot{x} = -\frac{g}{V^2}(V^2 + v^2)$, where g is the acceleration due to gravity.

ii Given H is the maximum height above the point of projection, show that $H = \frac{V^2 \ln 2}{2g}$.

iii Given T is the time taken to achieve maximum height, show that for $0 \le t \le T$,

$$t = \frac{V}{g}\left\{\frac{\pi}{4} - \tan^{-1}\left(\frac{v}{V}\right)\right\}$$

iv Given that $\frac{2V^2}{V^2+v^2} = 1 + \sin\left(\frac{2g}{V}t\right)$, show that the time for the particle to reach half its maximum height is $\frac{V}{2g}\sin^{-1}(\sqrt{2}-1)$ seconds.

136 Prove that a positive integer n is divisible by 3 if and only if the sum of its digits is divisible by 3.You may assume $10^p = 3q + 1$ for integral p, q.

137 Let α be a non-real root of the polynomial $z^7 = 1$ with the smallest possible argument.
Let $\theta = \alpha + \alpha^2 + \alpha^4$ and $\delta = \alpha^3 + \alpha^5 + \alpha^6$

i Explain why $\alpha^7 = 1$ and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$

ii Show that $\theta + \delta = -1$ and $\theta\delta = 2$ and hence write a quadratic equation whose roots are θ and δ.

iii Show that $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $\delta = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$

iv Write down α in modulus-argument form, and show that

$$\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2} \text{ and } \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$$

138 Let $I_n = \int_0^1 (1-t^2)^{\frac{n-1}{2}}\,dt$ for $n = 0, 1, 2, \ldots$

i Show that $t^2(1-t^2)^{\frac{n-3}{2}} = (1-t^2)^{\frac{n-3}{2}} - (1-t^2)^{\frac{n-1}{2}}$

ii Using integration by parts, show that $nI_n = (n-1)I_{n-2}$ for $n = 2, 3, 4, \ldots$

iii Let $J_n = nI_nI_{n-1}$ for $n = 1, 2, 3, \ldots$ By using the principle of mathematical induction, prove that $J_n = \frac{\pi}{2}$ for $n = 1, 2, 3, \ldots$

iv Briefly explain why $0 < I_n < I_{n-1}$ for $n = 1, 2, 3, \ldots$

v Deduce that $\sqrt{\frac{\pi}{2(n+1)}} < I_n < \sqrt{\frac{\pi}{2n}}$ for $n = 1, 2, 3 \ldots$

139 Let the interval AB, with $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, have $|AB| = 2r$ and midpoint $M(u, v, w)$. Let $\underset{\sim}{p} = \overrightarrow{OP}, \underset{\sim}{a} = \overrightarrow{OA}, \underset{\sim}{b} = \overrightarrow{OB}, \underset{\sim}{m} = \overrightarrow{OM}$.

Prove that the equation $\left(\underset{\sim}{p} - \underset{\sim}{a}\right) \cdot \left(\underset{\sim}{p} - \underset{\sim}{b}\right) = 0$ is equivalent to $(x-u)^2 + (y-v)^2 + (z-w)^2 = r^2$.

140 A particle is projected from the origin with speed V at an angle α to the horizontal. The particle is subject to both gravity and an air resistance proportional to its velocity, so that its respective horizontal and vertical components of acceleration while it is rising are given by:
$\ddot{x} = -k\dot{x}$ and $\ddot{y} = -g - k\dot{y}$

i Show that

$\boldsymbol{\alpha}$ $\dot{x} = V\cos\alpha\ e^{-kt}$

$\boldsymbol{\beta}$ $\dot{y} = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}$

ii Hence show that

$\boldsymbol{\alpha}$ $x = \frac{V\cos\alpha}{k}\left(1 - e^{-kt}\right)$

$\boldsymbol{\beta}$ $y = \left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t$

iii When the particle reaches its greatest height, show that it has travelled a horizontal distance of $\frac{V^2 \sin 2\alpha}{2(g + Vk\sin\alpha)}$.

141 A circle and a chord of that circle are drawn in a plane. Then a second circle, and chord of that circle, are added. Repeating this process, once there are n circles with chords drawn, prove that the regions in the plane divided off by the circles and chords can be coloured with three colours in such a way that no two regions sharing the same length of border are the same colour.

142 **i** Prove

$$\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} = i\tan\frac{\theta}{2}$$

ii Find the five roots of the equation $\omega^5 = 1$ and express your answers in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.

iii Hence show that the roots of the equation

$$\left(\frac{2+z}{2-z}\right)^5 = 1 \qquad (*)$$

are $2i\tan\left(\frac{k\pi}{5}\right)$, where $k = 0, \pm 1, \pm 2$.

iv By expressing the equation in part (c) $(*)$ in the form $z^5 + mz^3 + nz = 0$, show that

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$$

143 Use integration to prove

$$\int \sqrt{\frac{x}{x+1}}\,dx = \sqrt{x^2+x} - \ln(\sqrt{x} + \sqrt{x+1}) + c$$

You may assume

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}\,dx = \ln\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right|$$

144 U and V are two points with position vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ respectively. W, with position vector $\underset{\sim}{w}$. lies on UV produced such that $|\overrightarrow{UV}| = |\overrightarrow{VW}|$. The point S , with position vector $\underset{\sim}{s}$, lies on $\overrightarrow{OV}$ produced, such that $|\overrightarrow{OV}| = \lambda|\overrightarrow{VS}|$, and WS is perpendicular to UV. Prove

$$\lambda = \frac{2\underset{\sim}{u}\cdot\underset{\sim}{v} - \left|\underset{\sim}{v}\right|^2 - \left|\underset{\sim}{u}\right|^2}{\underset{\sim}{u}\cdot\underset{\sim}{v} - \left|\underset{\sim}{v}\right|^2}$$

145 A particle of unit mass is fired vertically upwards in a medium where the resistance to motion as magnitude kv^2 when the speed is v ms^{-1}. The particle has height x metres above the point of projection at time t seconds. The maximum height H metres is reached at time T seconds. The speed of projection U ms^{-1} is equal to the terminal velocity of a particle falling in the medium. The acceleration due to gravity has magnitude g ms^{-2}.

i Express U^2 in terms of g and k, and deduce that $\ddot{x} = -\frac{g}{U^2}(U^2 + v^2)$.

ii Show that $\frac{v}{U} = \tan\left(\frac{\pi}{4} - \frac{g}{U}t\right)$

iii Show that $\frac{x}{U} = \frac{U}{g}\ln\left[\sqrt{2}\left|\cos\left(\frac{\pi}{4} - \frac{g}{U}t\right)\right|\right]$

iv Show that at time $\frac{1}{2}T$ seconds $\frac{x}{U} = \frac{U}{2g}\ln\left[1 + \frac{1}{\sqrt{2}}\right]$ and calculate the percentage of the maximum height attained during the first half of the ascent time, giving your answer to the nearest 1%.

146 Let a, b, A and B be positive numbers.
i Prove that
$$\frac{ab}{AB} \leq \frac{1}{2}\left(\frac{a^2}{A^2} + \frac{b^2}{B^2}\right)$$
ii Let
$$A = \sqrt{\sum_{k=1}^{n} a_k^2} \text{ and } B = \sqrt{\sum_{k=1}^{n} b_k^2}$$
Where a_k and b_k are positive real numbers.
Use (i) to prove that
$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$
iii Let $S = x_1 + x_2 + x_3 + \ldots + x_n$, where $x_k > 0$ for all $1 \leq k \leq n$. Use (ii) to prove that
$$\frac{S}{S - x_1} + \frac{S}{S - x_2} + \frac{S}{S - x_3} + \ldots + \frac{S}{S - x_n} \geq \frac{n^2}{n - 1}$$

147 Given $z = \cos\alpha + i\sin\alpha$, where $\sin\alpha \neq 0$:
i Prove that $\frac{1}{1 - z\cos\alpha} = 1 + i\cot\alpha$
ii Hence, by considering $\sum_{k=0}^{\infty}(z\cos\alpha)^k$, deduce that the sum of the infinite series
$$\sin\alpha\cos\alpha + \sin 2\alpha\cos^2\alpha + \ldots + \sin k\alpha\, cos^k\alpha + \ldots = \cot\alpha$$

148 Prove
$$I_{2n} = \int_{-1}^{1} (1 - x^2)^n\, dx = \frac{2^{2n+1}(n!)^2}{(2n + 1)!}$$

149 The points $A(1,2,-1), B(0,1,1)$ and $C(-1,3,2)$ lie in three dimensional space.
i Find the vectors $\underset{\sim}{u} = \overrightarrow{AB}$ and $\underset{\sim}{v} = \overrightarrow{AC}$.
ii Find $\text{proj}_{\underset{\sim}{v}} \underset{\sim}{u}$.
iii Hence find the shortest distance from B to the line through A and C.
iv Using similar steps, prove that the shortest distance, d, from the point $B(\underset{\sim}{b})$ to the line through $A(\underset{\sim}{a})$ and $C(\underset{\sim}{c})$ is given by
$$d^2 = \left(\underset{\sim}{b} - \underset{\sim}{a}\right) \cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right) - \left[\frac{\left(\underset{\sim}{b} - \underset{\sim}{a}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right)}{\left|\underset{\sim}{c} - \underset{\sim}{a}\right|}\right]^2$$
v Hence find the shortest distance from C to AB using the points $A(1,2,-1), B(0,1,1)$ and $C(-1,3,2)$.

150 A particle is moving in simple harmonic motion about the origin, and at the end of three consecutive seconds its displacement is $x = 1, x = 5$ and $x = 5$. Prove that the period of the particle is
$$T = \frac{2\pi}{\arccos\left(\frac{3}{5}\right)}$$

1 LHS − RHS $= (a^2+b^2)(x^2+y^2)-(ax+by)^2$
$= a^2x^2+a^2y^2+b^2x^2+b^2y^2-a^2x^2-2abxy-b^2y^2$
$= a^2y^2-2abxy+b^2x^2$
$= (ay-bx)^2$
≥ 0 for real a,b,x,y
$\therefore (a^2+b^2)(x^2+y^2)\geq(ax+by)^2$

2
$\therefore z^3-z^2+9z-9=0$
$z^2(z-1)+9(z-1)=0$
$(z^2+9)(z-1)=0$
$(z+3i)(z-3i)(z-1)=0$
$z=1,\pm3i$
$\therefore p=1, q=3i, r=-3i$

3 $\int \frac{\sin(\tan\theta)}{\cos^2\theta}\,d\theta$
$= \int \sec^2\theta\sin(\tan\theta)\,d\theta$
$= -\cos(\tan\theta)+c$

4 $|\overrightarrow{AB}|=\sqrt{8^2+(-3)^2+5^2}=7\sqrt{2}$
$|\overrightarrow{AC}|=\sqrt{11^2+(-5)^2+(-1)^2}=7\sqrt{3}$
$|\overrightarrow{BC}|=\sqrt{3^2+(-2)^2+(-6)^2}=7$
$\therefore s=7$
$V=s^3=7^3=343$

5 **i**
$\frac{2\pi}{n}=10 \rightarrow n=\frac{\pi}{5}$
Let $x=10\cos\left(\frac{\pi t}{5}\right)$, so the particle is starting at its rightmost extremity. Find the time it takes to reach $x=6$:
$6=10\cos\left(\frac{\pi t}{5}\right)$
$\frac{3}{5}=\cos\left(\frac{\pi t}{5}\right)$
$t=\frac{5}{\pi}\cos^{-1}\left(\frac{3}{5}\right)$

ii
$\dot{x}=-2\pi\sin\left(\frac{\pi t}{5}\right)$
At $t=\frac{5}{\pi}\cos^{-1}\left(\frac{3}{5}\right)$
speed $=2\pi\sin\left(\frac{\pi}{5}\left(\frac{5}{\pi}\cos^{-1}\left(\frac{3}{5}\right)\right)\right)$
$=5.0265\ldots$
$=5\ \text{ms}^{-1}$ (nearest whole)

6 Let $P(n)$ represent the proposition.

$P(1)$ is true since $(1)\big((1)+1\big)\big((1)+2\big)(3(1)+5)=48=24(2)$.

If $P(k)$ is true for some arbitrary $k\geq1$ then $k(k+1)(k+2)(3k+5)=24m$ for integral m

RTP $P(k+1)$ $(k+1)(k+2)(k+3)(3k+8)=24p$ for integral p

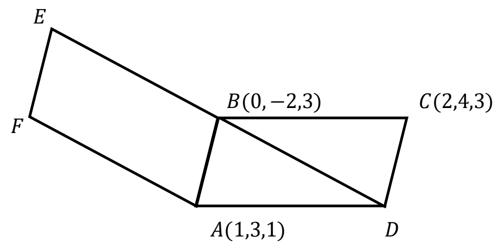
LHS $=(k+1)(k+2)(3k^2+17k+24)$
$=(k+1)(k+2)(3k^2+5k+12k+24)$
$=(k+1)(k+2)[k(3k+5)+12k+24]$
$=k(k+1)(k+2)(3k+5)+12k(k+1)(k+2)+24(k+1)(k+2)$
$=24m+12k(k+1)(k+2)+24(k+1)(k+2)$ from $P(k)$
$=24m+12k(2q)+24(2q)$ since $k+1, k+2$ are consecutive their product is even
$=24(m+kq+2q)$
$=24p$ since m,k,q are integral
= RHS
$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\geq1$ by induction

7 $(1+i)(1)$:
$(1+i)[(1-i)z+2w]=(1+i)[i-7]$
$2z+2(1+i)w=i-7-1-7i$
$2z+2(1+i)w=-8-6i$ (3)
$(1)-(3)$: $(i-2)w=8+6i$
$w=\frac{8+6i}{i-2}\times\frac{i+2}{i+2}$
$=\frac{8i+16-6+12i}{-1-4}$
$=-2-4i$
sub in (1): $2z+3i(-2-4i)=0$
$2z-6i+12=0$
$2z=-12+6i$
$z=-6+3i$

8 $xy-x=1$
$x(y-1)=1$
$x=\frac{1}{y-1}$
$\int_3^5 x\,dy=\int_3^5\frac{dy}{y-1}$
$=\Big[\ln(y-1)\Big]_3^5$
$=\ln4-\ln2$
$=\ln2$

9



i
$\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}$
$=\overrightarrow{OA}+\overrightarrow{BC}$
$=(1,3,1)+(2-0,4+2,3-3)$
$=(3,9,1)$

ii
$\overrightarrow{OE}=\overrightarrow{OB}+\overrightarrow{DB}$
$=(0,-2,3)+(0-3,-2-9,3-1)$
$=(-3,-13,5)$

iii
$\overrightarrow{OF}=\overrightarrow{OB}+\overrightarrow{BF}$
$=\overrightarrow{OB}+\overrightarrow{DA}$
$=(0,-2,3)+(1-3,3-9,1-1)$
$=(-2,-8,3)$

9… **iv**

$\overrightarrow{BF} = (-2-0, -8+2, 3-3)$
$= (-2, -6, 0)$
$\overrightarrow{CF} = (-2-2, -8-4, 3-3)$
$= (-4, -12, 0)$
$= 2\overrightarrow{BF}$
$\therefore B$ is the midpoint of $\overrightarrow{CF}$.

v

$\overrightarrow{DA} = (1-3, 3-9, 1-1) = (-2, -6, 0) = \overrightarrow{BF}$
$\therefore ADBF$ is a parallelogram since a pair of opposite sides are equal and parallel.

10

$$60 = 5v\cos\theta$$
$$v\cos\theta = 12$$
$$57.5 = -5(5^2) + 5v\sin\theta$$
$$v\sin\theta = 36.5$$
$$(v\cos\theta) + (v\sin\theta)^2 = v^2$$
$$v^2 = 12^2 + 36.5^2$$
$$v = 38.4\ \text{ms}^{-1}$$

11 Suppose $y > x$
$\therefore y(x^2+y^2) > x(x^2+y^2)$ since x, y real
$y^3 + x^2y > x^3 + xy^2$
$\therefore$ if $y^3 + yx^2 \leq x^3 + xy^2$ then $y \leq x$ by contrapositive.

Alternatively
$$y^3 + yx^2 \leq x^3 + xy^2$$
$$y(y^2+x^2) \leq x(y^2+x^2)$$
$$(y-x)(x^2+y^2) \leq 0$$
$\therefore y - x \leq 0$ since $x^2 + y^2 \geq 0$ for real x, y
$\therefore y \leq x$

12 Let $3 + i = r(\cos\theta + i\sin\theta)$ and $3 - i = r(\cos(-\theta) + i\sin(-\theta))$
$(3+i)^n + (3-i)^n$
$= (r(\cos\theta + i\sin\theta))^n + (r(\cos(-\theta) + i\sin(-\theta)))^n$
$= r^n(\cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta))$
$= r^n(\cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta))$
$= 2r^n\cos(n\theta)$ which is purely real.

13

$$\int \frac{\sec^2 x}{1-\tan^2 x}\,dx$$
$$= \int \frac{1+\tan^2 x}{1-\tan^2 x}\,dx$$
$$= \int \sec 2x\,dx$$
$$= \frac{1}{2}\ln|\tan 2x + \sec 2x| + c$$

14 The long diagonal of the prism is $\sqrt{2^2+8^2+16^2} = 18$.

The rectangular prism fits inside the sphere centred at origin with radius 9.

$\therefore |OP| \leq 9$.

15

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1 + \ln x$$
$$\frac{1}{2}v^2 = \int_1^x (1+\ln x)\,dx$$
$$v^2 = 2\int_1^x (1+\ln x)\,dx$$
$$= 2\left\{\Big[x(1+\ln x)\Big]_1^x - \int_1^x dx\right\}$$
$$= 2\left\{x(1+\ln x) - 1 - \Big[x\Big]_1^x\right\}$$
$$= 2\{x + x\ln x - 1 - x + 1\}$$
$$= 2x\ln x$$

$u = 1 + \ln x$	$\frac{dv}{dx} = 1$
$u' = \frac{1}{x}$	$v = x$

when $x = e^2$
$v^2 = 2e^2\ln e^2$
$= 4e^2$
$v = 2e \quad v > 0$ given initial conditions

16 Let $P(n)$ represent the proposition.

$P(1)$ is true since $x + \frac{1}{x} > 2\sqrt{x \cdot \frac{1}{x}} = 2$ since the arithmetic mean is greater than or equal to the geometric mean.

Similarly $P(2)$ is true since $x^2 + \frac{1}{x^2} > 2\sqrt{x^2 \cdot \frac{1}{x^2}} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then
$$x^k + x^{k-2} + x^{k-4} + \ldots + \frac{1}{x^{k-4}} + \frac{1}{x^{k-2}} + \frac{1}{x^k} \geq k+1$$

RTP $P(k+2)$
$$x^{k+2} + x^k + x^{k-2} + x^{k-4} + \ldots + \frac{1}{x^{k-4}} + \frac{1}{x^{k-2}} + \frac{1}{x^k} + \frac{1}{x^{k+2}} \geq k+3$$

$$\text{LHS} = x^{k+2} + x^k + x^{k-2} + x^{k-4} + \ldots + \frac{1}{x^{k-4}} + \frac{1}{x^{k-2}} + \frac{1}{x^k} + \frac{1}{x^{k+2}}$$
$$= x^k + x^{k-2} + x^{k-4} + \ldots + \frac{1}{x^{k-4}} + \frac{1}{x^{k-2}} + \frac{1}{x^k} + x^{k+2} + \frac{1}{x^{k+2}}$$
$$\geq k + 1 + x^{k+2} + \frac{1}{x^{k+2}} \quad \text{from } P(k)$$
$$\geq k + 1 + 2\sqrt{x^{k+2} \cdot \frac{1}{x^{k+2}}} \quad \text{AM} - \text{GM Inequality}$$
$$\geq k + 1 + 2$$
$$\geq k + 3$$
$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd and even $n \geq 1$ by induction

17 **i**

$$z = \frac{-\sqrt{3} \pm \sqrt{\left(\sqrt{3}\right)^2 - 4(i)(-1)}}{2(i)}$$
$$= \frac{-\sqrt{3} \pm \sqrt{3+4i}}{2i}$$

Let $(a+ib)^2 = 3 + 4i$
$\therefore a^2 - b^2 = 3 \quad 2ab = 4 \to ab = 2$
$a = \pm 2, b = \pm 1$

$$z = \frac{\sqrt{3} \pm (2+i)}{2i}$$
$$= \frac{2+\sqrt{3}+i}{2i}, \frac{-2+\sqrt{3}-i}{2i}$$
$$= \frac{1-\left(2+\sqrt{3}\right)i}{2}, \frac{-1+\left(2-\sqrt{3}\right)i}{2}$$
$$\therefore \alpha = \frac{1-\left(2+\sqrt{3}\right)i}{2}, \beta = \frac{-1+\left(2-\sqrt{3}\right)i}{2}$$

ii

$$\alpha\beta = \frac{c}{a} = -\frac{1}{i} \times \frac{i}{i} = -\frac{i}{-1} = i$$
$$\therefore\ \alpha^2\beta^2 + 1 = i^2 + 1 = 0$$

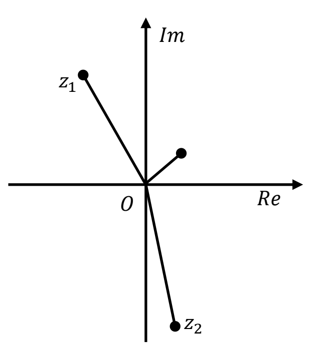
18
$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{2e^{\frac{2}{x^2}}}{x^3}\,dx$$
$$= -\frac{1}{2}\int_{\sqrt{2}}^{\sqrt{3}} (-4x^{-3})e^{2x^{-2}}\,dx$$
$$= -\frac{1}{2}\left[e^{\frac{2}{x^2}}\right]_{\sqrt{2}}^{\sqrt{3}}$$
$$= -\frac{1}{2}\left(e^{\frac{2}{3}} - e\right)$$
$$= \frac{e - e^{\frac{2}{3}}}{2}$$

19
$$x = \sec(t) - 1 \;\rightarrow\; \sec(t) = x + 1$$
$$y = \tan(t) - 2 \;\rightarrow\; \tan(t) = y + 2$$
$$\sec^2(t) = \tan^2(t) + 1$$
$$(x+1)^2 = (y+2)^2 + 1$$
$$(y+2)^2 = x^2 + 2x + 1 - 1$$
$$y + 2 = \sqrt{x^2 + 2x} \quad \text{since } x, y \ge 0 \text{ for } t \ge 0$$
$$y = \sqrt{x^2 + 2x} - 2$$

20
$$\dot{x} = V$$
$$x = Vt$$
$$t = \frac{x}{V}$$
$$\ddot{y} = -g$$
$$\dot{y} = -gt$$
$$y - H = -g\int_0^t t\,dt$$
$$y = -\frac{gt^2}{2} + H$$
$$= -\frac{gx^2}{2V^2} + H$$
Let $y = 0$
$$\frac{gx^2}{2V^2} = H$$
$$x^2 = \frac{2HV^2}{g}$$
$$x = V\sqrt{\frac{2H}{g}}$$

21
$$\frac{a^7 + a^7 + a^7 + a^7 + b^7 + b^7 + b^7}{7} \ge \sqrt[7]{a^{28}b^{21}}$$
$$\therefore \frac{4a^7 + 3b^7}{7} \ge a^4b^3$$
Similarly
$$\frac{4b^7 + 3c^7}{7} \ge b^4c^3$$
$$\frac{4c^7 + 3a^7}{7} \ge c^4a^3$$
Summing the above gives
$$a^7 + b^7 + c^7 \ge a^4b^3 + b^4c^3 + c^4a^3$$

22 **i**



The sum of vectors must add to zero, so when laid tip to tail they should start and end at zero.

22 … **ii**
$$z_1 = \cos\alpha + i\sin\alpha$$
$$z_2 = k(\cos\beta + i\sin\beta)$$
$$z_3 = (2-k)(\cos\gamma + i\sin\gamma)$$

iii
$$z_1 + z_2 + z_3 = 0$$
$$\therefore \cos\alpha + i\sin\alpha + k(\cos\beta + i\sin\beta) + (2-k)(\cos\gamma + i\sin\gamma) = 0$$
$$(\cos\alpha + k\cos\beta + (2-k)\cos\gamma) + i(\sin\alpha + k\sin\beta + (2-k)\sin\gamma) = 0$$
Equating real and imaginary components:
$$\cos\alpha + k\cos\beta + (2-k)\cos\gamma = 0$$
$$\sin\alpha + k\sin\beta + (2-k)\sin\gamma = 0$$

iv
$$\sin\alpha = -(k\sin\beta + (2-k)\sin\gamma)$$
$$\sin^2\alpha = k^2\sin^2\beta + 2k(2-k)\sin\beta\sin\gamma + (2-k)^2\sin^2\gamma$$
$$\cos\alpha = -(k\cos\beta + (2-k)\cos\gamma)$$
$$\cos^2\alpha = k^2\cos^2\beta + 2k(2-k)\cos\beta\cos\gamma + (2-k)^2\cos^2\gamma$$
$$\sin^2\alpha + \cos^2\alpha = 1$$
$$k^2(\sin^2\beta + \cos^2\beta) + 2k(2-k)(\cos\beta\cos\gamma + \sin\beta\sin\gamma)$$
$$+(2-k)^2(\sin^2\gamma + \cos^2\gamma) = 1$$
$$k^2 + 2k(2-k)\cos(\beta - \gamma) + (2-k)^2 = 1$$

v
$$2k(2-k)\cos(\beta-\gamma) = 1 - (2-k)^2 - k^2$$
$$\cos(\beta-\gamma) = \frac{1 - 4 + 4k - k^2 - k^2}{2k(2-k)}$$
$$= \frac{2k^2 - 4k + 3}{2k^2 - 4k}$$
$$= 1 + \frac{3}{2k^2 - 4k}$$
$$\therefore \left|1 + \frac{3}{2k^2 - 4k}\right| \le 1$$
$$-2 \le \frac{3}{2k^2 - 4k} \le 0$$

$$-2 \le \frac{3}{2k^2 - 4k}$$
$$-4k^2 + 8k \ge 3 \quad \text{since } 2k^2 - 4k < 0$$
$$4k^2 - 8k + 3 \le 0$$
$$(2k-1)(2k-3) \le 0$$
$$\frac{1}{2} \le k \le \frac{3}{2}$$
and
$$\frac{3}{2k^2 - 4k} \le 0$$
$$\therefore 2k^2 - 4k \le 0$$
$$2k(k-2) \le 0$$
$$0 \le k \le 2$$

$$\therefore \frac{1}{2} \le k \le \frac{3}{2}$$

23
$$\int (2x\operatorname{cosec} x - x^2\cot x\operatorname{cosec} x)\,dx$$
$$= \int\left(\frac{2x}{\sin x} - \frac{x^2\cos x}{\sin^2 x}\right)dx$$
$$= \int \frac{2x\sin x - x^2\cos x}{\sin^2 x}\,dx$$
$$= \int \frac{d}{dx}\left(\frac{x^2}{\sin x}\right)dx$$
$$= \frac{x^2}{\sin x} + c$$

24
$$\left(\left|\underset{\sim}{a}\right|\underset{\sim}{b} + \left|\underset{\sim}{b}\right|\underset{\sim}{a}\right)\cdot\left(\left|\underset{\sim}{a}\right|\underset{\sim}{b} - \left|\underset{\sim}{b}\right|\underset{\sim}{a}\right)$$
$$= \left|\underset{\sim}{a}\right|^2 \underset{\sim}{b}\cdot\underset{\sim}{b} - \left|\underset{\sim}{b}\right|^2 \underset{\sim}{a}\cdot\underset{\sim}{a}$$
$$= \left|\underset{\sim}{a}\right|^2\left|\underset{\sim}{b}\right|^2 - \left|\underset{\sim}{b}\right|^2\left|\underset{\sim}{a}\right|^2$$
$$= 0$$
$\therefore \left|\underset{\sim}{a}\right|\underset{\sim}{b} + \left|\underset{\sim}{b}\right|\underset{\sim}{a}$ is perpendicular to $\left|\underset{\sim}{a}\right|\underset{\sim}{b} - \left|\underset{\sim}{b}\right|\underset{\sim}{a}$

25 **i**

$x = C\cos 2t + D\sin 2t$
$\dot{x} = -2C\sin 2t + 2D\cos 2t$
$\ddot{x} = -4C\cos 2t - 4D\sin 2t$
$= -4(C\cos 2t + D\sin 2t)$
$= -2^2 x$

ii *α*

$$C\cos\frac{2\pi}{3} + D\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$
$$C\left(-\frac{1}{2}\right) + D\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$
$$-C + \sqrt{3}D = \sqrt{3} \qquad (1)$$
$$-2C\sin\frac{2\pi}{3} + 2D\cos\frac{2\pi}{3} = -5$$
$$-2C\left(\frac{\sqrt{3}}{2}\right) + 2D\left(-\frac{1}{2}\right) = -5$$
$$-\sqrt{3}C - D = -5 \qquad (2)$$

$\sqrt{3}(1) - (2)$:
$$4D = 8 \ \rightarrow\ D = 2$$
sub in (1):
$$C = \sqrt{3}(2) - \sqrt{3} \ \rightarrow\ C = \sqrt{3}$$

ii *β*

$$R = \sqrt{(2)^2 + \left(\sqrt{3}\right)^2} = \sqrt{7}$$

26 **i**

$$\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^2 \geq 0$$
$$\frac{x}{y} - 2 + \frac{y}{x} \geq 0$$
$$\frac{x}{y} + \frac{y}{x} \geq 2$$

ii

Similarly:
$$\frac{x}{w} + \frac{w}{x} \geq 2, \frac{x}{z} + \frac{z}{x} \geq 2, \frac{w}{y} + \frac{y}{w} \geq 2, \frac{w}{z} + \frac{z}{w} \geq 2, \frac{y}{z} + \frac{z}{y} \geq 2$$
Summing the 6 inequalities:
$$\frac{x}{y} + \frac{y}{x} + \frac{x}{w} + \frac{w}{x} + \frac{x}{z} + \frac{z}{x} + \frac{w}{y} + \frac{y}{w} + \frac{w}{z} + \frac{z}{w} + \frac{y}{z} + \frac{z}{y} \geq 12$$
$$\frac{x+y+z}{w} + \frac{w+y+z}{x} + \frac{w+x+z}{y} + \frac{w+x+y}{z} \geq 12$$

iii

Let $w + x + y + z = 1$
$$\therefore \frac{1-w}{w} + \frac{1-x}{x} + \frac{1-y}{y} + \frac{1-z}{z} \geq 12$$
$$\frac{1}{w} - 1 + \frac{1}{x} - 1 + \frac{1}{y} - 1 + \frac{1}{z} - 1 \geq 12$$
$$\frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 16$$

27

$$\therefore \phi = \omega \operatorname{cis}\frac{2\pi}{3}$$
$$(\omega + \phi)^2 = \left(\omega + \omega\operatorname{cis}\frac{2\pi}{3}\right)^2$$
$$= \omega^2\left(1 + \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^2$$
$$= \omega^2\left(1 - \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2$$
$$= \omega^2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2$$
$$= \omega^2\left(\frac{1}{4} + \frac{\sqrt{3}}{2}i - \frac{3}{4}\right)$$
$$= \omega^2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= \omega^2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$
$$= \omega \cdot \omega\operatorname{cis}\frac{2\pi}{3}$$
$$= \omega\phi$$

28

$$\int \frac{e^{\log_2 x}}{x}\,dx$$
$$= \int \frac{e^{\left(\frac{\ln x}{\ln 2}\right)}}{x}\,dx$$
$$= \int \frac{\left(e^{\ln x}\right)^{\frac{1}{\ln 2}}}{x}\,dx$$
$$= \int \frac{x^{\frac{1}{\ln 2}}}{x}\,dx$$
$$= \int x^{\frac{1}{\ln 2} - 1}\,dx$$
$$= \ln 2\, x^{\frac{1}{\ln 2}} + c$$

29 **i**

$$\underset{\sim}{r} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

ii

$$\underset{\sim}{r} = \begin{pmatrix} (1-\lambda)x_1 + \lambda x_2 \\ (1-\lambda)y_1 + \lambda y_2 \\ (1-\lambda)z_1 + \lambda z_2 \end{pmatrix}$$
$$\therefore x = (1-\lambda)x_1 + \lambda x_2$$
$$x - x_1 = \lambda(x_2 - x_1)$$
$$\lambda = \frac{x - x_1}{x_2 - x_1} \quad (1)$$
Similarly
$$\lambda = \frac{y - y_1}{y_2 - y_1} \quad (2) \qquad \lambda = \frac{z - z_1}{z_2 - z_1} \quad (3)$$
Equating (1), (2) and (3):
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

iii

$$\frac{x-3}{-2} = \lambda \ \rightarrow x = 3 - 2\lambda$$
$$\frac{y+2}{3} = \lambda \ \rightarrow y = -2 + 3\lambda$$
$$z - 1 = \lambda \rightarrow z = 1 + \lambda$$

$$\underset{\sim}{r} = \begin{pmatrix} 3 - 2\lambda \\ -2 + 3\lambda \\ 1 + \lambda \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 - 3 \\ 1 - (-2) \\ 2 - 1 \end{pmatrix}$$

The curve passes through $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, amongst many possible pairs of points.

30 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x - 2$$
$$\frac{1}{2}(v^2 - 1^2) = 2\int_0^x (2x-1)\,dx$$
$$v^2 - 1 = 4\left[x^2 - x\right]_0^x$$
$$v^2 = 4x^2 - 4x + 1$$
$$= (2x-1)^2$$

ii

$\therefore v = -(2x-1)$ given initial conditions
$\therefore \ddot{x} = 4x - 2 = -2(2x-1) = -2v$

iii

$$\therefore \frac{dv}{dt} = -2v$$
$$\frac{dt}{dv} = -\frac{1}{2v}$$
$$t = -\frac{1}{2}\int_1^v \frac{dv}{v}$$
$$2\text{t} = \left[\ln v\right]_v^1$$
$$2t = 0 - \ln v$$
$$v = e^{-2t}$$
$$\therefore x = \int_0^t e^{-2t}\,dt$$
$$= -\frac{1}{2}\left[e^{-2t}\right]_0^t$$
$$= -\frac{1}{2}(e^{-2t} - 1)$$
$$= \frac{1 - e^{-2t}}{2}$$

iv

As $t \to \infty$ $e^{-2t} \to 0$
$$\therefore \lim_{t\to\infty} = \frac{1-0}{2} = \frac{1}{2}$$
The particle approaches $x = \frac{1}{2}$ but never reaches it.

31 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{1^5}{5} + \frac{1^4}{2} + \frac{1^3}{3} - \frac{1}{30} = 1$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30} = m$ for integral m

RTP $P(k+1)$ $\frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{(k+1)}{30} = p$ for integral p

$$\text{LHS} = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{(k+1)}{30}$$
$$= \frac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{2} + \frac{k^3 + 3k^2 + 3k + 1}{3} - \frac{k+1}{30}$$
$$= \frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30} + \frac{5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{4k^3 + 6k^2 + 4k + 1}{2} + \frac{3k^2 + 3k + 1}{3} - \frac{1}{30}$$
$$= m + k^4 + 2k^3 + k + \frac{1}{5} + 2k^3 + 3k^2 + 2k + \frac{1}{2} + k^2 + k + \frac{1}{3} - \frac{1}{30} \text{ from } P(k)$$
$$= m + k^4 + 4k^3 + 4k + 1$$
$$= p \text{ since } m, k \text{ are integral}$$
$$= \text{RHS}$$
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

32 **i** $\boldsymbol{\alpha}$

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(z_2 - z_1)$$

$\boldsymbol{\beta}$

$$\tan\left(\frac{\alpha}{2}\right) = \frac{|AM|}{|MP|}$$
$$|MP| = \frac{|AM|}{\tan\left(\frac{\alpha}{2}\right)} = |AM|\cot\left(\frac{\alpha}{2}\right)$$
$$\overrightarrow{MP} = \frac{1}{2}(z_2 - z_1)\cot\left(\frac{\alpha}{2}\right)i$$

ii

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AM} + \overrightarrow{(MP)}$$
$$= z_1 + \frac{1}{2}(z_2 - z_1) + \frac{1}{2}(z_2 - z_1)\cot\left(\frac{\alpha}{2}\right)i$$
$$= \frac{1}{2}\left(1 - i\cot\frac{\alpha}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)z_2$$

33

$$\int \frac{1 + \sin 2x}{\cos x + \sin x}\,dx$$
$$= \int \frac{\sin^2 x + 2\sin x\cos x + \cos^2 x}{\sin x + \cos x}\,dx$$
$$= \int \frac{(\sin x + \cos x)^2}{\sin x + \cos x}\,dx$$
$$= \int (\sin x + \cos x)\,dx$$
$$= -\cos x + \sin x + c$$

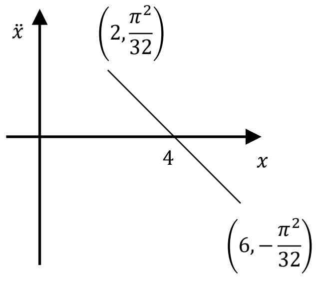
34

$$\sqrt{x^2 + y^2 + z^2} = 2\sqrt{(x-3)^2 + y^2 + z^2}$$
$$x^2 + y^2 + z^2 = 4x^2 - 24x + 36 + 4y^2$$
$$3x^2 - 24x + 3y^2 + 3z^2 + 36 = 0$$
$$x^2 - 8x + y^2 + z^2 + 12 = 0$$
$$x^2 - 8x + 16 + y^2 + z^2 = 4$$
$$(x-4)^2 + y^2 + z^2 = 2^2$$
Which is a sphere centred at $(4,0,0)$ with radius 2.

35 $T = 16 \rightarrow \frac{2\pi}{n} = 16 \rightarrow n = \frac{\pi}{8}$

$\ddot{x} = -n^2(x - c) = -\frac{\pi^2}{64}(x - 4)$

$x = 2 \rightarrow \ddot{x} = \frac{\pi^2}{32}, x = 4 \rightarrow \ddot{x} = 0, x = 6 \rightarrow \ddot{x} = -\frac{\pi^2}{32}$



36 Suppose $x = \frac{p}{q}$ for integral p, q is a solution and p and q have no common factor except 1 $*$

$\therefore \left(\frac{p}{q}\right)^3 + 3\left(\frac{p}{q}\right) + 5 = 0$

$p^3 + 3pq^2 + 5q^3 = 0$

Now the RHS is even

If p and q are odd then LHS = odd + odd + odd = odd which is a contradiction, so no solution.

If p is odd and q is even then LHS = odd + even + even = odd which is a contradiction, so no solution.

If p is even and q is odd then LHS = even + even + odd = odd which is a contradiction, so no solution.

If p and q are both even then they have a common factor of 2 which contradicts (*)

$\therefore$ there are no rational solutions to $x^3 + 3x + 5 = 0$

37 Splitting ΔOKL in half to create two right angled triangles, we can see that $|OL| = \sqrt{3}r$, so $|OM| = \sqrt{3}r$ as ΔOLM is equilateral.

$\angle KOL = \frac{\pi}{6}$ (angle sum of ΔOKL)

$\angle LOM = \frac{\pi}{3}$ (angle in an equilateral triangle)

$\therefore \angle KOM = \frac{\pi}{2}$

$\therefore OM$ is OK rotated anticlockwise by $\frac{\pi}{2}$ and stretched by $\sqrt{3}$.

$\therefore \beta = \alpha\sqrt{3}i$

$$\begin{aligned}\therefore 3\alpha^2 + \beta^2 &= 3\alpha^2 + \left(\alpha\sqrt{3}i\right)^2 \\ &= 3\alpha^2 - 3\alpha^2 \\ &= 0\end{aligned}$$

38 $\frac{d}{dx}\left(e^{\sqrt{2x^2-1}}\right) = \frac{1}{2}(2x^2 - 1)^{-\frac{1}{2}}(4x)e^{\sqrt{2x^2-1}} = \frac{2xe^{\sqrt{2x^2-1}}}{\sqrt{2x^2-1}}$

$$\begin{aligned}&\int \frac{xe^{\sqrt{2x^2-1}}\sin\left(e^{\sqrt{2x^2-1}}\right)}{\sqrt{2x^2-1}}dx \\ &= \frac{1}{2}\int \frac{2xe^{\sqrt{2x^2-1}}}{\sqrt{2x^2-1}}\sin\left(e^{\sqrt{2x^2-1}}\right)dx \\ &= -\frac{1}{2}\cos\left(e^{\sqrt{2x^2-1}}\right) + c\end{aligned}$$

39 $f(t)$ is a parabola which has the z-axis as its axis, is concave up, and is vertically above the line $y = x$.

$g(t)$ is one half of the same parabola, restricted to positive values of x and y, so in the first octant only.

$h(t)$ is $f(t)$ from $(1,1,1)$ to $(-1,-1,1)$.

40 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x + \frac{5}{(2-x)^2}$

$$\begin{aligned}\frac{1}{2}v^2 &= \int_0^x \left(-9x + \frac{5}{(2-x)^2}\right)dx \\ v^2 &= 2\left[-\frac{9x^2}{2} + \frac{5}{2-x}\right]_0^x \\ &= 2\left(\left(-\frac{9x^2}{2} + \frac{5}{2-x}\right) - \left(0 + \frac{5}{2}\right)\right) \\ &= -9x^2 + \frac{10}{2-x} - 5 \\ &= \frac{10 - 5(2-x) - 9x^2(2-x)}{2-x} \\ &= \frac{5x - 18x^2 + 9x^3}{2-x} \\ &= \frac{x(9x^2 - 18x + 5)}{2-x} \\ &= \frac{x(9x^2 - 15x - 3x + 5)}{2-x} \\ &= \frac{x(3x-1)(3x-5)}{2-x}\end{aligned}$$

$v = 0$ when $x = 0, \frac{1}{3}$ only, as $v^2 < 0$ for $x > \frac{1}{3}$.

The particle starts at $x = 0$ and comes to rest at $x = \frac{1}{3}$ before moving left again since $\ddot{x} < 0$. The particle moves between $x = 0$ and $x = \frac{1}{3}$.

41 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\sqrt{1} \leq \frac{1}{\sqrt{1}} \leq 2\sqrt{1} - 1 \rightarrow 1 \leq 1 \leq 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then
$\sqrt{k} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} \leq 2\sqrt{k} - 1$

$RTP\ \ P(k+1)$
$\sqrt{k+1} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$

$$\sqrt{k} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} \leq 2\sqrt{k} - 1 \qquad \text{from } P(k)$$

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$$

$$\frac{\sqrt{k} \times \sqrt{k+1} + 1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \frac{2\sqrt{k} \times \sqrt{k+1} + 1}{\sqrt{k+1}} - 1$$

$$\frac{\left(\sqrt{k} \times \sqrt{k} + 1\right)}{\sqrt{k+1}} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \frac{2\sqrt{k^2 + k} + 1}{\sqrt{k+1}} - 1$$

$$\frac{k+1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \frac{2\sqrt{k^2 + k + \frac{1}{2}} + 1}{\sqrt{k+1}} - 1$$

$$\frac{k+1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \frac{2\sqrt{k^2 + k + \frac{1}{2}} + 1}{\sqrt{k+1}} - 1$$

$$\sqrt{k+1} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \frac{2\sqrt{\left(k + \frac{1}{2}\right)^2} + 1}{\sqrt{k+1}} - 1$$

$$\sqrt{k+1} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \frac{2\left(k + \frac{1}{2}\right) + 1}{\sqrt{k+1}} - 1$$

$$\sqrt{k+1} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \frac{2(k+1)}{\sqrt{k+1}} - 1$$

$$\sqrt{k+1} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$$

$\therefore P(k) \Rightarrow P(k+1)$
$\therefore P(n)$ is true for $n \geq 1$ by induction

42 For $f(x) = e^{ix}$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \ldots$$

$$e^{ix} = e^{0i} + ie^{0i}x + \frac{i^2 e^{0i} x^2}{2!} + \frac{i^3 e^{0i} x^3}{3!} + \frac{i^4 e^{0i} x^4}{4!} + \ldots$$

$$= 1 + xi - \frac{x^2}{2!} - \frac{x^3}{3!}i + \frac{x^4}{4!} + \frac{x^5}{5!}i - \frac{x^6}{6!} - \frac{x^7}{7!}i + \ldots \qquad *$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \qquad (1)$$

For $g(x) = \cos x$

$$g(x) = g(0) + \frac{g'(0)}{1!}x + \frac{g''(0)}{2!}x^2 + \frac{g'''(a)}{3!}x^3 + \ldots$$

$$\cos x = \cos(0) - \sin(0)\,x - \frac{\cos(0)\,x^2}{2!} + \frac{\sin(0)\,x^3}{3!} + \frac{\cos(0)\,x^4}{4!} + \ldots$$

$$= 1 - 0x - \frac{x^2}{2!} + 0x^3 + \frac{x^4}{4!} + 0x^5 - \frac{x^6}{6!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad (2)$$

For $h(x) = \sin x$

$$h(x) = h(0) + \frac{h'(0)}{1!}x + \frac{h''(0)}{2!}x^2 + \frac{h'''(a)}{3!}x^3 + \ldots$$

$$\sin x = \sin(0) + \cos(0)\,x - \frac{\sin(0)\,x^2}{2!} - \frac{\cos(0)\,x^3}{3!} + \frac{\sin(0)\,x^4}{4!} + \ldots$$

$$= 0 + x + 0x^2 - \frac{x^3}{3!} + 0x^4 + \frac{x^5}{5!} + 0x^6 + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad (3)$$

So we have the following equations:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \qquad (1)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad (2)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad (3)$$

From $(1), (2)$ and (3) we see that $e^{ix} = \cos x + i \sin x$

43

$$\int \frac{\sin^3 x}{\cos x - 1}\,dx$$
$$= \int \frac{\sin x \sin^2 x}{\cos x - 1}\,dx$$
$$= \int \frac{\sin x\,(1-\cos^2 x)}{\cos x - 1}\,dx$$
$$= \int \frac{\sin x\,(1+\cos x)(1-\cos x)}{\cos x - 1}\,dx$$
$$= -\int (\sin x + \sin x \cos x)\,dx$$
$$= -\int \left(\sin x + \frac{1}{2}\sin 2x\right)dx$$
$$= \cos x + \frac{1}{4}\cos 2x + c$$

44

$$\left|\underset{\sim}{a}\right| = \sqrt{1^2+m^2+1^2} = \sqrt{2+m^2}$$
$$\left|\underset{\sim}{b}\right| = \sqrt{m^2+1^2+1^2} = \sqrt{2+m^2}$$
$$\cos\frac{\pi}{3} = \frac{(1)(m)+(m)(1)+(1)(1)}{\sqrt{2+m^2}\times\sqrt{2+m^2}}$$
$$\frac{1}{2} = \frac{2m+1}{2+m^2}$$
$$m^2+2 = 4m+2$$
$$m^2-4m = 0$$
$$m(m-4) = 0$$
$$m = 0, 4$$

45

i

$$\frac{dx}{dt} = \frac{10}{\sqrt{1-t^2}} + \frac{1}{(1-t)^2}$$
$$\Delta x = \int_0^{\frac{1}{2}} \left(\frac{10}{\sqrt{1-t^2}} + \frac{1}{(1-t)^2}\right)dt$$
$$= 10\left[\sin^{-1} t + \frac{1}{1-t}\right]_0^{\frac{1}{2}}$$
$$= 10\left(\left(\frac{\pi}{6}+2\right) - (0+1)\right)$$
$$= \frac{5(\pi+6)}{3}$$

ii

The velocity is an increasing function for the time interval, so the maximum velocity occurs when $t = \frac{1}{2}$

$$v_{\max} = \frac{10}{\sqrt{1-\left(\frac{1}{2}\right)^2}} + \frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{20}{\sqrt{3}} + 4 = \frac{20\sqrt{3}+12}{3}$$

46

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1})$$
$$< (a-b)\underbrace{(a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1})}_{n\text{ times}}$$
$$< na^{n-1}(a-b)$$

47

i $\text{LHS} = a(\cos\alpha + i\sin\alpha) + b(\cos\beta + i\sin\beta)$

$= (a\cos\alpha + b\cos\beta) + i(a\sin\alpha + b\sin\beta)$

$\text{RHS} = r(\cos\theta + i\sin\theta) = r\cos\alpha + ir\sin\alpha$

Equating real parts: $r\cos\theta = a\cos\alpha + b\cos\beta$

$$\cos\theta = \frac{a\cos\alpha + b\cos\beta}{r}$$

Equating imaginary parts: $r\sin\theta = a\sin\alpha + b\sin\beta$

$$\sin\theta = \frac{a\sin\alpha + b\cos\beta}{r}$$

ii

$r^2 = re^{i\theta} \times re^{-i\theta}$

$= \left(ae^{i\alpha} + be^{i\beta}\right)\left(ae^{-i\alpha} + be^{-i\beta}\right)$

$= a^2 + abe^{(\alpha-\beta)i} + abe^{-(\alpha-\beta)i} + b^2$

$= a^2 + b^2$

$+ ab\big(\cos(\alpha-\beta) + i\sin(\alpha-\beta) + \cos\big(-(\alpha-\beta)\big)$

$+ i\sin\big(-(\alpha-\beta)\big)\big)$

$= a^2 + b^2$

$+ ab(\cos(\alpha-\beta) + i\sin(\alpha-\beta) + \cos(\alpha-\beta)$

$- i\sin(\alpha-\beta))$

$= a^2 + b^2 + ab(2\cos(\alpha-\beta))$

$= a^2 + b^2 + 2ab\cos(\alpha-\beta)$

$\therefore r = \sqrt{a^2 + b^2 + 2ab\cos(\alpha-\beta)}$

iii

Let $z_1 = ae^{i\alpha}, z_2 = be^{i\beta}$

$|z_1| + |z_2| = a + b \quad (1)$

$|z_1 + z_2| = \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \quad \text{from (ii)}$

$\leq \sqrt{a^2+b^2+2ab} \quad \text{since } \cos\theta \leq 1$

$\leq \sqrt{(a+b)^2}$

$\leq a + b \quad (2)$

$\therefore$ from (1)and (2)$|z_1| + |z_2| \geq |z_1 + z_2|$

48

$$\int \frac{2}{x\ln 7}\,dx$$
$$= \frac{2}{\ln 7}\int \frac{dx}{x}$$
$$= \frac{2}{\ln 7}\ln|x| + c$$
$$= \frac{2\ln|x|}{\ln 7} + c$$
$$= \frac{\ln(x^2)}{\ln 7} + c$$
$$= \log_7 x^2 + c$$

49

Let the medians of OA, OB and AB be L, M and N respectively.

Let $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$

$\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right| = \left|\underset{\sim}{b} - \underset{\sim}{a}\right|$

$\overrightarrow{ON} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$

$\overrightarrow{ON}\cdot\overrightarrow{BA} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)\cdot\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

$= \frac{1}{2}\left(\underset{\sim}{a}\cdot\underset{\sim}{a} - \underset{\sim}{b}\cdot\underset{\sim}{b}\right)$

$= \frac{1}{2}\left(\left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{b}\right|^2\right)$

$= 0 \text{ since } \left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$

$\therefore ON$ is a median and an altitude, and similarly for AM and BL, so the medians of an equilateral triangle are also the altitudes.

50 **i**

Vertically:

$$mg = F\sin\theta + N \;\rightarrow\; N = 50 - F\sin\theta$$

Horizontally:

$$F\cos\theta = R = 0.2(50 - F\sin\theta) = 10 - 0.2\,F\sin\theta$$

$\therefore F\cos\theta + 0.2\,F\sin\theta = 10$

$$F = \frac{50}{5\cos\theta + \sin\theta}$$

ii

Let $5\cos\theta + \sin\theta = A\cos(\theta - \alpha)$

$$\rightarrow\; A = \sqrt{5^2 + 1^2} = \sqrt{26}$$

$$\therefore\; F = \frac{50}{\sqrt{26}\cos(\theta - \alpha)}$$

F is a minimum when $\cos(\theta - \alpha)$ is at its maximum value of 1

$F_{\min} = \frac{50}{\sqrt{26}}$ newtons.

51 Suppose $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$

$$\therefore \frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$$

$$\frac{1}{x+y} = \frac{y+x}{xy}$$

$$\therefore xy = (x+y)^2$$

$$xy = x^2 + 2xy + y^2$$

$$x^2 + xy + y^2 = 0$$

taking y as a constant $\Delta = y^2 - 4(1)(y^2)$

$$= -3y^2$$

$$\leq 0$$

$\therefore$ no solutions

$$\therefore \frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$$

52

$$x^2 - 2x + y^2 - 2y - 2 = 0, \qquad y \neq 1$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 - 4 = 0$$

$$(x-1)^2 + (y-1)^2 = 4$$

This is the circle centred at $(1,1)$ with radius 2, excluding the points $(3,1)$ and $(-1,1)$. The angle in a semicircle is $\frac{\pi}{2}$, so:

$z_1 = (3,1), z_2 = (-1,1)$ (the two can be swapped) and $\theta = \frac{\pi}{2}$.

53

$$\int \frac{dx}{\sec x - 1}$$

$$= \int \frac{\cos x}{1 - \cos x}\,dx$$

$$= \int \frac{\cos x}{1 - \cos x} \times \frac{1 + \cos x}{1 + \cos x}\,dx$$

$$= \int \frac{\cos x + \cos^2 x}{1 - \cos^2 x}\,dx$$

$$= \int \frac{\cos x + \cos^2 x}{\sin^2 x}\,dx$$

$$= \int \cos x\,(\sin x)^{-2}\,dx + \int \cot^2 x\,dx$$

$$= -\frac{1}{\sin x} + \int (\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\operatorname{cosec} x - \cot x - x + c$$

54 The planes collide at $t = 30$.

$30 = 2(30) - \alpha \;\rightarrow\; \alpha = 30$

$2t - 30 = 2(t - T) \rightarrow T = 15$

The second plane takes off 15 seconds after the first.

Alternatively:

The second plane flies twice as fast, so starts when the first plane has travelled half of the one and a half laps, so three quarters of a lap. $T = \frac{3}{4} \times 20 = 15$ seconds.

55 **i**

$$v\frac{dv}{dx} = -g - kv^2$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$H = -\int_{v_0}^{0} \frac{v}{g + kv^2}\,dv$$

$$= \frac{1}{2k}\Big[\ln(g + kv^2)\Big]_0^{v_0}$$

$$= \frac{1}{2k}\left(\ln(g + kv_0^2) - \ln g\right)$$

$$= \frac{1}{2k}\ln\left(\frac{g + kv_0^2}{g}\right)$$

$$= \frac{1}{2k}\ln\left(1 + \frac{kv_0^2}{g}\right)$$

ii

Let the new initial velocity be v_1.

$$\therefore 2H = \frac{1}{2k}\ln\left(1 + \frac{kv_1^2}{g}\right)$$

$$e^{4kH} = 1 + \frac{kv_1^2}{g}$$

$$v_1^2 = \frac{g}{k}\left(e^{4kH} - 1\right)$$

$$v_1 = \sqrt{\frac{g}{k}\left(e^{4kH} - 1\right)} \qquad \text{since } v_1 > 0$$

$$= \sqrt{\frac{g}{k}\left(e^{2kH} + 1\right)\left(e^{2kH} - 1\right)}$$

$$= \sqrt{\frac{g}{k}\left(e^{2kH} - 1\right)} \times \sqrt{e^{2kH} + 1} \qquad (1)$$

similarly:

$$v_0 = \sqrt{\frac{g}{k}\left(e^{2kH} - 1\right)} \qquad (2)$$

substituting (2) into (1):

$$v_1 = v_o\sqrt{e^{2kH} + 1}$$

$$= \left(e^{2kH} + 1\right)^{\frac{1}{2}} v_0$$

56 **i**

$$\frac{(n+9)^2(n+10)^2-n^2(n-1)^2}{4}$$
$$=\frac{(n^2+18n+81)(n^2+20n+100)-n^2(n^2-2n+1)}{4}$$
$$=\frac{n^4+20n^3+100n^2+18n^3+360n^2+1800n+81n^2+1620n+8100-n^4+2n^3-n^2}{4}$$
$$=\frac{40n^3+540n^2+3420n+8100}{4}$$
$$=10n^3+135n^2+855n+2025$$
$$=5(2n^3+27n^2+171n+405)$$
$$=5(2n^3+18n^2+90n+9n^2+81n+405)$$
$$=5(2n+9)(n^2+9n+45)$$

ii

Let $P(n)$ represent the proposition

$P(1)$ is true since $\text{LHS}=1^3=1; \text{RHS}=\frac{(1)^2}{4}(1+1)^2=1$

If $P(k)$ is true for some arbitrary $k\geq 1$ then $1^3+2^3+3^3+\ldots+k^3=\frac{k^2}{4}(k+1)^2$

RTP: $P(k+1)$: $\quad 1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{(k+1)^2}{4}(k+2)^2$

$$\text{LHS}=\frac{k^2}{4}(k+1)^2+(k+1)^3 \quad \text{from } P(k)$$
$$=\frac{(k+1)^2}{4}[k^2+4(k+1)]$$
$$=\frac{(k+1)^2}{4}(k+4k+4)$$
$$=\frac{(k+1)^2}{4}(k+2)^2$$
$$=\text{RHS}$$

$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\geq 1$ by induction.

iii

$$n^3+(n+1)^3+(n+2)^3+\ldots+(n+9)^3$$
$$=(1^3+2^3+3^3+\ldots+(n+9)^3)-(1^3+2^3+3^3+\ldots+(n-1)^3)$$
$$=\frac{(n+9)^2}{4}(n+10)^2-\frac{(n-1)^2}{4}n^2$$
$$=\frac{(n+9)^2(n+10)^2-n^2(n-1)^2}{4}$$
$$=5(2n+9)(n^2+9n+45) \quad \text{from (i)}$$

57 The question is most easily answered using the vector equation of a line.

i

$z=2i+1(1+3i-2i)=1+3i$

z is the point $(1,3)$

ii

When $k=0$ $z=z_1=2i$

For $0<k<1$ we have the interval between (but not including) z_1 and z_2, so between $(0,2)$ and $(1,3)$.

iii

The line through z_1 and z_2, $y=x+2$.

iv

$\alpha=2i, \beta=1+3i$ (the two can be swapped) and $\theta=\pi$.

58

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$$
$$=\int \frac{1}{x^3\left(\frac{1}{x^2}+1\right)^{\frac{3}{2}}}dx$$
$$=\int (x^{-2}+1)^{-\frac{3}{2}}(x^{-3})\,dx$$
$$=(x^{-2}+1)^{-\frac{1}{2}}+c$$
$$=\frac{1}{\sqrt{\frac{1}{x^2}+1}}+c$$
$$=\frac{x}{\sqrt{1+x^2}}+c$$

59 The point $(\sqrt{3},1)$ is on a circle of radius 2. The move clockwise swap sine and cosine, so

$x=2\sin(t+\alpha), y=2\cos(t+\alpha)$. When $t=0$ we want

$$x=2\sin\alpha=\sqrt{3} \;\rightarrow\; \alpha=\frac{\pi}{3}$$

$$\therefore x=2\sin\left(t+\frac{\pi}{3}\right), y=2\cos\left(t+\frac{\pi}{3}\right)$$

60 **i**

$\frac{dv}{dt} = -\lambda(c+v)$

$\frac{dt}{dv} = -\frac{1}{\lambda(c+v)}$

$T = -\frac{1}{\lambda}\int_u^0 \frac{dv}{c+v}$

$= -\frac{1}{\lambda}\Big[\ln(c+v)\Big]_u^0$

$= -\frac{1}{\lambda}\Big(\ln c - \ln(c+u)\Big)$

$= \frac{1}{\lambda}\ln\left(\frac{c+u}{c}\right)$

Let T_2 be the time until $v = \frac{u}{4}$

$T_2 = -\frac{1}{\lambda}\int_u^{\frac{u}{4}} \frac{dv}{c+v}$

$= \frac{1}{\lambda}\Big[\ln(c+v)\Big]_{\frac{u}{4}}^u$

$= \frac{1}{\lambda}\left(\ln(c+u) - \ln\left(c+\frac{u}{4}\right)\right)$

$= \frac{1}{\lambda}\ln\left(\frac{4c+4u}{4c+u}\right)$

Let $c = \frac{u}{8}$

$T_2 = \frac{1}{\lambda}\ln\left(\frac{\frac{u}{2}+4u}{\frac{u}{2}+u}\right)$

$= \frac{1}{\lambda}\ln 3$

$T = \frac{1}{\lambda}\ln\left(\frac{\frac{u}{8}+u}{\frac{u}{8}}\right)$

$= \frac{1}{\lambda}\ln 9$

$= \frac{2}{\lambda}\ln 3$

∴ if $c = \frac{1}{8}u$ the particle will reach a velocity of $\frac{1}{4}u$ in time $\frac{1}{2}T$.

ii

$t = -\frac{1}{\lambda}\int_u^v \frac{dv}{c+v}$

$= \frac{1}{\lambda}\Big[\ln(c+v)\Big]_u^v$

$\lambda t = \ln(c+v) - \ln(c+u)$

$\ln(c+v) = \ln(c+u) + \lambda t$

$c + v = (c+u)e^{\lambda t}$

$v = (c+u)e^{\lambda t} - c$

$= \left(\frac{u}{8}+u\right)e^{\lambda t} - \frac{u}{8}$

$8v = 9ue^{\lambda t} - u$

$8\frac{v}{u} = 9e^{\lambda t} - 1$

61 Suppose n is composite.

Let $n = pq$ where p, q are positive integers greater than 1

$2^n - 1$

$= 2^{pq} - 1$

$= (2^p)^q - 1^q$

$= \big((2^p) - 1\big)\big((2^p)^{q-1} + (2^p)^{q-2} + (2^p)^{q-3} + \ldots + 1\big)$

$= jk$ where j, k are integers greater than 1,

since p, q are integers greater than 1

∴ if n is composite then $2^n - 1$ is composite.

∴ if $2^n - 1$ is prime then n is prime, for $n > 1$ by contrapositive.

62 **i**

$\frac{1}{\omega^2} + \frac{1}{\omega} = \frac{\omega + \omega^2}{\omega^3}$

$= \omega + \omega^2$

$= 1 + \omega + \omega^2 - 1$

$= -1$

ii

$\frac{1+2\omega+3\omega^2}{2+3\omega+\omega^2} \times \frac{\omega}{\omega} + \frac{1+2\omega+3\omega^2}{3+\omega+2\omega^2} \times \frac{\omega^2}{\omega^2}$

$= \frac{1+2\omega+3\omega^2}{2\omega+3\omega^2+\omega^3} \times \omega + \frac{1+2\omega+3\omega^2}{3\omega^2+\omega^3+2\omega^4} \times \omega^2$

$= \frac{1+2\omega+3\omega^2}{2\omega+3\omega^2+1} \times \omega + \frac{1+2\omega+3\omega^2}{3\omega^2+1+2\omega} \times \omega^2$

$= \omega + \omega^2$

$= 1 + \omega + \omega^2 - 1$

$= -1$

63 **i**

$u = \frac{1}{2}x^{2n}$	$\frac{dv}{dx} = 2xe^{x^2}$
$\frac{du}{dx} = nx^{2n-1}$	$v = e^{x^2}$

$I_{2n+1} = \int_0^1 x^{2n+1}e^{x^2}\,dx$

$= \frac{1}{2}\Big[x^{2n}e^{x^2}\Big]_0^1 - n\int_0^1 x^{2n-1}e^{x^2}\,dx$

$= \frac{1}{2}(e-0) - nI_{2n-1}$

$= \frac{e}{2} - nI_{2n-1}$

ii

$I_5 = \frac{e}{2} - 2I_3$

$= \frac{e}{2} - 2\left(\frac{e}{2} - I_1\right)$

$= -\frac{e}{2} + 2\int_0^1 xe^{x^2}\,dx$

$= -\frac{e}{2} + \Big[e^{x^2}\Big]_0^1$

$= -\frac{e}{2} + e - 1$

$= \frac{e}{2} - 1$

64

$\left|\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}\right|^2$

$= \left(\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}\right)\cdot\left(\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}\right)$

$= \underset{\sim}{u}\cdot\underset{\sim}{u} + \underset{\sim}{u}\cdot\underset{\sim}{v} + \underset{\sim}{u}\cdot\underset{\sim}{w} + \underset{\sim}{u}\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\underset{\sim}{w} + \underset{\sim}{u}\cdot\underset{\sim}{w} + \underset{\sim}{v}\cdot\underset{\sim}{w} + \underset{\sim}{w}\cdot\underset{\sim}{w}$

$= \left|\underset{\sim}{u}\right|^2 + \left|\underset{\sim}{v}\right|^2 + \left|\underset{\sim}{w}\right|^2 + 2\left(\underset{\sim}{u}\cdot\underset{\sim}{v} + \underset{\sim}{u}\cdot\underset{\sim}{w} + \underset{\sim}{v}\cdot\underset{\sim}{w}\right)$

$= \left|\underset{\sim}{u}\right|^2 + \left|\underset{\sim}{v}\right|^2 + \left|\underset{\sim}{w}\right|^2 + 2\left(\underset{\sim}{u}\cdot\underset{\sim}{v} + \underset{\sim}{w}\cdot\left(\underset{\sim}{u} + \underset{\sim}{v}\right)\right)$

$= 2^2 + 3^2 + 1^2 + 2\big(2 + (-1)\big)$

$= 16$

$\therefore \left|\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}\right| = 4$

65

$$\ddot{x} = -g - kv$$
$$= -(g + kv)$$
$$\frac{dv}{dt} = -(g + kv)$$
$$\frac{dt}{dv} = -\frac{1}{g + kv}$$
$$T = -\int_v^0 \frac{1}{g + kv}\, dv$$
$$= \frac{1}{k}\Big[\ln(g + kv)\Big]_0^v$$
$$= \frac{1}{k}(\ln(g + kv) - \ln g)$$
$$= \frac{1}{k}\ln\frac{g + kv}{g}$$
$$e^{kT} = \frac{g + kv}{g}$$
$$ge^{kT} - g = kv$$
$$v = \frac{g\left(e^{kT} - 1\right)}{k} \qquad (*)$$
$$H = \frac{g}{k}\int_0^T \left(e^{kT} - 1\right) dT$$
$$= \frac{g}{k}\left[\frac{1}{k}e^{kT} - T\right]_0^T$$
$$= \frac{g}{k}\left(\left(\frac{1}{k}e^{kT} - T\right) - \left(\frac{1}{k} - 0\right)\right)$$
$$\therefore H = \frac{g}{k}\left(\frac{1}{k}e^{kT} - T - \frac{1}{k}\right)$$
$$kH = \frac{g}{k}e^{kT} - gT - \frac{g}{k}$$
$$\therefore kH + gT = \frac{g\left(e^{kT} - 1\right)}{k}$$
$$\therefore v = kH + gT \qquad \text{from } (*)$$

66 Let $P(n)$ represent the proposition

$P(1)$ is true since $\text{LHS} = 1; \text{RHS} = \frac{4(1)+3}{6}\sqrt{1} = \frac{7}{6}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$1 + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{k} \leq \frac{4k+3}{6}\sqrt{k}$$

RTP: $P(k+1)$

$$1 + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{k} + \sqrt{k+1} \leq \frac{4k+7}{6}\sqrt{k+1}$$

$$\text{LHS} \leq \frac{4k+3}{6}\sqrt{k} + \sqrt{k+1} \qquad \text{from } P(k)$$
$$= \frac{(4k+3)\sqrt{k} + 6\sqrt{k+1}}{6}$$
$$= \frac{\sqrt{(4k+3)^2 \times k} + 6\sqrt{k+1}}{6}$$
$$= \frac{\sqrt{16k^3 + 24k^2 + 9k} + 6\sqrt{k+1}}{6}$$
$$\leq \frac{\sqrt{16k^3 + 24k^2 + 9k + 1} + 6\sqrt{k+1}}{6}$$
$$= \frac{\sqrt{(k+1)(16k^2 + 8k + 1)} + 6\sqrt{k+1}}{6}$$
$$= \frac{\sqrt{(k+1)(4k+1)^2} + 6\sqrt{k+1}}{6}$$
$$= \frac{(4k+1)\sqrt{(k+1)} + 6\sqrt{k+1}}{6}$$
$$= \frac{4k+7}{6}\sqrt{k+1}$$
$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore$ $P(n)$ is true for $n \geq 1$ by induction.

67 **i**

Since the cubic equation is not $x^3 = 0$ the roots α, β, γ are non-zero.
The coefficients are real, so any non-real roots must occur in conjugate pairs.
In each equation the sum of squares is zero, and since the roots are non-zero this means that either: α, γ are real and β non-real, or β is real and α, γ are non-real.
Since by the conjugate root theorem any non-real roots must occur in conjugate pairs, $\therefore$ β is real and α and γ are not real

ii

$\alpha^2 = \gamma^2 = -\beta^2 = (\beta i)^2$ where β is purely real

$\alpha = \gamma = \pm\beta i$

$\therefore$ α, γ are purely imaginary.

iii

$$\alpha\beta\gamma = -8$$
$$-\beta^2 \times \beta = -8$$
$$\beta^3 = 8$$
$$\beta = 2$$
$$\therefore \alpha = 2i, \gamma = -2i$$
$$\alpha + \beta + \gamma = -A$$
$$A = -(2i + 2 - 2i)$$
$$= -2$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = B$$
$$B = (2i)(2) + (2i)(-2i) + (2)(-2i)$$
$$= 4$$

68

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\, dx = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\, dx$$
$$= \int_0^\pi \frac{(\pi - x)\sin x}{1 + \cos^2 x}\, dx$$
$$= \pi\int_0^\pi \frac{\sin x}{1 + \cos^2 x}\, dx$$
$$2\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\, dx = \pi\int_0^\pi \frac{\sin x}{1 + \cos^2 x}\, dx$$
$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\, dx = -\frac{\pi}{2}\Big[\tan^{-1}(\cos x)\Big]_0^\pi$$
$$= -\frac{\pi}{2}(\tan^{-1}(-1) - \tan^{-1}(1))$$
$$= -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$
$$= \frac{\pi^2}{4}$$

69 Let $\underset{\sim}{b} = \lambda\underset{\sim}{i} + \lambda\underset{\sim}{j} + \lambda\underset{\sim}{k}$

$$\underset{\sim}{c} = \underset{\sim}{a} - \underset{\sim}{b}$$
$$= (2 - \lambda)\underset{\sim}{i} + (3 - \lambda)\underset{\sim}{j} + (1 - \lambda)\underset{\sim}{k}$$
$$\underset{\sim}{b} \cdot \underset{\sim}{c} = 0$$
$$\lambda(2 - \lambda) + \lambda(3 - \lambda) + \lambda(1 - \lambda) = 0$$
$$2\lambda - \lambda^2 + 3\lambda - \lambda^2 + \lambda - \lambda^2 = 0$$
$$-3\lambda^2 + 6\lambda = 0$$
$$-3\lambda(\lambda - 2) = 0$$
$$\lambda = 2 \ \ (\lambda \neq 0)$$
$$\therefore \underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}, \underset{\sim}{c} = \underset{\sim}{j} - \underset{\sim}{k}$$

70 i

$$\ddot{x} = -g - \frac{gv^2}{k^2}$$
$$= -\left(g + \frac{gv^2}{k^2}\right)$$
$$\therefore \ddot{x} = -\frac{g}{k^2}(k^2 + v^2)$$

ii

$$v\frac{dv}{dx} = -\frac{g}{k^2}(k^2 + v^2)$$
$$\frac{dv}{dx} = -\frac{g(k^2 + v^2)}{vk^2}$$
$$\frac{dx}{dv} = -\frac{k^2}{g} \times \frac{v}{k^2 + v^2}$$
$$x_{\text{max}} = -\frac{k^2}{g}\int_u^0 \frac{v}{k^2 + v^2}\,dv$$
$$= \frac{k^2}{2g}\Big[\ln(k^2 + v^2)\Big]_0^u$$
$$= \frac{k^2}{2g}\left(\ln(k^2 + u^2) - \ln(k^2)\right)$$
$$= \frac{k^2}{2g}\ln\left(\frac{k^2 + u^2}{k^2}\right)$$
$$= \frac{k^2}{2g}\ln\left(1 + \frac{u^2}{k^2}\right)$$

71 i

$$\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$$
$$a - 2\sqrt{ab} + b \geq 0$$
$$a + b \geq 2\sqrt{ab}$$
$$\frac{a + b}{2} \geq \sqrt{ab}$$

ii

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

$\therefore a_1 a_2 a_3 \ldots a_n$

Let $a_1 = 1, a_2 = 2, a_3 = 3, \ldots, a_n = n$

$a_1 a_2 a_3 \ldots a_n = 1 \times 2 \times 3 \times \ldots \times n$

$= n!$ (2)

$a_1 + a_2 + a_3 + \ldots + a_n = 1 + 2 + 3 + \ldots + n$

$= \frac{n}{2}(1 + n)$ (3)

sub (2), (3) in (1)

$$n! \leq \left(\frac{\frac{n}{2}(1 + n)}{n}\right)^n$$
$$n! \leq \left(\frac{n + 1}{2}\right)^n$$

72 i

$a\bar{z}^3 + b\bar{z}^2 + c\bar{z} + d$

$= a\overline{z^3} + b\overline{z^2} + c\bar{z} + d$

$= \overline{az^3 + bz^2 + cz + d}$

$= \bar{0}$

$= 0$

$\therefore \bar{z} = x - iy$ is also a root.

ii

$1 - 2i$ is also a root.

$$\alpha + \beta + \gamma = -\frac{b}{a}$$
$$1 + 2i + 1 - 2i + \gamma = 6$$
$$\gamma = 4$$

The roots of the equation are $1 \pm 2i$ and $4.$

73

$$\int \frac{\cos x\,(2a\sin x + b)}{(\sin x - \alpha)(\sin x - \beta)}\,dx$$
$$= \int \frac{\cos x\,(2a\sin x + b)}{\sin^2 x - (\alpha + \beta)\sin x + \alpha\beta}\,dx$$
$$= \int \frac{\cos x\,(2a\sin x + b)}{\sin^2 x + \frac{b}{a}\sin x + \frac{c}{a}}\,dx$$
$$= \int \frac{a\cos x\,(2a\sin x + b)}{a\sin^2 x + b\sin x + c}\,dx$$
$$= a\int \frac{2a\sin x\cos x + b\cos x}{a\sin^2 x + b\sin x + c}\,dx$$
$$= a\ln|a\sin^2 x + b\sin x + c| + c_1$$

74

$$\sqrt{a}\sin t = -(x - 1)$$
$$\sin t = -\frac{x - 1}{\sqrt{a}}$$
$$\frac{1}{b}\cos t = -(y - 1)$$
$$\cos t = -b(y - 1)$$
$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{x - 1}{\sqrt{a}}\right)^2 + \left(b(y - 1)\right)^2 = 1$$
$$(x - 1)^2 + ab^2(y - 1)^2 = 1$$
$$(x - h)^2 + (y - k)^2 = r^2$$
$$\therefore \text{if } ab^2 = 1$$

if the path is a circle centred at (1,1) with radius 1

75 i

Taking clockwise motion as positive:

$$(m + 3m)\ddot{x} = (3mg - kv - T) + (T - kv - mg)$$
$$4m\ddot{x} = 2mg - 2kv$$
$$\ddot{x} = \frac{mg - kv}{2m}$$

ii

Let $\ddot{x} = 0, v = V$

$$\therefore 0 = \frac{mg - kV}{2m}$$
$$kV = mg$$
$$V = \frac{mg}{k}$$

iii

$$\frac{dv}{dt} = \frac{mg - kv}{2m}$$
$$\frac{dt}{dv} = \frac{2m}{mg - kv}$$
$$t = \int_0^v \frac{2m}{mg - kv}\,dv$$
$$= \frac{2m}{k}\int_v^0 \frac{-k}{mg - kv}\,dv$$
$$= \frac{2m}{k}\Big[\ln(mg - kv)\Big]_v^0$$
$$= \frac{2m}{k}\left(\ln mg - \ln(mg - kv)\right)$$
$$= \frac{2m}{k}\ln\left(\frac{mg}{mg - kv}\right)$$

iv

$$t = \frac{2m}{k}\ln\left(\frac{mg}{mg - k\left(\frac{mg}{2k}\right)}\right)$$
$$= \frac{2m}{k}\ln 2$$
$$= \frac{m}{k}\ln 4$$
$$= \frac{V}{g}\ln 4$$

76 **i**

Let the consecutive integers be $n, n+1, n+2$ for $n \geq 1$

Let $P(n) = n(n+1)(n+2) = n(n^2+3n+2) = n^3 + 3n^2 + 2n$

$P(1)$ is true since $1^3 + 3(1)^2 + 2(1) = 6$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $k^3 + 3k^2 + 2k = 6m$ for integral m

RTP: $P(k+1)$ $\quad (k+1)^3 + 3(k+1)^2 + 2(k+1) = 6p$ for integral p

LHS $= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2$

$= k^3 + 3k^2 + 2k + 3k^2 + 9k + 6$

$= 6m + 3(k^2 + 3k + 2)$ from $P(k)$

$= 6m + 3(k+1)(k+2)$

$= 6m + 3(2q)$ for integral q since $k+1, k+2$ are consecutive

$= 6(m+q)$

$= 6p$ for integral p since m, q integral

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

Hence the product of three consecutive positive integers is divisible by 6, by induction.

ii

$(k-1)^3 + k^3 + (k+1)^3$

$= k^3 - 3k^2 + 3k - 1 + k^3 + k^3 + 3k^2 + 3k + 1$

$= 3k^3 + 6k$

$= 3k(k^2 + 2)$

$= 3k(k^2 + 3k + 2 - 3k)$

$= 3k\big((k+1)(k+2) - 3k\big)$

$= 3k(k+1)(k+2) - 9k^2$

$= 3(6m) - 9k^2$ for integral m since $k, k+1, k+2$ are consecutive

$= 9(2m - k^2)$

77 Let $w = a + ib, z = x + iy$

$w = \dfrac{z+3}{z} \times \dfrac{\bar{z}}{\bar{z}}$

$= \dfrac{z\bar{z} + 3\bar{z}}{z\bar{z}}$

$= \dfrac{|z|^2 + 3\bar{z}}{|z|^2}$

$= \dfrac{4 + 3\bar{z}}{4}$

$= 1 + \dfrac{3}{4}\bar{z}$

Which is a circle centred at $(1,0)$ with radius $\frac{3}{4} \times 2 = \frac{3}{2}$

78 Consider the circle $x^2 + y^2 = r^2$. In the first quadrant we have $y = (r^2 - x^2)^{\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2}(r^2 - x^2)^{-\frac{1}{2}}(-2x) = -\dfrac{x}{\sqrt{r^2 - x^2}}$

$\therefore \ell = \displaystyle\int_0^r \sqrt{1 + \left(\frac{x^2}{r^2 - x^2}\right)}\,dx$

$= \displaystyle\int_0^r \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}}\,dx$

$= \displaystyle\int_0^r \frac{r}{\sqrt{r^2 - x^2}}\,dx$

$= r\left[\sin^{-1}\left(\dfrac{x}{r}\right)\right]_0^r$

$= r\left(\dfrac{\pi}{2} - 0\right)$

$= \dfrac{\pi r}{2}$

$\therefore C = 4\ell = 4\left(\dfrac{\pi r}{2}\right) = 2\pi r$

79 Let the medians of OA, AB and OC be L, M and N respectively.

$L = (0,3,6), M = (9,21,6)$ and $N = (9,18,0)$

$\overrightarrow{AN} = (9,12,-12), \overrightarrow{OM} = (9,21,6), \overrightarrow{BL} = (-18,-33,6)$

Let the intersection of OM and BL be $X(a,b,c)$.

$$\overrightarrow{OX} = \mu\overrightarrow{OM}$$
$$(a,b,c) = \mu(9,21,6)$$
$$\overrightarrow{BX} = \lambda\overrightarrow{BL}$$
$$(a-18, b-36, c) = \lambda(-18,-33,6)$$
$$a = 9\mu = 18 - 18\lambda \qquad (1)$$
$$b = 21\mu = 36 - 33\lambda \qquad (2)$$
$$c = 6\mu = 6\lambda \;\rightarrow\; \mu = \lambda$$
$$\text{sub in (2)}:\; 21\mu = 36 - 33\mu \rightarrow \mu = \frac{2}{3}$$
$$\therefore X(6,14,4)$$
$$\overrightarrow{AX} = (6,8,-8)$$
$$= \frac{2}{3}(9,12,-12)$$
$$= \frac{2}{3}\overrightarrow{AN}$$

$\therefore A, X$ and N are collinear.

Since X lies on OM, AN and BL the three medians are concurrent.

80 **i**

$t = \dfrac{x}{v\cos\theta}$

$y = -\dfrac{1}{2}g\left(\dfrac{x}{v\cos\theta}\right)^2 + v\left(\dfrac{x}{v\cos\theta}\right)\sin\theta$

$= -\dfrac{gx^2}{2v^2}\sec^2\theta + x\tan\theta$

Let $x = d, y = h$

$$h = -\frac{gd^2}{2v^2}\sec^2\theta + d\tan\theta$$

$$d\tan\theta - h = \frac{gd^2\sec^2\theta}{2v^2}$$

$$v^2 = \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$$

ii

maximum height occurs at the axis of symmetry

$$t = -\frac{v\sin\theta}{2\left(-\frac{1}{2}g\right)} = \frac{v\sin\theta}{g}$$

$y_{\text{max}} = -\dfrac{1}{2}g\left(\dfrac{v\sin\theta}{g}\right)^2 + v\left(\dfrac{v\sin\theta}{g}\right)\sin\theta$

$$= \frac{v^2\sin^2\theta}{2g} + \frac{v^2\sin^2\theta}{g}$$
$$= \frac{v^2\sin^2\theta}{2g}$$
$$= \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)} \times \frac{\sin^2\theta}{2g}$$
$$= \frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$$

iii

$$h = \frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$$
$$4dh\tan\theta - 4h^2 = d^2\tan^2\theta$$
$$d^2\tan^2\theta - 4dh\tan\theta - 4h^2 = 0$$
$$(d\tan\theta - 2h)^2 = 0$$
$$\tan\theta = \frac{2h}{d}$$

81 Let $P(n)$ represent the proposition.

$P(0)$ is true since $\text{LHS} = \cos(2^0\alpha) = \cos\alpha$; $\text{RHS} = \frac{\sin(2\alpha)}{2\sin\alpha} = \frac{2\sin\alpha\cos\alpha}{2\sin\alpha} = \cos\alpha$

If $P(k)$ is true for some arbitrary $k \geq 0$ then

$$\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^k\alpha = \frac{\sin(2^{k+1}\alpha)}{2^{k+1}\sin\alpha}$$

RTP $P(k+1)$

$$\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^k\alpha\cos 2^{k+1}\alpha = \frac{\sin(2^{k+2}\alpha)}{2^{k+2}\sin\alpha}$$

$$\begin{aligned}\text{LHS} &= \cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^k\alpha\cos 2^{k+1}\alpha\\ &= \frac{\sin(2^{k+1}\alpha)}{2^{k+1}\sin\alpha}\times\cos 2^{k+1}\alpha \qquad \text{from } P(k)\\ &= \frac{2\sin(2^{k+1}\alpha)\cos 2^{k+1}\alpha}{2^{k+2}\sin\alpha}\\ &= \frac{\sin 2^{k+2}\alpha}{2^{k+2}\sin\alpha}\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 0$ by induction

82

$$\begin{aligned}(z_1+z_2+z_3)^2 &= z_1^2+z_2^2+z_3^2+2(z_1z_2+z_1z_3+z_2z_3)\\ z_1^2+z_2^2+z_3^2 &= (z_1+z_2+z_3)^2-2(z_1z_2+z_1z_3\\ &= 0-2z_1z_2z_3\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)\\ &= -2z_1z_2z_3(\overline{z_1}+\overline{z_2}+\overline{z_3})\\ &= -2z_1z_2z_3\overline{(z_1+z_2+z_3)}\\ &= -2z_1z_2z_3(0)\\ &= 0\end{aligned}$$

83 Since $\tan^{-1}x$ is odd and $1+\sin^2 x$ is even then $\frac{\tan^{-1}x}{1+\sin^2 x}$ is odd.

$\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd

$\therefore \int_{-1}^{1}\frac{\tan^{-1}x}{1+\sin^2 x}dx = 0$

84

$$\begin{aligned}2\cos\theta-\cos 2\theta &= 1\\ 2\cos\theta-(2\cos^2\theta-1) &= 1\\ 2\cos^2\theta-2\cos\theta &= 1\\ 2\cos\theta(\cos\theta-1) &= 0\\ \cos\theta = 0 \quad \cos\theta &= 1\\ \theta = \frac{\pi}{2},\frac{3\pi}{2} \quad \theta &= 0,2\pi\\ \text{when } \theta = \frac{\pi}{2} \quad y &= 2\sin\frac{\pi}{2}-\sin\left(2\times\frac{\pi}{2}\right)\end{aligned}$$

This is in the first quadrant, so $P(1,2)$.

85 **i**

$$\begin{aligned}\ddot{x} &= g-kv\\ 0 &= g-kV_T\\ V_T &= \frac{g}{k}\end{aligned}$$

85 … **ii**

$$\begin{aligned}\frac{dv}{dt} &= g-kv\\ \frac{dt}{dv} &= \frac{1}{g-kv}\\ t &= \int_0^{\frac{g}{2k}}\frac{1}{g-kv}dv\\ &= -\frac{1}{k}\left[\ln(g-kv)\right]_0^{\frac{g}{2k}}\\ &= \frac{1}{k}\left(\ln g-\ln\left(g-k\left(\frac{g}{2k}\right)\right)\right)\\ &= \frac{1}{k}\ln\frac{g}{\frac{g}{2}}\\ &= \frac{1}{k}\ln 2\end{aligned}$$

iii

$$\begin{aligned}v\frac{dv}{dx} &= g-kv\\ \frac{dv}{dx} &= \frac{g-kv}{v}\\ \frac{dx}{dv} &= \frac{v}{g-kv}\\ x &= \int_0^{\frac{g}{2k}}\frac{v}{g-kv}dv\\ &= \int_0^{\frac{g}{2k}}\frac{-\frac{1}{k}(g-kv)+\frac{g}{k}}{g-kv}dv\\ &= \int_0^{\frac{g}{2k}}\left(-\frac{1}{k}-\frac{g}{k^2}\times\frac{-k}{g-kv}\right)dv\\ &= \left[-\frac{v}{k}-\frac{g}{k^2}\ln(g-kv)\right]_0^{\frac{g}{2k}}\\ &= \left(\left(-\frac{g}{2k^2}-\frac{g}{k^2}\ln\left(g-\frac{g}{2}\right)\right)-\left(0-\frac{g}{k^2}\ln g\right)\right)\\ &= -\frac{g}{2k^2}+\frac{g}{k^2}\left(\ln g-\ln\frac{g}{2}\right)\\ &= \frac{g}{k^2}\left(\ln 2-\frac{1}{2}\right)\end{aligned}$$

86 Suppose $\frac{a+c}{\sqrt{a^2+c^2}}-\frac{b+c}{\sqrt{b^2+c^2}} \leq 0$

$$\begin{aligned}\therefore \frac{a+c}{\sqrt{a^2+c^2}} &\leq \frac{b+c}{\sqrt{b^2+c^2}}\\ (a+c)\sqrt{b^2+c^2} &\leq (b+c)\sqrt{a^2+c^2}\\ (a+c)^2(b^2+c^2) &\leq (b+c)^2(a^2\\ (a^2+2ac+c^2)(b^2+c^2) &\leq (b^2+2bc+c^2)(a^2+c^2)\\ 2ab^2c+2ac^3 &\leq 2a^2bc+2bc^3\\ ab^2+ac^2 &\leq a^2b+bc^2\\ ab^2-a^2b+ac^2-bc^2 &\leq 0\\ ab(b-a)+c^2(a-b) &\leq 0\\ c^2(a-b)-ab(a-b) &\leq 0\\ (c^2-ab)(a-b) &\leq 0 \quad \#\end{aligned}$$

This is a contradiction since $c^2 > ab$ and $a > b$, so $(c^2-ab)(a-b) > 0$

Hence $\frac{a+c}{\sqrt{a^2+c^2}}-\frac{b+c}{\sqrt{b^2+c^2}} > 0$

87

$$\begin{aligned}\frac{2\bar{z}(1-2i)}{5}+\frac{iz}{1+2i} &= 2-3i\\ \frac{2\bar{z}(1-2i)}{5}+\frac{iz}{1+2i}\times\frac{1-2i}{1-2i} &= 2-3i\\ \frac{2\bar{z}(1-2i)}{5}+\frac{iz+2z}{1+4} &= 2-3i\\ 2\bar{z}-4i\bar{z}+iz+2z &= 10\end{aligned}$$

$2(x-iy)-4i(x-iy)+i(x+iy)+2(x+iy)$
$= 10-15i$

$$\begin{aligned}2x-2yi-4xi-4y+xi-y+2x+2yi &= 10\\ 4x-5y-3xi &= 10\\ \therefore -3x &= -15\\ 4x-5y &= 10 \rightarrow y\end{aligned}$$

$\therefore z = 5+2i$

88

$$\int_0^1 (x^2-a)^2\,dx$$

$$= \int_0^1 (x^4-2ax^2+a^2)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{2a}{3}x^3 + a^2x\right]_0^1$$

$$= \left(\frac{1}{5} - \frac{2a}{3} + a^2\right) - (0)$$

$$= a^2 - \frac{2a}{3} + \frac{1}{5}$$

Minimum when $a = -\frac{-\frac{2}{3}}{2(1)} = \frac{1}{3}$

$$\min = \left(\frac{1}{3}\right)^2 - \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{5} = \frac{4}{45}$$

89

$$\cos^3 t = \frac{x}{2} \rightarrow \cos^2 t = \left(\frac{x}{2}\right)^{\frac{2}{3}} = \sqrt[3]{\frac{x^2}{4}}$$

$$\sin^3 t = \frac{y}{2} \rightarrow \sin^2 t = \left(\frac{y}{2}\right)^{\frac{2}{3}} = \sqrt[3]{\frac{y^2}{4}}$$

$\cos^2 t + \sin^2 t = 1$

$$\sqrt[3]{\frac{x^2}{4}} + \sqrt[3]{\frac{y^2}{4}} = 1$$

90

i

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha$

$t = \dfrac{x}{V\cos\alpha}$

$$\ddot{y} = -g$$

$$\dot{y} - V\sin\alpha = -g\int_0^t dt$$

$$\dot{y} = V\sin\alpha - gt$$

$$y = \int_0^t (V\sin\alpha - gt)\,dt$$

$$= Vt\sin\alpha - \frac{gt^2}{2}$$

$$= V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$$

$$= x\tan\alpha - \frac{gx^2}{2V^2}(\tan^2\alpha + 1)$$

$$= -\frac{gx^2}{2V^2}\tan^2\alpha + x\tan\alpha - \frac{gx^2}{2V^2}$$

Let $x = m, y = n$

$$n = -\frac{gm^2}{2V^2}\tan^2\alpha$$

$$-2nV^2 = gm^2\tan^2\alpha$$

$$gm^2\tan^2\alpha - 2mV^2\tan\alpha + gm^2 + 2nV^2 = 0$$

ii

There are two solutions if $\Delta > 0$

$\therefore (-2mV^2)^2 - 4(gm^2)(gm^2 + 2nV^2) > 0$

$$4m^2V^4 - 4g^2m^4 - 8gm^2nV^2 > 0$$

$$V^4 - 2gV^2n - g^2m^2 > 0$$

$$V^4 - 2V^2gn + g^2n^2 > g^2m^2 + g^2n^2$$

$$(V^2 - gn)^2 > g^2(m^2 + n^2)$$

91

i

Let $P(1)$ represent the proposition.

$P(1)$ is true since $\frac{a_1-\sqrt{2}}{a_1+\sqrt{2}} = \frac{1-\sqrt{2}}{1+\sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^1 = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{1-1}}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$\frac{a_k - \sqrt{2}}{a_k + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}$$

RTP $P(k+1)$

$$\frac{a_{k+1} - \sqrt{2}}{a_{k+1} + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^k}$$

$$\text{LHS} = \frac{a_{k+1} - \sqrt{2}}{a_{k+1} + \sqrt{2}}$$

$$= \frac{\frac{1}{2}\left(a_k + \frac{2}{a_k}\right) - \sqrt{2}}{\frac{1}{2}\left(a_k + \frac{2}{a_k}\right) + \sqrt{2}}$$

$$= \frac{a_k + \frac{2}{a_k} - 2\sqrt{2}}{a_k + \frac{2}{a_k} + 2\sqrt{2}}$$

$$= \frac{a_k^2 + 2 - 2\sqrt{2}a_k}{a_k^2 + 2 + 2\sqrt{2}a_k}$$

$$= \frac{a_k^2 - 2\sqrt{2}a_k + 2}{a_k^2 + 2\sqrt{2}a_k + 2}$$

$$= \frac{\left(a_k - \sqrt{2}\right)^2}{\left(a_k + \sqrt{2}\right)^2}$$

$$= \left(\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}\right)^2$$

$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^k}$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

ii

as $n \to \infty$,

$\left(\dfrac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}} \to 0$

$\dfrac{a_n - \sqrt{2}}{a_n + \sqrt{2}} \to 0$

$a_n - \sqrt{2} \to 0$

$a_n \to \sqrt{2}$

92

$2z^3 = z + \bar{z}$

$2z^3 = 2\text{Re}(z)$

$z^3 = \text{Re}(z)$

$z = 0, \pm 1$ are all solutions since their cube equals their real component.

93

$$\int_0^a [f(a-x)+f(a+x)]dx$$

$$= \int_0^a f(a-x)\,dx + \int_0^a f(x+a)\,dx$$

$$= \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

since $\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$ and

shifting $f(x+a)$ a units to the right to become $f(x)$

$$= \int_0^{2a} f(x)\,dx$$

ANSWER (B)

94

$\arccos(t)$ is defined for $-1 \le t \le 1$
$1+\sin(\pi t)$ is defined for all t
$\sqrt{-t}$ is defined for $t \le 0$
The curve is defined for $-1 \le t \le 0$.

$$\underset{\sim}{r}(-1) = \begin{pmatrix} \arccos(-1) \\ 1+\sin(-\pi) \\ \sqrt{-(-1)} \end{pmatrix} = \begin{pmatrix} \pi \\ 1 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r}(0) = \begin{pmatrix} \arccos(0) \\ 1+\sin(0) \\ \sqrt{-(0)} \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} \\ 1 \\ 0 \end{pmatrix}$$

95

i

$$\frac{dV}{dt} = -RV^2$$
$$\frac{dt}{dV} = -\frac{1}{RV^2}$$
$$t = -\frac{1}{R}\int_u^V \frac{1}{V^2}dV$$
$$= \frac{1}{R}\left[\frac{1}{V}\right]_u^V$$
$$Rt = \frac{1}{V} - \frac{1}{u}$$
$$\frac{Rut+1}{u} = \frac{1}{V}$$
$$V = \frac{u}{Rut+1}$$

ii

$$\frac{dv}{dt} = -(g+Rv^2)$$
$$\frac{dt}{dv} = -\frac{1}{g+Rv^2}$$
$$T = -\int_u^0 \frac{dv}{g+Rv^2}dv$$
$$= \frac{1}{\sqrt{R}}\int_0^u \frac{\sqrt{R}}{(\sqrt{g})^2 + (\sqrt{R}v)^2}dv$$
$$= \frac{1}{\sqrt{R}}\left[\frac{1}{\sqrt{g}}\tan^{-1}\left(\frac{\sqrt{R}v}{\sqrt{g}}\right)\right]_0^u$$
$$= \frac{1}{\sqrt{Rg}}\tan^{-1}\left(\sqrt{\frac{R}{g}}u\right)$$
$$\frac{1}{V} = \frac{Ru\frac{1}{\sqrt{Rg}}\tan^{-1}\left(\sqrt{\frac{R}{g}}u\right)+1}{u}$$
$$= \sqrt{\frac{R}{g}}\tan^{-1}\left(\sqrt{\frac{R}{g}}u\right) + \frac{1}{u}$$
$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + \frac{1}{u} \quad \text{since } Ra^2 = g \ \rightarrow \ \frac{R}{g} = \frac{1}{a^2}$$

96

i

$(\sqrt{x}-\sqrt{y})^2 \ge 0$
$x - 2\sqrt{xy} + y \ge 0$
$x + y \ge 2\sqrt{xy}$ (1)

ii

Similarly
$x + z \ge 2\sqrt{xz}$ (2)
$y + z \ge 2\sqrt{yz}$ (3)
$(1)\times(2)\times(3)$:
$(x+y)(x+z)(y+z) \ge 8\sqrt{x^2y^2z^2}$
$(x+y)(x+z)(y+z) \ge 8xyz$

iii

Let $x = S-a, y = S-b, z = S-c$ in (ii)

$$\therefore (2S-a-b)(2S-a-c)(2S-b-c) \ge 8(S-a)(S-b)(S-c)$$
$$(c)(b)(a) \ge 8(S-a)(S-b)(S-c)$$
$$(S-a)(S-b)(S-c) \le \frac{abc}{8}$$
$$\therefore S(S-a)(S-b)(S-c) \le \frac{abc\times S}{8}$$
$$\left(\sqrt{S(S-a)(S-b)(S-c)}\right)^2 \le \frac{abc\times\frac{1}{2}(a+b+c)}{8}$$
$$\therefore A^2 \le \frac{(a+b+c)abc}{16}$$

97

i

$$|\alpha|^2 + |\beta|^2 - |\alpha-\beta|^2 = \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)\left(\overline{\alpha-\beta}\right)$$
$$= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)(\bar{\alpha}-\bar{\beta})$$
$$= \alpha\bar{\alpha} + \beta\bar{\beta} - \alpha\bar{\alpha} + \alpha\bar{\beta} + \bar{\alpha}\beta - \beta\bar{\beta}$$
$$= \bar{\alpha}\beta + \overline{\bar{\alpha}\beta}$$
$$= 2\text{Re}(\bar{\alpha}\beta)$$

ii

Using the cosine rule:

$$|\alpha-\beta|^2 = |\alpha|^2 + |\beta|^2 - 2|\alpha||\beta|\cos\theta$$
$$2|\alpha||\beta|\cos\theta = |\alpha|^2 + |\beta|^2 - |\alpha-\beta|^2$$
$$= 2\text{Re}(\bar{\alpha}\beta) \quad \text{from (i)}$$
$$\therefore |\alpha||\beta|\cos\theta = \text{Re}\left(\alpha\bar{\beta}\right)$$

98

i

Let $u = \frac{1}{x} \ \rightarrow \ \frac{du}{dx} = -\frac{1}{x^2} = -u^2 \quad dx = -\frac{du}{u^2}$

$$\int_{\frac{1}{a}}^a \frac{f(x)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}dx$$
$$= \int_a^{\frac{1}{a}} \frac{f\left(\frac{1}{u}\right)}{\left(\frac{1}{u}\right)\left(f\left(\frac{1}{u}\right)+f(u)\right)}\left(-\frac{du}{u^2}\right)$$
$$= \int_{\frac{1}{a}}^a \frac{f\left(\frac{1}{u}\right)}{u\left(f(u)+f\left(\frac{1}{u}\right)\right)}du$$
$$= \int_{\frac{1}{a}}^a \frac{f\left(\frac{1}{x}\right)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}dx$$

98 ... **ii**

$$I=\int_{\frac{1}{2}}^{2}\frac{\sin x}{x\left(\sin x+\sin\frac{1}{x}\right)}dx$$
$$=\int_{\frac{1}{2}}^{2}\frac{\sin\frac{1}{x}}{x\left(\sin x+\sin\frac{1}{x}\right)}dx$$
$$\therefore 2I=\int_{\frac{1}{2}}^{2}\frac{\sin x+\sin\frac{1}{x}}{x\left(\sin x+\sin\frac{1}{x}\right)}dx$$
$$I=\frac{1}{2}\int_{\frac{1}{2}}^{2}\frac{dx}{x}$$
$$=\frac{1}{2}\Big[\ln x\Big]_{\frac{1}{2}}^{2}$$
$$=\frac{1}{2}\left(\ln 2-\ln\left(\frac{1}{2}\right)\right)$$
$$=\frac{1}{2}(\ln 2-(-\ln 2))$$
$$=\ln 2$$

99

$$\left(\underset{\sim}{a}+2\underset{\sim}{b}\right)\cdot\left(5\underset{\sim}{a}-4\underset{\sim}{b}\right)=0$$
$$5\,\underset{\sim}{a}\cdot\underset{\sim}{a}+6\,\underset{\sim}{a}\cdot\underset{\sim}{b}-8\,\underset{\sim}{b}\cdot\underset{\sim}{b}=0$$
$$5\left|\underset{\sim}{a}\right|^2+6\,\underset{\sim}{a}\cdot\underset{\sim}{b}-8\left|\underset{\sim}{b}\right|^2=0$$
$$5+6\,\underset{\sim}{a}\cdot\underset{\sim}{b}-8=0$$
$$6\,\underset{\sim}{a}\cdot\underset{\sim}{b}=3$$
$$\underset{\sim}{a}\cdot\underset{\sim}{b}=\frac{1}{2}$$
$$\cos\theta=\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|}$$
$$=\frac{\frac{1}{2}}{1\times 1}$$
$$\theta=60°$$

100 **i**

$$\dot{x}=V\cos 45°$$
$$=\frac{V}{\sqrt{2}}$$
$$x=\frac{Vt}{\sqrt{2}}$$
$$t=\frac{\sqrt{2}x}{V}$$
$$\ddot{y}=-g$$
$$\dot{y}-V\sin 45°=-g\int_0^t dt$$
$$\dot{y}=\frac{V}{\sqrt{2}}-gt$$
$$y=\int_0^t\left(\frac{V}{\sqrt{2}}-gt\right)dt$$
$$=\frac{Vt}{\sqrt{2}}-\frac{gt^2}{2}$$

ii

$$y=\frac{V}{\sqrt{2}}\left(\frac{\sqrt{2}x}{V}\right)-\frac{g}{2}\left(\frac{\sqrt{2}x}{V}\right)^2$$
$$=x-\frac{gx^2}{V^2}$$

iii

Let $y=0$

$$\therefore x-\frac{gx^2}{V^2}=0$$
$$x\left(1-\frac{gx}{V^2}\right)=0$$
$$x=0,\frac{V^2}{g}$$

$\therefore$ the range is $\frac{V^2}{g}$.

100 ... **iv α**

The two posts most be equidistant from the points of projection and impact, so the range is $2b$ plus the distance between the posts.

$$\therefore\frac{V^2}{g}=2b+12a^2$$

iv β

Let $x=b, y=8a^2$ in $y=x-\frac{gx^2}{V^2}$

$$8a^2=b-\frac{gb^2}{V^2}$$

v

$$\frac{V^2}{g}=2b+12a^2$$
$$2b=\frac{V^2}{g}-12a^2$$
$$b=\frac{V^2}{2g}-6a^2$$
$$\therefore 8a^2=\frac{V^2}{2g}-6a^2-\frac{g}{V^2}\left(\frac{V^2}{2g}-6a^2\right)^2$$
$$14a^2=\frac{V^2}{2g}-\frac{g}{V^2}\left(\frac{V^4}{4g^2}-\frac{6a^2V^2}{g}+36a^4\right)$$
$$56a^2V^2g^2=2gV^4-g(V^4-24a^2gV^2+144a^4g^2)$$
$$56a^2V^2g^2=2gV^4-gV^4+24a^2g^2V^2-144a^4g^3$$
$$gV^4-32a^2g^2V^2-144a^4g^3=0$$
$$V^4-32a^2gV^2-144a^4g^2=0$$
$$V^2=\frac{32a^2g\pm\sqrt{(-32a^2g)^2-4(1)(-144a^4g^2)}}{2(1)}$$
$$=\frac{32a^2g\pm 40a^2g}{2}$$
$$=36a^2g\ \text{ since } V^2>0$$
$$V=6a\sqrt{g}$$

101 **i**

$$T_n=(4n-1)^2$$

ii

$$S_{2n}=A_n-B_n$$
$$=[1^2+5^2+9^2+\ldots+(4n-3)^2]$$
$$-[3^2+7^2+11^2+\ldots+(4n-1)^2]$$
$$=(1^2-3^2)+(5^2-7^2)$$
$$+(9^2-11^2)+\ldots+\left((4n-3)^2-(4n-1)^2\right)$$
$$=-8-24-40+\ldots+(16n^2-24n+9-16n^2+8n$$
$$=-8-24-40+\ldots-8(2n-1)$$
$$=\frac{n}{2}\left(-8-8(2n-1)\right)$$ (sum of an arithmetic series)
$$=-8n^2$$

iii

$$101^2-103^2+105^2-107^2+\ldots+1993^2-1995^2$$
$$=(1^2-3^2+5^2-7^2+9^2-11^2+\ldots+97^2-99^2+101^2$$
$$-103^2+105^2-107^2+\ldots+1993^2-1995^2)$$
$$-(1^2-3^2+5^2-7^2+9^2-11^2+\ldots+97^2-99^2)$$
$$=S_{2(499)}-S_{2(25)}$$
$$=-8(499^2-25^2)$$
$$=-1\,987\,008$$

102 i

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$
$$= \cos^3\theta + 3\cos^2\theta\sin\theta\, i$$

Equating real components:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$= \cos^3\theta - 3\cos\theta\,(1-\cos^2\theta)$$
$$= 4\cos^3\theta - 3\cos\theta$$

ii

$$8x^3 - 6x - 1 = 0$$
$$8x^3 - 6x = 1$$
$$4x^3 - 3x = \frac{1}{2}$$
$$\text{Let } x = \cos\theta$$
$$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$$
$$\therefore \cos 3\theta = \frac{1}{2} \text{ from (i)}$$

$\therefore 8x^3 - 6x - 1 = 0$ has solutions in the form $x = \cos\theta$ where $\cos 3\theta = \frac{1}{2}$

iii

$$3\theta = \cos^{-1}\left(\frac{1}{2}\right), 2\pi - \cos^{-1}\left(\frac{1}{2}\right), 2\pi + \cos^{-1}\left(\frac{1}{2}\right)$$
$$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$
$$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$
$$\therefore x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$$

iv

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)$$
$$= \cos\left(\frac{\pi}{9}\right)\left(-\cos\left(\frac{7\pi}{9}\right)\right)\left(-\cos\left(\frac{5\pi}{9}\right)\right)$$
$$= \cos\frac{\pi}{9} \times \cos\frac{5\pi}{9} \times \cos\frac{7\pi}{9}$$
$$= -\frac{d}{a}$$
$$= \frac{1}{8}$$

103 i

$$\int x^m \log_e^n x\, dx$$

$u = \log_e^n x$	$\frac{dv}{dx} = x^m$
$\frac{du}{dx} = n\log_e^{n-1} x \times \frac{1}{x}$	$v = \frac{x^{m+1}}{m+1}$

$$= \frac{x^{m+1}\log_e^n x}{m+1} - \frac{n}{m+1}\int x^m \log_e^{n-1} x\, dx$$

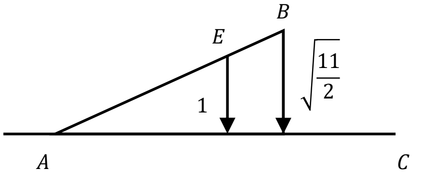
ii

$$I_{m,n} = \frac{x^{m+1}\log_e^n x}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

$$\int x^3\log_e^3 x\, dx = I_{3,3}$$
$$= \frac{x^4\log_e^3 x}{4} - \frac{3}{4} I_{3,2}$$
$$= \frac{x^4\log_e^3 x}{4} - \frac{3}{4}\left(\frac{x^4\log_e^2 x}{4} - \frac{1}{2} I_{3,1}\right)$$
$$= \frac{x^4\log_e^3 x}{4} - \frac{3x^4\log_e^2 x}{16} + \frac{3}{8}\left(\frac{x^4\log_e x}{4}\right.$$
$$= \frac{x^4\log_e^3 x}{4} - \frac{3x^4\log_e^2 x}{16} + \frac{3x^4\log_e x}{32}$$
$$= \frac{x^4\log_e^3 x}{4} - \frac{3x^4\log_e^2 x}{16} + \frac{3x^4\log_e x}{32}$$

104

$$d^2 = (1,0,-3)\cdot(1,0,-3) - \left[\frac{(1,0,-3)\cdot(0,2,-2)}{\sqrt{0^2+2^2+(-2)^2}}\right]^2$$
$$= 1 + 0 + 9 - \left[\frac{0+0+6}{\sqrt{8}}\right]^2$$
$$= 10 - \frac{36}{8}$$
$$= \frac{11}{2}$$
$$d = \sqrt{\frac{11}{2}}$$



$$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$$
$$= \overrightarrow{OA} + k\overrightarrow{AB}$$
$$= \overrightarrow{OA} + \frac{1}{d}\overrightarrow{AB}$$
$$= (2,1,4) + \sqrt{\frac{2}{11}}(1,0,-3)$$
$$= \left(2 + \sqrt{\frac{2}{11}}, 1, 4 - 3\sqrt{\frac{2}{11}}\right)$$
$$= \left(\frac{22+\sqrt{22}}{11}, 1, \frac{44-3\sqrt{22}}{11}\right)$$

105 i

$$\frac{d}{dx}\left[\sqrt{bx-x^2} + \frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right)\right]$$
$$= \frac{1}{2}(bx-x^2)^{-\frac{1}{2}}(b-2x) + \frac{b}{2}\left(-\frac{1}{\sqrt{1-\left(\frac{2x-b}{b}\right)^2}}\right)\left(\frac{2}{b}\right)$$
$$= \frac{b-2x}{2\sqrt{bx-x^2}} - \frac{1}{\sqrt{1-\frac{(2x-b)^2}{b^2}}}$$
$$= \frac{b-2x}{2\sqrt{bx-x^2}} - \frac{b}{\sqrt{b^2-4x^2+4xb-b^2}}$$
$$= \frac{b-2x}{2\sqrt{bx-x^2}} - \frac{b}{2\sqrt{bx-x^2}}$$
$$= -\frac{x}{\sqrt{bx-x^2}}$$
$$= -\sqrt{\frac{x^2}{bx-x^2}}$$
$$= -\sqrt{\frac{x}{b-x}}$$

ii

$$m\ddot{x} = -\frac{\mu m}{x^2}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{\mu}{x^2}$$
$$\frac{1}{2}v^2 = -\mu\int_b^x \frac{dx}{x^2}$$
$$v^2 = 2\mu\left[\frac{1}{x}\right]_b^x$$
$$= 2\mu\left(\frac{1}{x} - \frac{1}{b}\right)$$
$$= 2\mu\left(\frac{b-x}{x}\right)$$

105 … iii

$$\frac{dx}{dt}=-\sqrt{2\mu\left(\frac{b-x}{x}\right)}$$
$$\frac{dt}{dx}=-\frac{1}{\sqrt{2\mu}}\times\sqrt{\frac{x}{b-x}}$$
$$t=-\frac{1}{\sqrt{2\mu}}\int_b^{\frac{b}{2}}\sqrt{\frac{x}{b-x}}\,dx$$
$$=\frac{1}{\sqrt{2\mu}}\left[\sqrt{bx-x^2}+\frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right)\right]_b^{\frac{b}{2}}$$
$$=\frac{1}{\sqrt{2\mu}}\left(\left(\sqrt{\frac{b^2}{2}-\frac{b^2}{4}}+\frac{b}{2}\cos^{-1}(0)\right)\right.$$
$$\left.-\left(\sqrt{b^2-b^2}+\frac{b}{2}\cos^{-1}\left(\frac{2b-b}{b}\right)\right)\right)$$
$$=\frac{1}{\sqrt{2\mu}}\left(\left(\frac{b}{2}+\frac{b}{2}\right)-(0+0)\right)$$
$$=\frac{b}{\sqrt{2\mu}}$$

106 i

Let $P(1)$ represent the proposition.

$P(1)$ is true since $a_{1+1}=a_1(a_1+1)=1(1+1)=$
$$1+\sum_{r=1}^{1}a_r^2=1+(a_1)^2=1+1^2=2$$

If $P(k)$ is true for some arbitrary $k\geq 1$ then
$$a_{k+1}=1+\sum_{r=1}^{k}a_r^2$$

RTP $P(k+1)$
$$a_{k+2}=1+\sum_{r=1}^{k+1}a_r^2$$

$$\begin{aligned}\text{LHS}&=a_{k+2}\\&=a_{k+1}(a_{k+1}+1)\\&=(a_{k+1})^2+a_{k+1}\\&=(a_{k+1})^2+1+\sum_{r=1}^{k}a_r^2\ \text{ from } P(k)\\&=1+\sum_{r=1}^{k+1}a_r^2\\&=\text{RHS}\end{aligned}$$
$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\geq 1$ by induction

ii

$$\begin{aligned}&(2a_{n+1}+1)^2\\&=4a_{n+1}^2+4a_{n+1}+1\\&=(2a_{n+1})^2+4a_n(a_n+1)+1\\&=4a_n^2+4a_n+1+(2a_{n+1})^2\\&=(2a_n+1)^2+(2a_{n+1})^2\end{aligned}$$

iii

$$(2a_{n+1}+1)^2-(2a_n+1)^2=(2a_{n+1})^2$$
$$\sum_{r=1}^{n}(2a_{r+1}+1)^2-(2a_r+1)^2=\sum_{r=1}^{n}(2a_{r+1})^2$$
$$(2a_{n+1}+1)^2-(2a_1+1)^2=\sum_{r=2}^{n+1}(2a_r)^2$$
$$\therefore(2a_{n+1}+1)^2=(2a_1+1)^2+\sum_{r=2}^{n+1}(2a_r)^2$$

106 … iv

$$\begin{aligned}a_2&=a_1(a_1+1)=1(1+1)=2\\a_3&=a_2(a_2+1)=2(2+1)=6\\a_4&=a_3(a_3+1)=6(6+1)=42\\a_5&=a_4(a_4+1)=42(42+1)=1806\end{aligned}$$

v

$$\begin{aligned}a_5&=1+\sum_{r=1}^{4}a_r^2\\&=1+a_1^2+a_2^2+a_3^2+a_4^2\\&=1+1^2+2^2+6^2+42^2\end{aligned}$$

vi

$$(2a_1+1)^2+\sum_{r=2}^{5}(2a_r)^2=(2a_5+1)^2\text{ from (iii)}$$
$$(2a_1+1)^2+(2a_2)^2+(2a_2)^2+(2a_2)^2+(2a_2)^2=(2a_5+1)^2$$
$$(2(1)+1)^2+\big(2(2)\big)^2+\big(2(6)\big)^2+\big(2(42)\big)^2+\big(2(1806)\big)^2=(2(1806)+1)^2$$
$$3^2+4^2+12^2+84^2+3612^2=3613^2$$

107 i

$$\begin{aligned}&(\cos\theta+i\sin\theta)^4=\cos 4\theta+i\sin 4\theta\\&=\cos^4\theta+4\cos^3\theta\sin\theta\, i-6\cos^2\theta\sin^2\theta\\&\qquad-4\cos\theta\sin^3\theta\, i+\sin^4\theta\end{aligned}$$
Equating real components:
$$\begin{aligned}\cos 4\theta&=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta\\&=\cos^4\theta-6\cos^2\theta\,(1-\cos^2\theta)+(1-\cos^2\theta)^2\\&=\cos^4\theta-6\cos^2\theta+6\cos^4\theta+1-2\cos^2\theta\\&=8\cos^4\theta-8\cos^2\theta+1\end{aligned}$$

ii

$$\begin{aligned}16x^4-16x^2+1&=0\\2(8x^4-8x^2+1)-1&=0\\8x^4-8x^2+1&=\frac{1}{2}\\\text{Let } x&=\cos\theta\\8\cos^4\theta-8\cos^2\theta+1&=\frac{1}{2}\\\cos 4\theta&=\frac{1}{2}\\4\theta&=\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3},\frac{11\pi}{3}\\\theta&=\frac{\pi}{12},\frac{5\pi}{12},\frac{7\pi}{12},\frac{11\pi}{12}\\\therefore x&=\cos\frac{\pi}{12},\cos\frac{5\pi}{12},\cos\frac{7\pi}{12},\cos\frac{11\pi}{12}\\&=\cos\frac{\pi}{12},\cos\frac{5\pi}{12},-\cos\frac{5\pi}{12},-\cos\frac{\pi}{12}\end{aligned}$$
$\therefore$ the roots are $x=\cos\frac{\pi}{12}, x=-\cos\frac{\pi}{12}, x=\cos\frac{5\pi}{12}$ and $x=-\cos\frac{5\pi}{12}$.

iii

$$\begin{aligned}x^2&=\frac{16\pm\sqrt{16^2-4(16)(1)}}{2(16)}\\&=\frac{16\pm\sqrt{192}}{32}\\&=\frac{16\pm 8\sqrt{3}}{32}\\&=\frac{2\pm\sqrt{3}}{4}\end{aligned}$$
Since $\cos\theta$ is positive and decreasing in the first quadrant:
$$\cos\frac{5\pi}{12}=\sqrt{\frac{2-\sqrt{3}}{4}}=\frac{\sqrt{2-\sqrt{3}}}{2}$$

108 **i**

$I_n = \int_0^1 (1-x^2)^n \, dx$

$u = (1-x^2)^n$	$\frac{dv}{dx} = 1$
$\frac{du}{dx} = n(1-x^2)^{(n-1)}(-2x)$	$v = x$

$= \left[x(1-x^2)^n\right]_0^1 + 2n\int_0^1 x^2(1-x^2)^{n-1}\, dx$
$= 0 - 0 + 2nJ_{n-1}$
$= 2nJ_{n-1}$

ii

$\therefore I_n = -2n\int_0^1 (1-x^2-1)(1-x^2)^{n-1}\, dx$
$= -2n\int_0^1 (1-x^2)^n\, dx + 2n\int_0^1 (1-x^2)^{n-1}\, dx$
$I_n = -2nI_n + 2nI_{n-1}$
$(2n+1)I_n = 2nI_{n-1}$
$I_n = \frac{2n}{2n+1} I_{n-1}$

iii

$I_n - I_{n+1} = \int_0^1 (1-x^2)^n\, dx - \int_0^1 (1-x^2)^{n+1}\, dx$
$= \int_0^1 [(1-x^2)^n - (1-x^2)^{n+1}]\, dx$
$= \int_0^1 [(1-x^2)^n(1-1+x^2)]\, dx$
$= \int_0^1 x^2(1-x^2)^n\, dx$
$= J_n$
$\therefore J_n = I_n - 2(n+1)J_n$ from (i)
$(2n+3)J_n = I_n$
$J_n = \frac{1}{2n+3} I_n$

iv

$J_n = \frac{1}{2n+3}(2nJ_{n-1})$
$= \frac{2n}{2n+3} J_{n-1}$

109 If $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are collinear then

$\underset{\sim}{b} - \underset{\sim}{a} = k\left(\underset{\sim}{c} - \underset{\sim}{a}\right)$
$(k-1)\underset{\sim}{a} + \underset{\sim}{b} - k\underset{\sim}{c} = 0$
Let $\lambda = k-1, \mu = 1, \nu = -k$
$\lambda + \mu + \nu = k - 1 + 1 - k$
$= 0$
$\therefore$ if $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ they must satisfy $\underset{\sim}{a} + \mu\underset{\sim}{b} + \nu\underset{\sim}{c} = 0$ where $\lambda + \mu + \nu = 0$

110 **i**

$v\frac{dv}{dx} = -\frac{B}{M}v^2$
$\frac{dv}{dx} = -\frac{B}{M}v$
$\frac{dx}{dv} = -\frac{M}{B} \times \frac{1}{v}$
$D_1 = -\frac{M}{B}\int_V^U \frac{dv}{v}$
$= \frac{M}{B}\left[\ln v\right]_U^V$
$= \frac{M}{B}(\ln V - \ln U)$
$= \frac{M}{B}\ln\left(\frac{V}{U}\right)$

110 **...** **ii**

$v\frac{dv}{dx} = -\frac{A+Bv^2}{M}$
$\frac{dv}{dx} = -\frac{A+Bv^2}{Mv}$
$\frac{dx}{dv} = -\frac{Mv}{A+Bv^2}$
$D_2 = -M\int_U^0 \frac{v}{A+Bv^2}\, dv$
$= \frac{M}{2B}\int_0^U \frac{2Bv}{A+Bv^2}\, dv$
$= \frac{M}{2B}\left[\ln(A+Bv^2)\right]_0^U$
$= \frac{M}{2B}(\ln(A+BU^2) - \ln A)$
$= \frac{M}{2B}\ln\left(\frac{A+BU^2}{A}\right)$
$= \frac{M}{2B}\ln\left[1 + \frac{B}{A}U^2\right]$

iii

$D = D_1 + D_2$
$= \frac{M}{B}\ln\left(\frac{V}{U}\right) + \frac{M}{2B}\ln\left[1 + \frac{B}{A}U^2\right]$
$= \frac{100\,000}{125}\ln\left(\frac{90}{60}\right) + \frac{100\,000}{2\times 125}\ln\left(1 + \frac{125}{75\,000}\times 60^2\right)$
$= 1102.74\ldots$
$= 1103$ m (nearest metre)

111 **i**
Let $u = \ln x$ for $x > 0$
$\therefore e^{\ln x} \geq 1 + \ln x$
$x \geq 1 + \ln x$
$\ln x \leq x - 1$

ii
Let $x = \frac{c_1}{\mu}$
$\ln\left(\frac{c_1}{\mu}\right) \leq \frac{c_1}{\mu} - 1 \quad (1)$
Similarly
$\ln\left(\frac{c_2}{\mu}\right) \leq \frac{c_2}{\mu} - 1 \quad (2), \qquad \ln\left(\frac{c_3}{\mu}\right) \leq \frac{c_3}{\mu} - 1 \quad (3), \ldots, \ln\left(\frac{c_n}{\mu}\right) \leq \frac{c_n}{\mu} - 1 \quad (n)$
Summing inequalities (1) to (n):

$$\ln\left(\frac{c_1}{\mu}\right) + \ln\left(\frac{c_2}{\mu}\right) + \ln\left(\frac{c_3}{\mu}\right) + \ldots + \ln\left(\frac{c_n}{\mu}\right) \leq \frac{c_1}{\mu} - 1 + \frac{c_2}{\mu} - 1 + \frac{c_3}{\mu} - 1 + \ldots + \frac{c_n}{\mu} - 1$$
$$\ln\left(\frac{c_1}{\mu} \times \frac{c_2}{\mu} \times \frac{c_3}{\mu} \times \ldots \times \frac{c_n}{\mu}\right) \leq \left(\frac{c_1}{\mu} + \frac{c_2}{\mu} + \frac{c_3}{\mu} + \ldots + \frac{c_n}{\mu}\right) - n$$
$$\therefore \log_e\left(\frac{c_1 c_2 c_3 \ldots c_n}{\mu^n}\right) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\mu} - n$$

iii
$$\log_e\left(\frac{c_1 c_2 c_3 \ldots c_n}{\mu^n}\right) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\mu} - n$$
$$\log_e(c_1 c_2 c_3 \ldots c_n) - \log_e(\mu^n) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\mu} - n$$
$$\log_e(c_1 c_2 c_3 \ldots c_n) - n\log_e(\mu) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\mu} - n$$
$$\log_e(c_1 c_2 c_3 \ldots c_n) - n\log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right) \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}} - n$$
$$\log_e(c_1 c_2 c_3 \ldots c_n) - n\log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right) \leq n - n$$
$$\log_e(c_1 c_2 c_3 \ldots c_n) \leq n\log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right)$$
$$\frac{1}{n}\log_e(c_1 c_2 c_3 \ldots c_n) \leq \log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right)$$
$$\log_e \sqrt[n]{c_1 c_2 c_3 \ldots c_n} \leq \log_e\left(\frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}\right)$$
$$\therefore \sqrt[n]{c_1 c_2 c_3 \ldots c_n} \leq \frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}$$

iv
$$\therefore c_1 + c_2 + c_3 + \ldots + c_n \geq n\sqrt[n]{c_1 c_2 c_3 \ldots c_n}$$
$$\frac{101}{103} + \frac{103}{105} + \frac{105}{107} + \ldots + \frac{197}{199} + \frac{199}{101} \geq 50\sqrt[50]{\frac{101}{103} \times \frac{103}{105} \times \frac{105}{107} \times \ldots \times \frac{197}{199} \times \frac{199}{101}}$$
$$\geq 50\sqrt[50]{1}$$
$$\geq 50$$

112 **i**
$(z^{2n} + z^{2n-2} + z^{2n-4} + \ldots + z^{-2n+2} + z^{-2n})\sin\theta$
$= [(z^{2n} + z^{-2n}) + (z^{2n-2} + z^{-2n+2}) + \ldots + (z^2 + z^{-2}) + 1]\sin\theta$
$= \left[2\cos(2n\theta) + 2\cos\big((2n-2)\theta\big) + \ldots + 2\cos 2\theta + 1\right]\sin\theta$
$= 2\cos(2n\theta)\sin\theta + 2\cos\big((2n-2)\theta\big)\sin\theta + \ldots + 2\cos 2\theta\sin\theta + \sin\theta$
$= (\sin(2n+1)\theta - \sin(2n-1)\theta) + (\sin(2n-1)\theta - \sin(2n-3)\theta) + \ldots + (\sin 3\theta - \sin\theta) + \sin\theta$
$= \sin(2n+1)\theta$

ii
Let $n = 3$ in (i):
$$(z^6 + z^4 + z^2 + 1 + z^{-2} + z^{-4} + z^{-6})\sin\theta = \sin 7\theta$$
$$z^6 + z^{-6} + z^4 + z^{-4} + z^2 + z^{-2} + 1 = \frac{\sin 7\theta}{\sin\theta}$$
$$2\cos 6\theta + 2\cos 4\theta + 2\cos 2\theta + 1 = \frac{\sin 7\theta}{\sin\theta}$$
$$2(4\cos^3 2\theta - 3\cos 2\theta) + 2(2\cos^2 2\theta - 1) + 2\cos 2\theta + 1 = \frac{\sin 7\theta}{\sin\theta}$$
$$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \frac{\sin 7\theta}{\sin\theta}$$

112 … **iii**

$$8x^3+4x^2-4x-1=0$$
Let $x=\cos 2\theta$
$$8\cos^3 2\theta+4\cos^2 2\theta-4\cos 2\theta-1=0$$
$$\therefore \frac{\sin 7\theta}{\sin\theta}=0$$
$$\therefore \sin 7\theta=0,\qquad \sin\theta\neq 0$$
$$7\theta=0,\pi,2\pi,3\pi,\ldots$$
$$\theta=\cancel{0},\frac{\pi}{7},\frac{2\pi}{7},\ldots$$
$\therefore \cos\frac{2\pi}{7}$ is one root of the equation.

113

$$\begin{aligned}I_n&=\int_0^1\sqrt{x}(1-x)^n\,dx\\&=\frac{2}{3}\left[x^{\frac{3}{2}}(1-x)^n\right]_0^1+\frac{2n}{3}\int_0^1 x^{\frac{3}{2}}(1-x)^{n-1}\,dx\\&=\frac{2}{3}(0-0)+\frac{2n}{3}\int_0^1 x\sqrt{x}(1-x)^{n-1}\,dx\\&=-\frac{2n}{3}\int_0^1(1-x-1)\sqrt{x}(1-x)^{n-1}\,dx\\&=-\frac{2n}{3}\int_0^1\sqrt{x}(1-x)^n\,dx+\frac{2n}{3}\int_0^1\sqrt{x}(1-x)^{n-1}\,dx\\&=-\frac{2n}{3}I_n+\frac{2n}{3}I_{n-1}\end{aligned}$$

$u=(1-x)^n$	$\frac{dv}{dx}=x^{\frac{1}{2}}$
$\frac{du}{dx}=n(1-x)^{n-1}(-1)$	$v=\frac{2}{3}x^{\frac{3}{2}}$

$$\frac{2n+3}{3}I_n=\frac{2n}{3}I_{n-1}$$
$$I_n=\frac{2n}{2n+3}I_{n-1}$$
$$\begin{aligned}I_3&=\frac{2(3)}{2(3)+3}I_2\\&=\frac{2}{3}\left(\frac{2(2)}{2(2)+3}\right)I_1\\&=\frac{8}{21}\left(\frac{2(1)}{2(1)+3}\right)I_0\\&=\frac{16}{105}\int_0^1 x^{\frac{1}{2}}\,dx\\&=\frac{16}{105}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1\\&=\frac{32}{315}\end{aligned}$$

114 Let $\overrightarrow{OA}=\underset{\sim}{a},\overrightarrow{OB}=\underset{\sim}{b}$ etc.

$\underset{\sim}{p}=\frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}\right),\underset{\sim}{q}=\frac{1}{2}\left(\underset{\sim}{b}+\underset{\sim}{c}\right),\underset{\sim}{r}=\frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{c}\right),\underset{\sim}{l}=\frac{1}{2}\left(\underset{\sim}{d}+\underset{\sim}{e}\right),\underset{\sim}{m}=\frac{1}{2}\left(\underset{\sim}{e}+\underset{\sim}{f}\right),\underset{\sim}{n}=\frac{1}{2}\left(\underset{\sim}{f}+\underset{\sim}{d}\right)$

The medians of ΔABC are:

$\overrightarrow{BR}=\frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{c}\right)-\underset{\sim}{b}=\frac{1}{2}\left(\underset{\sim}{a}-2\underset{\sim}{b}+\underset{\sim}{c}\right)$

$\overrightarrow{AQ}=\frac{1}{2}\left(\underset{\sim}{b}+\underset{\sim}{c}\right)-\underset{\sim}{a}=\frac{1}{2}\left(-2\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)$

$\overrightarrow{CP}=\frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}\right)-\underset{\sim}{c}=\frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}-2\underset{\sim}{c}\right)$

Let $\overrightarrow{EF}=\lambda\overrightarrow{BR}=\frac{\lambda}{2}\left(\underset{\sim}{a}-2\underset{\sim}{b}+\underset{\sim}{c}\right)$

$\overrightarrow{FD}=\lambda\overrightarrow{CP}=\frac{\lambda}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}-2\underset{\sim}{c}\right)$

$\overrightarrow{DE}=\lambda\overrightarrow{AQ}=\frac{\lambda}{2}\left(-2\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)$

$$\begin{aligned}\overrightarrow{EN}&=\overrightarrow{EF}+\overrightarrow{FN}\\&=\overrightarrow{EF}+\frac{1}{2}\overrightarrow{FD}\\&=\frac{\lambda}{2}\left(\underset{\sim}{a}-2\underset{\sim}{b}+\underset{\sim}{c}\right)+\frac{\lambda}{4}\left(\underset{\sim}{a}+\underset{\sim}{b}-2\underset{\sim}{c}\right)\\&=\frac{3\lambda}{4}\left(\underset{\sim}{a}-\underset{\sim}{b}\right)\\&=\frac{3\lambda}{4}\overrightarrow{BA}\end{aligned}$$

Similarly $\overrightarrow{DM}=\frac{3\lambda}{4}\overrightarrow{AC},\overrightarrow{FL}=\frac{3\lambda}{4}\overrightarrow{CB}$

$\therefore$ the medians of triangle DEF are parallel to the sides of triangle ABC

115 **i**

$m\ddot{x} = -mg - kmx$
$\ddot{x} = -g - kx$

ii

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -(g + kx)$$
$$\frac{1}{2}v^2 = -\int_{-a}^{x}(g + kx)\,dx$$
$$v^2 = 2\left[gx + \frac{1}{2}kx^2\right]_x^{-a}$$
$$= 2\left(\left(-ga + \frac{1}{2}ka^2\right) - \left(gx + \frac{1}{2}kx^2\right)\right)$$
$$= (ka^2 - 2ga) - (kx^2 + 2gx)$$
$$= \left(ka^2 - 2ga + \frac{g^2}{k}\right) - \left(kx^2 + 2gx + \frac{g^2}{k}\right)$$
$$= k\left(\left(a - \frac{g}{k}\right)^2 - \left(x + \frac{g}{k}\right)^2\right)$$

iii

$$x = \left(\frac{g}{k} - a\right)\cos(\sqrt{k}t) - \frac{g}{k}$$
$$\dot{x} = -\sqrt{k}\left(\frac{g}{k} - a\right)\sin(\sqrt{k}t)$$
$$\ddot{x} = -k\left(\frac{g}{k} - a\right)\cos(\sqrt{k}t)$$

consider:

$-g - kx$
$$= -g - k\left(\left(\frac{g}{k} - a\right)\cos(\sqrt{k}t) - \frac{g}{k}\right)$$
$$= -g - k\left(\frac{g}{k} - a\right)\cos(\sqrt{k}t) + g$$
$$= -k\left(\frac{g}{k} - a\right)\cos(\sqrt{k}t)$$
$= \ddot{x}$
$\therefore \ddot{x} = -g - kx$
$\therefore$ the motion is described by $x = \left(\frac{g}{k} - a\right)\cos(\sqrt{k}t) - \frac{g}{k}$

116 **i**

There are 9 choices for the first digit (since 0 is excluded), and for each digit after that there are 9 choices (since the previous choice is excluded). So there are 9 choices for each of n digits, so $B(n) = 9^n$

ii

There are 5 odd digits, 1, 3, 5, 7 and 9.
If the first k digits form an odd blue integer then there are only four choices for the last digit, giving $4 \times O(k)$ ways for a $k + 1$ digit odd blue integer to be formed from an odd k digit blue integer.
If the first k digits form an even blue integer then there are five choices for the last digit, giving $5 \times E(k)$ ways for a $k + 1$ digit odd blue integer to be formed from an even k digit blue integer.
$\therefore O(k + 1) = 4 \times O(k) + 5 \times E(k)$

iii

Let $P(n)$ represent the proposition
$P(1)$ is true since $O(1) = \frac{9^1 + (-1)^{1-1}}{2} = 5$ and there are 5 odd digits
If $P(k)$ is true for some arbitrary $k \geq 1$ then $O(k) = \frac{9^k + (-1)^{k-1}}{2}$
RTP: $P(k + 1)$: $O(k + 1) = \frac{9^{k+1} + (-1)^k}{2}$

$\text{LHS} = O(k + 1)$
$$= 4 \times O(k) + 5 \times E(k)$$
$$= 4 \times O(k) + 5\left(9^k - O(k)\right)$$
$$= 5 \times 9^k - O(k)$$
$$= 5 \times 9^k - \frac{9^k + (-1)^{k-1}}{2} \quad \text{from } P(k)$$
$$= \frac{10 \times 9^k - 9^k - (-1)^{k-1}}{2}$$
$$= \frac{9 \times 9^k + (-1)^k}{2}$$
$$= \frac{9^{k+1} + (-1)^k}{2}$$
$= \text{RHS}$
$\therefore P(k) \Rightarrow P(k + 1)$
Hence $P(n)$ is true for $n \geq 1$ by induction.

116 ... **iv**

$$E(n) = 9^5 - O(n)$$
$$= 9^n - \frac{9^n + (-1)^{n-1}}{2}$$
$$= \frac{(2 \times 9^n - 9^n - (-1)^{n-1})}{2}$$
$$= \frac{9^n + (-1)^n}{2}$$

117 **i**

$$z^5 = 1$$
$$z^5 - 1 = 0$$
$$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$
$$\therefore (\omega - 1)(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$$
$$\therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0 \quad \text{since } \omega \text{ is non-real}$$

ii

$$\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1$$
$$= \omega^2 + 2 + \frac{1}{\omega^2} + \omega + \frac{1}{\omega} - 1$$
$$= \omega^2 + 1 + \frac{1}{\omega^2} + \omega + \frac{1}{\omega}$$
$$= \frac{\omega^4 + \omega^2 + 1 + \omega^3 + \omega}{\omega^2}$$
$$= \frac{0}{\omega^2}$$
$$= 0$$

iii

$$\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$
$$\frac{1}{\omega} = \overline{\omega} = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right) = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} \text{since } |\omega| = 1$$
$$\omega + \overline{\omega} = 2\cos\frac{2\pi}{5}$$
$$\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$$
$$\left(2\cos\frac{2\pi}{5}\right)^2 + \left(2\cos\frac{2\pi}{5}\right) - 1 = 0$$
$$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0$$
$$\cos\frac{2\pi}{5} = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$$
$$= \frac{-2 \pm \sqrt{20}}{8}$$
$$= \frac{-1 \pm \sqrt{5}}{4}$$

Now $0 < \frac{2\pi}{5} < \frac{\pi}{2} \therefore \cos\frac{2\pi}{5} > 0$

$$\therefore \cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$$

118 **i**

$$\int \frac{dx}{5-4\cos x}$$
$$= \int \frac{1}{5-\frac{4(1-t^2)}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int \frac{dt}{5+5t^2-4+4t^2}$$
$$= 2\int \frac{dt}{9t^2+1}$$
$$= \frac{2}{3}\int \frac{3}{(3t)^2+1}dt$$
$$= \frac{2}{3}\tan^{-1} 3t + c$$
$$= \frac{2}{3}\tan^{-1}\left(3\tan\frac{x}{2}\right) + c$$

$t = \tan\frac{x}{2}$ $dx = \frac{2dt}{1+t^2}$

ii

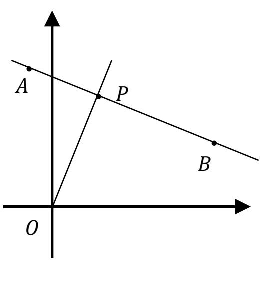
$$\int_0^\pi \frac{\cos x}{5-4\cos x}dx$$
$$= \int_0^\pi \frac{-\frac{1}{4}(5-4\cos x)+\frac{5}{4}}{5-4\cos x}dx$$
$$= -\frac{1}{4}\int_0^\pi dx + \frac{5}{4}\int_0^\pi \frac{dx}{5-4\cos x}$$
$$= -\frac{1}{4}\Big[x\Big]_0^\pi + \frac{5}{4}\left(\frac{\pi}{3}\right)$$
$$= -\frac{\pi}{4} + \frac{5\pi}{12}$$
$$= \frac{\pi}{6}$$

iii

$$u_{n+1} + u_{n-1} - \frac{5}{2}u_n$$
$$= \int_0^\pi \frac{\cos\big((n+1)x\big) + \cos\big((n-1)x\big) - \frac{5}{2}\cos nx}{5-4\cos x}dx$$
$$= \int_0^\pi \frac{\cos nx\cos x - \sin nx \sin x + \cos nx \cos x + \sin nx \sin x - \frac{5}{2}\cos nx}{5-4\cos x}dx$$
$$= \int_0^\pi \frac{2\cos nx\cos x - \frac{5}{2}\cos nx}{5-4\cos x}dx$$
$$= \frac{1}{2}\int_0^\pi \frac{\cos nx\,(4\cos x - 5)}{5-4\cos x}dx$$
$$= -\frac{1}{2}\int_0^\pi \cos nx\, dx$$
$$= -\frac{1}{2}\left[\frac{1}{n}\sin nx\right]_0^\pi$$
$$= -\frac{1}{2n}(0-0)$$
$$= 0$$

119

$$\text{Let } \underset{\sim}{p} = \underset{\sim}{a} + \lambda\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$$
$$\underset{\sim}{p}\cdot\left(\underset{\sim}{a}-\underset{\sim}{b}\right) = 0$$
$$\left(\underset{\sim}{a} + \lambda\left(\underset{\sim}{a}-\underset{\sim}{b}\right)\right)\cdot\left(\underset{\sim}{a}-\underset{\sim}{b}\right) = 0$$
$$\underset{\sim}{a}\cdot\left(\underset{\sim}{a}-\underset{\sim}{b}\right) + \lambda\left(\underset{\sim}{a}-\underset{\sim}{b}\right)\cdot\left(\underset{\sim}{a}-\underset{\sim}{b}\right) = 0$$
$$\underset{\sim}{a}\cdot\left(\underset{\sim}{a}-\underset{\sim}{b}\right) + \lambda\left|\underset{\sim}{a}-\underset{\sim}{b}\right|^2 = 0$$
$$\lambda\left|\underset{\sim}{a}-\underset{\sim}{b}\right|^2 = \underset{\sim}{a}\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)$$
$$\lambda = \frac{\underset{\sim}{a}\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)}{\left|\underset{\sim}{a}-\underset{\sim}{b}\right|^2}$$
$$\therefore \underset{\sim}{p} = \underset{\sim}{a} + \frac{\underset{\sim}{a}\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)}{\left|\underset{\sim}{a}-\underset{\sim}{b}\right|^2}\left(\underset{\sim}{a}-\underset{\sim}{b}\right)$$
$$= \underset{\sim}{a} - \frac{\left[\underset{\sim}{a}\cdot\left(\underset{\sim}{a}-\underset{\sim}{b}\right)\right]\left(\underset{\sim}{a}-\underset{\sim}{b}\right)}{\left|\underset{\sim}{a}-\underset{\sim}{b}\right|^2}$$



120 **i**

$$m\ddot{x} = P - R$$
$$\frac{dv}{dt} = \frac{10 - (5 + 3v)}{m}$$
$$= \frac{5 - 3v}{m}$$
$$\frac{dt}{dv} = \frac{m}{5 - 3v}$$
$$t = m\int_{v_0}^{v} \frac{dv}{5 - 3v}$$
$$\frac{t}{m} = -\frac{1}{3}\Big[\ln(5 - 3v)\Big]_{v_0}^{v}$$
$$-\frac{3t}{m} = \ln(5 - 3v) - \ln(5 - 3v_0)$$
$$\ln(5 - 3v) = \ln(5 - 3v_0) - \frac{3t}{m}$$
$$5 - 3v = (5 - 3v_0)e^{-\frac{3t}{m}}$$
$$3v = 5 - 5e^{-\frac{3t}{m}} + 3v_o e^{-\frac{3t}{m}}$$
$$v = \frac{5}{3}\left(1 - e^{-\frac{3t}{m}}\right) + v_o e^{-\frac{3t}{m}}$$

ii

Let $\ddot{x} = 0, v = V_T$

$$0 = \frac{5 - 3V_T}{m}$$
$$3V_T = 5$$
$$V_T = \frac{5}{3}$$

iii

$$v\frac{dv}{dx} = \frac{5 - 3v}{m}$$
$$\frac{dv}{dx} = \frac{5 - 3v}{mv}$$
$$\frac{dx}{dv} = \frac{mv}{5 - 3v}$$
$$\Delta x = \int_{v_0}^{v_1} \frac{mv}{5 - 3v} dv$$
$$= \int_{v_0}^{v_1} \frac{-\frac{m}{3}(5 - 3v) + \frac{5m}{3}}{5 - 3v} dv$$
$$= \int_{v_0}^{v_1} \left(-\frac{m}{3} - \frac{5m}{9} \times \left(-\frac{3}{5 - 3v}\right)\right) dv$$
$$= -\left[\frac{mv}{3} + \frac{5m}{9}\ln(5 - 3v)\right]_{v_0}^{v_1}$$
$$= \left(\frac{mv_0}{3} + \frac{5m}{9}\ln(5 - 3v_0)\right) - \left(\frac{mv_1}{3} + \frac{5m}{9}\ln(5 - 3v_1)\right)$$
$$= \frac{m}{9}\Big[3v_0 - 3v_1 + 5\ln(5 - 3v_0) - 5\ln(5 - 3v_0)\Big]$$
$$= \frac{m}{9}\left[3(v_0 - v_1) + 5\ln\left(\frac{5 - 3v_0}{5 - 3v_1}\right)\right]$$

121 **i**

$$(b + d)^n = b^n + \binom{n}{1}b^{n-1}d + \binom{n}{2}b^{n-2}d^2 + \ldots + \binom{n}{n}d^n$$
$$a^n = b^n + nb^{n-1}d + \binom{n}{2}b^{n-2}d^2 + \ldots + \binom{n}{n}d^n$$
$$a^n - b^n - nb^{n-1}d = d^2\left(\binom{n}{2}b^{n-2} + \ldots + \binom{n}{n}d^{n-2}\right)$$
$$a^n - b^{n-1}(b + nd) = d^2\left(\binom{n}{2}b^{n-2} + \ldots + \binom{n}{n}d^{n-2}\right)$$

$\therefore a^n - b^{n-1}(b + nd)$ is divisible by d^2.

ii

$a^n - (a - d)^{n-1}(a - d + nd)$
$= a^n - (a - d)^{n-1}(a + (n - 1)d)$
$= a^n - (a - d)^{n-1} \times T_n$
$= a^n - l(a - d)^{n-1}$ is divisible by d^2

iii

Let $d = 3, a = 5, n = 682$
$\therefore l = a + (n - 1)d = 5 + 681 \times 3 = 2048$
$\therefore a^n - l(a - d)^{n-1} = 5^{682} - 2048(5 - 3)^{681}$
$= 5^{682} - 2^{11} \times 2^{681}$
$= 5^{682} - 2^{692}$
$\therefore\ 5^{682} - 2^{692}$ is divisible by $3^2 = 9$

122 **i**

$$z^n + z^{-n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$$
$$= 2\cos(n\theta)$$

ii

$$u^3 = \left(z + \frac{1}{z}\right)^3$$
$$= z^3 + 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z^2}\right) + \left(\frac{1}{z}\right)^3$$
$$= z^3 + 3z + 3\left(\frac{1}{z}\right) + \frac{1}{z^3}$$
$$= z^3 + \frac{1}{z^3} + 3u$$
$$z^3 + \frac{1}{z^3} = u^3 - 3u$$

iii

$$1 + \cos 10\theta = 1 + \cos\big(2(5\theta)\big)$$
$$= 1 + 2\cos^2 5\theta - 1$$
$$= 2\cos^2 5\theta$$
$$= \frac{1}{2}(2\cos 5\theta)^2$$
$$= \frac{1}{2}(z^5 + z^{-5})^2$$
$$= \frac{1}{2}(u^5 - 5u^3 + 5u)^2$$
$$= \frac{1}{2}\left((2\cos\theta)^5 - 5(2\cos\theta)^3 + 5(2\cos\theta)\right)^2$$
$$= \frac{1}{2}(32\cos^5\theta - 40\cos^3\theta + 10\cos\theta)^2$$
$$= 2(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)^2$$

123 **i**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \frac{1}{2}\sin x}$$
$$= \int_0^1 \frac{1}{1 + \frac{t}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$t = \tan\frac{x}{2} \qquad dx = \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{dt}{1 + t^2 + t}$$
$$= 2\int_0^1 \frac{dt}{t^2 + t + \frac{1}{4} + \frac{3}{4}}$$
$$= 2\int_0^1 \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \frac{4}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$$
$$= \frac{4}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$
$$= \frac{4}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$
$$= \frac{2\pi}{3\sqrt{3}}$$

ii

$$\int_0^a f(2a - x)\,dx$$
$$= \int_{2a}^a f(u)\,(-du)$$

$$u = 2a - x \qquad du = -dx \qquad dx = -du$$

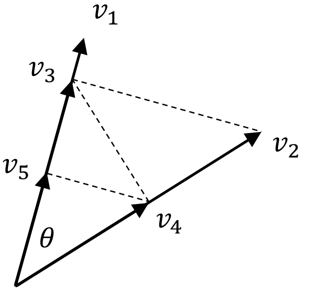
$$= \int_a^{2a} f(u)\,du$$
$$= \int_a^{2a} f(x)\,dx$$

$$\int_0^{2a} f(x)\,dx$$
$$= \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$
$$= \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx$$
$$= \int_0^a [f(x) + f(2a - x)]\,dx$$

123 ... **iii**

$$\int_0^{\pi} \frac{x\,dx}{1+\frac{1}{2}\sin x}$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{x}{1+\frac{1}{2}\sin x} + \frac{\pi - x}{1+\frac{1}{2}\sin(\pi - x)} \right) dx$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{x}{1+\frac{1}{2}\sin x} + \frac{\pi - x}{1+\frac{1}{2}\sin x} \right) dx$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{1+\frac{1}{2}\sin x} \right) dx$$
$$= \pi \int_0^{\frac{\pi}{2}} \frac{dx}{1+\frac{1}{2}\sin x}$$
$$= \pi \left(\frac{2\pi}{3\sqrt{3}} \right) \text{ from } (i)$$
$$= \frac{2\pi^2}{3\sqrt{3}}$$

124 The vectors alternate back and forth from $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$.



$$\cos\theta = \frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2}{\left|\underset{\sim}{v}_1\right| \times \left|\underset{\sim}{v}_2\right|} = \frac{10}{3 \times 4} = \frac{5}{6}$$
$$\cos\theta = \frac{\left|\underset{\sim}{v}_{k-1}\right|}{\left|\underset{\sim}{v}_k\right|}$$
$$\left|\underset{\sim}{v}_{k-1}\right| = \frac{5}{6}\left|\underset{\sim}{v}_k\right|$$
$$S = \sum_{n=1}^{\infty} \left|\underset{\sim}{v}_n\right|$$
$$= \left|\underset{\sim}{v}_1\right| + \left|\underset{\sim}{v}_2\right| + \left|\underset{\sim}{v}_3\right| + \left|\underset{\sim}{v}_4\right| + \dots$$
$$= 3 + 4 + 4\left(\frac{5}{6}\right) + 4\left(\frac{5}{6}\right)^2 + \dots$$
$$= 3 + \frac{4}{1 - \frac{5}{6}}$$
$$= 27$$

125 **i** $\boldsymbol{\alpha}$

$\ddot{x} = g - kv$

$\boldsymbol{\beta}$

$$\frac{dv}{dt} = g - kv$$
$$\frac{dt}{dv} = \frac{1}{g - kv}$$
$$t = \int_0^v \frac{dv}{g - kv}$$
$$= -\frac{1}{k}\Big[\ln(g - kv)\Big]_0^v$$
$$-kt = \ln(g - kv) - \ln g$$
$$\ln(g - kv) = \ln g - kt$$
$$g - kv = ge^{-kt}$$
$$kv = g(1 - e^{-kt})$$
$$v = \frac{g}{k}(1 - e^{-kt})$$

$\boldsymbol{\gamma}$

Let $\ddot{x} = 0, v = V_T$

$$0 = g - kV_T$$
$$V_T = \frac{g}{k}$$

ii $\boldsymbol{\alpha}$

$\ddot{x} = -(g + kv)$

$\boldsymbol{\beta}$

$$\frac{dv}{dt} = -(g + kv)$$
$$\frac{dt}{dv} = -\frac{1}{g + kv}$$
$$t = -\int_u^0 \frac{dv}{g + kv}$$
$$t = \frac{1}{k}\Big[\ln(g + kv)\Big]_0^u$$
$$= \frac{1}{k}(\ln(g + ku) - \ln g)$$
$$= \frac{1}{k}\ln\left(\frac{g + ku}{g}\right)$$
$$v = \frac{g}{k}\left(1 - \exp\left(-k\left(\frac{1}{k}\ln\left(\frac{g + ku}{g}\right)\right)\right)\right)$$
$$= \frac{g}{k}\left(1 - \exp\left(-\ln\left(\frac{g + ku}{g}\right)\right)\right)$$
$$= \frac{g}{k}\left(1 - \frac{g}{g + ku}\right)$$
$$= \frac{g}{k}\left(\frac{ku}{g + ku}\right)$$
$$= \frac{g}{k}\left(\frac{u}{\frac{g}{k} + u}\right)$$
$$= \frac{V_T u}{V_T + u}$$

126

$$\sqrt{b} \le \frac{\sqrt{a} + \sqrt{b}}{2} \le \sqrt{a}$$
$$\frac{1}{\sqrt{a}} \le \frac{2}{\sqrt{a} + \sqrt{b}} \le \frac{1}{\sqrt{b}} \quad \text{since all positive}$$
$$\frac{1}{a} \le \frac{4}{(\sqrt{a} + \sqrt{b})^2} \le \frac{1}{b} \quad \text{since all positive}$$
$$b(\sqrt{a} + \sqrt{b})^2 \le 4ab \le a(\sqrt{a} + \sqrt{b})^2$$
$$b((\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}))^2 \le 4ab(\sqrt{a} - \sqrt{b})^2 \le a\left((\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})\right)^2$$
$$b(a - b)^2 \le 4ab(a - 2\sqrt{ab} + b) \le a(a - b)^2$$
$$b(a - b)^2 \le 4ab(a + b) - 8ab\sqrt{ab} \le a(a - b)^2$$
$$\frac{(a - b)^2}{8a} \le \frac{a + b}{2} - \sqrt{ab} \le \frac{(a - b)^2}{8b}$$

127 **i**

$$z_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$
$$z_1^r = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^r \quad \text{for } r = 1,2,\ldots,n$$
$$= \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} \quad \text{for } k = 1,2,\ldots,n$$

Since z_1^r covers all possible roots of $z^n = 1$ it is a primitive root.

ii

$$z_5 = \cos\frac{2\pi\times 5}{6} + i\sin\frac{2\pi\times 5}{6}$$
$$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$
$$z_1^r = \left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)^r \quad \text{for } r = 1,2,..,6$$
$$\theta = \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{15\pi}{3}, \frac{20\pi}{3}, \frac{25\pi}{3}, \frac{30\pi}{3}$$
$$= \frac{5\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{6\pi}{3}$$
$$= \frac{2\pi k}{6} \quad \text{for } k = 1,2,\ldots,6$$

Since z_5^r covers all possible roots of $z^6 = 1$ it is a primitive root.

iii

If $k = qh$ is a factor of $n = qh$ then multiples of k will include $\frac{1}{h}$ of the numbers $1,2,\ldots,n$. For z_k to be a primitive root all numbers need to be included, so $h = 1$, since $\frac{1}{1} = 1$.

128 **i**

$$I_1 = \int_0^{\frac{1}{2}} \frac{dx}{1+4x^2}$$
$$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{2}{1+(2x)^2}\,dx$$
$$= \frac{1}{2}\Big[\tan^{-1} 2x\Big]_0^{\frac{1}{2}}$$
$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$
$$= \frac{\pi}{8}$$

ii

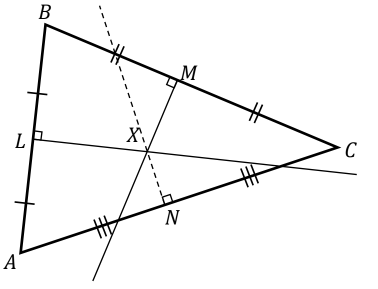
$$I_n = \int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^n} = \int_0^{\frac{1}{2}} (1+4x^2)^{-n}\,dx$$
$$= \left[\frac{x}{(1+4x^2)^n}\right]_0^{\frac{1}{2}} + 2n\int_0^{\frac{1}{2}} \frac{4x^2}{(1+4x^2)^{n+1}}\,dx$$
$$= \frac{1}{2^{n+1}} - 0 + 2n\int_0^{\frac{1}{2}} \frac{1+4x^2-1}{(1+4x^2)^{n+1}}\,dx$$
$$= \frac{1}{2^{n+1}} + 2n\int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^n}\,dx - 2n\int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^{n+1}}\,dx$$
$$I_n = \frac{1}{2^{n+1}} + 2nI_n - 2nI_{n+1}$$
$$(1-2n)I_n = -2nI_{n+1} + \frac{1}{2^{n+1}}$$
$$I_n = \frac{2nI_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)}$$

$u = (1+4x^2)^{-n}$	$\frac{dv}{dx} = 1$
$\frac{du}{dx} = -n(1+4x^2)^{-(n+1)}(8x)$	$v = x$

iii

$$I_n = \frac{2nI_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)}$$
$$I_n - \frac{1}{2^{n+1}(1-2n)} = \frac{2nI_{n+1}}{2n-1}$$
$$I_{n+1} = \frac{2n-1}{2n}I_n + \frac{1}{n2^{n+2}}$$
$$I_3 = \frac{2(2)-1}{2(2)}I_2 + \frac{1}{(2)2^{(2)+2}}$$
$$= \frac{3}{4}I_2 + \frac{1}{32}$$
$$= \frac{3}{4}\left(\frac{2(1)-1}{2(1)}I_1 + \frac{1}{(1)2^{(1)+2}}\right) + \frac{1}{32}$$
$$= \frac{3}{4}\left(\frac{1}{2}\times\frac{\pi}{8} + \frac{1}{8}\right) + \frac{1}{32}$$
$$= \frac{3}{4}\left(\frac{\pi+2}{16}\right) + \frac{1}{32}$$
$$= \frac{3\pi+8}{64}$$

129 In ΔABC, let L and M be the midpoints of AB, BC and ACrespectively, and X be the point of intersection of the perpendiculars from L and M as shown in the diagram.



Let $\underset{\sim}{a} = \overrightarrow{OA}, \underset{\sim}{b} = \overrightarrow{OB}$ etc

$$\left(\underset{\sim}{x} - \underset{\sim}{l}\right)\cdot\left(\underset{\sim}{a} - \underset{\sim}{b}\right) = 0$$
$$\underset{\sim}{x}\cdot\underset{\sim}{a} - \underset{\sim}{x}\cdot\underset{\sim}{b} - \underset{\sim}{l}\cdot\left(\underset{\sim}{a} - \underset{\sim}{b}\right) = 0$$
$$\underset{\sim}{x}\cdot\underset{\sim}{a} - \underset{\sim}{x}\cdot\underset{\sim}{b} - \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)\cdot\left(\underset{\sim}{a} - \underset{\sim}{b}\right) = 0$$
$$\underset{\sim}{x}\cdot\underset{\sim}{a} - \underset{\sim}{x}\cdot\underset{\sim}{b} - \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)\cdot\left(\underset{\sim}{a} - \underset{\sim}{b}\right) = 0$$
$$\underset{\sim}{x}\cdot\underset{\sim}{a} - \underset{\sim}{x}\cdot\underset{\sim}{b} - \frac{1}{2}\left(\underset{\sim}{a}\cdot\underset{\sim}{a} - \underset{\sim}{b}\cdot\underset{\sim}{b}\right) = 0$$
$$\underset{\sim}{x}\cdot\underset{\sim}{a} - \underset{\sim}{x}\cdot\underset{\sim}{b} = \frac{1}{2}\left(\left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2\right) \quad (1)$$
$$\text{similarly } \underset{\sim}{x}\cdot\underset{\sim}{c} - \underset{\sim}{x}\cdot\underset{\sim}{b} = \frac{1}{2}\left(\left|\underset{\sim}{c}\right|^2 + \left|\underset{\sim}{b}\right|^2\right) \quad (2)$$
$$(1)-(2)\text{: } \underset{\sim}{x}\cdot\left(\underset{\sim}{a} - \underset{\sim}{c}\right) = \frac{1}{2}\left(\left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{c}\right|^2\right) \quad (3)$$

$$\begin{aligned}\left(\underset{\sim}{x} - \underset{\sim}{n}\right)\cdot\left(\underset{\sim}{a} - \underset{\sim}{c}\right) &= \underset{\sim}{x}\cdot\left(\underset{\sim}{a} - \underset{\sim}{c}\right) - \underset{\sim}{n}\cdot\left(\underset{\sim}{a} - \underset{\sim}{c}\right)\\ &= \underset{\sim}{x}\cdot\left(\underset{\sim}{a} - \underset{\sim}{c}\right) - \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)\cdot\left(\underset{\sim}{a} - \underset{\sim}{c}\right)\\ &= \underset{\sim}{x}\cdot\left(\underset{\sim}{a} - \underset{\sim}{c}\right) - \frac{1}{2}\left(\underset{\sim}{a}\cdot\underset{\sim}{a} - \underset{\sim}{c}\cdot\underset{\sim}{c}\right)\\ &= \frac{1}{2}\left(\left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{c}\right|^2\right) - \frac{1}{2}\left(\left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{c}\right|^2\right)\\ &= 0\end{aligned}$$

$\therefore NX \perp AC$ so the three perpendicular bisectors of ΔABC are concurrent at X.

130 **i**

For downward motion $\ddot{x} = g - kv^2$

Let $\ddot{x} = 0, v = V$

$$0 = g - kV^2$$
$$kV^2 = g$$

ii

For upward motion

$$\begin{aligned}\ddot{x} &= -(g + kv^2)\\ &= -g\left(1 + \frac{kv^2}{g}\right)\\ &= -g\left(1 + \frac{kv^2}{kV^2}\right)\\ &= -g\left(1 + \frac{v^2}{V^2}\right)\end{aligned}$$

iii

$$v\frac{dv}{dx} = -g\left(1 + \frac{v^2}{V^2}\right)$$
$$= -\frac{g(V^2 + v^2)}{V^2}$$
$$\frac{dv}{dx} = -\frac{g(V^2 + v^2)}{V^2 v}$$
$$\frac{dx}{dv} = -\frac{V^2 v}{g(V^2 + v^2)}$$
$$H = -\frac{V^2}{g}\int_{\frac{V}{3}}^{0}\frac{v}{V^2 + v^2}\,dv$$
$$= \frac{V^2}{2g}\left[\ln(V^2 + v^2)\right]_0^{\frac{V}{3}}$$
$$= \frac{V^2}{2g}\left(\ln\left(V^2 + \frac{V^2}{9}\right) - \ln V^2\right)$$
$$= \frac{V^2}{2g}\left(\ln\left(\frac{10V^2}{9}\right) - \ln V^2\right)$$
$$= \frac{V^2}{2g}\ln\left(\frac{10}{9}\right)$$

iv

$$v\frac{dv}{dy} = g\left(1 - \frac{v^2}{V^2}\right)$$
$$= \frac{g(V^2 - v^2)}{V^2}$$
$$\frac{dv}{dy} = \frac{g(V^2 - v^2)}{V^2 v}$$
$$\frac{dy}{dv} = \frac{V^2 v}{g(V^2 - v^2)}$$
$$y = \frac{V^2}{g}\int_0^v \frac{v}{V^2 - v^2}\,dv$$
$$= -\frac{1}{2}\times\frac{V^2}{g}\int_0^v \frac{-2v}{V^2 - v^2}\,dv$$
$$= -\frac{V^2}{2g}\left[\ln(V^2 - v^2)\right]_0^v$$
$$= \frac{V^2}{2g}(\ln(V^2) - \ln(V^2 - v^2))$$
$$= \frac{V^2}{2g}\ln\left(\frac{V^2}{V^2 - v^2}\right)$$

v

Let $y = H$ and $v = U$

$$\frac{V^2}{2g}\ln\left[\frac{V^2}{V^2 - U^2}\right] = \frac{V^2}{2g}\ln\left(\frac{10}{9}\right)$$
$$\frac{V^2}{V^2 - U^2} = \frac{10}{9}$$
$$9V^2 = 10V^2 - 10U^2$$
$$10U^2 = V^2$$
$$\frac{V^2}{U^2} = 10$$
$$\frac{V}{U} = \sqrt{10}$$

131 **i**

$T(k,x) - T(k,x+1)$
$= \dfrac{k!}{x(x+1)(x+2)\dots(x+k)} - \dfrac{k!}{(x+1)(x+2)\dots(x+k+1)}$
$= \dfrac{k!\,(x+k+1) - k!\,x}{x(x+1)(x+2)\dots(x+k)(x+k+1)}$
$= \dfrac{(k+1)!}{x(x+1)(x+2)\dots(x+k)(x+k+1)}$
$= T(k+1,x)$

ii

Let $P(n,x)$ represent the proposition

$P(1,x)$ is true since $T(1,x) = \frac{^1C_0}{x} - \frac{^1C_1}{x+1} = \frac{1}{x} - \frac{1}{x+1} = \frac{x+1-x}{x(x+1)} = \frac{1!}{x(x+1)}$

If $P(k,x)$ is true for some arbitrary $x \ge 1$ then

$T(k,x) = \dfrac{^kC_0}{x} - \dfrac{^kC_1}{x+1} + \dfrac{^kC_2}{x+2} - \dots + (-1)^k \dfrac{^kC_k}{x+k} = \dfrac{k!}{x(x+1)(x+2)\dots(x+k)}$

RTP $T(k+1,x)$:

$T(k+1,x) = \dfrac{^{k+1}C_0}{x} - \dfrac{^{k+1}C_1}{x+1} + \dfrac{^{k+1}C_2}{x+2} - \dots + (-1)^k \dfrac{^kC_k}{x+k} + (-1)^{k+1} \dfrac{^{k+1}C_{k+1}}{x+k+1}$
$= \dfrac{(k+1)!}{x(x+1)(x+2)\dots(x+k+1)}$

$\text{LHS} = \dfrac{^{k+1}C_0}{x} - \dfrac{^{k+1}C_1}{x+1} + \dfrac{^{k+1}C_2}{x+2} - \dots + (-1)^k \dfrac{^{k+1}C_k}{x+k} + (-1)^{k+1} \dfrac{^{k+1}C_{k+1}}{x+k+1}$
$= \dfrac{^kC_0}{x} - \dfrac{^kC_0 + {^kC_1}}{x+1} + \dfrac{^kC_1 + {^kC_2}}{x+2} - \dots + (-1)^k \dfrac{^kC_{k-1} + {^kC_k}}{x+k} + (-1)^{k+1} \dfrac{^{k+1}C_{k+1}}{x+k+1}$
$= \left(\dfrac{^kC_0}{x} - \dfrac{^kC_1}{x+1} + \dfrac{^kC_2}{x+2} - \dots + (-1)^k \dfrac{^kC_k}{x+k}\right) -$
$\left(\dfrac{^kC_0}{x+1} - \dfrac{^kC_1}{x+2} + \dfrac{^kC_2}{x+3} \dots + (-1)^k \dfrac{^kC_{k-1}}{x+k} + (-1)^{k+1} \dfrac{^kC_k}{x+k+1}\right)$
$= T(k,x) - T(k,x+1)$
$= T(k+1,x)$ from (i)
$= \dfrac{(k+1)!}{x(x+1)(x+2)\dots(x+k+1)}$

$\therefore P(k,x) \Rightarrow P(k+1,x)$
$\therefore P(n,x)$ is true for $n \ge 1$ by induction

iii

$\dfrac{^nC_0}{1} - \dfrac{^nC_1}{3} + \dfrac{^nC_2}{5} - \dots + (-1)^n \dfrac{^nC_n}{2n+1}$
$= \dfrac{1}{2}\left(\dfrac{^nC_0}{1/2} - \dfrac{^nC_1}{1 + 1/2} + \dfrac{^nC_2}{2 + 1/2} - \dots + (-1)^n \dfrac{^nC_n}{n + 1/2}\right)$
$= \dfrac{1}{2} \times T\left(n, \dfrac{1}{2}\right)$
$= \dfrac{1}{2} \times \dfrac{n!}{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\dots\left(\frac{1}{2}+n\right)}$
$= \dfrac{1}{2} \times \dfrac{2^{n+1}n!}{1 \times 3 \times 5 \times \dots \times (2n+1)}$
$= \dfrac{2^n n!}{1 \times 3 \times 5 \times \dots \times (2n+1)}$

132 $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$
$= \cos^3\theta + 3\cos^2\theta\sin\theta\, i - 3\cos\theta\sin^2\theta - \sin^3\theta\, i$
$= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta) + i[3(1 - \sin^2\theta)\sin\theta - \sin^3\theta]$
$= 4\cos^3\theta - 3\cos\theta + i(3 - 4\sin^3\theta)$

Equating real components:

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$8x^3 - 6x - \sqrt{2} = 0$
$8x^3 - 6x = \sqrt{2}$
$4x^3 - 3x = \dfrac{1}{\sqrt{2}}$

Let $x = \cos\theta$

$4\cos^3\theta - 3\cos\theta = \dfrac{1}{\sqrt{2}}$
$\cos 3\theta = \dfrac{1}{\sqrt{2}}$
$3\theta = \dfrac{\pi}{4}, \dfrac{7\pi}{4}, \dfrac{9\pi}{4}$
$\theta = \dfrac{\pi}{12}, \dfrac{7\pi}{12}, \dfrac{3\pi}{4}$
$\therefore x = \cos\dfrac{\pi}{12}, \cos\dfrac{7\pi}{12}, \cos\dfrac{3\pi}{4}$
$= 0.966, -0.259, -0.707$

133 **i**

$$
\begin{aligned}
(2k)!! &= (2k)(2k-2)(2k-4)\ldots 2\\
&= 2(k)2(k-1)2(k-2)\ldots 2(1)\\
&= 2^k k!
\end{aligned}
$$

ii

$$
\begin{aligned}
&(2k+1)!!\,2^k k!\\
&= [(2k+1)(2k-1)(2k-3)\ldots 1][2(k)2(k-1)2(k-2)\ldots 2(1)]\\
&= [(2k+1)(2k-1)(2k-3)\ldots 1][(2k)(2k-2)(2k-4)\ldots 2]\\
&= (2k+1)(2k)(2k-1)(2k-2)(2k-3)(2k-4)\ldots 2\cdot 1\\
&= (2k+1)!\\
&\therefore (2k+1)!! = \frac{(2k+1)!}{2^k k!}
\end{aligned}
$$

iii

$$
\begin{aligned}
I_{2n} &= \int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx\\
&= -\left[\cos x \sin^{2n-1} x\right]_0^{\frac{\pi}{2}} + (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n-2} x \cos^2 x\, dx\\
&= -(0-0) + (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n-2} x\,(1-\sin^2 x)\, dx\\
&= (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n-2} x\, dx - (2n-1)\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx\\
I_{2n} &= (2n-1)I_{2n-2} - (2n-1)I_{2n}\\
2nI_{2n} &= (2n-1)I_{2n-2}\\
I_{2n} &= \frac{2n-1}{2n} I_{2n-2}\\
&= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2} I_{2n-4}\\
&= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\frac{2n-5}{2n-4} I_{2n-6}\\
&= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\frac{2n-5}{2n-4}\times\ldots\times\frac{1}{2} I_0\\
&= \frac{(2n-1)!!}{(2n-2)!!}\int_0^{\frac{\pi}{2}} dx\\
&= \frac{\frac{(2n)!}{2^n n!}}{2^n n!}\times\left[x\right]_0^{\frac{\pi}{2}}\\
&= \frac{(2n)!}{2^{2n}(n!)^2}\left(\frac{\pi}{2}-0\right)\\
&= \frac{\pi(2n)!}{2^{2n+1}(n!)^2}
\end{aligned}
$$

$u = \sin^{2n-1} x$	$\frac{dv}{dx} = \sin x$
$\frac{du}{dx} = (2n-1)\sin^{2n-2} x \cos x$	$v = -\cos x$

134

$$
\begin{aligned}
\frac{y}{x} &= \frac{\frac{3at^2}{1+t^3}}{\frac{3at}{1+t^3}} = t\\
\therefore x &= \frac{3a\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^3}\\
x &= \frac{\frac{3ay}{x}}{1+\frac{y^3}{x^3}}\\
x &= \frac{3ax^2y}{x^3+y^3}\\
x(x^3+y^3) &= 3ax^2y\\
x^3+y^3 &= 3axy
\end{aligned}
$$

135 **i**

On the downward flight:

$$\ddot{x} = g - kv^2$$
$$\therefore 0 = g - kV^2$$
$$k = \frac{g}{V^2}$$

On the upward flight:

$$\begin{aligned}\ddot{x} &= -g - kv^2 \\ &= -g - \left(\frac{g}{V^2}\right)v^2 \\ &= -g\left(1 + \frac{v^2}{V^2}\right) \\ &= -\frac{g}{V^2}(V^2 + v^2)\end{aligned}$$

ii

$$v\frac{dv}{dx} = -\frac{g}{V^2}(V^2 + v^2)$$
$$\frac{dv}{dx} = -\frac{g(V^2 + v^2)}{V^2 v}$$
$$\frac{dx}{dv} = -\frac{V^2 v}{g(V^2 + v^2)}$$
$$\begin{aligned}H &= -\int_V^0 \frac{V^2 v}{g(V^2 + v^2)}\,dv \\ &= -\frac{V^2}{g}\int_V^0 \frac{v}{V^2 + v^2}\,dv \\ &= \frac{V^2}{2g}\Big[\ln(V^2 + v^2)\Big]_0^V \\ &= \frac{V^2}{2g}\left(\ln(2V^2) - \ln(V^2)\right) \\ &= \frac{V^2 \ln 2}{2g}\end{aligned}$$

iii

$$\frac{dv}{dt} = -\frac{g}{V^2}(V^2 + v^2)$$
$$\frac{dt}{dv} = -\frac{V^2}{g(V^2 + v^2)}$$
$$\begin{aligned}t &= -\int_V^v \frac{V^2}{g(V^2 + v^2)}\,dv \\ &= \frac{V^2}{g}\int_v^V \frac{1}{V^2 + v^2}\,dv \\ &= \frac{V}{g}\left[\tan^{-1}\frac{v}{V}\right]_v^V \\ &= \frac{V}{g}\left(\frac{\pi}{4} - \tan^{-1}\frac{v}{V}\right)\end{aligned}$$

iv

$$\begin{aligned}\frac{H}{2} &= -\int_V^v \frac{V^2 v}{g(V^2 + v^2)}\,dv \\ \frac{V^2 \ln 2}{4g} &= \frac{V^2}{g}\int_v^V \frac{v}{V^2 + v^2}\,dv \\ \frac{\ln 2}{4} &= \frac{1}{2}\Big[\ln(V^2 + v^2)\Big]_v^V \\ \frac{\ln 2}{2} &= \ln 2V^2 - \ln(V^2 + v^2) \\ \ln\sqrt{2} &= \ln\left(\frac{2V^2}{V^2 + v^2}\right) \\ \frac{2V^2}{V^2 + v^2} &= \sqrt{2} \\ \therefore 1 + \sin\left(\frac{2g}{V}t\right) &= \sqrt{2} \\ \sin\left(\frac{2g}{V}t\right) &= \sqrt{2} - 1 \\ \frac{2g}{V}t &= \sin^{-1}(\sqrt{2} - 1) \\ t &= \frac{V}{2g}\sin^{-1}(\sqrt{2} - 1)\ \text{ s}\end{aligned}$$

136 Let $n = a_0 + a_1 \times 10^1 + a_2 \times 10^2 + \ldots + a_k \times 10^k$ for integral $a_0, a_1, a_2 \ldots a_k > 0$.

$\therefore n = a_0 + a_1 \times (3q_1 + 1) + a_2 \times (3q_2 + 1) + \ldots + a_k \times (3q_k + 1)$ for integral $q_1, q_2 \ldots q_k > 0$

$\therefore n = (a_0 + a_1 + a_2 + \ldots + a_k) + 3(a_2q_2 + \ldots + a_kq_k)$

$= (a_0 + a_1 + a_2 + \ldots + a_k) + 3m$ for integral m since a_i, q_i are integral

If n is divisible by 3, let $n = 3j$ for integral j.

$\therefore (a_0 + a_1 + a_2 + \ldots + a_k) + 3m = 3j$

$a_0 + a_1 + a_2 + \ldots + a_k = 3(j - m)$

$= 3t$ for integral t since j, m are integral

$\therefore$ if n is divisible by 3 then the sum of its digits is divisible by 3

Conversely, if the sum of its digits is divisible by 3

$a_0 + a_1 + a_2 + \ldots + a_k = 3s$ for integral s

$\therefore n = 3s + 3m$

$= 3r$ for integral r since s, m are integral

$\therefore$ if the sum of its digits is divisible by 3 the sum of its digits is divisible by 3 then n is divisible by 3

$\therefore$ a positive integer n is divisible by 3 if and only if the sum of its digits is divisible by 3.

137 **i**

$\alpha^7 = 1$ since α is a root of $z^7 = 1$

$$z^7 = 1$$
$$z^7 - 1 = 0$$
$$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$$
$$\therefore (\alpha - 1)(\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) = 0$$
$$\therefore 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0 \quad \text{since } \alpha \neq 1 \text{ as it is non} - \text{real}$$

ii

$$\theta + \delta = \alpha + \alpha^2 + \alpha^4 + \alpha^3 + \alpha^5 + \alpha^6$$
$$= \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6$$
$$= 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 - 1$$
$$= 0 - 1$$
$$= -1$$
$$\theta\delta = (\alpha + \alpha^2 + \alpha^4)(\alpha^3 + \alpha^5 + \alpha^6)$$
$$= \alpha^4 + \alpha^6 + \alpha^7 + \alpha^5 + \alpha^7 + \alpha^8 + \alpha^7 + \alpha^9 + \alpha^{10}$$
$$= \alpha^4 + \alpha^6 + 1 + \alpha^5 + 1 + \alpha + 1 + \alpha^2 + \alpha^3$$
$$= 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + 2$$
$$= 0 + 2$$
$$= 2$$
$$(z - \theta)(z - \delta) = 0$$
$$z^2 - (\theta + \delta)z + \theta\delta = 0$$
$$z^2 + z + 2 = 0$$

iii

$$z = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm i\sqrt{7}}{2}$$

$\therefore \theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $\delta = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$

iv

$$\alpha = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$$

$$\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$$
$$\alpha + \alpha^2 + \alpha^4 = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$$
$$\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} + \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7} = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$$

Equating real and imaginary components:

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2} \rightarrow \therefore \cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$$
$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = -\frac{\sqrt{7}}{2} \rightarrow \therefore \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$$

138 **i**

$(1-t^2)^{\frac{n-3}{2}}-(1-t^2)^{\frac{n-1}{2}}$
$=(1-t^2)^{\frac{n-3}{2}}\left(1-(1-t^2)\right)$
$=t^2(1-t^2)^{\frac{n-3}{2}}$

ii

$$\begin{aligned}I_n&=\int_0^1(1-t^2)^{\frac{n-1}{2}}dt\\&=\left[t(1-t^2)^{\frac{n-1}{2}}\right]_0^1+(n-1)\int_0^1 t^2(1-t^2)^{\frac{(n-3)}{2}}dt\\&=0+(n-1)\int_0^1\left((1-t^2)^{\frac{n-3}{2}}-(1-t^2)^{\frac{n-1}{2}}\right)dt\\&=(n-1)\int_0^1(1-t^2)^{\frac{n-3}{2}}dt-(n-1)\int_0^1(1-t^2)^{\frac{n-1}{2}}dt\end{aligned}$$

$I_n=(n-1)I_{n-2}-(n-1)I_n$
$nI_n=(n-1)I_{n-2}$

$u=(1-t^2)^{\frac{n-1}{2}}$	$\frac{dv}{dt}=1$
$\frac{du}{dx}=\frac{n-1}{2}(1-t^2)^{\frac{n-3}{2}}(-2t)$	$v=t$

iii

Let $P(n)$ represent the proposition.
$P(1)$ is true since

$$\begin{aligned}J_1&=I_1I_0\\&=\int_0^1 dt\times\int_0^1(1-t^2)^{-\frac{1}{2}}dt\\&=\Big[t\Big]_0^1\times\Big[\sin^{-1}t\Big]_0^1\\&=(1-0)\times\left(\frac{\pi}{2}-0\right)\\&=\frac{\pi}{2}\end{aligned}$$

If $P(k)$ is true for arbitrary $k\geq 1$ then
$J_k=kI_kI_{k-1}=\frac{\pi}{2}$
RTP: $P(k+1)$ $J_{k+1}=(k+1)I_{k+1}I_k=\frac{\pi}{2}$
$(k+1)I_{k+1}I_k$
$=kI_{k-1}I_k$ from (ii)
$=\frac{\pi}{2}$ from $P(k)$
$\therefore P(k)\Rightarrow P(k+1)$
$\therefore P(n)$ is true for $n\geq 1$ by induction

iv

$0\leq 1-t^2\leq 1$ for $0\leq t\leq 1$
$\therefore\ 0\leq (1-t^2)^{\frac{n-1}{2}}\leq (1-t^2)^{\frac{n-2}{2}}$
$\therefore 0<\int_0^1(1-t^2)^{\frac{n-1}{2}}dt<\int_0^1(1-t^2)^{\frac{n-2}{2}}dt$
$0<I_n<I_{n-1}$

v

$$\begin{aligned}J_n=nI_nI_{n-1}&=\frac{\pi}{2}\\I_nI_{n-1}&=\frac{\pi}{2n}\\I_n^2&<\frac{\pi}{2n}\ \text{from (iv)}\\I_n&<\sqrt{\frac{\pi}{2n}}\\J_{n+1}=(n+1)I_{n+1}I_n&=\frac{\pi}{2}\\I_{n+1}I_n&=\frac{\pi}{2(n+1)}\\I_n^2&>\frac{\pi}{2n}\ \text{from (iv)}\\\therefore\ \sqrt{\frac{\pi}{2(n+1)}}&<I_n^2\\\therefore\ \sqrt{\frac{\pi}{2(n+1)}}&<I_n<\sqrt{\frac{\pi}{2n}}\end{aligned}$$

139

$$|AB| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$\therefore r = \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$(u,v,w) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

$$\left(\underset{\sim}{p}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{p}-\underset{\sim}{b}\right) = 0$$
$$\underset{\sim}{p}\cdot\underset{\sim}{p}-\underset{\sim}{p}\cdot\left(\underset{\sim}{a}+\underset{\sim}{b}\right)+\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$$
$$x^2+y^2+z^2-\underset{\sim}{p}\cdot\left(2\underset{\sim}{m}\right)+x_1x_2+y_1y_2+z_1z_2 = 0$$
$$x^2+y^2+z^2-2\left(x\left(\frac{x_1+x_2}{2}\right)+y\left(\frac{y_1+y_2}{2}\right)+z\left(\frac{z_1+z_2}{2}\right)\right)+x_1x_2+y_1y_2+z_1z_2 = 0$$
$$x^2-2x\left(\frac{x_1+x_2}{2}\right)+x_1x_2+y^2-2y\left(\frac{y_1+y_2}{2}\right)+y_1y_2+z^2-2z\left(\frac{z_1+z_2}{2}\right)+z_1z_2 = 0$$
$$x^2-2x\left(\frac{x_1+x_2}{2}\right)+\left(\frac{x_1+x_2}{2}\right)^2+y^2-2y\left(\frac{y_1+y_2}{2}\right)+\left(\frac{y_1+y_2}{2}\right)^2+z^2-2z\left(\frac{z_1+z_2}{2}\right)+\left(\frac{z_1+z_2}{2}\right)^2$$
$$= \left(\frac{x_1-x_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2+\left(\frac{z_1-z_2}{2}\right)^2$$
$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2+\left(z-\frac{z_1+z_2}{2}\right)^2 = \frac{1}{4}\left((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right)$$
$$(x-u)^2+(y-v)^2+(z-w)^2 = r^2$$

140 **i** $\boldsymbol{\alpha}$

$$\frac{d\dot{x}}{dt} = -k\dot{x}$$
$$\frac{dt}{d\dot{x}} = -\frac{1}{k\dot{x}}$$
$$t = -\frac{1}{k}\int_{V\cos\alpha}^{\dot{x}}\frac{dx}{\dot{x}}$$
$$-kt = \Big[\ln\dot{x}\Big]_{V\cos\alpha}^{\dot{x}}$$
$$-kt = \ln\dot{x}-\ln(V\cos\alpha)$$
$$\ln\dot{x} = \ln(V\cos\alpha)-kt$$
$$\dot{x} = V\cos\alpha\ e^{-kt}$$

$\boldsymbol{\beta}$

$$\frac{d\dot{y}}{dt} = -g-k\dot{y}$$
$$\frac{dt}{d\dot{y}} = -\frac{1}{g+k\dot{y}}$$
$$t = -\int_{V\sin\alpha}^{\dot{y}}\frac{d\dot{y}}{g+k\dot{y}}$$
$$-t = \frac{1}{k}\Big[\ln(g+k\dot{y})\Big]_{V\sin\alpha}^{\dot{y}}$$
$$-kt = \ln(g+k\dot{y})-\ln(g+kV\sin\alpha)$$
$$\ln(g+k\dot{y}) = \ln(g+kV\sin\alpha)-kt$$
$$g+k\dot{y} = (g+kV\sin\alpha)e^{-kt}$$
$$k\dot{y} = (g+kV\sin\alpha)e^{-kt}-g$$
$$\dot{y} = \left(\frac{g}{k}+V\sin\alpha\right)e^{-kt}-\frac{g}{k}$$

ii $\boldsymbol{\alpha}$

$$\frac{dx}{dt} = V\cos\alpha\ e^{-kt}$$
$$x = V\cos\alpha\int_0^t e^{-kt}\,dt$$
$$= -\frac{V\cos\alpha}{k}\Big[e^{-kt}\Big]_0^t$$
$$= -\frac{V\cos\alpha}{k}\left(e^{-kt}-1\right)$$
$$= \frac{V\cos\alpha}{k}\left(1-e^{-kt}\right)$$

$\boldsymbol{\beta}$

$$\frac{dy}{dt} = \left(\frac{g}{k}+V\sin\alpha\right)e^{-kt}-\frac{g}{k}$$
$$y = \int_0^t\left(\left(\frac{g}{k}+V\sin\alpha\right)e^{-kt}-\frac{g}{k}\right)dt$$
$$= \left[-\frac{1}{k}\left(\frac{g}{k}+V\sin\alpha\right)e^{-kt}-\frac{gt}{k}\right]_0^t$$
$$= \left(-\frac{1}{k}\left(\frac{g}{k}+V\sin\alpha\right)e^{-kt}-\frac{gt}{k}\right)-\left(-\frac{1}{k}\left(\frac{g}{k}+V\sin\alpha\right)-0\right)$$
$$= \left(\frac{g}{k^2}+\frac{V\sin\alpha}{k}\right)\left(1-e^{-kt}\right)-\frac{g}{k}t$$

140 … **iii**

Let $\dot{y} = 0$

$$\left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k} = 0$$

$$\frac{g}{k}\left(1 - e^{-kt}\right) = V\sin\alpha\ e^{-kt}$$

$$1 - e^{-kt} = \frac{kV\sin\alpha}{g}e^{-kt} \qquad (1)$$

$$\frac{\text{kV}\sin\alpha + g}{g}e^{-kt} = 1$$

$$e^{-kt} = \frac{g}{kV\sin\alpha + g} \qquad (2)$$

sub $(1), (2)$ in ii α:

$$x = \frac{V\cos\alpha}{k} \times \frac{kV\sin\alpha}{g} \times \frac{g}{kV\sin\alpha + g}$$

$$= \frac{V^2\sin\alpha\cos\alpha}{kV\sin\alpha + g}$$

$$= \frac{V^2\sin 2\alpha}{2(g + kV\sin\alpha)}$$

141 Let $P(n)$ represent the proposition.

$P(1)$ is true since when one circle and a chord of that circle are drawn in a plane the two segments can be coloured in the first two colours, and the outside of the circle in the third colour. Say red, blue and yellow respectively (Fig 1).

If $P(k)$ is true for some arbitrary $k \geq 1$ then when k circles and chords are drawn on a plane then the regions created can be coloured with three colours in such a way that no two regions sharing the same length of border are the same colour.

RTP $P(k+1)$ when $k+1$ circles and chords are drawn on a plane then the regions created can be coloured with three colours in such a way that no two regions sharing the same length of border are the same colour.

To the k lines and/or circles from $P(k)$ add a circle and a chord to that circle (Fig 2):

- For regions intersected by the minor segment change yellow to red, red to blue and blue to yellow.
- For regions intersected by the major segment change the colours in the reverse order, so red to yellow, yellow to blue and blue to red (Fig 3).
- There are three cases to consider:
    - regions that are outside the new circle do not change colour, so no regions sharing the same border are the same colour, from $P(k)$.
    - any region that is cut by one segment has the outside stay the same colour and the inside change colour, so no regions sharing the same border are the same colour.
    - any region that is cut by two segments has the outside stay the same colour and the two inside regions change colour in opposite directions, so no regions sharing the same border are the same colour.

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction

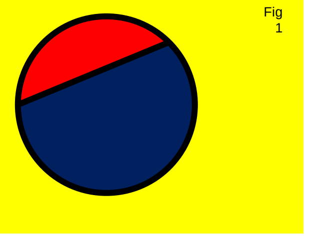


Fig 1

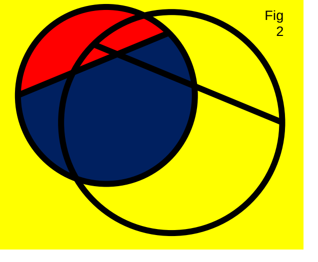


Fig 2

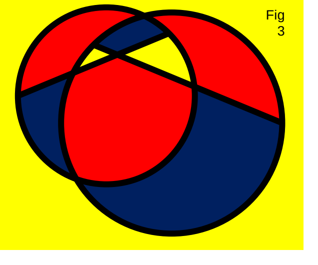


Fig 3

142 **i**

$$\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1}$$

$$= \frac{\frac{1-t^2}{1+t^2} + \frac{2ti}{1+t^2} - 1}{\frac{1-t^2}{1+t^2} + \frac{2ti}{1+t^2} + 1}$$

$$= \frac{1 - t^2 + 2ti - 1 - t^2}{1 - t^2 + 2ti + 1 + t^2}$$

$$= \frac{2ti - 2t^2}{2 + 2ti}$$

$$= \frac{t(i - t)}{(1 + ti)} \times \frac{1 - ti}{1 - ti}$$

$$= \frac{ti + t^2 - t^2 + t^3 i}{1 + t^2}$$

$$= \frac{it(1 + t^2)}{1 + t^2}$$

$$= it$$

$$= i\tan\frac{\theta}{2}$$

142 …

ii

$\omega = 1, \cos\left(\pm\frac{2\pi}{5}\right) + i\sin\left(\pm\frac{2\pi}{5}\right), \cos\left(\pm\frac{4\pi}{5}\right) + i\sin\left(\pm\frac{4\pi}{5}\right)$

***iii**

Let $\omega = \frac{2+z}{2-z}$

$$2\omega - \omega z = 2 + z$$
$$z(\omega+1) = 2(\omega-1)$$
$$z = 2\left(\frac{\omega-1}{\omega+1}\right)$$
$$= 2i\tan\frac{\theta}{2} \quad \text{for } \omega = \cos\theta + i\sin\theta$$
$$= 2i\tan\left(\frac{k\pi}{5}\right) \quad \text{for } k = 0, \pm1, \pm2$$

iv

$$\left(\frac{2+z}{2-z}\right)^5 = 1$$
$$(2+z)^5 = (2-z)^5$$
$$32 + 5(2^4)z + 10(2^3)z^2 + 10(2^2)z^3 + 5(2)z^4 + z^5$$
$$= 32 - 5(2^4)z + 10(2^3)z^2 - 10(2^2)z^3 + 5(2)z^4 - z^5$$
$$10(2^4)z + 20(2^2)z^3 + 2z^5 = 0$$
$$80z + 40z^3 + z^5 = 0$$
$$z^5 + 40z^3 + 80z = 0$$
$$z(z^4 + 40z^2 + 80) = 0$$

The product of the roots excluding $z = 0$ is 80

$$\therefore 2i\tan\left(-\frac{\pi}{5}\right) \times 2i\tan\left(-\frac{2\pi}{5}\right) \times 2i\tan\left(\frac{\pi}{5}\right) \times 2i\tan\left(\frac{\pi}{5}\right) = 80$$
$$-2\tan\left(\frac{\pi}{5}\right) \times \left(-2\tan\left(\frac{2\pi}{5}\right)\right) \times 2\tan\left(\frac{\pi}{5}\right) \times 2\tan\left(\frac{\pi}{5}\right) = 80$$
$$\left(\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\right)^2 = 5$$
$$\therefore \tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5} \text{ since } \tan\frac{\pi}{5}, \tan\frac{2\pi}{5} > 0$$

143

$$\int\sqrt{\frac{x}{x+1}}\,dx$$
$$= \int\sqrt{\frac{x}{x+1}} \times \sqrt{\frac{x}{x}}\,dx$$
$$= \int\frac{x}{\sqrt{x^2+x}}\,dx$$
$$= \int\frac{x}{\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}}}\,dx$$
$$= \int\frac{x}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\,dx$$
$$= \int\frac{x+\frac{1}{2}-\frac{1}{2}}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\,dx$$
$$= \int\frac{x+\frac{1}{2}}{\sqrt{x^2+x}}\,dx - \frac{1}{2}\int\frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\,dx$$
$$= \frac{1}{2}\int(2x+1)(x^2+x)^{-\frac{1}{2}}\,dx - \frac{1}{2}\int\frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\,dx$$
$$= \frac{1}{2} \times 2(x^2+x)^{\frac{1}{2}} - \frac{1}{2}\ln\left|x + \frac{1}{2} + \sqrt{x^2+x}\right| + c$$
$$= \sqrt{x^2+x} - \frac{1}{2}\ln\left|x + \frac{1}{2} + \sqrt{x^2+x}\right| + c$$
$$= \sqrt{x^2+x} - \frac{1}{2}\ln\left|\left(\sqrt{\frac{x}{2}} + \sqrt{\frac{x+1}{2}}\right)^2\right| + c$$
$$= \sqrt{x^2+x} - \ln\left|\sqrt{\frac{x}{2}} + \sqrt{\frac{x+1}{2}}\right| + c$$
$$= \sqrt{x^2+x} - \ln\left(\sqrt{x} + \sqrt{x+1}\right) + \ln\sqrt{2} + c$$
$$= \sqrt{x^2+x} - \ln\left(\sqrt{x} + \sqrt{x+1}\right) + c$$

144 $\underset{\sim}{w} = \overrightarrow{OV} + \overrightarrow{VW} = \overrightarrow{OV} + \overrightarrow{UV} = \underset{\sim}{v} + \underset{\sim}{v} - \underset{\sim}{u} = 2\underset{\sim}{v} - \underset{\sim}{u}$

$\underset{\sim}{s} = (1+\lambda)\overrightarrow{OV} = (1+\lambda)\underset{\sim}{v}$

$\overrightarrow{SW} = \underset{\sim}{w} - \underset{\sim}{s} = 2\underset{\sim}{v} - \underset{\sim}{u} - (1+\lambda)\underset{\sim}{v} = (1-\lambda)\underset{\sim}{v} - \underset{\sim}{u}$

$\overrightarrow{SW} \cdot \overrightarrow{UV} = 0$

$$\left((1-\lambda)\underset{\sim}{v} - \underset{\sim}{u}\right) \cdot \left(\underset{\sim}{v} - \underset{\sim}{u}\right) = 0$$

$$(1-\lambda)\underset{\sim}{v} \cdot \underset{\sim}{v} - (1-\lambda)\underset{\sim}{v} \cdot \underset{\sim}{u} - \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{u} = 0$$

$$(1-\lambda)\left|\underset{\sim}{v}\right|^2 + (\lambda - 2)\underset{\sim}{u} \cdot \underset{\sim}{v} + \left|\underset{\sim}{u}\right|^2 = 0$$

$$\lambda\left(\underset{\sim}{u} \cdot \underset{\sim}{v} - \left|\underset{\sim}{v}\right|^2\right) = 2\underset{\sim}{u} \cdot \underset{\sim}{v} - \left|\underset{\sim}{v}\right|^2 - \left|\underset{\sim}{u}\right|^2$$

$$\lambda = \frac{2\underset{\sim}{u} \cdot \underset{\sim}{v} - \left|\underset{\sim}{v}\right|^2 - \left|\underset{\sim}{u}\right|^2}{\underset{\sim}{u} \cdot \underset{\sim}{v} - \left|\underset{\sim}{v}\right|^2}$$

145 **i**

When falling $\ddot{x} = mg - kv^2$.

Let $\ddot{x} = 0, m = 1, v = V_T$

$$0 = g - kV_T^2$$

$$V_T^2 = \frac{g}{k}$$

$$V_T = \sqrt{\frac{g}{k}}$$

$$\therefore U^2 = \left(\sqrt{\frac{g}{k}}^2\right) = \frac{g}{k}$$

When rising

$$\ddot{x} = -(g + kv^2) = -\left(g + \frac{g}{U^2}v^2\right) = -\frac{g}{U^2}(U^2 + v^2)$$

ii

$$\frac{dv}{dt} = -\frac{g}{U^2}(U^2 + v^2)$$

$$\frac{dt}{dv} = -\frac{U^2}{g} \times \frac{1}{U^2 + v^2}$$

$$t = -\frac{U^2}{g}\int_U^v \frac{dv}{U^2 + v^2}$$

$$\frac{gt}{U^2} = \frac{1}{U}\left[\tan^{-1}\left(\frac{v}{U}\right)\right]_v^U$$

$$\frac{gt}{U} = \tan^{-1} 1 - \tan^{-1}\left(\frac{v}{U}\right)$$

$$\tan^{-1}\left(\frac{v}{U}\right) = \frac{\pi}{4} - \frac{gt}{U}$$

$$\frac{v}{U} = \tan\left(\frac{\pi}{4} - \frac{gt}{U}\right)$$

iii

$$\therefore \frac{dx}{dt} = U\tan\left(\frac{\pi}{4} - \frac{gt}{U}\right)$$

$$x = U\int_0^t \tan\left(\frac{\pi}{4} - \frac{gt}{U}\right)dt$$

$$= -\frac{U^2}{g}\int_0^t \left(-\frac{g}{U}\right)\tan\left(\frac{\pi}{4} - \frac{gt}{U}\right)dt$$

$$= \frac{U^2}{g}\left[\ln\left|\cos\left(\frac{\pi}{4} - \frac{gt}{U}\right)\right|\right]_0^t$$

$$= \frac{U^2}{g}\left(\ln\left|\cos\left(\frac{\pi}{4} - \frac{gt}{U}\right)\right| - \ln\left|\cos\left(\frac{\pi}{4}\right)\right|\right)$$

$$= \frac{U^2}{g}\left(\ln\left|\cos\left(\frac{\pi}{4} - \frac{gt}{U}\right)\right| + \ln\sqrt{2}\right)$$

$$= \frac{U^2}{g}\ln\left[\sqrt{2}\left|\cos\left(\frac{\pi}{4} - \frac{gt}{U}\right)\right|\right]$$

$$\therefore \frac{x}{U} = \frac{U}{g}\ln\left[\sqrt{2}\left|\cos\left(\frac{\pi}{4} - \frac{g}{U}t\right)\right|\right]$$

iv

Let $v = 0$

$$\tan\left(\frac{\pi}{4} - \frac{g}{U}T\right) = 0 \text{ from (ii)}$$

$$\frac{\pi}{4} - \frac{g}{U}T = 0$$

$$T = \frac{U\pi}{4g}$$

$$\therefore \frac{1}{2}T = \frac{U\pi}{8g}$$

At $t = \frac{1}{2}T$:

$$\frac{x}{U} = \frac{U}{g}\ln\left[\sqrt{2}\left|\cos\left(\frac{\pi}{4} - \frac{g}{U}\left(\frac{U\pi}{8g}\right)\right)\right|\right]$$

$$= \frac{U}{g}\ln\left[\sqrt{2}\cos\left(\frac{\pi}{8}\right)\right]$$

$$= \frac{U}{g}\ln\left[\sqrt{2}\sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{4}\right)}\right]$$

$$= \frac{U}{g}\ln\left[\sqrt{2}\sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)}\right]$$

$$= \frac{U}{g}\ln\left[\sqrt{\left(1 + \frac{1}{\sqrt{2}}\right)}\right]$$

$$= \frac{U}{2g}\ln\left[1 + \frac{1}{\sqrt{2}}\right]$$

At $t = \frac{1}{2}T$:

$$\frac{H}{U} = \frac{U}{g}\ln\left[\sqrt{2}\left|\cos\left(\frac{\pi}{4} - \frac{g}{U}\left(\frac{U\pi}{4g}\right)\right)\right|\right]$$

$$= \frac{U}{g}\ln\left[\sqrt{2}\cos(0)\right]$$

$$= \frac{U}{2g}\ln 2$$

$$\frac{x}{H} = \frac{U}{2g}\ln\left[1 + \frac{1}{\sqrt{2}}\right] \div \frac{U}{2g}\ln 2$$

$$= \ln\left[1 + \frac{1}{\sqrt{2}}\right] \div \ln 2$$

$$= 0.77155\ldots$$

$$= 77\%$$

The particle gains 77% of the total height within the first half of the time.

146 **i**

$$\left(\frac{a}{A}-\frac{b}{B}\right)^2 \geq 0$$
$$\frac{a^2}{A^2}-\frac{2ab}{AB}+\frac{b^2}{B^2} \geq 0$$
$$\frac{a^2}{A^2}+\frac{b^2}{B^2} \geq \frac{2ab}{AB}$$
$$\therefore \frac{ab}{AB} \leq \frac{1}{2}\left(\frac{a^2}{A^2}+\frac{b^2}{B^2}\right)$$

ii

$$\frac{a_k b_k}{AB} \leq \frac{1}{2}\left(\frac{a_k^2}{A^2}+\frac{b_k^2}{B^2}\right) \text{ from (i)}$$
$$\therefore \sum_{k=1}^{n}\frac{a_k b_k}{AB} \leq \frac{1}{2}\sum_{k=1}^{n}\left(\frac{a_k^2}{A^2}+\frac{b_k^2}{B^2}\right)$$
$$\frac{1}{AB}\sum_{k=1}^{n} a_k b_k \leq \frac{1}{2}\left(\frac{\sum_{k=1}^{n} a_k^2}{A^2}+\frac{\sum_{k=1}^{n} b_k^2}{B^2}\right)$$
$$\frac{1}{AB}\sum_{k=1}^{n} a_k b_k \leq \frac{1}{2}\left(\frac{A^2}{A^2}+\frac{B^2}{B^2}\right)$$
$$\sum_{k=1}^{n} a_k b_k \leq AB$$
$$\sum_{k=1}^{n} a_k b_k \leq \left(\sqrt{\sum_{k=1}^{n} a_k^2}\right)\left(\sqrt{\sum_{k=1}^{n} b_k^2}\right)$$
$$\therefore \left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

iii

Let $a_k = \sqrt{\frac{S}{S-x_k}}$, $b_k = \sqrt{\frac{S-x_k}{S}}$

$$\therefore \left(\sum_{k=1}^{n}\sqrt{\frac{S}{S-x_k}}\times\sqrt{\frac{S-x_k}{S}}\right)^2 \leq \left(\sum_{k=1}^{n}\frac{S}{S-x_k}\right)\left(\sum_{k=1}^{n}\frac{S-x_k}{S}\right)$$
$$\left(\sum_{k=1}^{n} 1\right)^2 \leq \left(\sum_{k=1}^{n}\frac{S}{S-x_k}\right)\times\frac{1}{S}\left(nS-(x_1+x_2+x_3+\ldots+x_k)\right)$$
$$n^2 \leq \left(\sum_{k=1}^{n}\frac{S}{S-x_k}\right)\times\frac{1}{S}(nS-S)$$
$$n^2 \leq \left(\sum_{k=1}^{n}\frac{S}{S-x_k}\right)(n-1)$$
$$\therefore \frac{S}{S-x_1}+\frac{S}{S-x_2}+\frac{S}{S-x_3}+\ldots+\frac{S}{S-x_n} \geq \frac{n^2}{n-1}$$

147 **i**

$$\frac{1}{1-z\cos\alpha}$$
$$=\frac{1}{1-(\cos\alpha+i\sin\alpha)\cos\alpha}$$
$$=\frac{1}{1-\cos^2\alpha-i\sin\alpha\cos\alpha}$$
$$=\frac{1}{\sin^2\alpha-i\sin\alpha\cos\alpha}$$
$$=\frac{1}{\sin\alpha(\sin\alpha-i\cos\alpha)}\times\frac{\sin\alpha+i\cos\alpha}{\sin\alpha+i\cos\alpha}$$
$$=\frac{\sin\alpha+i\cos\alpha}{\sin\alpha(\sin^2\alpha+\cos^2\alpha)}$$
$$=\frac{\sin\alpha+i\cos\alpha}{\sin\alpha}$$
$$=1+i\cot\alpha$$

147 ... **ii**

$$\sum_{k=0}^{\infty}(z\cos\alpha)^k = \frac{a}{1-r} = \frac{1}{1-z\cos\alpha} = 1 + i\cot\alpha \quad (1)$$

$$\begin{aligned}\sum_{k=0}^{\infty}(z\cos\alpha)^k &= \sum_{k=0}^{\infty}\big((\cos\alpha + i\sin\alpha)\cos\alpha\big)^k \\ &= \sum_{k=0}^{\infty}(\cos k\alpha + i\sin k\alpha)\cos^k\alpha \\ &= \sum_{k=0}^{\infty}\cos k\alpha\cos^k\alpha + i\sum_{k=0}^{\infty}\sin k\alpha\cos^k\alpha \quad (2)\end{aligned}$$

Equating the imaginary components of (1) and (2):

$$\sum_{k=0}^{\infty}\sin k\alpha\cos^k\alpha = \cot\alpha$$

148

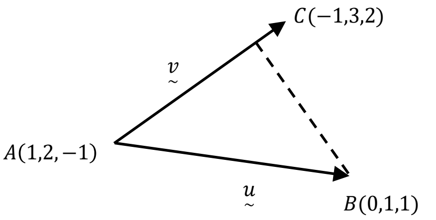
$$\begin{aligned}I_{2n} &= \int_{-1}^{1}(1-x^2)^n\,dx \\ &= \Big[x(1-x^2)^n\Big]_{-1}^{1} + 2n\int_{-1}^{1}x^2(1-x^2)^{n-1}\,dx \\ &= 0 - 2n\int_{-1}^{1}(1-x^2-1)(1-x^2)^{n-1}\,dx \\ &= -2n\int_{-1}^{1}(1-x^2)^n\,dx + 2n\int_{-1}^{1}(1-x^2)^{n-1}\,dx \\ I_{2n} &= -2nI_{2n} + 2nI_{2n-2} \\ (2n+1)I_{2n} &= 2nI_{2n-2} \\ I_{2n} &= \frac{2n}{2n+1}I_{2n-2} \\ &= \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}I_{2n-4} \\ &= \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-3}I_{2n-6} \\ &= \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-3}\times\ldots\times\frac{2}{3}I_0 \\ &= \frac{2n}{2n+1}\times\frac{2n}{2n}\times\frac{2n-2}{2n-1}\times\frac{2n-2}{2n-2}\times\frac{2n-4}{2n-3}\times\frac{2n-4}{2n-4}\times\ldots\times\frac{2}{3}\times\frac{2}{2}\times\int_{-1}^{1}dx \\ &= \frac{\big(2(n)2(n-1)2(n-2)\ldots2(1)\big)^2}{(2n+1)!}\Big[x\Big]_{-1}^{1} \\ &= \frac{2^{2n}(n!)^2}{(2n+1)!}\big(1-(-1)\big) \\ &= \frac{2^{2n+1}(n!)^2}{(2n+1)!}\end{aligned}$$

$u = (1-x^2)^n$	$\frac{dv}{dx} = 1$
$\frac{du}{dx} = n(1-x^2)^{n-1}(-2x)$	$v = x$

149 **i**

$\underset{\sim}{u} = \big(0-1, 1-2, 1-(-1)\big) = (-1,-1,2)$

$\underset{\sim}{v} = \big(-1-1, 3-2, 2-(-1)\big) = (-2,1,3)$



ii

$$\begin{aligned}\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u} &= \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\left|\underset{\sim}{v}\right|^2}\underset{\sim}{v}\\ &= \frac{(-1)(-2)+(-1)(1)+(2)(3)}{(-2)^2+1^2+3^2}(-2,1,3)\\ &= \frac{1}{2}(-2,1,3)\\ &= \left(-1,\frac{1}{2},\frac{3}{2}\right)\end{aligned}$$

iii

$$\begin{aligned}\underset{\sim}{u}-\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u} &= \left(-1-(-1), -1-\frac{1}{2}, 2-\frac{3}{2}\right)\\ &= \left(0,-\frac{3}{2},\frac{1}{2}\right)\\ \left|\underset{\sim}{u}-\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}\right| &= \sqrt{0^2+\left(-\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2}\\ &= \sqrt{\frac{5}{2}}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

iv

$$\begin{aligned}\text{proj}_{\overrightarrow{AC}}\overrightarrow{AB} &= \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|AC|^2}\overrightarrow{AC}\\ &= \frac{\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{a}\right)}{\left|\underset{\sim}{c}-\underset{\sim}{a}\right|^2}\left(\underset{\sim}{c}-\underset{\sim}{a}\right)\\ \left|\text{proj}_{\overrightarrow{AC}}\overrightarrow{AB}\right|^2 &= \left(\frac{\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{a}\right)}{\left|\underset{\sim}{c}-\underset{\sim}{a}\right|^2}\right)^2\left|\underset{\sim}{c}-\underset{\sim}{a}\right|^2\\ &= \left[\frac{\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{a}\right)}{\left|\underset{\sim}{c}-\underset{\sim}{a}\right|}\right]^2\\ \left|\overrightarrow{AB}\right|^2 &= \left|\underset{\sim}{b}-\underset{\sim}{a}\right|^2\\ &= \left(\underset{\sim}{b}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\\ d^2 &= \left|\overrightarrow{AB}\right|^2-\left|\text{proj}_{\overrightarrow{AC}}\overrightarrow{AB}\right|^2\\ &= \left(\underset{\sim}{b}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)-\left[\frac{\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{a}\right)}{\left|\underset{\sim}{c}-\underset{\sim}{a}\right|}\right]^2\end{aligned}$$

v

$$\begin{aligned}d^2 &= \left(\underset{\sim}{c}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{a}\right)-\left[\frac{\left(\underset{\sim}{c}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)}{\left|\underset{\sim}{b}-\underset{\sim}{a}\right|}\right]^2\\ &= (-2,1,3)\cdot(-2,1,3)-\left[\frac{(-2,1,3)\cdot(-1,-1,2)}{\sqrt{(-1)^2+(-1)^2+2^2}}\right]^2\\ &= 4+1+9-\left[\frac{2-1+6}{\sqrt{6}}\right]^2\\ &= 14-\frac{49}{6}\\ &= \frac{35}{6}\\ \therefore d &= \sqrt{\frac{35}{6}}\\ &= \frac{\sqrt{210}}{6}\end{aligned}$$

150 Let the times be $t = 0, 1, 2$ and $x = a\cos(nt + \alpha)$.
At $t = 0$ $a\cos\alpha = 1 \rightarrow \cos\alpha = \frac{1}{a}$
At $t = 1$ $a\cos(n + \alpha) = 5 \rightarrow \cos(n + \alpha) = \frac{5}{a}$
$\therefore \cos(n + \alpha) = 5\cos\alpha$

Due to symmetry the particle will be at its rightmost position at $t = \frac{3}{2}$.
At $t = \frac{3}{2}$ $a\cos\left(\frac{3}{2}n + \alpha\right) = a \rightarrow \frac{3}{2}n + a = 0 \rightarrow \alpha = -\frac{3}{2}n$

$$\begin{aligned}
\therefore \cos\left(n - \frac{3}{2}n\right) &= 5\cos\left(-\frac{3}{2}n\right) \\
\cos\left(-\frac{n}{2}\right) &= 5\cos\left(-\frac{3}{2}n\right) \\
\cos\left(\frac{n}{2}\right) &= 5\cos\left(\frac{3}{2}n\right) \\
\cos\left(\frac{n}{2}\right) &= 5\cos\left(n + \frac{n}{2}\right) \\
\cos\left(\frac{n}{2}\right) &= 5\left[\cos n\cos\left(\frac{n}{2}\right) - \sin n\sin\left(\frac{n}{2}\right)\right] \\
\cos\left(\frac{n}{2}\right) &= 5\left[\cos 2\left(\frac{n}{2}\right)\cos\left(\frac{n}{2}\right) - \sin 2\left(\frac{n}{2}\right)\sin\left(\frac{n}{2}\right)\right] \\
\cos\left(\frac{n}{2}\right) &= 5\left[\left(2\cos^2\left(\frac{n}{2}\right) - 1\right)\cos\left(\frac{n}{2}\right) - 2\sin\left(\frac{n}{2}\right)\cos\left(\frac{n}{2}\right)\sin\left(\frac{n}{2}\right)\right] \\
\cos\left(\frac{n}{2}\right) &= 5\left[2\cos^3\left(\frac{n}{2}\right) - \cos\left(\frac{n}{2}\right) - 2\sin^2\left(\frac{n}{2}\right)\cos\left(\frac{n}{2}\right)\right] \\
\cos\left(\frac{n}{2}\right) &= 5\cos\left(\frac{n}{2}\right)\left[2\cos^2\left(\frac{n}{2}\right) - 1 - 2\left(1 - \cos^2\left(\frac{n}{2}\right)\right)\right] \\
\frac{1}{5} &= 4\cos^2\left(\frac{n}{2}\right) - 3 \\
\cos^2\left(\frac{n}{2}\right) &= \frac{4}{5} \\
\cos n &= 2\cos^2\left(\frac{n}{2}\right) - 1 \\
&= 2\left(\frac{4}{5}\right) - 1 \\
&= \frac{3}{5} \\
n &= \arccos\left(\frac{3}{5}\right) \\
T &= \frac{2\pi}{n} \\
&= \frac{2\pi}{\arccos\left(\frac{3}{5}\right)}
\end{aligned}$$