HSC Mathematics Extension 2

Steve Howard

A textbook for the Mathematics Extension 2 (2017) NSW Syllabus

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About the Author

Steve is a Mathematics Teacher at Cowra High School, a medium sized rural high school in Central West NSW, where he has taught for over 25 years. He has also taught gifted and talented students online through xsel and Aurora College.

He has a particular love and passion for Mathematics Extension 2, writing this textbook and the 1000 Extension 2 Revision Questions - see howardmathematics.com. Both the textbook and revision questions are completely free, although students may like to sign up to his course that includes fully recorded lessons that cover the entire Extension 2 course, including fully recorded solutions to every question in the textbook. Much cheaper than tutoring!

In his spare time Steve writes commercial online professional development courses for teachers covering the new Mathematics Advanced, Extension 1 and 2 syllabuses through TTA. The online courses for teachers cover this material in much greater depth and include hundreds of recorded examples. For the courses currently offered see https://ttacreativemarketi.wixsite.com/stevehowardtta

Steve loves finding more efficient techniques for solving mathematical questions, by trawling through other teachers' solutions or making up his own approaches when there must be a better way. Many of the approaches you see here are unique, with emphases on both understanding the work, plus little tricks to help students succeed in exams.

Steve did 4 Unit Mathematics as a student way back in 1988, gaining a mark of 198/200 and training as an actuary. Working in an office in the city wasn't for him, so he went back to uni to retrain as a teacher then headed to the country, where he lives in an owner built mud house, with chickens, goats and rescued native birds (which you will sometimes hear in his recordings)!!

Contents

Introduction

Welcome! I hope all students and teachers find this book useful and enjoyable in your journey in Extension 2 Maths. This textbook and the matching ‘1000 Revision Questions’ were first released for the 2020 HSC, and will be continually changed and updated over time - the two digital books work best together. For free downloads of the latest versions please visit howardmathematics.com.

This textbook for the new syllabus evolved from the textbook I wrote for the old syllabus which itself evolved over a few years from recorded lessons I was making for my classes, and sharing with a few others around the state. I was originally using some of the commercially published textbooks for the old course, plus one with restricted availability, and found they weren’t meeting the needs of my students or myself. This textbook is the end result of that. It is mainly a passion project, but indirectly earns me a bit of pocket money through school copyright licencing and increasing traffic to my commercial online professional development courses for teachers through TTA.

Some of the features of this textbook that students and teachers may find useful:

- The questions are in the style of past HSC questions where possible, so students are learning the right type of questions throughout the course.
- The questions are graded in difficulty so students can work at their own level.
- Each exercise starts with Basic questions that match the examples, so students can ease themselves into the new concepts if they need to. More confident students can skip these first questions and start with Medium.
- All questions have fully worked solutions.
- There are lots of diagrams that help understand concepts that would normally only be dealt with algebraically.
- There are many hints and tips on how to more easily answer questions, and to see the way past tricks in exam questions.
- Some of the chapters have appendices, where extra content that may sometimes be useful is available. This content is not necessary to succeed in the course, but can help high ability students eliminate mistakes in exams or is sometimes just plain interesting!

The textbook is paired with ‘1000 Revision Questions in Mathematics Extension 2’. The first 500 questions are arranged topic by topic matching the chapters from the textbook to help students study and revise, while the last 500 questions are from mixed topics to help students prepare for their Trials and the HSC.

The course is currently set out in 42 lessons. The aim is for you to be able to finish the course as soon as possible so that you can have months of revision before the HSC to master the content.

If you find any mistakes, or have any ideas that would make either the textbook or the revision questions better, please contact me via email below.

Cheers
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Thanks to the following teachers and students for letting me know errors or improvements:
Gavin S, Matthew D, Drew S, Brailey S, Charlene C, Ian B, Glenn B, Daniel P, Kingsley H, Abi R, Chris B.

And to my current or past students who have earned lots of Mars Bars by picking up errors:
Alan, Brendan, Kurt, Andrew, Sam, Xavier, Luke, El, Sean and Aeryc.

Chapter 1
The Nature of Proof

MEX-P1: The Nature of Proof
All topics in Extension 2 rely on your ability to construct logically sound and convincing proofs, so it is a fitting topic with which we begin this textbook. In the Nature of Proof we will focus on the logic of proofs, and use alternative ways to prove arguments where direct proof or mathematical induction are not appropriate.

The topic tests our ability to use mathematical language and plain language to reason and communicate, promoting clear, simple and logical thought processes.

Lessons
The Nature of Proof is covered in 6 lessons.

1.1 The Language of Proof and Simple Proofs
1.2 Proof by Contrapositive
1.3 Proof by Contradiction
1.4 Equivalence and Disproofs
1.5 Inequality Proofs
1.6 Arithmetic Mean - Geometric Mean Inequality

Revision Questions
In '1000 Revision Questions', the revision book that goes with this textbook you will find the following questions matching this chapter:

- Revision Exercise 1
  60 graded questions on this topic only
- Revision Exercises 7 (Basic), 8 (Medium) and 9 (Challenging)
  Another 60 questions mixed through other topics for when you finish the course.

Don't forget to do any questions from the exercises in this textbook you haven't done.

1.1 LANGUAGE OF PROOF AND SIMPLE PROOFS

In Lesson 1 we look at the language of proof and simple proofs, covering:

- How should we use the Language of Proof?
- Statements
- Implication
- Quantifiers
- Other Terminology
- Symbols and Set Notation
- Simple Proofs Involving Numbers

HOW SHOULD WE USE THE LANGUAGE OF PROOF?

You should make sure that you know what the symbols that we cover in this lesson mean, but use them sparingly in your own proofs. It may be tempting to cram as many of the symbols at any possible point in a proof, yet if we look at the solutions to the NESA sample questions and many university level references to proofs for first year students they use no symbols whatsoever, instead using whole words and sentences!

Using the symbols can save you time, but they make your proof harder to read and thus less convincing for your reader. Use them sparingly! Never forget that a good proof needs to convince the reader of the truth of the argument, and preferably explain to them why it is true.

STATEMENTS

A statement is an assertion that can be true or false but not both. Some examples are:

- 6 is an even number (which is a true statement)
- 6 is an odd number (which is a false statement)
- The square of a number is even (which is a false statement since it is not true for all numbers).
- 6 is an even number **and** 3 is an odd number (which is true since both parts are true).
- 6 is an even number **and** 4 is an odd number (which is false since one or more parts are false).
- 6 is an even number **or** 4 is an odd number (which is true since one or more parts is true).
- 6 is an odd number **or** 4 is an odd number (which is false since neither part is true).

We often use the letters P, Q, R or S as shorthand to represent a statement. If there is a variable used in the statement then we often add it in brackets or as a subscript, such as

$P(n)$: The sum of the first n positive integers is $\frac{n(n+1)}{2}$.

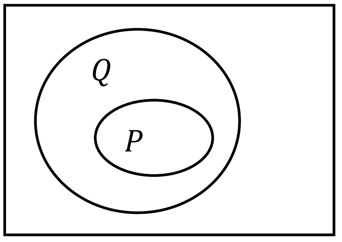
If there is a number used for the variable in the question we often use it, such as

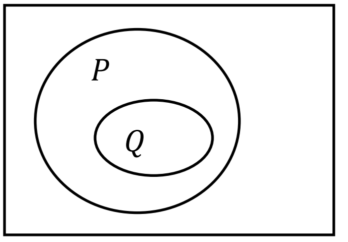
$P(10)$: The sum of the first 10 positive integers is $\frac{10\times 11}{2}$

IMPLICATION

To say that P implies Q means that **if** P is true **then** Q must be true. It is an if-then statement. We are not saying that Q is true, just promising that if P is true then Q must be true.

We can see how implication looks like using an Euler Diagram at left:





If you are in the P ellipse then you must be in the Q ellipse as well, so $P \Rightarrow Q$ (if P is true then Q must be true).

We can also write $P \Leftarrow Q$ if Q implies P (P is implied by Q) if the relationship is reversed, as seen at right.

Example 1

Consider the statements:

P: n is a positive integer.

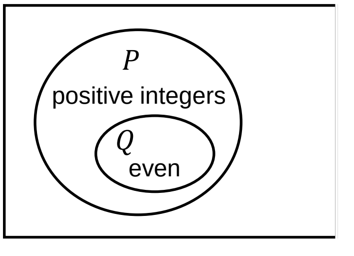
Q: n is an even number greater than 0

Which of the following is true? There is more than one correct answer.

a $P \Rightarrow Q$ **b** $P \Leftarrow Q$ **c** $Q \Rightarrow P$ **d** P implies Q **e** P is implied by Q

Solution

Consider the Euler diagram at right, where every even number is also an integer.



a False Since if a number is an integer it might not be even

b True Since all even numbers are integers

c True Since a number must be an integer if it is even

d False Since if a number is an integer it might not be even

e True Since a number must be an integer if it is even

QUANTIFIERS

For all ∀

Some statements are true or false for all values of a variable, and we can abbreviate this with the symbol ∀, an upside down A.

For example, ‘$\forall \text{ real numbers } x, x^2 \geq 0$’ means that the square of any real number must be non-negative.

There exists ∃

The term ‘there exists’ indicates that there is at least one number for which the statement is true.

We represent the words with a back to front E.

For example ‘$\exists x$ for which x^2 is odd’ is saying there is at least one value of x for which x^2 is odd.

Example 2

If m is an integer, add the most relevant quantifier to the start of the statement to make it true as often as possible.

a $m, m^2 \geq 0$ **b** $m, m = 0$ **c** $m, \frac{m}{2}$ is integral **d** $m, -1 \leq \sin m \leq 1$

Solution

a ∀ Since the square of all integers are non-negative
b ∃ Since only one integer is equal to zero
c ∃ Since only the even integers are divisible by 2
d ∀ Since the sine ratio is always from -1 to 1

SIMPLE PROOFS INVOLVING NUMBERS

We can prove some simple results using direct proofs, rather than mathematical induction. Many of these results could also be proved by induction, but the direct proofs we will use here are simpler.

Example 3

If m is odd and n is even, prove mn is even.

Solution

Let $m = 2k + 1$ and $n = 2j$ for integral j, k

$$\begin{aligned} mn &= (2k+1) \times 2j \\ &= 2(2jk + j) \\ &= 2p \qquad \text{where } p \text{ is integral since } j \text{ and } k \text{ are integral} \end{aligned}$$

$\therefore mn$ is even □

Example 4

If m is a multiple of 3, prove m^2 is a multiple of 9.

Solution

Let $m = 3k$ where k is integral

$$\begin{aligned} \therefore m^2 &= (3k)^2 \\ &= 9k^2 \\ &= 9j \qquad \text{where } j \text{ is integral since } k \text{ is integral} \end{aligned}$$

$\therefore m^2$ is a multiple of 9 □

Example 5

Prove that the sum of any two consecutive numbers is always odd.

Solution

Let the consecutive numbers be k and $k + 1$ for integral k

$S = k + k + 1$

$= 2k + 1$ which is odd □

EXERCISE 1.1

BASIC

1 Consider the statements:

P: n is a multiple of 9. Q: n is a multiple of 3.

Which of the following is true? There is more than one correct answer.

a) $P \Rightarrow Q$ b) $P \Leftarrow Q$ c) $Q \Rightarrow P$ d) P implies Q

2 If m is a positive integer, add the most relevant quantifier to the start of the statement to make it true as often as possible.

a) __ m, $2m$ is even   b) __ m, $m^2 = 4$

c) __ m, $m^2 \leq 2$ d) __ m, $1 + \cos m \geq 0$

3 If m and n are odd, prove
a mn is odd
b $m + n$ is even

4 If m is a multiple of 4, prove m^2 is a multiple of 16

5 Prove that the sum of any two consecutive numbers equals the difference of their squares.

6 If m is even, prove m^2 is even.

7 Prove that the product of any three consecutive numbers is even.

8 Prove that the sum of any four consecutive numbers is even.

MEDIUM

9 Given $a^k - b^k = (a-b)\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 ... + b^{k-1}\right)$ prove
a $\frac{3^k}{2}$ always has a remainder of 1.
b $3^{2n} - 1$ is divisible by 4

10 If $a + b = 2$ prove $a^2 + 2b = b^2 + 2a$

11 Prove the expression $a^3 - a + 1$ is odd for all positive integer values of a.

12 Prove $n^2 - 1$ is divisible by 3 if n is not a multiple of 3.

13 Prove $(2^m - 1)^2 - 1$ is divisible by 2^{m+1}

14 **a** Given a is integral and not divisible by 5, prove the remainder when a^2 is divided by 5 is either 1 or 4
b Hence given that a, b are integral and not divisible by 5, prove that $a^4 - b^4$ is divisible by 5.

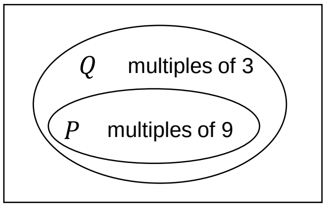
15 Prove $(k^3 - k)(2k^2 + 5k - 3)$ is divisible by 5 without using induction. You may assume that the product of 5 consecutive numbers is divisible by 5.

16 A triangular number is in the form $T = \frac{k(k+1)}{2}$. Prove the square of any odd positive integer greater than 1 is of the form $8T + 1$ where T is a triangular number.

17 Prove that an irrational number raised to the power of an irrational number can be rational, by considering $\sqrt{2}^{\sqrt{2}}$. You may assume $\sqrt{2}$ is irrational.

SOLUTIONS - EXERCISE 1.1

1 Consider the Euler diagram at right, where every multiple of 9 is also a multiple of 3.



a True Since if a number is a multiple of 9 it must also be a multiple of 3
b False Since a number is not necessarily a multiple of 9 if it is a multiple of 3
c False Since if a number is a multiple of 3 it is not necessarily a multiple of 9
d True Since if a number is a multiple of 9 it must also be a multiple of 3

2 **a** $\forall$ Since twice any integer is even
b $\exists$ Since only two integers (± 2) when squared give 4
c $\exists$ Since only $0, \pm 1$ when squared give an answer less than or equal to 2
d $\forall$ Since the cosine ratio is always from -1 to 1

3 **a** Let $m = 2j + 1$ and $n = 2k + 1$ for integral j, k

$mn = (2j + 1) \times (2k + 1)$

$= 4jk + 2j + 2k + 1$

$= 2(2jk + j + k) + 1$

$= 2p + 1$ for integral p

$\therefore mn$ is odd $\square$

b Let $m = 2j + 1$ and $n = 2k + 1$ for integral j, k

$mn = (2j + 1) + (2k + 1)$

$= 2j + 2k + 12$

$= 2(j + k + 1)$

$= 2p$ for integral p

$\therefore m + n$ is even $\square$

4 Let $m = 4k$ where k is integral

$\therefore m^2 = (4k)^2$

$= 16k^2$

$= 16p$ for integral p

$\therefore m^2$ is a multiple of 16 $\square$

5 Let the consecutive numbers be k and $k + 1$ for integral k

$(k + 1)^2 - k^2$

$= k^2 + 2k + 1 - k^2$

$= 2k + 1$

$= k + (k + 1)$ $\square$

6 Let $m = 2k$ for integral k

$\therefore m^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$

$= 2p$ for integral p

$\therefore$ If m is even then m^2 is even $\square$

7 Let the consecutive numbers be $k-1, k$ and $k+1$ for integral k
$P = k(k-1)(k+1) = k^3 - k$

Case 1 - k is even
Let $k = 2m$ for integral m
$\therefore P = (2m)^3 - 2m$
$= 2(4m^3 - m)$
$= 2p$ for integral p
$\therefore$ true if k is even

Case 2 - k is odd
Let $k = 2m+1$ for integral m
$\therefore P = (2m+1)^3 - (2m+1)$
$= 8m^3 + 12m^2 + 6m + 1 - 2m - 1$
$= 2(4m^3 + 6m^2 + 2m)$
$= 2p$ for integral p
$\therefore$ true if k is odd
$\therefore$ The product of any three consecutive numbers is even □

8 Let the consecutive numbers be $k, k+1, k+2$ and $k+3$ for integral k

$S = k + k + 1 + k + 2 + k + 3$

$= 4k + 6$

$= 2(2k+3)$

$= 2p$ for integral p

$\therefore$ The sum of any four consecutive numbers is always even. □

9 **a** $3^k = 3^k - 1^k + 1$
$= (3-1)\left(3^{k-1} + 3^{k-2} \times 1 + 3^{k-3} \times 1^2 ... + 1^{k-1}\right) + 1$
$= 2\left(3^{k-1} + 3^{k-2} \times 1 + 3^{k-3} \times 1^2 ... + 1^{k-1}\right) + 1$
$= 2p + 1$ for integral p
$\therefore \frac{3^k}{2}$ always has remainder 1 □

b $3^{2n} - 1 = (3^n - 1)(3^n + 1)$

$= (3-1)(3^{n-1} + 3^{n-2} + ... + 3^0)(3^n + 1)$

$= 2(3^{n-1} + 3^{n-2} + ... + 3^0)(2k)$ for integral k since 3^n is odd $3^n + 1$ is even

$= 4k(3^{n-1} + 3^{n-2} + ... + 3^0)$

$= 4p$ for integral p

$\therefore 3^{2n} - 1$ is divisible by 4 □

10 $\text{LHS} - \text{RHS} = a^2 - b^2 + 2b - 2a$
$= (a+b)(a-b) - 2(a-b)$
$= (a-b)(a+b-2)$
$= (a-b)(0) \quad \text{since } a+b=2$
$= 0$
$\therefore a^2 - b^2 + 2b - 2a = 0$
$\therefore a^2 + 2b = b^2 + 2a$ □

Alternatively
$\text{LHS} = a^2 + 2b$
$= (2-b)^2 + 2b \quad \text{since } a+b=2$
$= 4 - 4b + b^2 + 2b$
$= b^2 + 4 - 2b$
$= b^2 + 2(2-b)$
$= b^2 + 2a$
$= \text{RHS}$ □

11 $a^3 - a + 1$
$= a(a^2-1) + 1$
$= a(a+1)(a-1) + 1$
$= 2k + 1$ where k is integral,
since we have proved that the product of three consecutive numbers is even.
$\therefore a^3 - a + 1$ is odd for all positive integer values of a. □

12 Let $n = 3k \pm 1$ for integral k
$n^2 - 1 = (3k \pm 1)^2 - 1$
$= 9k^2 \pm 6k + 1 - 1$
$= 9k^2 \pm 6k$
$= 3k(3k \pm 2)$
$= 3p \quad \text{for integral } p$
$\therefore n^2 - 1$ is divisible by 3 if n is not a multiple of 3 □

13 $(2^m - 1)^2 - 1 = (2^m - 1 + 1)(2^m - 1 - 1)$
$= 2^m(2^m - 2)$
$= 2^m \cdot 2(2^{m-1} - 1)$
$= 2^{m+1}(2^{m-1} - 1)$
$\therefore (2^m - 1)^2 - 1$ is divisible by 2^{m+1} □

14 **a** Let $a = 5k + m$, where $m = 1, 2, 3$ or 4 and k is integral
$a^2 = (5k+m)^2$
$= 25k^2 + 20km + m^2$
$= 5(5k^2 + 4km) + m^2$
Now $m^2 = 1, 4, 9$ or 16
$\therefore a^2 = 5(5k^2+4km)+1, 5(5k^2+4km)+4, 5(5k^2+4km+1)+4$ or
$5(5k^2+4km+3)+1$
$\therefore a^2 = 5j+1, 5j+4 \text{ where } j \text{ is integral}$ □

b $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$
Case 1 - $a^2 = 5j + r, b^2 = 5i + r \quad r = 1 \text{ or } 4$ and j, i integral, have the same remainder when divided by 5
$a^4 - b^4$
$= \big(5j + r - (5i + r)\big)(5j + r + 5i + r)$
$= 5(j - i)(5j + 5i + 2r)$
So $a^4 - b^4$ is divisible by 5.
Case 2 - $a^2 = 5j + 1, b^2 = 5i + 4$
$a^4 - b^4$
$= \big((5j+1) - (5i+4)\big)\big((5j+1) + (5i+4)\big)$
$= (5j - 5i - 3)(5j + 5i + 5)$
$= 5(5j - 5i - 3)(j + i + 1)$
So $a^4 - b^4$ is divisible by 5.

$\therefore a^4 - b^4$ is divisible by 5. □

15 $(k^3-k)(2k^2+5k-3)$
$= k(k^2-1)(2k^2+6k-k-3)$
$= k(k+1)(k-1)\big(2k(k+3)-(k+3)\big)$
$= k(k+1)(k-1)(2k-1)(k+3)$
$= k(k+1)(k-1)\left(2(k+2)-5\right)(k+3)$
$= 2(k-1)k(k+1)(k+2)(k+3)-5(k-1)k(k+1)(k+3)$
$= 2(5m)-5n \qquad m, n$ integral, since the product of 5 consecutive numbers is divisible by 5
$= 5(2m-n)$ □

16 $(2k+1)^2$

$= 4k^2+4k+1$

$= 8\left(\dfrac{k^2+k}{2}\right)+1$

$= 8\left(\dfrac{k(k+1)}{2}\right)+1$

$= 8T+1$ □

17 Case 1 - $\sqrt{2}^{\sqrt{2}}$ is rational

$\sqrt{2}^{\sqrt{2}}$ is an irrational number to the power is an irrational number, so if $\sqrt{2}^{\sqrt{2}}$ is rational then we have proved the problem.

Case 2 - $\sqrt{2}^{\sqrt{2}}$ is irrational

Consider $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$, which in this case is an irrational number to the power is an irrational number.

$$\begin{aligned}\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} &= \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} \\ &= \sqrt{2}^{2} \\ &= 2\end{aligned}$$

which is rational, so we have proved the problem.

∴ whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational we have proved that an irrational number raised to the power of an irrational number can be rational. □

In Lesson 2 we look at the first of two methods of indirect proof - Proof by Contrapositive. We will cover:

- Converse
- Negation
- Contrapositive
- Proof by Contrapositive

CONVERSE

The converse of a statement 'If P then Q' is 'If Q then P'. The statements can be represented as 'the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$', or 'the converse of $P \Rightarrow Q$ is $P \Leftarrow Q$'.

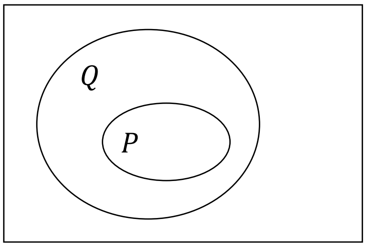
For example:

Statement: 'If a number is even it is an integer'
Converse: 'If a number is an integer it is even'

The converse of a true statement is not necessarily true, which is a major source of improperly constructed proofs.

A common fault involves square roots. For example the converse of the statement 'if $x = 4$ then $x^2 = 16$' is the statement 'if $x^2 = 16$ then $x = 4$'. Now the original statement is true, but the converse is false since it does not include $x = -4$ as a possible solution.

We can see that the converse is not necessarily true using an Euler Diagram:



$P \Rightarrow Q$ is true since all of P lies within Q, but $Q \Rightarrow P$ is not necessarily correct as not all of Q lies within P.

The only time that it is appropriate to prove a converse is as one half of a proof where we are proving equivalence.

NEGATION

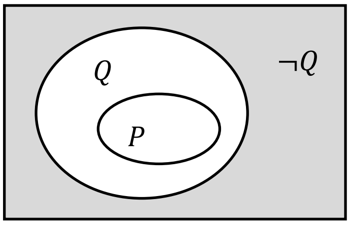
If P is a statement then the statement 'not P' is called the negation of P. The negation of P is denoted by $\neg P$ or $\sim P$.

For example the negation of the statement 'x is even' would be 'x is not even' which we could also write as 'x is odd' if x is a positive integer. The negation can be true or false, as the original statement could be false or true.

Negation is an important technique that we will use in indirect proofs - proof by contrapositive and proof by contradiction.

We can see negations using an Euler Diagram:

We can see that $\neg Q$ is the area outside the Q ellipse.



Note that $\neg P$ is not shown on this diagram to avoid confusion, but includes everything outside the P ellipse, which includes parts of Q and all of $\neg Q$.

When we negate equality statements or inequalities, the negation involves everything else - similar to the concept of the complement in probability.

Original Statement	Negation
$=$	$\neq$
$>$	$\leq$
$\geq$	$<$
$<$	$\geq$
$\leq$	$>$

For example, the negation of $x > 2$ is $x \leq 2$.

Example 1

Find the negation of the following:

a $x = 2$ **b** $x < 2$ **c** $x \geq 2$

Solution

a $x \neq 2$ **b** $x \geq 2$ **c** $x < 2$

NEGATING STATEMENTS INVOLVING ‘FOR ALL’ OR ‘THERE EXISTS’

When we negate statements involving ‘For All’ or ‘There exists’, then the original statement and its negation swap the two terms, as well as negating any other part of the statement as shown above.

For example:
The negation of ‘$\forall$ real numbers $x, x^2 \geq 0$’ is ‘$\exists$ a real number $x, x^2 < 0$’. The original statement is saying that $x^2 \geq 0$ for every value of x, while the negation is just saying, hang on, there is at least one value where that isn’t true.

Notice the negation is not saying that it the original statement is false for every value of x, just for one or more. In this case the original statement is true and its negation is false.

The negation of ‘$\exists$ a real number $x, x^2 - 4 = 0$’ is ‘$\forall$ real numbers $x, x^2 - 4 \neq 0$’. The original statement is saying that $x^2 - 4 = 0$ for at least one value of x, while the negation is just saying, hang on, there are no values of x for which it is true, in other words $x^2 - 4 \neq 0$ for every value of x.

Notice the negation is not saying that there is one or more values of x for which $x^2 - 4 \neq 0$. Again the original statement is true and its negation is false.

Example 2

Find the negation of the following:

a $\forall$ integers x, $2x$ is even

b $\exists$ a real number $x, x = 4m$ for integral m

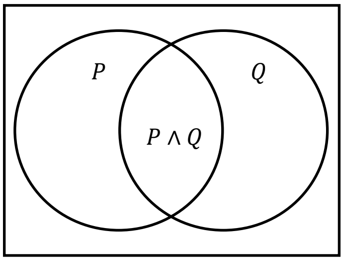
Solution

a $\exists$ integers x, $2x$ is odd

b $\forall$ real numbers $x, x \neq 4m$ for integral m

NEGATING COMPOUND STATEMENTS (AND/OR)

Say we have to negate a compound statement like 'x is even and less than 10', let's look at the Euler Diagram and find the complement. So we break the compound statement into two statements, '$P: x$ is even', and '$Q: x$ is less than 10'. The original statement is '$R: P \wedge Q$' using the 'and' proof symbol, or represented by the intersection below.



$\wedge$ = 'and'
$\vee$='or'

Now the complement of $P \wedge Q$ is everything outside the intersection, which we can most easily write as '$\neg P \vee \neg Q$' using the 'or' proof symbol. So the negation of 'x is even and less than 10' is 'x is not even or not less than 10', so 'x is odd or greater than or equal to 10'.

To negate a compound statement, negate each of the original statements and swap 'and' for 'or'.

You might come across this as DeMorgan's Laws in some texts.

Example 3

Find the negation of the following:

a $\forall$ integers $x, 2x$ is even and $2x+1$ is odd

b $\exists$ a real number $x, x^2 = 9$ or $x > 2$

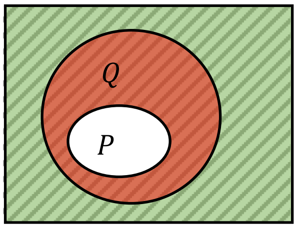
Solution

a $\exists$ integers $x, 2x$ is odd or $2x+1$ is even

b $\forall$ a real number $x, x^2 \neq 9$ and $x \leq 2$

CONTRAPOSITIVE

The contrapositive of the conditional statement 'If P then Q' is 'If not Q then not P', so the negation of Q implies the negation of P. Using symbols the contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. The contrapositive is true if and only if the statement itself is also true.



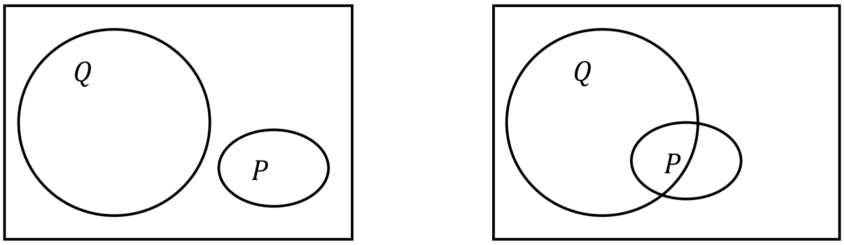
$\neg Q$ green shading

$\neg P$ diagonal stripes

It is hard to draw this clearly on an Euler diagram, but we can see above that if $P \Rightarrow Q$ then P is inside Q, so $\neg Q$ (shaded in green) must all be within $\neg P$ (diagonal stripes).

So '$\neg Q \Rightarrow \neg P$' is true when '$P \Rightarrow Q$' is true.

Now if the original statement is false, then P is either completely outside Q (below left) or overlaps Q (below right).



In each case $\neg Q$ is not completely inside $\neg P$. So when '$P \Rightarrow Q$' is false then '$\neg Q \Rightarrow \neg P$' is also false.

So we have seen is that if:

- $P \Rightarrow Q$ is true then $\neg Q \Rightarrow \neg P$ is true
- $P \Rightarrow Q$ is false then $\neg Q \Rightarrow \neg P$ is false.

We can say that:

> The original statement and its contrapositive are logically equivalent, so to prove a conditional statement we can prove its contrapositive instead.

Example 4

Find the contrapositive of the following:

a If x is even then $x + 1$ is odd.

b If $x^2 = 25$ then $x = \pm 5$

Solution

a If $x + 1$ is even then x is odd.

b If $x \neq \pm 5$ then $x^2 \neq 25$

PROOF BY CONTRAPOSITIVE

Now the syllabus states that students need to ‘use proof by contradiction’, but there is no similar statement about using proof by contrapositive. The only mention of contrapositive is that students ‘understand that a statement is equivalent to its contrapositive’ as we have just seen.

Proof by Contrapositive is generally considered easier than proof by contradiction, so I would certainly get students to use it and expect to see it in exams. It is just a pity the wording of the syllabus isn’t clearer!

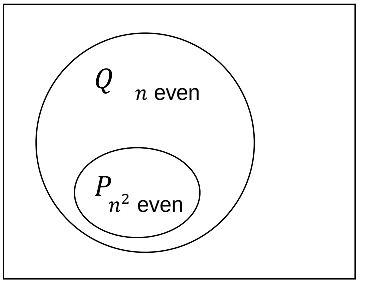
In general we should always try direct proof first, then proof by contrapositive (if it is a conditional statement), then proof by contradiction as a last resort.

Example 5

Prove by contrapositive for integral n that if n^2 is even then n is even.

Solution

Suppose n is odd.
Let $n = 2k + 1$ for integral k.
$\therefore n^2 = (2k+1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
$= 2j + 1$ for integral j since $2, k$ are integral
$\therefore$ if n is odd then n^2 is odd.
$\therefore$ if n^2 is even then n is even by contrapositive.



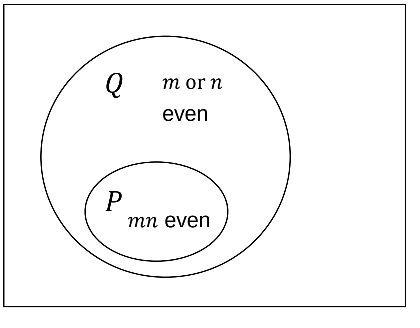
Example 6

Prove by contrapositive that if mn is even then m or n must be even.

Solution

Note that we are negating a compound statement, so ‘or’ is replaced with ‘and’ and ‘even’ is replaced with ‘odd’.

Suppose m is odd and n is odd.
Let $m = 2k + 1, n = 2j + 1$ for integral k, j
$mn = (2k+1)(2j+1)$
$= 4kj + 2k + 2j + 1$
$= 2(2kj + k + j) + 1$
$= 2p + 1$ for integral p since $2, j, k$ are integral
$\therefore$ if m and n are both odd then mn is odd.
$\therefore$ if mn is even then m or n must be even.



EXERCISE 1.2

BASIC

1 Find the negation of the following:

a $x \leq 1$ **b** $x = 1$ **c** $x > 1$

2 Find the negation of the following:

a $\exists$ an integer $x, 2x = x^2$ **b** $\forall$ real numbers $x, x < 2m + 1$ for integral m

3 Find the negation of the following:

a $\forall$ integers $x, 2x$ is prime or $2x+1$ is odd **b** $\exists$ a real number $x, x^2 - 7 = 9$ and $x > 2$

4 Find the contrapositive of the following:

a If x is prime then $2x + 1$ is composite. **b** If $x^2 = 1$ then $\frac{1}{x} = 1$

5 Prove by contrapositive for integral n that if n^2 is odd then n is odd.

6 Prove by contrapositive for m, n positive integers that if mn is divisible by 5 then m or n must be divisible by 5.

7 Prove by contrapositive that if m is an integer and m^2 is not divisible by 4 then m is odd.

MEDIUM

8 Prove by contrapositive that if $\frac{mn}{2}$ is integral then m or n must be even.

1 **a** $x > 1$ **b** $x \neq 1$ c) $x \leq 1$

2 **a** $\forall$ integers x, $2x \neq x^2$ **b** $\exists$ real numbers x, $x \geq 2m + 1$for integral m

3 **a** $\exists$ an integer x, $2x$ is not prime and $2x$+1 is even **b** $\forall$ real numbers x, $x^2 - 7 \neq 9$ or $x \leq 2$

4 **a** If $2x + 1$ is not composite then x is not prime **b** If $\frac{1}{x} \neq 1$ then $x^2 \neq 1$.

5 Suppose n is even.

Let $n = 2k$ for integral k.

$\therefore n^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$

$= 2p$ for integral p

$\therefore$ if n is even then n^2 is even.

$\therefore$ if n^2 is odd then n is odd by contrapositive.

6 Suppose neither m nor n are divisible by 5.

Let $m = 5j + p$ and $n = 5k + q$ for integral j, k and $p, q = 1,2,3$ or 4.

$mn = (5j + p)(5k + q)$

$= 25jk + 5jq + 5kp + pq$

$= 5(5jk + jq + kp) + pq$

$= 5c + pq$ for integral c since j, k, p, q are integral

Now pq=1, 2, 3, 4, 6, 8, or 12, so none a multiple of 5, so mn is not a multiple of 5.

$\therefore$ if m and n are not divisible by 5 then mn is not divisible by 5.

$\therefore$ if mn is divisible by 5 then m or n must be divisible by 5 by contrapositive.

7 Suppose m is even.

Let $m = 2k$ for integral k

$m^2 = (2k)^2$

$= 4k^2$

$= 4p$ for integral p

$\therefore$ if m is even then m^2 is divisible by 4.

$\therefore$ if m^2 is not divisible by 4 then m must be odd by contrapositive.

8 Suppose m is odd and n is odd.

Let $m = 2k + 1, n = 2j + 1$ for integral k, j

$$\frac{mn}{2} = \frac{(2k+1)(2j+1)}{2}$$

$$= \frac{4kj + 2k + 2j + 1}{2}$$

$$= 2kj + k + j + \frac{1}{2}$$

Which is not integral since j and k are integral

$\therefore$ if m and n are both odd then $\frac{mn}{2}$ is not integral.

$\therefore$ if $\frac{mn}{2}$ is integral then m or n must be even.

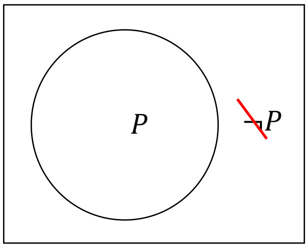
In Lesson 3 we look at the second of two methods of indirect proof - Proof by Contradiction. We will cover:

- Proof by Contradiction

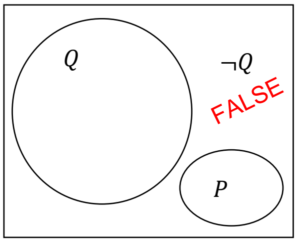
PROOF BY CONTRADICTION

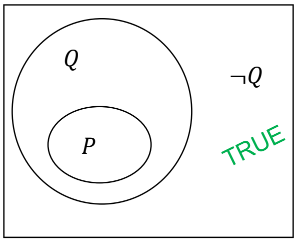
We have seen that a statement must be either true or false, so its negation must be false or true respectively.

Using our Euler diagram again, if $\neg P$ is false then the only option remaining is P.



For conditional proofs of the form $P \Rightarrow Q$ we start by assuming $P \Rightarrow \neg Q$ (diagram below left) and find the contradiction there. Since the negation is false then $P \Rightarrow Q$, illustrated in the diagram below right where the P ellipse must be inside the Q ellipse.





So we can prove that the original statement is true by proving its negation is false. This is called proof by contradiction. It gets its name since the negation normally leads to a fact that contradicts some other fact - either a fact that we assumed earlier, or a fact reached simultaneously. For example a number cannot be both odd and even, or both rational and irrational.

Now we started by assuming that the negation is correct, which we weren't sure about. We then followed a series of steps that we know are correct, then reached a contradiction. This means the only step that can have caused the contradiction is the negation, which must be wrong.

Example 1

Prove by contradiction that if n is an odd integer then n^2 is odd.

Solution

Suppose n is an odd integer and n^2 is even. *

Let $n = 2k + 1$ for integral k

$\therefore n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2p + 1$ for integral p since $2, k$ are integral

$\therefore n^2$ is odd #

Which contradicts (*) since n^2 cannot be both odd and even, hence if n is an odd integer then n^2 is odd.

Example 2

Prove $\sqrt{2} + \sqrt{3} < \sqrt{10}$ by contradiction

Solution

Suppose $\sqrt{2} + \sqrt{3} \geq \sqrt{10}$ $\quad *$

$$\left(\sqrt{2} + \sqrt{3}\right)^2 \geq 10 \qquad \text{since } \sqrt{2}, \sqrt{3}, \sqrt{10} > 0$$

$$2 + 2\sqrt{6} + 3 \geq 10$$

$$2\sqrt{6} \geq 5$$

$$\sqrt{24} \geq \sqrt{25}$$

$$24 \geq 25 \qquad \#$$

Which is a contradiction, so $\sqrt{2} + \sqrt{3} < \sqrt{10}$

Example 3

Prove that $\sqrt{2}$ is irrational.

Solution

Suppose that $\sqrt{2}$ is rational.

$\therefore \sqrt{2} = \frac{p}{q}$ where p, q are integers with no common factor except 1 (*) $2q^2 = p^2$.

Now 2 is even

$\therefore 2q^2$ is even

$\therefore p^2$ is even

$\therefore p$ is even

Let $p = 2k$ for some integer k.

$\therefore 2q^2 = 4k^2$

$q^2 = 2k^2$

Now $2k^2$ is even

$\therefore q^2$ is even

$\therefore q$ is even. #

This contradicts (*), since if p and q are both even they have a common factor of 2, hence $\sqrt{2}$ is irrational.

EXERCISE 1.3

BASIC

1 Prove by contradiction that if n is an even integer then n^2 is even.

2 Prove by contradiction that if n is an integer and $n^2 - 1$ is even then n is odd.

MEDIUM

3 Prove $\sqrt{5} + \sqrt{7} < 5$ by contradiction

4 Prove that $\sqrt{2}$ is irrational.

5 Prove that $\log_2 5$ is irrational.

6 Prove by contradiction that if a, b are integral and $a + b \leq 5$ then $a \leq 2$ or $b \leq 2$.

7 Prove by contradiction that there are no integers m, n which satisfy $4n + 8m = 102$

CHALLENGING

8 Prove by contradiction that the square root of π is also irrational.

9 Prove $\sin x + \cos x \geq 1$ for all $0 \leq x \leq \frac{\pi}{2}$ by contradiction.

10 If a is rational and b is irrational, prove $a + b$ is irrational.

11 Prove that for positive integers a, b and $a > 1$ that either b is not divisible by a or $b + 1$ is not divisible by a.

12 Prove that there are no positive integers x, y such that $x^2 - y^2 = 1$.

1 Suppose n is an even integer and n^2 is odd. *

Let $n = 2k$ for integral k

$\therefore n^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$

$= 2p \quad$ for integral p

$\therefore n^2$ is even #

Which contradicts (*) since n^2 cannot be both odd and even, hence if n is an even integer then n^2 is even.

2 Suppose $n^2 - 1$ is even and n is even. *

Let $n = 2k$ for integral k

$\therefore n^2 - 1 = (2k)^2 - 1$

$= 4k^2 - 1$

$= 2(2k^2 - 1) + 1$

$= 2p + 1 \quad$ for integral p

$\therefore n^2 - 1$ is odd #

Which contradicts (*) since $n^2 - 1$ cannot be both odd and even, hence if $n^2 - 1$ is even then n is odd.

3 Suppose $\sqrt{5} + \sqrt{7} \geq 5$

$\left(\sqrt{5} + \sqrt{7}\right)^2 \geq 25 \quad$ since $\sqrt{5}, \sqrt{7}, 5 > 0$

$5 + 2\sqrt{35} + 7 \geq 25$

$2\sqrt{35} \geq 13$

$\sqrt{140} \geq 13$

$140 \geq 169$ #

Which is a contradiction, so $\sqrt{5} + \sqrt{7} < 5$

4 Suppose that $\sqrt{2}$ is rational.

$\therefore \sqrt{2} = \frac{p}{q}$ where p, q are integers with no common factor except 1 (*) $2q^2 = p^2$.

Now 2 is even
$\therefore 2q^2$ is even
$\therefore p^2$ is even
$\therefore p$ is even

Let $p = 2k$ for some integer k.
$\therefore 2q^2 = 4k^2$
$q^2 = 2k^2$

Now $2k^2$ is even
$\therefore q^2$ is even
$\therefore q$ is even. #
This contradicts (*), since if p and q are both even they have a common factor of 2, hence $\sqrt{2}$ is irrational.

5 Suppose that $\log_2 5$ is rational.

$\therefore \log_2 5 = \frac{p}{q}$ where p, q are integers with no common factor except 1

$q \log_2 5 = p$

$\log_2 5^q = p$

$5^q = 2^p$ #

Now the LHS is odd and the RHS is even which is a contradiction, hence $\log_2 5$ is irrational.

6 Suppose $a + b \leq 5$ and $a > 2$ and $b > 2$ *

$\therefore a + b \geq 3 + 3 \quad \text{since } a, b \text{ integral}$

≥ 6 #

Which contradicts (*) since $a + b$ cannot be ≤ 5 and ≥ 6, hence $a \leq 2$ or $b \leq 2$.

7 Suppose m, n are integers which do satisfy $4n + 8m = 102$

$\therefore 4(n + 2m) = 102$

$4p = 4 \times 25 + 2 \quad \text{for integral } p$ #

Which is a contradiction since the LHS is a multiple of 4 but the RHS is not, hence there are no integers m, n which satisfy $4n + 8m = 102$

8 Suppose that $\sqrt{\pi}$ is rational.

$\therefore \sqrt{\pi} = \frac{p}{q}$ where p, q are integers with no common factor except 1

$\pi q^2 = p^2$ #

Now the RHS is an integer but the LHS is not since π is irrational which is a contradiction, hence the square root of the irrational number m is also irrational.

9 Suppose $\sin x + \cos x < 1$

$$\therefore (\sin x + \cos x)^2 < 1 \quad \text{since } \sin x, \cos x \geq 0 \text{ in the given domain}$$

$$\sin^2 x + 2\sin x\cos x + \cos^2 x < 1$$

$$1 + 2\sin x\cos x < 1$$

$$2\sin x\cos x < 0 \qquad \#$$

But $\sin x, \cos x > 0$ for $0 \leq x \leq \frac{\pi}{2}$ so this is a contradiction, hence $\sin x + \cos x \geq 1$.

10 Suppose by contradiction that a is rational, b irrational and $a + b$ rational (*).

Let $a = \frac{p}{q}, a + b = \frac{j}{k}$ for integral p, q, j, k

$$\therefore \frac{p}{q} + b = \frac{j}{k}$$

$$b = \frac{j}{k} - \frac{p}{q}$$

$$= \frac{jq - kp}{kq}$$

$$= \frac{m}{n} \text{ for integral } m, n \text{ since } p, q, j, k \text{ are integral}$$

$\therefore b$ is rational #

This contradicts (*) since b cannot be rational and irrational, $\therefore$ if a is rational and b is irrational, then $a + b$ is irrational

11 Suppose by contradiction that b and $b + 1$ are both divisible by a.

Let $b = ma \quad (1)$ and $b + 1 = na \quad (2)$ for integral m, n.

$\therefore ma + 1 = na$

$na - ma = 1$

$$n - m = \frac{1}{a} \qquad \#$$

This is a contradiction since the LHS is an integer but the RHS is not since $a > 1$, $\therefore$ for positive integers a, b and $a > 1$ b is not divisible by a or $b + 1$ is not divisible by a.

12 Suppose by contradiction that $x^2 - y^2 = 1$ has solutions x, y positive integers (*)

$$\therefore (x+y)(x-y) = 1$$

$$x + y = 1 \text{ and } x - y = 1 \text{ since } x \text{ and } y \text{ are integers} \quad \#$$

This contradicts (*) since $x + y = 1$ has no positive integral solutions, so there are no positive integers x, y such that $x^2 - y^2 = 1$

In Lesson 4 we look at equivalent statements and disproof, covering:

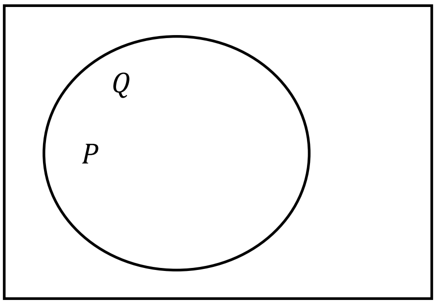
- Equivalent Statements (if and only if)
- Proving Equivalence
- Disproof, examples and counterexamples

EQUIVALENT STATEMENTS (IF AND ONLY IF)

If there is only one condition P that can result in Q, then we can say 'Q if and only if P' which we write as 'Q iff P'.

It is interesting to note that quite often when we write P if Q we could write P iff Q, as a lot of our relationships are equivalent.

Now if we think of the Euler Diagram, the P ellipse must have grown to fill all of the Q ellipse, as only P can lead to Q. The two ellipses are now the same, or equivalent.



Since the two ellipses are the same, then we can also say 'P iff Q' is also correct. Since either statement implies the other we can write the iff statements as '$P \Leftrightarrow Q$'.

In equivalent statements we can swap the two statements around and still create a true statement, as the original and its converse are both true.

For example: P: x^2 is odd Q: x is odd

Now $P \Rightarrow Q$ and $Q \Rightarrow P$ are both true, so we can write $P \Leftrightarrow Q$.

We can say this because the only way x^2 could be odd is if x is odd, and the only way x can be odd is if x^2 is odd.

Let's look at an **incorrect** example: '$x^2 = 4$ iff $x = 2$'

Now this is false, because $x = -2$ would also cause $x^2 = 4$ to be true.

PROVING EQUIVALENCE

To prove a statement in the form $P \Leftrightarrow Q$ we have to prove 2 cases:

- Case 1: $P \Rightarrow Q$
- Case 2: $Q \Rightarrow P$ (which we can also write as $P \Leftarrow Q$)

It is best to write them out under two headings as above, or as two separate paragraphs with 'conversely' before we prove $Q \Rightarrow P$.

Example 1

Prove x is odd if and only if x^2 is odd

Solution

If x is odd, let $x = 2k + 1$ for integral k.

$$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^3 + 2k) + 1 \\ &= 2j + 1 \text{ for integral } j \text{ since } k \text{ is integral} \end{aligned}$$

$\therefore$ if x is odd then x^2 is odd

Conversely, we will show that if x^2 is odd then x is odd using proof by contrapositive.

Suppose x is even

Let $x = 2k$ for integral k

$$\begin{aligned} x^2 &= 4k^2 \\ &= 2(2k^2) \\ &= 2p \text{ for integral } p \text{ since } 2, k \text{ are integral} \end{aligned}$$

$\therefore$ if x is even x^2 is even, hence if x^2 is odd then x is odd by contrapositive.

$\therefore$ x is odd if and only if x^2 is odd $\square$

DISPROOF, EXAMPLES AND COUNTEREXAMPLES

Commonly we will be given a statement and told to prove it is true, but there are times when we are given a statement that is false, or that could be true or false, and have to disprove it.

Not being able to prove a statement is true is not enough to prove it is false.

There are different techniques that we can use to disprove a statement, depending on whether it applies to all numbers (∀) or to at least one (∃).

If we are given a statement that might be true or false we will have to read the question carefully and use our judgement as to how we should approach it.

DISPROVING UNIVERSAL STATEMENTS

To disprove a statement that is said to apply for all numbers we can:

- Use one counterexample. A counterexample is an example where the statement is false.
- Use a direct proof to show that the negation of the statement is true for at least one value, so the statement is false.
- Use a direct proof of the statement to lead to a contradiction which proves that it is false. This is a type of proof by contradiction, but we do not need to negate the statement.
- We can use Proof by Cases to show that it is false for at least one case.
- use a proof by contrapositive to prove either $\neg Q \Rightarrow P$ or $\neg Q \Rightarrow (P \vee \neg P)$ so the statement is false

∧ = 'and' ∨='or'

For example, to disprove '∀ odd positive integers $x,\ x+1$ is odd' we could:
- Use a counterexample by letting $x = 1$. Since $x + 1 = 2$ is even, the statement is false.
- Use a direct proof by cases to show that if x is odd that $x + 1$ must be even, which is a contradiction and the statement is false.
- Use a proof by contrapositive to show that if $x + 1$ is even then x must be odd, and the statement must be false.

The counterexample is the best method of disproving a universal statement, if you can find one easily.

Also remember that even though one counterexample is sufficient to prove a universal statement is false, no number of examples are sufficient to prove a universal statement is true, as we have seen in False Proofs by Mathematical Induction in Extension 1.

Example 2

Prove the following statement is false:

If $a^2 - b^2 > 0$, where a and b are real, then $a - b > 0$

Solution

This is a universal statement, that is easiest to disprove using a counterexample.

Let $a = -2, b = 0$

$(-2)^2 - 0^2$ is positive yet $-2 - 0$ is negative, so the statement is false.

Alternatively we can show

$$a^2 - b^2 > 0$$

$$(a + b)(a - b) > 0$$

$$\therefore a + b > 0 \text{ and } a - b < 0 \quad \text{or} \quad a + b < 0 \text{ and } \boldsymbol{a - b < 0}$$

∴ the statement is false, since it is false if $a + b < 0$.

Example 3

Prove the following statement is false: There are no prime numbers divisible by 5

Solution

5 is prime and it is divisible by 5, so the statement is false

To disprove a statement that applies to at least one number we cannot use a counterexample, as it is perfectly acceptable that there are many numbers for which the statement is false, and yet the statement itself is true. We can:

- Use a direct proof to show that the negation of the statement is true for all values, so the original statement is false for all values.
- Use a direct proof of the statement for all values that leads to a contradiction, so the statement is false. If using Proof by Cases all cases must cause a contradiction.
- Use a proof by contrapositive for all values to prove $\neg Q \Rightarrow P$ so the statement is false

Example 4

Prove the following statement is false: $\exists$ a real number x, $x^2 + 2x + 5 < 0$

Solution

Here we will prove that the statement is false for all real numbers.

$\mathrm{a} > 0$

$\Delta = 2^2 - 4(1)(5) < 0$

The quadratic is positive definite, so the statement is false.

Example 5

Prove the following statement is false: There is a Pythagorean Triad where all numbers are odd.

Solution

Let the triad of odd numbers be a, b and c, such that $a = 2i + 1$, $b = 2j + 1$ and $c = 2k + 1$ for i, j, k integral.

$$\therefore (2i+1)^2 + (2j+1)^2 = (2k+1)^2$$
$$4i^2 + 4i + 1 + 4j^2 + 4j + 1 = 4k^2 + 4k + 1$$
$$4(i^2 + i + j^2 + j) + 1 = 4(k^2 + k)$$
$$4m + 1 = 4n \text{ for integral } m, n \text{ since } i, j, k \text{ are integral}$$

Now the RHS is a multiple of 4 yet the LHS isn't so we have a contradiction, so there is no Pythagorean Triad where all numbers are odd.

Example 6

Prove or disprove the following statement is false:

The sum of the squares of two consecutive even numbers is divisible by 8

Solution

We will disprove it by contradiction for all real numbers.

Let the two consecutive even numbers be $2k$ and $2k+2$ for integral k.

$(2k)^2+(2k+2)^2$

$= 4k^2+4k^2+8k+4$

$= 8k^2+8k+4$

$= 4(2k^2+2k+1)$

The statement is false, as the sum of the squares of two consecutive even numbers is divisible by 4 but not 8.

Example 7

Prove or disprove the following statement is false: $\exists$ a real number n such that $3^n+4^n=5^n$

Solution

Let $x=2$

$3^2+4^2=5^2$

The statement is true.

This is a rare example where an example does prove a statement, as it is not a universal statement.

EXERCISE 1.4

BASIC

1 Prove x is even if and only if x^2 is even.

2 Prove the following statement is false: If $a - b > 0$, where a, b are real, then $a^2 - b^2 > 0$

3 Prove the following statement is false: There are no prime numbers divisible by 7

4 Prove the following statement is false: ∃ a real number x, $-x^2 + 2x - 2 \geq 0$

5 Prove the following statement is false: There is a Pythagorean Triad where the two smallest numbers are even and the largest number is odd.

6 Prove or disprove the following statement: The sum of the squares of three consecutive even numbers is divisible by 4

7 Prove or disprove the following statement: ∃ a real number n such that $3^n + 4^n < 5^n$

MEDIUM

8 Prove for integral x, x^2 is divisible by 9 if and only if x is a multiple of 3.

9 Prove that if m, n are integers that $m^2 - n^2$ is even iff at least one of the sum and difference of m and n are even.

10 Prove the following statement is false: $|2x + 5| \leq 9 \Rightarrow |x| \leq 4$

11 Prove or disprove that if x and y are irrational and $x \neq y$, then xy is irrational.

12 Prove that a number is divisible by 6 if and only if it is divisible by 2 and 3.

13 Prove that the sum of two integers is even if and only if they have the same parity (both odd or both even).

CHALLENGING

14 Prove that a number is divisible by 4 if and only if the last two digits are a multiple of 4.

15 Prove that a three digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.

1 If x is even, let $x = 2k$ for integral k.
$x^2 = (2k)^2$
$= 4k^2$
$= 2(2k^2)$
$= 2p$ for integral p
$\therefore$ if x is even then x^2 is even

Conversely, we will show that if x^2 is even then x is even using proof by contrapositive.
Suppose x is odd
Let $x = 2k + 1$ for integral k
$x^2 = (2k + 1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
$= 2p + 1$ for integral p since k is integral
$\therefore$ if x is odd x^2 is odd, hence if x^2 is even then x is even by contrapositive.
$\therefore x$ is even if and only if x^2 is even $\square$

2 Let $a = -2$ $b = -3$
$\therefore a - b = -2 - (-3) = 1 > 0$
$(-2)^2 - (-3)^2 < 0$, so the statement is false.

3 7 is prime and it is divisible by 7, so the statement is false.

4 $a < 0$
$\Delta = 2^2 - 4(-1)(-2) < 0$
The quadratic is negative definite, so the statement is false.

5 Here we will use a proof by contradiction to prove it is false for all real numbers.

Let the triad of odd numbers be a, b and c, such that $a = 2i$,
$b = 2j$ and $c = 2k + 1$ for i, j, k integral.

$\therefore (2i)^2 + (2j)^2 = (2k + 1)^2$
$4i^2 + 4j^2 = 4k^2 + 4k + 1$
$4(i^2 + j^2) = 4(k^2 + k) + 1$
$4m = 4n + 1$ for integral m, n since i, j, k are integral

Now the LHS is a multiple of 4 yet the RHS isn't so we have a contradiction, so there is no Pythagorean Triad where the two smallest numbers are even and the largest number is odd.

6 Let the three consecutive even numbers be $2k, 2k + 2$ and $2k + 4$ for integral k.

$(2k)^2 + (2k + 2)^2 + (2k + 4)^2$
$= 4k^2 + 4k^2 + 8k + 4 + 4k^2 + 16k + 16$
$= 12k^2 + 24k + 20$
$= 4(3k^2 + 6k + 5)$
$= 4p$ for integral p since k is integral
The statement is true.

7 Let $n = 3$ $3^3 + 4^3 = 27 + 64 = 91 < 5^3$
The statement is true.

8 Prove that if x^2 is divisible by 9 then x is a multiple of 3 by contrapositive
Suppose x is not a multiple of 3
Let $x = 3k + j$ for integral k and $j = 1,2$
$\therefore x^2 = (3k + j)^2$
$= 9k^2 + 6jk + j^2$
$= 3(3k^2 + 2k) + j^2$
$= 3p + 1 \text{ or } 3p + 4$ for integral p since k is integral, which are not multiples of 9
$\therefore$ if x is not a multiple of 3 then x^2 is not a multiple of 9
$\therefore$ if x^2 is a multiple of 9 then x is a multiple of 3 by contrapositive.

Conversely, if x is a multiple of 3 let $x = 3j$ for integral j
$x^2 = (3j)^2$
$= 9j^2$
$= 9p$ for integral p since j is integral
$\therefore x^2$ is divisible by 9.

$\therefore x^2$ is divisible by 9 if and only if x is a multiple of 3 □

9 If $m^2 - n^2$ is even then $(m + n)(m - n)$ is even, using the difference of two squares.
$\therefore$ At least one of $m + n$ and $m - n$ is even, since two odd numbers have an odd product,
$\therefore$ If $m^2 - n^2$ is even at least one of the sum and difference of m and n are even.

Conversely, if at least one of $m + n$ and $m - n$ are even then $(m + n)(m - n)$ is even, since the product of two even numbers or an even and an odd number is even.
$\therefore m^2 - n^2$ is even, using the difference of two squares.
$\therefore$ If at least one of the sum and difference of m and n are even then $m^2 - n^2$ is even.

$\therefore m^2 - n^2$ is even iff at least one of the sum and difference of m and n are even. □

10 Let $x = -6$
$|2(-6) + 5| = 7 \leq 9$ yet $|-6| > 4$, so the statement is false.

11 Let $x = \sqrt{2}, y = 2\sqrt{2} \therefore xy = 2 \times 2 = 4$
$\therefore$ the statement is false.

12 Let x be divisible by 6
$\therefore x = 6m$ for integral m
$\therefore x = 2 \times 3 \times m$
$\therefore$ if a number is divisible by 6 then it is divisible by 2 and 3.

Conversely, we will use contrapositive to show that if a number is not divisible by 6 then it is not divisible by 2 and 3.
Let $x = 6m + k$ where k is not a multiple of 6
$= 2 \times 3 \times \left(m + \frac{k}{6}\right)$
$\neq 2 \times 3 \times p$ for integral p since k is not a multiple of 6
$\therefore$ if a number is not divisible by 6 it is not divisible by 2 and 3.
$\therefore$ if a number is divisible by 2 and 3 then it is divisible by 6

$\therefore$ a number is divisible by 6 if and only if it is divisible by 2 and 3

13 Let $a = 2m + j, b = 2n + j$ for integral m, n and $j = 0,1$
$a + b = 2m + 2n + 2j$
$= 2(m + n + j)$
$= 2p$ for integral p
$\therefore a + b$ is even if a, b have the same parity
Conversely, we will show by contradiction that if two numbers have the same parity then their sum must be even.
Suppose a, b have opposite parity and their sum is even (*)
Let $a = 2m + j, b = 2n + k$ for integral m, n and $j, k = 0,1$ and $j \neq k$
$a + b = 2m + 2n + j + k$
$= 2(m + n) + 1$
$= 2p + 1$ for integral p
$\therefore a + b$ is odd #
This contradicts (*) as $a + b$ cannot be odd and even.
$\therefore$ if two numbers have the same parity then their sum must be even.
$\therefore$ the sum of two integers is even if and only if they have the same parity (both odd or both even)

14 Let the number be $x = 100a + 10b + c$ where a, b, c are positive integers and $b, c \leq 9$
If the last two digits are a multiple of 4 then $10b + c = 4m$ for integral m
$\therefore x = 4(25a) + 4m$
$= 4(25a + m)$
$= 4p$ for integral p since a, m are integral
$\therefore$ if the last two digits are a multiple of 4 then the number is divisible by 4.
Conversely, we will show by contrapositive that if a number is divisible by 4 then the last two digits are a multiple of 4.
If the last two digits are not a multiple of 4 then $10b + c = 4m + k$ for integral m, k with k not a multiple of 4.
$\therefore x = 4(25a) + 4m + k$
$= 4(25a + m) + k$
$\neq 4p$ for integral p since a, m are integral and k is not a multiple of 4
$\therefore$ if the last two digits are not a multiple of 4 then the number is not divisible by 4.
$\therefore$ if the number is divisible by 4 then the last two digits are a multiple of 4
$\therefore$ a number is divisible by 4 if and only if the last two digits are a multiple of 4.

15 Let the number be $x = 100a + 10b + c$ where a, b, c are positive integers and $\mathrm{a}, b, c \leq 9$
If the sum of the digits is divisible by 9 then $a + b + c = 9m$ for integral m
$\therefore x = 100a + 10b + c$
$= 99a + a + 9b + b + c$
$= 9(11a + b) + a + b + c$
$= 9p + 9m$ for integral p since a, b are integral
$= 9q$ for integral q since p, m are integral
$\therefore$ if the sum of the digits is divisible by 9 then the three digit number is divisible by 9
Conversely, we will show by contrapositive that if the three digit number is divisible by 9 then the sum of the digits is 9.
Suppose the sum of the digits is not divisible by 9 then $a + b + c = 9m + k$ for integral m, and k not a multiple of 9
$\therefore x = 100a + 10b + c$
$= 99a + a + 9b + b + c$
$= 9(11a + b) + a + b + c$
$= 9p + 9m + k$ for integral p since a, b are integral
$\neq 9q$ for integral q since p, m are integral and k is not a multiple of 9
$\therefore$ if the sum of the digits is not divisible by 9 then the three digit number is not divisible by 9
$\therefore$ if the three digit sum is divisible by 9 then the sum of the digits is divisible by 9
$\therefore$ a three digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.

In Lesson 5 we look at Inequalities, covering:

- Inequalities
- Setting Out the Proof
- The Basics of Inequalities
- Equality
- Properties of Positive Numbers
- Geometrical Analogies
- Miscellaneous Tips
- The Triangle Inequality
- The Square of a Real Number is Non-negative

INEQUALITIES

Inequalities are a great source of hard yet simple proofs, that stretch the mind and separate out the very top students. In the old syllabus these formed the basis of the hardest marks on average of any topic, and yet the final solution was always short and simple.

Why are they so hard? Firstly there is generally little if any scaffolding to provide hints as to how to solve them. Secondly students must have superb algebra skills, and a deep understanding of any limitations to methods they might be considering.

Let's have another look at how to set out a proof before we delve into inequalities.

SETTING OUT THE PROOF

Before we start looking at inequalities, a quick reminder on how to set out proofs, which is particularly important in inequality proofs.

In the In Depth Extension 1 Year 12 Part I course we looked in detail at how a well formed proof should be set out. This is essential in proving inequalities. As a brief reminder, we said

> A proof is an argument that convinces one of your peers that a statement is always true. A better proof also convinces a suspicious peer that it is true, and helps a less able peer understand why it is true.

In order to make a convincing argument you are best to follow one of two types of method below.

Type 1 - Inequality

- Start with an inequality or equation known to be true (an axiom* or a result given or already proved)
- Progress in clear and logical steps <u>towards</u> the result to be proved.

* An axiom is a basic assumption we accept to be true. Many of the proofs rely on the identity that the square of a real number must be zero or positive - which we can write as $\mathbb{R}^2 \geq 0$.

If you are stuck remember that you can use working out paper to work from the conclusion to a true statement, then write it out in the correct order as your answer.

Type 2 - Expression

- Start by finding an expression which equals $\text{LHS} - \text{RHS}$
- Progress in clear and logical steps until it simplifies to an expression that is positive or negative.
- Interpret this result as $\text{LHS} > \text{RHS}$ or $\text{LHS} < \text{RHS}$ respectively.

WHY DO WE SWAP THE INEQUALITY SIGN AT TIMES AND NOT AT OTHERS?

An inequality is a statement showing the order of two (or three) expressions. When we perform an operation on each expression the order might stay the same or it might be reversed, so we might leave the sign alone or reverse it.

In summary:

- leave the sign alone if you perform an operation that matches a monotonic increasing function, such as addition, subtraction, multiplication by a positive number, or division by a positive number.
- swap the sign if
    - you perform an operation that matches a monotonic decreasing function such as multiplying by a negative number or dividing by a negative number
    - you swap sides (or reverse all three parts)
- get more information if you wish to perform an operation that matches a function that is not monotonic or has discontinuities, such as taking reciprocals or trigonometric ratios.

Example 1

When each of the following operations is performed on each side of an inequality, state whether the sign stays the same, changes, or whether we need more information.

a multiplying by 3 **b** dividing by -2 **c** taking the tangent

d taking the negative reciprocal **e** taking the square root

Solution

a stays the same — Multiplying by a positive is an increasing function

b swaps — Dividing by a negative is a decreasing function

c more information — The tangent is a discontinuous function

d more information — The reciprocal matches the hyperbola which is a discontinuous function

e stays the same, assuming both sides are positive

The square root is an increasing function for positive values

Example 2

If $a > b > 0$ for real a and b, prove that $2^{-a^2} < 2^{-b^2}$

Solution

Method 1

$\text{LHS} - \text{RHS}$

$= 2^{-a^2} - 2^{-b^2}$

$= \dfrac{1}{2^{a^2}} - \dfrac{1}{2^{b^2}}$

$= \dfrac{2^{b^2} - 2^{a^2}}{2^{a^2 b^2}}$

$< 0 \quad \text{since } a > b > 0$

$\therefore 2^{-a^2} < 2^{-b^2} \quad \square$

Method 2

$a > b$

$a^2 > b^2 \quad \text{since } a > b > 0$

$-a^2 < -b^2$

$2^{-a^2} < 2^{-b^2}$

EQUALITY

We are sometimes interested in what values of the variables can cause the two expressions to be equal - we call this equality. This will often be when the variables equal each other.

Example 3

Given $a + 4b \geq 4\sqrt{ab}$ for real a and b, for what values will equality occur?

Solution

Let $a + 4b = 4\sqrt{ab}$

$a^2 + 8ab + 16b^2 = 16ab$

$a^2 - 8ab + 16b^2 = 0$

$(a - 4b)^2 = 0$

$a - 4b = 0$

$a = 4b$

Equality will occur whenever $a = 4b$ □

PROPERTIES OF POSITIVE NUMBERS

The syllabus has not specified that $a, b > 0$, but in reality many questions will specify that all values are positive. Many consequences of this are more common sense than rules.

Example 4

If $a > b > 0$ for real a and b, prove that $(a + 1)(a - 1) > (b + 1)(b - 1)$

Solution

Method 1

$\text{LHS} - \text{RHS}$

$= (a + 1)(a - 1) - (b + 1)(b - 1)$

$= (a^2 - 1) - (b^2 - 1)$

$= a^2 - b^2$

$> 0 \quad \text{since } a > b > 0$

$\therefore (a + 1)(a - 1) > (b + 1)(b - 1)$ □

Method 2 (work backwards on working out paper first, then rewrite in the correct order)

$a > b$

$a^2 > b^2 \quad \text{since } a > b > 0$

$a^2 - 1 > b^2 - 1$

$(a + 1)(a - 1) > (b + 1)(b - 1)$ □

Example 5

If $a, b, c > 0$ prove that $(a+b+c)^2 - (ab+bc+ac) \geq 0$

Solution

Remembering from Polynomials that

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ac)$$

$$\begin{aligned}
\text{LHS} &= (a+b+c)^2 - (ab+bc+ac) \\
&= a^2+b^2+c^2+2(ab+bc+ac)-(ab+bc+ac) \\
&= a^2+b^2+c^2+ab+bc+ac \\
&\geq 0+0+0+0+0+0 \qquad \text{since } a,b,c>0 \\
&\geq 0
\end{aligned}$$

$\therefore (a+b+c)^2 - (ab+bc+ac) \geq 0$ □

SANDWICH THEOREM

If the value of a function must lie between the values of two other functions (for example $0 < f(x) < e^{-x}$), and at some point the two outer functions approach each other, then the function in the middle must also approach this value. So in the example below as $x \to \infty$ we see that $y = e^{-x}$ approaches 0, which is the lower function. This means that $f(x) \to 0$ as well.

Example 6

Prove that $\frac{\sin x}{x} \to 0$ as $x \to \infty$

Solution

$-1 \leq \sin x \leq 1$

$\therefore -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$

As $x \to \infty, \frac{1}{x} \to 0$

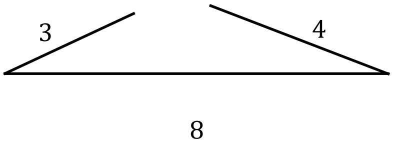
$\therefore$ as $x \to \infty \quad 0 \leq \frac{\sin x}{x} \leq 0$

$\therefore \frac{\sin x}{x} \to 0$ as $x \to \infty$ □

TRIANGLE INEQUALITY

For three straight lines to form a triangle any two sides must add to more than the other side.

Not convinced? Try making a triangle with sides $3, 4$ and 8



So we can see that regardless the values of x and y, that

$$\boxed{|x| + |y| \geq |x + y|}$$

We use the triangle inequality in situations other than triangles - if we are using real numbers or vectors the 'sides' of the triangle can lay flat, which is why we need to use $\geq$.

We will use a much easier proof of this rule in Vectors.

Proof 7

Prove $|x| + |y| \geq |x + y|$ for all real numbers x, y.

Solution

Case 1: $x, y \geq 0$
$|x| + |y|$
$= x + y \quad \text{for } x, y \geq 0$
$= |x + y|$
$\therefore \text{ true for } x, y \geq 0$

Case 2: $x, y \leq 0$
$|x| + |y|$
$= -x - y \quad \text{for } x, y \leq 0$
$= -(x + y)$
$= |x + y|$
$\therefore \text{ true for } x, y \leq 0$

Case 3: $x \geq 0, y < 0$ without loss of generality *
$|x| + |y|$
$= x - y \quad \text{for } x > 0, y < 0$
$= |x - y| \quad \text{for } x > 0, y < 0$
$> |x + y| \quad \text{for } x > 0, y < 0$
$\therefore \text{ true for } x, y > 0$

$\therefore |x| + |y| \geq |x + y|$ for all real numbers x, y.

* Without loss of generality means we could prove a similar result which swapped x and y in the same way.

Example 8

Prove $|a^2 - ab| + |ab - b^2| \geq (a+b)(a-b)$ for $a > b > 0$

Solution

$|a^2 - ab| + |ab - b^2|$

$\geq |(a^2 - ab) + (ab - b^2)|$ by the triangle inequality

$= |a^2 - b^2|$

$= a^2 - b^2$ since $a > b$

$= (a+b)(a-b)$

$\therefore |a^2 - ab| + |ab - b^2| \geq (a+b)(a-b)$ □

Example 9

Prove $|a - b| + |c - b| \geq a - c$ for $a > b > c > 0$

Solution

$|a - b| + |c - b|$

$= |a - b| + |b - c|$

$\geq |(a - b) + (b - c)|$ by the triangle inequality

$= |a - c|$

$= a - c$ since $a > c$

$\therefore |a - b| + |c - b| \geq a - c$

Example 10

A triangle has sides $x, 10, 12$. What are the possible values of x?

Solution

If x is the smallest side then $x + 10 > 12$, so $x > 2$.

If x is the longest side then $10 + 12 > x$, so $x < 22$

$\therefore 2 < x < 22$.

THE SQUARE OF A REAL NUMBER IS NON-NEGATIVE

A fundamental property of real numbers is that when we square them the result is non-negative, so zero or positive. We can write this as $\mathbb{R}^2 \geq 0$, although $\mathbb{R}$ is set notation and not in the syllabus.

We will use this property extensively with the arithmetic mean - geometric mean inequality, but here are a couple of other applications.

Example 11

If a, b, c are real prove that $(a+b+c)^2 - (ab+bc+ac) \geq 0$

Solution

$$\begin{aligned}
\text{LHS} &= a^2+b^2+c^2-(ab+bc+ca) \\
&= \frac{1}{2}(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2) \\
&= \frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right) \\
&\geq \frac{1}{2}(0+0+0) \\
&= 0
\end{aligned}$$

$\therefore (a+b+c)^2-(ab+bc+ac) \geq 0 \quad \square$

Example 12

If $x, y > 0$ prove that $x^4+y^4 \geq x^3y+xy^3$

Solution

$$\begin{aligned}
\text{LHS} - \text{RHS} &= x^4-x^3y+y^4-xy^3 \\
&= x^3(x-y)+y^3(y-x) \\
&= x^3(x-y)-y^3(x-y) \\
&= (x^3-y^3)(x-y) \\
&= (x-y)(x^2+xy+y^2)(x-y) \\
&= (x-y)^2(x^2+xy+y^2) \\
&\geq 0 \quad \text{since both expressions are non−negative}
\end{aligned}$$

$\therefore x^4+y^4 \geq x^3y+xy^3 \quad \square$

EXERCISE 1.5

BASIC

1 When each of the following operations is performed on each side of an inequality, state whether the sign stays the same, changes, or whether we need more information.

a adding -3 **b** dividing by 2 **c** multiplying by -1

d taking the sine **e** taking the natural logarithm

2 If $a > b > 0$ for real a and b, prove that $-\frac{1}{a} > -\frac{1}{b}$

3 Given $16a + 9b \geq 24\sqrt{ab}$ for real a and b, for what values will equality occur?

4 If $a > b > 0$ for real a and b, prove that $(2 + a)(2 - a) < (2 + b)(2 - b)$

5 **a** If $a, b, c > 0$ prove that $(a + b + c)^2 - (ab + bc + ac) \geq 0$

b If a, b, c are real prove that $a^2 + b^2 + c^2 - (ab + bc + ac) \geq 0$

6 Prove that $e^{-x} \sin x \to 0$ as $x \to \infty$, given $-e^{-x} \leq e^{-x} \sin x \leq e^{-x}$

7 Prove $|a^2 - 2ab| + |4b^2 - 2ab| \geq (a + 2b)(a - 2b)$ for $a > 2b > 0$

8 Prove $|a - b| + |b - a| > 0$ for $a > b > 0$.

9 A triangle has sides $x, 7, 5$. What are the possible values of x?

10 If $x, y > 0$ prove that $x^5 + y^5 \geq x^4y + xy^4$

MEDIUM

11 If $a = b + c$ for $a, b, c > 0$, prove $a^2 - bc > b^2 + c^2$

12 If $a, b > 0$ prove $\mathrm{a}^3 - ab^2 - a^2b + b^3 \geq 0$

13 Given $x > \sin x$ for $x > 0$, prove $\pi x - x^2 > \sin^2 x$ for $0 < x < \pi$

14 Two sides of an isosceles triangle are 4 cm and 6 cm. Prove the third side can be 4 cm or 6 cm.

15 If $a > b$ for real a and b, prove that $a^2 + b^2 + 8 \geq 4(a + b)$

16 Show that for any real x, $(x-3)(x+1)(x-5)(x+3)+40 \geq 0$

17 Let $a, b, x, y > 0$. Prove that $(a^2-b^2)(x^2-y^2) \leq (ax-by)^2$

CHALLENGING

18 If $a, b > 0$, prove that

$$\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}} \geq \sqrt{a}+\sqrt{b}$$

19 Prove $a^2+b^2+c^2+d^2+e^2 \geq a(b+c+d+e)$

20

a By substituting $a=2x, b=y$ in $(a-b)^2 \geq 0$ prove $(x+y)\left(\frac{1}{x}+\frac{4}{y}\right) \geq 9$ for $x, y > 0$

b By substituting $a=\sqrt{x}, b=\sqrt{y}$ in $(a-b)^2 \geq 0$ then $a=\frac{1}{\sqrt{x}}, b=\sqrt{\frac{4}{y}}$ prove

$(x+y)\left(\frac{1}{x}+\frac{4}{y}\right) \geq 8$.

c The results in (a) and (b) are different but both correct. Now $(x+y)\left(\frac{1}{x}+\frac{4}{y}\right)$ cannot be smaller than 9. What have we done in (b) that has caused us to set a lower limit than is needed?

1 a) stays the same
b) stays the same
c) swaps
d) more information
e) stays the same, assuming both sides are positive

2 Method 1

LHS − RHS

$= \frac{1}{b} - \frac{1}{a}$

$= \frac{a-b}{ab}$

$> 0 \quad$ since $a > b > 0$

$\therefore -\frac{1}{a} > -\frac{1}{b} \quad \square$

Method 2

$a > b$

$\frac{1}{a} < \frac{1}{b} \quad$ since $a > b > 0$

$-\frac{1}{a} > -\frac{1}{b} \quad \square$

3 Let $16a + 9b = 24\sqrt{ab}$

$256a^2 + 288ab + 81b^2 = 576ab$

$256a^2 - 288ab + 81b^2 = 0$

$(16a - 9b)^2 = 0$

$16a - 9b = 0$

$a = \frac{9b}{16}$

Equality will occur whenever $a = \frac{9b}{16}$

4 Method 1

LHS − RHS

$= (2+a)(2-a) - (2+b)(2-b)$

$= (4-a^2) - (4-b^2)$

$= b^2 - a^2$

$< 0 \quad$ since $a > b > 0$

$\therefore (2+a)(2-a) < (2+b)(2-b) \quad \square$

Method 2 (work backwards on working out paper first, then rewrite in the correct order)

$a > b$

$a^2 > b^2 \quad$ since $a > b > 0$

$-a^2 < -b^2$

$4 - a^2 < 4 - b^2$

$(2+a)(2-a) < (2+b)(2-b) \quad \square$

5 **a** LHS $= (a+b+c)^2 - (ab+bc+ac)$

$= a^2 + b^2 + c^2 + 2(ab+bc+ac) - (ab+bc+ac)$

$= a^2 + b^2 + c^2 + ab + bc + ac$

$\geq 0 + 0 + 0 + 0 + 0 + 0 \qquad$ since $a, b, c > 0$

≥ 0

$\therefore (a+b+c)^2 - (ab+bc+ac) \geq 0 \quad \square$

b LHS $= a^2 + b^2 + c^2 - (ab+bc+ca)$

$= \frac{1}{2}(a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2)$

$= \frac{1}{2}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$

$\geq \frac{1}{2}(0+0+0)$

$= 0$

$\therefore (a+b+c)^2 - (ab+bc+ac) \geq 0 \quad \square$

6 $e^{-x} \to 0$ as $x \to \infty$
$\therefore 0 \leq e^{-x} \sin x \leq 0 \; as \, x \to \infty$
$\therefore e^{-x} \sin x \to 0$ as $x \to \infty$ □

7 $|a^2 - 2ab| + |4b^2 - 2ab|$
$= |a^2 - 2ab| + |2ab - 4b^2|$
$\geq |(a^2 - 2ab) + (2ab - 4b^2)|$ by the triangle inequality
$= |a^2 - 4b^2|$
$= a^2 - 4b^2$ since $a > 2b > 0$
$= (a + 2b)(a - 2b)$
$\therefore |a^2 - 2ab| + |4b^2 - 2ab| \geq (a + 2b)(a - 2b)$ □

8 $|a - b| + |b - a|$

$> |(a - b) + (b - a)|$ by the triangle inequality and since a>b

$= |0|$

$= 0$

$\therefore |a - b| + |b - a| > 0$ for $a > b > 0$ □

9 If x is the smallest side then $x + 5 > 7$, so $x > 2$.
If x is the longest side then $5 + 7 > x$, so $x < 12$
$\therefore 2 < x < 12$.

10 $\text{LHS} - \text{RHS} = x^5 - x^4y + y^5 - xy^4$
$= x^4(x - y) + y^4(y - x)$
$= x^4(x - y) - y^4(x - y)$
$= (x^4 - y^4)(x - y)$
$= (x^2 + y^2)(x^2 - y^2)(x - y)$
$= (x^2 + y^2)(x + y)(x - y)(x - y)$
$= (x^2 + y^2)(x + y)(x - y)^2$
≥ 0 since all three expressions are non−negative
$\therefore x^5 + y^5 \geq x^4y + xy^4$ □

11 $a = b + c$

$a^2 = b^2 + 2bc + c^2$

$a^2 - 2bc = b^2 + c^2$

$a^2 - bc > b^2 + c^2$ since $bc > 0$ □

12 $\text{LHS} = a(a^2 - b^2) - b(a^2 - b^2)$

$= (a - b)(a^2 - b^2)$

$= (a - b)(a + b)(a - b)$

$= (a - b)^2(a + b)$

≥ 0 since $\mathbb{R}^2 \geq 0$ and $a, b > 0$ □

13 $x > \sin x$ (1)

$\pi - x > \sin(\pi - x)$ also from (1) since $\pi - x > 0$ in the domain

$\therefore \pi - x > \sin x$ since $\sin(\pi - x) = \sin x$ (2)

(1) × (2):

$\pi x - x^2 > \sin^2 x$ □

14 If the sides are $4, 4$ and 6 then the sides satisfy the triangle inequality in any order: $4 + 4 > 6,\ \ 4 + 6 > 4,\ \ 6 + 4 > 4$

If the sides are $4, 6$ and 6 then the sides also satisfy the triangle inequality in any order: $4 + 6 > 6,\ \ 6 + 4 > 6,\ \ 6 + 6 > 4$

The third side of the isosceles triangle can be 4 cm or 6 cm.

15
$$\begin{aligned}
\text{LHS} - \text{RHS} &= a^2 + b^2 - 4a - 4b + 8 \\
&= a^2 - 4a + 4 + b^2 - 4b + 4 \\
&= (a-2)^2 + (b-2)^2 \\
&\geq 0 + 0 \\
&= 0
\end{aligned}$$
$\therefore a^2 + b^2 + 8 \geq 4(a + b)$ □

16
$$\begin{aligned}
\text{LHS} - \text{RHS} &= (x-3)(x+1)(x-5)(x+3) + 40 \\
&= (x^2 - 2x - 3)(x^2 - 2x - 15) + 40 \\
&= (x^2 - 2x - 9 + 6)(x^2 - 2x - 9 - 6) + 40 \\
&= (x^2 - 2x - 9)^2 - 6^2 + 40 \\
&= (x^2 - 2x - 9)^2 + 4
\end{aligned}$$
> 0 since the expressions are non−negative and positive respectively

$\therefore (x-3)(x+1)(x-5)(x+3) + 40 > 0$ □

17
$$\begin{aligned}
\text{LHS} - \text{RHS} &= (a^2 - b^2)(x^2 - y^2) - (ax - by)^2 \\
&= a^2x^2 - a^2y^2 - b^2x^2 + b^2y^2 - a^2x^2 + 2abxy - b^2y^2 \\
&= -(a^2y^2 - 2abxy + b^2x^2) \\
&= -(ay - bx)^2 \\
&\leq 0
\end{aligned}$$
$\therefore (a^2 - b^2)(x^2 - y^2) \leq (ax - by)^2$ □

18

$$\begin{aligned}\text{LHS}-\text{RHS} &= \frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}}-\sqrt{a}-\sqrt{b}\\ &= \frac{a\sqrt{a}+b\sqrt{b}-a\sqrt{b}-b\sqrt{a}}{\sqrt{ab}}\\ &= \frac{\sqrt{a}(a-b)-\sqrt{b}(a-b)}{\sqrt{ab}}\\ &= \frac{(a-b)\left(\sqrt{a}-\sqrt{b}\right)}{\sqrt{ab}}\\ &= \frac{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)}{\sqrt{ab}}\\ &= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2\left(\sqrt{a}+\sqrt{b}\right)}{\sqrt{ab}}\\ &\geq 0\end{aligned}$$

$$\therefore \frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}} \geq \sqrt{a}+\sqrt{b} \quad \square$$

19

$$\begin{aligned}\text{LHS}-\text{RHS} &= a^2+b^2+c^2+d^2+e^2-a(b+c+d+e)\\ &= \left(\frac{a^2}{4}-ab+b^2\right)+\left(\frac{a^2}{4}-ac+c^2\right)+\left(\frac{a^2}{4}-ad+d^2\right)+\left(\frac{a^2}{4}-ae+e^2\right)\\ &= \left(\frac{a}{2}-b\right)^2+\left(\frac{a}{2}-c\right)^2+\left(\frac{a}{2}-d\right)^2+\left(\frac{a}{2}-e\right)^2\\ &\geq 0\end{aligned}$$

$$\therefore a^2+b^2+c^2+d^2+e^2 \geq a(b+c+d+e)$$

20 **a**

$$\begin{aligned}(2x-y)^2 &\geq 0\\ 4x^2-4xy+y^2 &\geq 0\\ 4x^2+5xy+y^2 &\geq 9xy\\ (x+y)(4x+y) &\geq 9xy\\ \frac{(x+y)(4x+y)}{xy} &\geq 9\\ (x+y)\left(\frac{1}{x}+\frac{4}{y}\right) &\geq 9 \qquad (1)\end{aligned}$$

b

$$\begin{aligned}(\sqrt{x}-\sqrt{y})^2 &\geq 0\\ x-2\sqrt{xy}+y &\geq 0\\ x+y &\geq 2\sqrt{xy} \qquad (2)\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{\sqrt{x}}-\frac{2}{\sqrt{y}}\right)^2 &\geq 0\\ \frac{1}{x}-\frac{4}{\sqrt{xy}}+\frac{4}{y} &\geq 0\\ \frac{1}{x}+\frac{4}{y} &\geq \frac{4}{\sqrt{xy}} \qquad (3)\end{aligned}$$

$(2)\times(3)$:

$$\therefore (x+y)\left(\frac{1}{x}+\frac{4}{y}\right) \geq 8$$

c The minimum value of each inequality occurs at equality, so $y = x$ in (2) and $y = 4x$ in (3). Neither of these matches equality in (1), which is $y = 2x$, which causes the disparity.

In Lesson 6 we look at the Arithmetic Mean - Geometric Mean Inequality, covering:

- The Four Means
- Arithmetic Mean and Geometric Mean Inequality

THE FOUR MEANS

When we talk about the mean of a set of data we are normally talking about the Arithmetic Mean (AM) - the sum of the scores divided by the number of scores. We also call this the average.

$$\mathrm{AM} = \frac{x_1 + x_2 + x_3 + \ldots x_n}{n}$$

The arithmetic mean is a useful measure of centre when the scores are close together, but is affected by very large values when the numbers are spread.

There are many other types of mean, each with a different purpose. We will use the Geometric mean extensively in the rest of this lesson, plus harder questions sometimes use the Harmonic Mean and Quadratic Mean without naming them.

The geometric mean of a set of data is the n^{th} root of their product. The numbers must all have the same sign to be able to find a geometric mean. It is more useful than the arithmetic mean when finding the centre of scores that are widely spread, as large scores don't distort the results as much.

$$\mathrm{GM} = \sqrt[n]{x_1 \times x_2 \times x_3 \times \ldots \times x_n}$$

The harmonic mean of a set of data is the reciprocal of the mean of the sum of the reciprocals of the scores. It is useful when finding the average of two or more rates. The term harmonic mean is not used in the syllabus.

$$\mathrm{HM} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

For two scores the Harmonic Mean can be rearranged to give $\frac{x_1 x_2}{x_1 + x_2}$, which students will sometimes see in questions.

The quadratic mean of a set of data is the square root of the average of the sum of the squares of the scores. It is related to standard deviation. The term quadratic mean is not used in the syllabus, but students will sometimes see questions that involve the expression.

$$\text{QM} = \sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}}$$

We will not mention the terms Harmonic Mean or Quadratic Mean again, although we will come across questions involving the expressions.

ARITHMETIC MEAN - GEOMETRIC MEAN INEQUALITY

The arithmetic mean of a set of numbers is always greater than or equal to their geometric mean, with equality when all numbers are equal.

$$\frac{x_1 + x_2 + x_3 + \ldots x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$$

We will look at many proofs involving this property, all restricted to positive real numbers. We will abbreviate this inequality as AM-GM at times.

The syllabus states that students should

- ‘establish and use the relationship between the arithmetic mean and geometric mean for two non-negative numbers’, then
- ‘prove further results involving inequalities by logical use of previously obtained inequalities’.

At a guess we can interpret this to mean we will use the AM-GM Inequality:

- for 2 variables in easy and hard questions
- for 2 variables at a time for 3 or 4 numbers (multiple applications of the AM-GM Inequality)
- for 3 or 4 variables in simple applications

Proof 1

For $a, b \geq 0$ prove $\frac{a+b}{2} \geq \sqrt{ab}$ using the square of the difference of the square roots of a and b.

Solution

Method 1

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$
$$a - 2\sqrt{ab} + b \geq 0$$
$$a + b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab} \quad \square$$

Method 2

$$(a-b)^2 \geq 0$$
$$a^2 - 2ab + b^2 \geq 0$$
$$a^2 + 2ab + b^2 \geq 4ab$$
$$(a+b)^2 \geq 4ab$$
$$a + b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$

In Method 1, which is the preferred method, we start by squaring the difference of the square roots of the two variables. We will use this technique in harder examples to come, rather than the traditional solution (Method 2), as it is often quicker than substituting into the AM-GM Inequality.

We could make it even shorter by using $\left(\sqrt{\frac{a}{2}}-\sqrt{\frac{b}{2}}\right)^2 \geq 0$, but I find the version used in the proof more useful overall.

Method 1

Starting with the axiom $\mathbb{R}^2 \geq 0$.

The expression in the brackets should be the difference of the square roots of the two terms in the result to be proved. For example if we want to prove $\frac{x^2+y^2}{2} > xy$ then we start with $(x-y)^2 \geq 0$, as x and y are the square roots of x^2 and y^2 respectively.

Solving from the axiom is simpler and more convincing.

Method 2

Starting with $\frac{a+b}{2} \geq \sqrt{ab}$

You can only do this if this result has been proven in part (i) of the question, or if you are told you may assume it. You then make appropriate substitutions for a and b.

If a question already uses the pronumerals you will need in the final answer, state the axiom/ known result using different pronumerals to make it easier to understand.

Method 3

Starting with an expression equal to $\mathrm{LHS} - \mathrm{RHS}$, and simplifying it until it is positive or negative, then interpreting the result as proving that which is to be proved.

Method 4

On working out paper work from the conclusion to a true statement, then write the steps out in reverse order as your actual solution.

Example 2

For $a > 0$ prove $a + \frac{1}{a} \geq 2$

Solution

Method 1

$$\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0$$

$$a - 2 + \frac{1}{a} \geq 0$$

$$a + \frac{1}{a} \geq 2 \qquad \square$$

Method 2

Let $x = a, y = \frac{1}{a}$ in $\frac{x+y}{2} \geq \sqrt{xy}$

$$\therefore \frac{a + \frac{1}{a}}{2} \geq \sqrt{a \times \frac{1}{a}}$$

$$a + \frac{1}{a} \geq 2 \qquad \square$$

Method 3

$$\begin{aligned} \mathrm{LHS} - \mathrm{RHS} &= a - 2 + \frac{1}{a} \\ &= \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \\ &\geq 0 \end{aligned}$$

$$\therefore a - 2 + \frac{1}{a} \geq 0$$

$$\therefore a + \frac{1}{a} \geq 2 \qquad \square$$

Method 4

On working out paper:

$$a + \frac{1}{a} \geq 2$$

$$a - 2 + \frac{1}{a} \geq 0$$

$$\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0$$

Now rewrite like Method 1

For the examples to follow we will only use Methods 1 and 2, reserving the other methods as back up for questions where we cannot see how to progress with the first two methods.

Example 3

For $a, b, c > 0$ prove $ac + \frac{b}{c} \geq 2\sqrt{ab}$

Solution

Method 1

$$\left(\sqrt{ac} - \sqrt{\frac{b}{c}}\right)^2 \geq 0$$

$$ac - 2\sqrt{ab} + \frac{b}{c} \geq 0$$

$$\therefore ac + \frac{b}{c} \geq 2\sqrt{ab} \quad \square$$

Method 2

Let $m = ac,\ n = \frac{b}{c}$ in $\frac{m+n}{2} \geq \sqrt{mn}$

$$\therefore \frac{ac + \frac{b}{c}}{2} \geq \sqrt{ac \cdot \frac{b}{c}}$$

$$\therefore ac + \frac{b}{c} \geq 2\sqrt{ab} \quad \square$$

Example 4

Prove $x \geq 2\sqrt{x-1}$ for $x \geq 1$

Solution

If we look closely we see this can be rewritten as $x - 1 + 1 \geq 2\sqrt{(x-1)(1)}$, which is the AM-GM with $x - 1$ and 1.

The best solution uses Method 1, the square of the difference of the square roots. It is simpler and more convincing than the alternatives to follow.

Method 1

$$\left(\sqrt{x-1} - 1\right)^2 \geq 0$$

$$x - 1 - 2\sqrt{x-1} + 1 \geq 0$$

$$x \geq 2\sqrt{x-1} \text{ for } x \geq 1 \quad \square$$

Method 2

Let $a = x - 1$ and $b = 1$ in AM-GM

$$\therefore \frac{(x-1) + (1)}{2} \geq \sqrt{(x-1)(1)}$$

$$\frac{x}{2} \geq \sqrt{x-1}$$

$$\therefore x \geq 2\sqrt{x-1} \text{ for } x \geq 1 \quad \square$$

Example 5

For $a, b > 0$ prove $(a+b)^2 \geq 4ab$

Solution

We could start with Method 1 or 2 and square our answer, but when we have to prove that one expression is greater than or equal to another expression, it is often easiest to use Method 3, investigating $\text{LHS} - \text{RHS}$, and in this case try and prove it is ≥ 0. Alternatively we could use Method 4.

Method 3

LHS − RHS

$= (a+b)^2 - 4ab$

$= a^2 + 2ab + b^2 - 4ab$

$= a^2 - 2ab + b^2$

$= (a-b)^2$

≥ 0 since $\mathbb{R}^2 \geq 0$

$\therefore (a+b)^2 - 4ab \geq 0$

$\therefore (a+b)^2 \geq 4ab$ $\square$

Method 4

Working out paper

$$\begin{aligned} (a+b)^2 &\geq 4ab \\ a^2 + 2ab + b^2 &\geq 4ab \\ a^2 - 2ab + b^2 &\geq 0 \\ (a-b)^2 &\geq 0 \end{aligned}$$

Actual Solution:

$$\begin{aligned} (a-b)^2 &\geq 0 \\ a^2 - 2ab + b^2 &\geq 0 \\ a^2 + 2ab + b^2 &\geq 4ab \\ (a+b)^2 &\geq 4ab \quad \square \end{aligned}$$

In harder questions we use the AM - GM inequality multiple times. These types of questions would often be part (ii) of a question where you had already proved the general result or similar.

Example 6

Given $\frac{a+b}{2} \geq \sqrt{ab}$, prove $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq x + y + z$ for $x, y, z > 0$

Solution

We could start with Method 1 or 2 and square our answer, but when we have to prove that one expression is greater than or equal to another expression, it is often easiest to use Method 3, investigating $\text{LHS} - \text{RHS}$, and in this case try and prove it is ≥ 0. Alternatively we could use Method 4.

$$\frac{\frac{xy}{z} + \frac{yz}{x}}{2} \geq \sqrt{\frac{xy^2z}{zx}} \qquad \text{(AM}-\text{GM)}$$

$$\frac{1}{2}\left(\frac{xy}{z} + \frac{yz}{x}\right) \geq y \qquad (1)$$

Similarly

$$\frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y}\right) \geq z \qquad (2)$$

$$\frac{1}{2}\left(\frac{xy}{z} + \frac{zx}{y}\right) \geq x \qquad (3)$$

$(1) + (2) + (3)$:

$$\frac{1}{2}\left(\frac{xy}{z} + \frac{yz}{x}\right) + \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y}\right) + \frac{1}{2}\left(\frac{xy}{z} + \frac{zx}{y}\right) \geq y + z + x$$

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq x + y + z \qquad \square$$

Example 7

Prove $x^2 + y^2 + z^2 \geq x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2}$

for $x, y, z > 0$

Solution

Notice that the RHS is asymmetrical as there is no $\mathrm{z}\sqrt{\mathrm{x}^2 + \mathrm{y}^2}$ term, so we only need 2 applications of AM-GM.

$$\frac{x^2 + (y^2 + z^2)}{2} \geq \sqrt{x^2(y^2 + z^2)} \qquad (\mathrm{AM} - \mathrm{GM})$$

$$\frac{x^2 + (y^2 + z^2)}{2} \geq x\sqrt{y^2 + z^2} \qquad (1)$$

Similarly

$$\frac{y^2 + (x^2 + z^2)}{2} \geq y\sqrt{x^2 + z^2} \qquad (2)$$

$(1) + (2)$:

$$x^2 + y^2 + z^2 \geq x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2} \qquad \square$$

PROVING THE AM-GM INEQUALITY FOR 3 OR 4 VARIABLES

We can prove the result for 3 or 4 variables most easily by:

- substituting the arithmetic mean of a and b, and the arithmetic mean of c and d into the AM-GM for two variables, to prove the result for 4 variables.
- substitute the arithmetic mean of a, b and c for d in the AM-GM for four variables to prove it for 3 variables.

We can prove these results from the axiom as well, but it is a bit longer.

Proof 8

For $a, b, c, d \geq 0$ prove $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$

Solution

Best solution - substitute the arithmetic mean of the first two variables and the last two variables into the AM-GM for two variables.

Let $x = \frac{a+b}{2}, y = \frac{c+d}{2}$ in $\frac{x+y}{2} \geq \sqrt{xy}$

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \sqrt{\frac{a+b}{2} \times \frac{c+d}{2}}$$

$$\therefore \frac{a+b+c+d}{4} \geq \sqrt{\sqrt{ab} \times \sqrt{cd}} \qquad \text{since } \frac{(a+b)}{2} \geq \sqrt{ab}, \frac{c+d}{2} \geq \sqrt{cd}$$

$$\therefore \frac{a+b+c+d}{4} \geq \sqrt[4]{abcd} \qquad \square$$

Example 9

If $a, b, c, d > 0$ then prove $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$

Solution

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}}{4} \geq \sqrt[4]{\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{a}} \quad (\mathrm{AM-GM})$$

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}}{4} \geq 1$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4 \ \square$$

Proof 10

For $a, b, c \geq 0$ prove $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$

Solution

Best solution - substitute the arithmetic mean of the first three variables for d in the AM-GM for four variables.

Let $w = a,\ x = b, y = c, z = \frac{a+b+c}{3}$ in $\frac{w+x+y+z}{4} \geq \sqrt[4]{wxyz}$

$$\therefore \frac{a+b+c+\left(\frac{a+b+c}{3}\right)}{4} \geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)}$$

$$\frac{\frac{4}{3}(a+b+c)}{4} \geq (abc)^{\frac{1}{4}}\left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{1} \geq (abc)^{\frac{1}{4}}\left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}} \geq (abc)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}\times\frac{4}{3}} \geq (abc)^{\frac{1}{4}\times\frac{4}{3}}$$

$$\therefore \frac{a+b+c}{3} \geq \sqrt[3]{abc} \qquad \square$$

Example 11

If $a, b, c > 0$ then prove $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$

Solution

$$\frac{a^3 + a^3 + b^3}{3} \geq \sqrt[3]{a^6b^3}$$

$$\therefore \frac{2a^3 + b^3}{3} \geq a^2b \quad (1)$$

Similarly

$$\frac{2b^3 + c^3}{3} \geq b^2c \qquad (2) \qquad \frac{2c^3 + a^3}{3} \geq c^2a \qquad (3)$$

$(1) + (2) + (3)$: $\qquad a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a \quad \square$

EXERCISE 1.6

BASIC

1 For $a,b \geq 0$ prove $\frac{a+b}{2} \geq \sqrt{ab}$ by starting with $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$

2 For $a,b \geq 0$ prove $\frac{a+b}{2} \geq \sqrt{ab}$ by starting with $(a-b)^2 \geq 0$

3 For $x \neq 0$ prove $x^2 + \frac{1}{x^2} \geq 2$

4 Prove $x^2 \geq 2\sqrt{(x-1)(x+1)}$ for $x \geq 1$

5 For $a,b > 0$ prove $(a+2b)^2 \geq 8ab$

MEDIUM

6 For $a,b,c > 0$ prove $a^2c^2 + \frac{b^2}{c^2} \geq 2ab$

7 Given $\frac{a+b}{2} \geq \sqrt{ab}$, prove $\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ for $x,y,z > 0$

8 Prove $\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{1}{yz}\sqrt{x^2y^2+z^4} + \frac{1}{xz}\sqrt{y^2z^2+x^4}$ for $x,y,z > 0$

9 For $a,b,c,d \geq 0$ prove $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$

10 If $a,b,c,d > 0$ then prove $\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{d} + \frac{d^3}{a} \geq 4\sqrt{abcd}$

11 For $a,b,c \geq 0$ prove $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$

12 If $a,b,c > 0$ then prove $a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$

13 Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2a_3\ldots a_n = 1$.
Prove that $(a_1^2+a_1)(a_2^2+a_2)\ldots(a_n^2+a_n) \geq 2^n$

14 i) For $a,b,c \geq 0$ prove $a^2+b^2+c^2 \geq ab+bc+ca$

ii) Hence prove $(a+b+c)^2 \geq 3(ab+bc+ca)$

15 Let $a+b=1$ prove that $a^4+b^4 \geq \frac{1}{8}$

16 Let $a,b,c>0$ such that $abc=1$, prove that $a^2+b^2+c^2 \geq a+b+c$

17 If $a,b,c>0$ then prove $a^4+b^4+c^4 \geq a^2bc+b^2ca+c^2ab$

18 If $a,b,c>0$ satisfy $abc=1$, prove
$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \geq 3$$

19 Prove the Harmonic Mean ≤ the Geometric Mean ≤ the Arithmetic Mean ≤ the Quadratic Mean for 2 numbers, ie:
$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

20 **a** For $a,b,c>0$ prove $3(a^2+b^2+c^2) \geq (a+b+c)^2 \geq 3(ab+bc+ac)$
b If $a+b+c=3$, hence prove that $a^2+b^2+c^2+ab+bc+ca \geq 6$

1

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$
$$a - 2\sqrt{ab} + b \geq 0$$
$$a + b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab} \quad \square$$

2

$$(a-b)^2 \geq 0$$
$$a^2 - 2ab + b^2 \geq 0$$
$$a^2 + 2ab + b^2 \geq 4ab$$
$$(a+b)^2 \geq 4ab$$
$$a + b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$

3 Method 1

$$\left(x - \frac{1}{x}\right)^2 \geq 0$$
$$x^2 - 2 + \frac{1}{x^2} \geq 0$$
$$x^2 + \frac{1}{x^2} \geq 2 \quad \square$$

Method 2

Let $a = x^2, b = \frac{1}{x^2}$ in $\frac{a+b}{2} \geq \sqrt{ab}$

$$\therefore \frac{x^2 + \frac{1}{x^2}}{2} \geq \sqrt{x^2 \times \frac{1}{x^2}}$$
$$x^2 + \frac{1}{x^2} \geq 2 \quad \square$$

Method 3

$$\text{LHS} - \text{RHS} = x^2 - 2 + \frac{1}{x^2}$$
$$= \left(x - \frac{1}{x}\right)^2$$
$$\geq 0$$
$$\therefore x^2 - 2 + \frac{1}{x^2} \geq 0$$
$$\therefore x^2 + \frac{1}{x^2} \geq 2 \quad \square$$

Method 4

On working out paper:

$$x^2 + \frac{1}{x^2} \geq 2$$
$$x^2 - 2 + \frac{1}{x^2} \geq 0$$
$$\left(x - \frac{1}{x}\right)^2 \geq 0$$

Now rewrite like Method 1

4

$$\left(\sqrt{x^2-1} - 1\right)^2 \geq 0$$
$$x^2 - 1 - 2\sqrt{x^2-1} + 1 \geq 0$$

$x^2 \geq 2\sqrt{(x-1)(x+1)}$ for $x \geq 1$ $\square$

5 LHS − RHS

$$= (a+2b)^2 - 8ab$$
$$= a^2 + 4ab + 4b^2 - 8ab$$
$$= a^2 - 4ab + 4b^2$$
$$= (a-2b)^2$$

≥ 0 since $\mathbb{R}^2 \geq 0$

$$\therefore (a+2b)^2 - 8ab \geq 0$$
$$\therefore (a+2b)^2 \geq 8ab \quad \square$$

6 Method 1

$$\left(ac - \frac{b}{c}\right)^2 \geq 0$$
$$a^2c^2 - 2ab + \frac{b^2}{c^2} \geq 0$$
$$\therefore a^2c^2 + \frac{b^2}{c^2} \geq 2ab \quad \square$$

Method 2

Let $m = a^2c^2, y = \frac{b^2}{c^2}$ in $\frac{m+n}{2} \geq \sqrt{mn}$

$$\therefore \frac{a^2c^2 + \frac{b^2}{c^2}}{2} \geq 2\sqrt{a^2c^2 \cdot \frac{b^2}{c^2}}$$
$$= 2\sqrt{a^2b^2}$$
$$= 2ab$$
$$\therefore a^2c^2 + \frac{b^2}{c^2} \geq 2ab \quad \square$$

7
$$\frac{\frac{x}{yz}+\frac{y}{xz}}{2}\geq\sqrt{\frac{xy}{xyz^2}}\qquad \text{(AM−GM)}$$
$$\frac{1}{2}\left(\frac{x}{yz}+\frac{y}{xz}\right)\geq\frac{1}{z}\qquad (1)$$

Similarly
$$\frac{1}{2}\left(\frac{x}{yz}+\frac{z}{xy}\right)\geq\frac{1}{y}\qquad (2)$$
$$\frac{1}{2}\left(\frac{y}{xz}+\frac{z}{xy}\right)\geq\frac{1}{x}\qquad (3)$$

$(1)+(2)+(3)$:

$$\frac{1}{2}\left(\frac{x}{yz}+\frac{y}{xz}\right)+\frac{1}{2}\left(\frac{x}{yz}+\frac{z}{xy}\right)+\frac{1}{2}\left(\frac{y}{xz}+\frac{z}{xy}\right)\geq\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\qquad \square$$

8
$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2}\geq\sqrt{\frac{x^2}{y^2}\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}\qquad \text{(AM − GM)}$$
$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2}\geq\sqrt{\frac{x^2}{y^2}\left(\frac{x^2y^2+z^4}{x^2z^2}\right)}$$
$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2}\geq\frac{1}{yz}\sqrt{x^2y^2+z^4}\qquad (1)$$

Similarly
$$\frac{\frac{y^2}{z^2}+\left(\frac{x^2}{y^2}+\frac{z^2}{x^2}\right)}{2}\geq\frac{1}{xz}\sqrt{y^2z^2+x^4}\qquad (2)$$

$(1)+(2)$:
$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq\frac{1}{yz}\sqrt{x^2y^2+z^4}+\frac{1}{xz}\sqrt{y^2z^2+x^4}\qquad \square$$

9
Let $x=\frac{a+b}{2}, y=\frac{c+d}{2}$ in $\frac{x+y}{2}\geq\sqrt{xy}$

$$\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2}\geq\sqrt{\frac{a+b}{2}\times\frac{c+d}{2}}$$

$$\therefore\frac{a+b+c+d}{4}\geq\sqrt{\sqrt{ab}\times\sqrt{cd}}\qquad \text{since } \frac{(a+b)}{2}\geq\sqrt{ab}, \frac{c+d}{2}\geq\sqrt{cd}$$

$$\therefore\frac{a+b+c+d}{4}\geq\sqrt[4]{abcd}\qquad \square$$

10

$$\frac{\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{d}+\frac{d^3}{a}}{4} \geq \sqrt[4]{\frac{a^3}{b}\times\frac{b^3}{c}\times\frac{c^3}{d}\times\frac{d^3}{a}} \quad (\mathrm{AM}-\mathrm{GM})$$

$$\frac{\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{d}+\frac{d^3}{a}}{4} \geq \sqrt[4]{a^2b^2c^2d^2}$$

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{d}+\frac{d^3}{a} \geq 4\sqrt{abcd} \quad \square$$

11

Let $w=a,\ x=b, y=c, z=\frac{a+b+c}{3}$ in $\frac{w+x+y+z}{4} \geq \sqrt[4]{wxyz}$

$$\therefore \frac{a+b+c+\left(\frac{a+b+c}{3}\right)}{4} \geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)}$$

$$\frac{\frac{4}{3}(a+b+c)}{4} \geq (abc)^{\frac{1}{4}}\left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{1} \geq (abc)^{\frac{1}{4}}\left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}} \geq (abc)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}\times\frac{4}{3}} \geq (abc)^{\frac{1}{4}\times\frac{4}{3}}$$

$$\therefore \frac{a+b+c}{3} \geq \sqrt[3]{abc} \quad \square$$

12

$$\frac{a^3+a^3+b^3}{3} \geq \sqrt[3]{a^6b^3}$$

$$\therefore \frac{2a^3+b^3}{3} \geq a^2b \quad (1)$$

Similarly

$$\frac{2b^3+a^3}{3} \geq b^2c \qquad (2) \qquad \frac{2c^3+a^3}{3} \geq c^2a \qquad (3)$$

$(1)+(2)+(3)$: $\qquad a^3+b^3+c^3 \geq a^2b+b^2c+c^2a \quad \square$

13 $(a_1 - \sqrt{a_1})^2 \geq 0$

$a_1^2 - 2a_1\sqrt{a_1} + a_1 \geq 0$

$a_1^2 + a_1 \geq 2a_1\sqrt{a_1} \quad (1)$

Similarly for a_2 to a_n $\quad (2)\ldots(n)$

$(1) \times (2) \times (3) \times \ldots \times (n)$:

$$\begin{aligned}\therefore (a_1^2 + a_1)(a_2^2 + a_2)(a_3^2 + a_3)\ldots(a_n^2 + a_n) &\geq 2a_1\sqrt{a_1} \times 2a_2\sqrt{a_2} \times 2a_3\sqrt{a_3} \times \ldots \times 2a_n\sqrt{a_n} \\ &\geq 2^n(a_1a_2a_3\ldots a_n)\sqrt{a_1a_2a_3\ldots a_n} \\ &\geq 2^n \quad \square\end{aligned}$$

14 **i** $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab \quad (1)$

Similarly:

$a^2 + c^2 \geq 2ac \quad (2)$

$b^2 + c^2 \geq 2bc \quad (3)$

$(1) + (2) + (3)$:

$2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$

$a^2 + b^2 + c^2 \geq ab + bc + ca \quad \square$

Alternatively

$a^2 + b^2 + c^2 - ab - bc - ca$

$= \dfrac{1}{2}(a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2)$

$= \dfrac{1}{2}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$

≥ 0

$\therefore a^2 + b^2 + c^2 \geq ab + bc + ca \quad \square$

ii $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\geq ab + bc + ca + 2(ab + bc + ca)$ from (i).

$\geq 3(ab + bc + ca) \quad \square$

15 $(a^2 - b^2)^2 \geq 0$

$a^4 - 2a^2b^2 + b^4 \geq 0$

$2(a^4 + b^4) - (a^2 + b^2)^2 \geq 0$

$\therefore a^4 + b^4 \geq \dfrac{(a^2 + b^2)^2}{2} \quad (1)$

Similarly

$a^2 + b^2 \geq \dfrac{(a+b)^2}{2}$

$\therefore a^2 + b^2 \geq \dfrac{1^2}{2}$

$a^2 + b^2 \geq \dfrac{1}{2} \quad (2)$

From (1) and (2):

$a^4 + b^4 \geq \dfrac{\left(\frac{1}{2}\right)^2}{2}$

$a^4 + b^4 \geq \dfrac{1}{8} \quad \square$

16 $a+b+c \geq 3\sqrt[3]{abc}$

$a+b+c \geq 3 \qquad (1)$

$$\frac{a^2+1}{2} \geq \sqrt{a^2 \cdot 1}$$
$$\geq a$$

Similarly

$$\frac{b^2+1}{2} \geq b \qquad \frac{c^2+1}{2} \geq c$$

Summing the inequalities:

$$\frac{a^2+1}{2}+\frac{b^2+1}{2}+\frac{c^2+1}{2} \geq a+b+c$$
$$a^2+b^2+c^2+3 \geq 2(a+b+c)$$
$$a^2+b^2+c^2 \geq 2(a+b+c)-3$$
$$\geq 2(a+b+c)-(a+b+c) \quad \text{from (1)}$$
$$\geq a+b+c$$

17 $$\frac{a^4+a^4+b^4+c^4}{4} \geq \sqrt[4]{a^8b^4c^4}$$

$$\therefore \frac{2a^4+b^4+c^4}{4} \geq a^2bc$$

Similarly

$$\frac{2b^4+a^4+c^4}{4} \geq b^2ac$$
$$\frac{2c^4+a^4+b^4}{4} \geq c^2ab$$

Summing the above gives

$a^4+b^4+c^4 \geq a^2bc+b^2ca+c^2ab \quad \square$

18 $$\frac{1+ab}{1+a}=\frac{abc+ab}{abc+a}=\frac{ab(c+1)}{a(bc+1)}=\frac{b(c+1)}{bc+1} \quad (3)$$

Similarly

$$\frac{1+bc}{1+b}=\frac{c(a+1)}{ac+1} \quad (2)$$
$$\frac{1+ac}{1+c}=\frac{a(b+1)}{ab+1} \quad (3)$$

$$\text{LHS}=\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c}$$
$$=\frac{1}{2}\left(\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c}+\frac{b(c+1)}{bc+1}+\frac{c(a+1)}{ac+1}+\frac{a(b+1)}{ab+1}\right)$$
$$\geq \frac{1}{2}\left(6\times\sqrt[6]{\frac{1+ab}{1+a}\cdot\frac{1+bc}{1+b}\cdot\frac{1+ac}{1+c}\cdot\frac{b(c+1)}{bc+1}\cdot\frac{c(a+1)}{ac+1}\cdot\frac{a(b+1)}{ab+1}}\right)$$
$$\geq \frac{1}{2}\times 6\times\sqrt{1}$$
$$\geq 3 \quad \square$$

19

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$
$$a - 2\sqrt{ab} + b \geq 0$$
$$a + b \geq 2\sqrt{ab}$$
$$\sqrt{ab} \leq \frac{a+b}{2} \quad (1)$$

$$(1) \times \sqrt{ab}: \quad ab \leq \frac{(a+b)\sqrt{ab}}{2}$$
$$\frac{2ab}{a+b} \leq \sqrt{ab} \quad (2)$$

$$\left(\frac{a}{2}-\frac{b}{2}\right)^2 \geq 0$$
$$\frac{a^2-2ab+b^2}{4} \geq 0$$
$$\frac{a^2-2ab+b^2}{4} + \frac{(a+b)^2}{4} \geq \frac{(a+b)^2}{4}$$
$$\frac{2a^2+2b^2}{4} \geq \frac{(a+b)^2}{4}$$
$$\frac{(a+b)^2}{4} \leq \frac{a^2+b^2}{2}$$
$$\sqrt{\frac{(a+b)^2}{4}} \leq \sqrt{\frac{a^2+b^2}{2}}$$
$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \quad (3)$$

From $(2), (1)$ and (3):

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \qquad \square$$

Alternative for GM and QM

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

$$\left(\frac{a+b}{2}\right)^2 - \left(\sqrt{\frac{a^2+b^2}{2}}\right)^2$$
$$= \frac{a^2+2ab+b^2}{4} - \frac{a^2+b^2}{2}$$
$$= -\frac{a^2-2ab+b^2}{4}$$
$$= -\left(\frac{a-b}{2}\right)^2$$
$$\leq 0$$
$$\therefore \left(\frac{a+b}{2}\right)^2 \leq \left(\sqrt{\frac{a^2+b^2}{2}}\right)^2$$
$$\therefore \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

20 **a**

$3(a^2+b^2+c^2) - (a+b+c)^2$
$= 3a^2+3b^2+3c^2 - \left((a^2+b^2+c^2) + 2(ab+bc+ca)\right)$
$= 2(a^2+b^2+c^2) - 2(ab+bc+ca)$
$= (a^2-2ab+b^2) + (b^2-2bc+c^2) + (c^2-2ca+a^2)$
$= (a-b)^2 + (b-c)^2 + (c-a)^2$
≥ 0
$\therefore 3(a^2+b^2+c^2) \geq (a+b+c)^2 \quad (1)$

$(a+b+c)^2 - 3(ab+bc+ac)$
$= a^2+b^2+c^2+2(ab+ca+ca) - 3(ab+bc+ca)$
$= a^2+b^2+c^2 - (ab+bc+ca)$
$= \dfrac{a^2-2ab+b^2}{2} + \dfrac{b^2-2bc+c^2}{2} + \dfrac{c^2-2ac+a^2}{2}$
$= \dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{2}$
≥ 0
$\therefore (a+b+c)^2 \geq 3(ab+bc+ac) \quad (2)$

From (1) and (2):
$3(a^2+b^2+c^2) \geq (a+b+c)^2 \geq 3(ab+bc+ac)$ $\square$

20 **b**

$$
\begin{aligned}
\text{LHS} - \text{RHS} &= a^2 + b^2 + c^2 + ab + bc + ca - 6 \\
&= \frac{a^2 + 2ab + b^2}{2} + \frac{b^2 + 2bc + c^2}{2} + \frac{c^2 + 2ac + a^2}{2} - 2(a + b + c) \\
&= \frac{(a+b)^2 + (b+c)^2 + (c+a)^2}{2} - \frac{12}{2} \quad \text{since } a + b + c = 3 \\
&= \frac{(3-c)^2 + (3-a)^2 + (3-b)^2}{2} - \frac{12}{2} \\
&= \frac{\left((3-c)^2 - 4\right) + \left((3-a)^2 - 4\right) + \left((3-b)^2 - 4\right)}{2} \\
&= \frac{(5-c)(1-c) + (5-a)(1-a) + (5-b)(1-b)}{2} \\
&= \frac{(5 - 6c + c^2) + (5 - 6a + a^2) + (5 - 6b + b^2)}{2} \\
&= \frac{(a^2 + b^2 + c^2) - 6(a + b + c) + 15}{2} \\
&= \frac{a^2 + b^2 + c^2 - 18 + 15}{2} \\
&= \frac{a^2 + b^2 + c^2 - 3}{2}
\end{aligned}
$$

Now $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$ from (a)

$$
\begin{aligned}
\therefore a^2 + b^2 + c^2 &\geq \frac{(a+b+c)^2}{3} \\
&\geq \frac{3^3}{3} \\
&\geq 3
\end{aligned}
$$

$$\geq 0$$

$\therefore a^2 + b^2 + c^2 + ab + bc + ca \geq 6 \quad \square$

Chapter 2
Complex Numbers

MEX-N1 Introduction to Complex Numbers
MEX-N2 Using Complex Numbers

Complex Numbers is a great topic as:

- it provides us with a completely new type of number system to wrap our heads around
- it closely links geometry and algebra, but in unexpected ways
- it introduces us to a branch of mathematics that seems to be purely theoretical yet has massively important practical uses
- it stretches our imaginations as we try to visualise how numbers that seemingly don't exist behave under different conditions

Lessons
Complex Numbers is covered in 10 lessons.

Revision Questions
In '1000 Revision Questions', the revision book that goes with this textbook you will find the following questions matching this chapter:

- Revision Exercise 2
  100 graded questions on this topic only
- Revision Exercises 7 (Basic), 8 (Medium) and 9 (Challenging)
  Another 100 questions mixed through other topics for when you finish the course.

Don't forget to do any questions from the exercises in this textbook you haven't done.

2.1 INTRODUCTION TO COMPLEX NUMBERS

In Lesson 1 we get a basic overview of complex numbers that will help us through the later lessons in the topic. The first lesson of complex numbers is mainly a general chat to help us attain understanding of the topic to follow, with only a few exercises at the end. We will cover:

- How Complex Numbers fit with other Number Systems
- Imaginary Numbers
- Definition of i
- Geometric Explanation of i
- Now Isn't $i = \sqrt{-1}$?
- Surd Laws
- Real World Uses of Complex Numbers
- The Complex Plane
- Powers of i
- Powers of -1

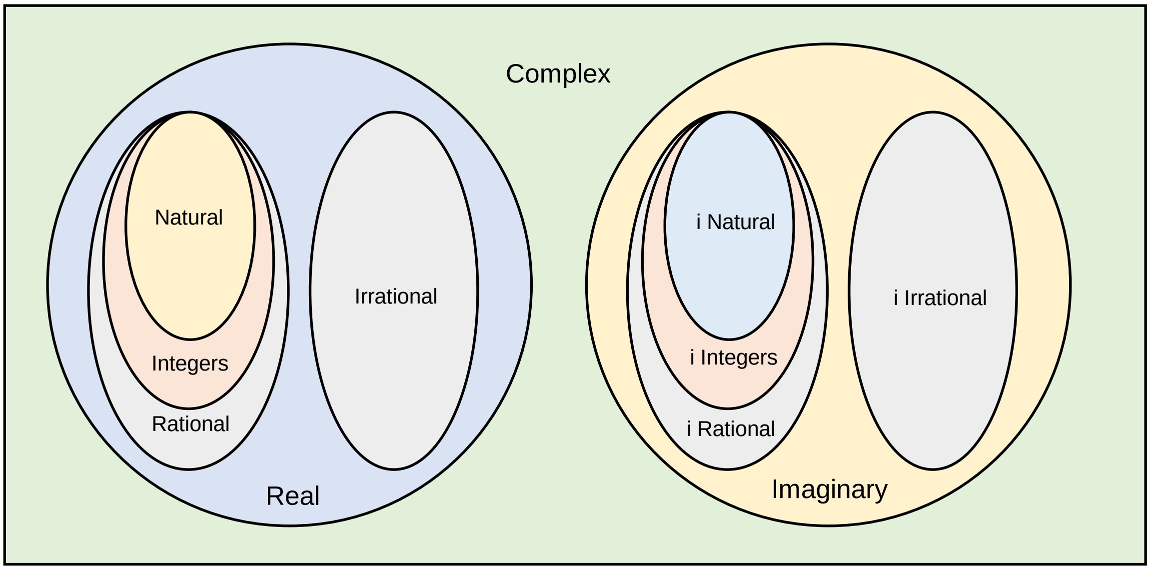
HOW COMPLEX NUMBERS FIT WITH OTHER NUMBER SYSTEMS

We first learned about Natural Numbers $\mathbb{N}$ (1,2,3…) when we were growing up. Over the years we then added numbers like zero and negative numbers to create the Integers $\mathbb{Z}$, then fractions (including terminating or recurring decimals) to create the Rational Numbers $\mathbb{Q}$. All of the rational numbers can be written as fractions.

The next step was to learn that there are numbers that cannot be written as fractions, such as π or $\sqrt{2}$. These are the Irrational Numbers (non recurring decimals), which together with Rational Numbers, form the larger group Real Numbers $\mathbb{R}$. A number cannot be both Rational and Irrational.

You are about to learn that there is another set of numbers, Imaginary Numbers $\mathbb{I}$, that together with Real Numbers form the larger group Complex Numbers $\mathbb{C}$. Just like Real Numbers, Imaginary Numbers can also be rational or irrational.

A number cannot be Real and Imaginary, but it can have a real and an imaginary part, just like the real number $1+\sqrt{3}$ has a rational and irrational part.



So the number 1, the first number we learned about as children, is Natural, an Integer, Rational, Real and Complex, but it is not Irrational or Imaginary.

IMAGINARY NUMBERS

Although what we now know as Imaginary Numbers have only been commonly accepted for a few hundred years, they were first discovered by the Ancient Greeks.

French mathematician and philosopher Rene Descartes ('I think, therefore I am') created the Cartesian Plane (which allows us to link geometry and algebra), and also derisively named numbers involving the square root of negatives as 'imaginary'. It is a poor choice of name that misleads us to this day.

He didn't think negative numbers existed either and called solutions which were negative 'false'!

We should also note that the 'complex' in Complex Numbers means 'made of parts' rather than 'complicated' - another misleading name! A complex number is made of a real part and an imaginary part.

So keep in mind that complex numbers are not hard and they do exist.

Imaginary numbers are a bit like irrational numbers, negative numbers and even zero - they are non-tangible numbers that have useful properties. You cannot hold $\sqrt{2}$ marbles, -2 marbles, 0 marbles or $\sqrt{-2}$ marbles in your hand, but irrational, negative, zero and imaginary numbers allow us to solve more difficult problems.

At various points in history in different cultures all of these non-tangible numbers have been seen as fictitious or useless!

DEFINITION OF i

The definition of i from the syllabus is as a solution of the equation $x^2 = -1$, so $i^2 = -1$.

The definition is not the more commonly recognised $i = \sqrt{-1}$, which is not mentioned in the syllabus at all. It is no big deal if you use both definitions for i, as we will certainly gain a better understanding from $i = \sqrt{-1}$. We will look at this more closely later in the lesson.

It is important to note that i is similar to a surd, so there are some similarities between the properties of complex numbers and surds, but not all of the rules for complex numbers follow the surd rules.

So let's look at a few examples of using complex numbers algebraically only, before we look at a geometric explanation to improve our understanding. We generally use z to represent a complex variable, as opposed to x which will generally be reserved for a real variable.

Example 1

a Factorise $z^2 + 9$

b Find the square roots of -9.

Solution

a

$z^2 + 9$

$= z^2 - 9i^2$

$= z^2 - (3i)^2$

$= (z + 3i)(z - 3i)$

b

Let z be a square root of -9

$$\therefore z^2 = -9$$

$$z^2 + 9 = 0$$

$$\therefore (z + 3i)(z - 3i) = 0$$

$$z = \pm 3i$$

Check: $(3i)^2 = 9i^2 = 9 \times (-1) = -9$

$(-3i)^2 = 9i^2 = 9 \times (-1) = -9$

Example 2

a Factorise $z^2 + 4z + 10$

b Solve $z^2 + 4z + 10 = 0$

Solution

a

$z^2 + 4z + 10$

$= z^2 + 4z + 4 + 6$

$= (z + 2)^2 + 6$

$= (z + 2)^2 - 6i^2$

$= (z + 2 + \sqrt{6}i)(z + 2 - \sqrt{6}i)$

b

$$z^2 + 4z + 10 = 0$$

$$\left(z + 2 + \sqrt{6}i\right)\left(z + 2 - \sqrt{6}i\right) = 0$$

$$z = -2 \pm \sqrt{6}i$$

Alternatively

$$z = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 10}}{2 \times 1}$$

$$= \frac{-4 \pm \sqrt{-24}}{2}$$

$$= \frac{-4 \pm 2\sqrt{6}i}{2}$$

$$= -2 \pm \sqrt{6}i$$

$\sqrt{-24} = \sqrt{4} \times \sqrt{6} \times \sqrt{-1}$

$= 2\sqrt{6}i$

Example 3

Factorise $z^2 + 4iz + 5$

a by completing the square

b by factorising

Solution

a

$z^2 + 4iz + 5$

$= z^2 + 4iz + (2i)^2 - (2i)^2 + 5$

$= (z + 2i)^2 + 4 + 5$

$= (z + 2i)^2 + 9$

$= (z + 2i)^2 - 9i^2$

$= (z + 2i + 3i)(z + 2i - 3i)$

$= (z + 5i)(z - i)$

b

$z^2 + 4iz + 5$

$= z^2 + 4iz - 5i^2$

$= (z + 5i)(z - i)$

GEOMETRIC EXPLANATION OF i

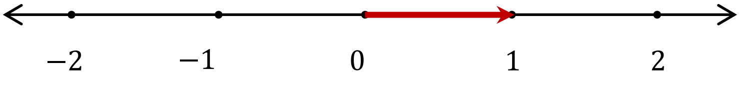
We have just used i in calculations, without really understanding what it is. Let's have a look at a geometric explanation of i to improve our understanding of what we are doing.

Up until now the one dimensional number line has been adequate for all our work with real numbers, but it has also restricted our thinking unnecessarily.

Let's have a look at the equation $x^2 = -1$, and make sure we understand what squaring, the minus sign and 1 really mean.

THE NUMBER 1

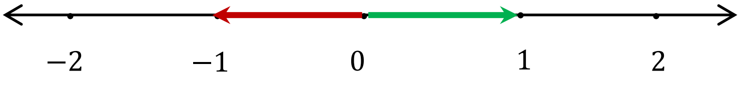
Well the 1 is easy - it just means one unit more than zero, which we can represent on the number line as one unit to the right, or by an arrow (vector) stretching from 0 to 1.



THE MINUS SIGN

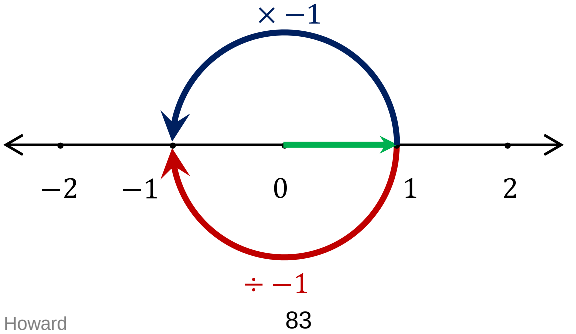
Now with the minus sign we see a restriction to our thinking caused by the number line.

When we place a minus in front of any real number then the only one dimensional operation we can perform is a reflection about 0, so -1 is the reflection (or opposite) of 1.



But what if we weren't restricted to movement in one dimension? Is there another operation that could take us to the same place?

The answer is yes, because a rotation either anticlockwise or clockwise through $180°$ would get us to the same point.



Now to be consistent with our work from other topics, we will define multiplying by -1 as causing an anticlockwise rotation by $180°$. We can also define dividing by -1 as causing a clockwise rotation by $180°$.

We can view -1 as $1 \times (-1)$, where 1 is our starting point and multiplying by -1 is the operation of rotating $180°$ anticlockwise.

SQUARING

Squaring normally means that a number is multiplied by itself, but if x represents an operation then it means that the same operation will be repeated twice in a row.

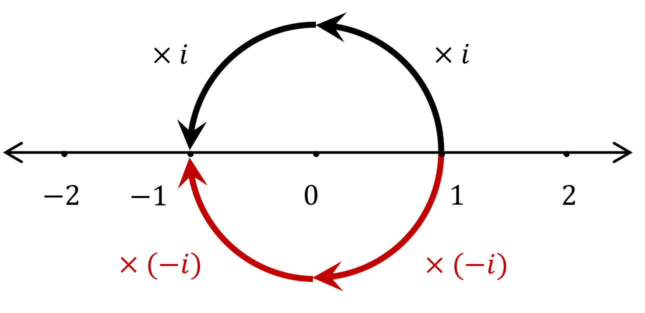
To solve $x^2 = -1$, we first view the equation as $1 \times x \times x = -1$. So starting at 1, what operation can we perform twice to end up at -1?

Reflection about 0 will not work, as reflecting twice brings us back to our starting point of 1.

There are two possible solutions:

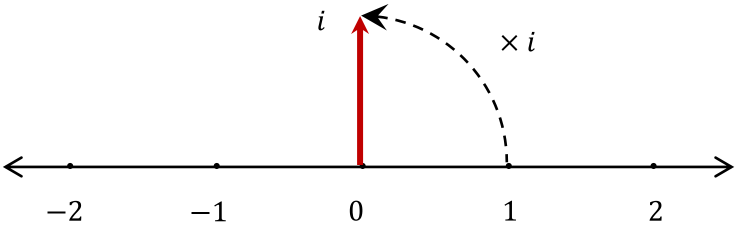
- x could represent a $90°$ anticlockwise rotation
- x could represent a $90°$ clockwise rotation

Now either one is just as valid to be the definition of i, but by tradition we let i represent the anticlockwise rotation. There is no reason we couldn't have defined it the other way around, but it is more consistent with our work in other topics.



So if the anticlockwise rotation is i, what is the clockwise rotation? It is $-i$, as $(-i)^2 = i^2 = -1$. The two solutions to $x^2 = -1$ are i and $-i$.

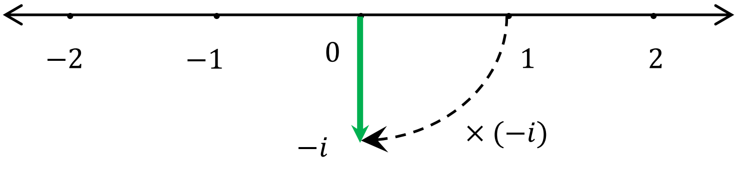
Now $i = 1 \times i$, so starting with the vector to 1 and rotating it $90°$ anticlockwise, we see that i must be a number that is off the number line, 1 unit above 0.



So i can either rotate a complex number by $\frac{\pi}{2}$ if it is multiplied or divided, or move it one unit perpendicular to the number line if it is added or subtracted.

It is the ease with which complex numbers can either move objects at right angles or rotate them that makes them so important in science.

Similarly $-i$ is one unit beneath 0.



Later in the lesson we will extend the number line to form the Complex Plane, but I always like showing i as a point somewhere off the number line so that we can better see the link between the number line and the Complex Plane, rather than being confused between the Cartesian Plane and the Complex Plane.

SURD LAWS

We mentioned that complex numbers act like surds in some cases.

It is important to note that our surd laws only work with complex numbers when certain restrictions are put in place. Since these restrictions are beyond the course it is safest to simply avoid the surd laws.

As an example of how the surd laws don't always work with complex numbers, consider this false proof:

$$\begin{aligned}-1 &= i^2 \\ &= i \times i \\ &= \sqrt{-1} \times \sqrt{-1} \\ &= \sqrt{(-1) \times (-1)} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

REAL WORLD USES OF COMPLEX NUMBERS

Like many concepts in mathematics complex numbers have been found to have uses far beyond those they were initially created for.

Complex Numbers can be used:

- As a quicker way to solve questions that could still be solved using only real numbers - this is the most common application
- To simplify calculations for determining the position of objects that are rotating or being transformed
- To solve equations where the first and last steps are purely real but intermediary steps can only be solved using complex numbers
- In a small number of cases to model situations that could not be handled by real numbers

Complex numbers are particularly useful as:

- Circular motion can be modelled quite easily, so the study of any object that that involves waves or oscillations, such as electricity, electromagnetism, earthquakes, soundwaves and light waves is simpler.
- Anything with two real components can be represented as one complex number, with one real number assigned to the real component and the other to the imaginary, such as the state of an electric circuit which is defined by voltage V and current I, and can be represented by $z = V + Ii$ (normally using j instead of i to avoid confusion).
- Anything with two components that are at right angles to each other can be represented as one complex number, since the real and imaginary components are at right angles, such as a force being split into its vertical and horizontal components
- Objects which repeatedly change direction can be modelled by imaginary numbers, since $i^2 = -1$ represents a change of direction, such as for alternating current in electricity
- Probabilities for predicting the positions of particles in quantum physics or the cycles in the share markets can be modelled using complex numbers

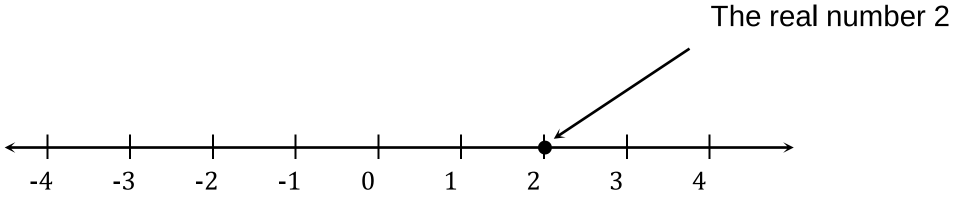
Most technology relies on complex numbers for its invention or to speed up calculations. If you are taking a selfie (CCD) on your phone while texting (signal), then switch to a game stored on your phone (magnetic disk and graphics), under a fluorescent light (particle control), then you are surrounded by advances that would not be possible without complex numbers.

We have barely scratched the surface of how we use complex numbers in the real world.

THE COMPLEX PLANE

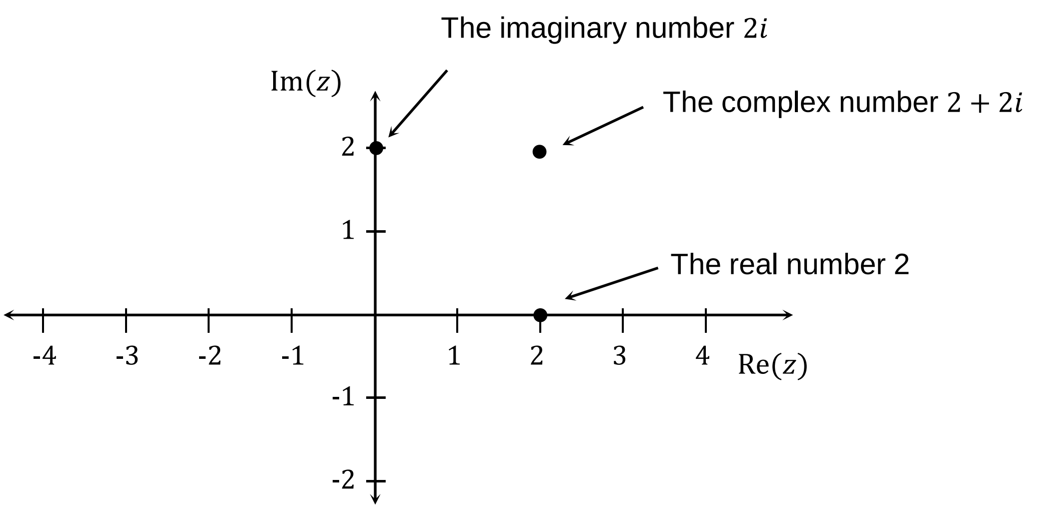
We have seen that imaginary numbers are above or below the number line - this is the basis of the Complex Plane, also known as the Argand Diagram. Let's look at it in more detail.

The Number Line is one dimensional and includes all real numbers. Positive numbers are to the right and negative numbers to the left.

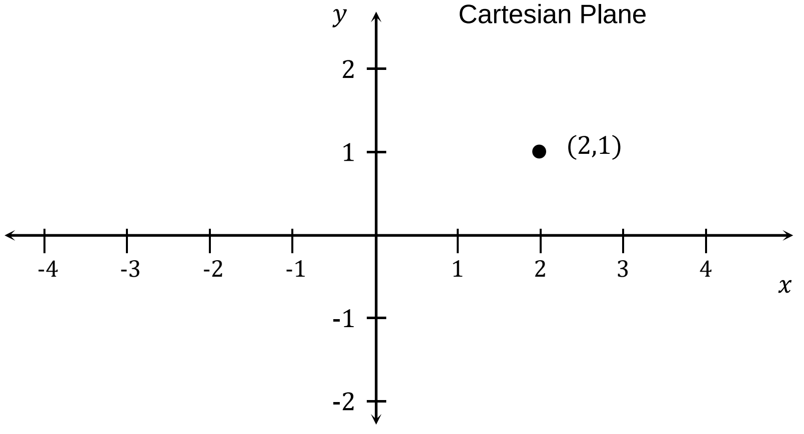


The Complex Plane starts with the Number Line, now called the Real axis and marked $\text{Re}(z)$.

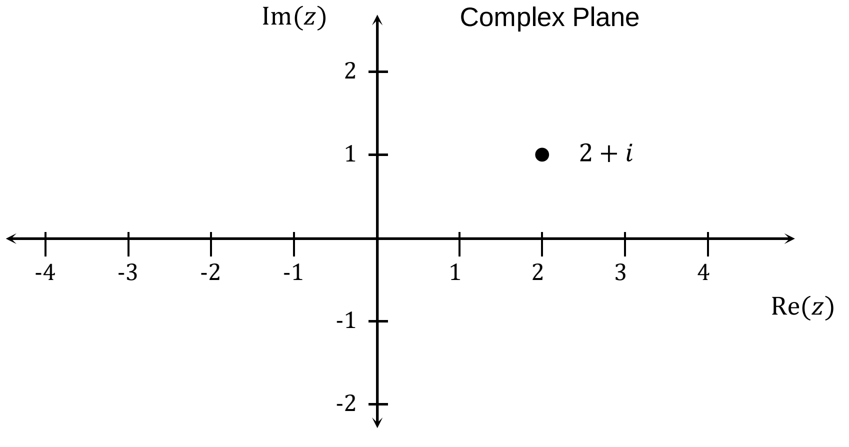
A vertical axis passes through 0, representing imaginary numbers, and is marked $\text{Im}(z)$. Positive multiples of i are above the line with negative multiples below.



Although the Complex Plane looks like a Cartesian Plane, it is more closely related to the number line and there are some important differences.



Each point on a Cartesian Plane represents two different real numbers that are somehow related, like the point $(2,1)$ shown at left. This could be the input and output of a function, like $y = x^2 - 3$, or the distance travelled and time taken.



Each point on the Complex Plane represents one number which has a real and an imaginary part, so $(2,1)$ now represents $2 + i$.

Where the two axes cross on the Complex Plane is the number zero, rather than the pair of zeros $(0,0)$, although we may still refer to it as the origin.

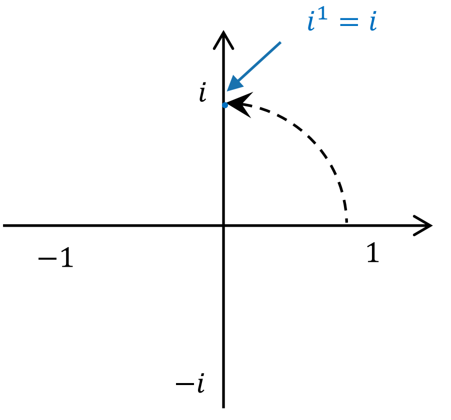
The Complex Plane has lots of interesting properties, and although it is in many ways more related to a number line, it shares properties with the Cartesian Plane and the Unit Circle.

POWERS OF i

Powers of i can be simplified algebraically or geometrically. Algebraically we can use index laws to break the powers into i, i^2 and i^4, noting $i^2 = -1$ and $i^4 = (i^2)^2 = 1$. We see a cyclical pattern repeating every four integers, $i, -1, -i, 1$, representing the intersections of the axes with the unit circle.

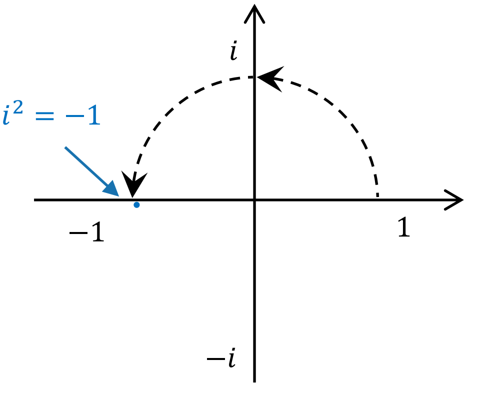
$i^1 = i$

It represents one $90°$ turn



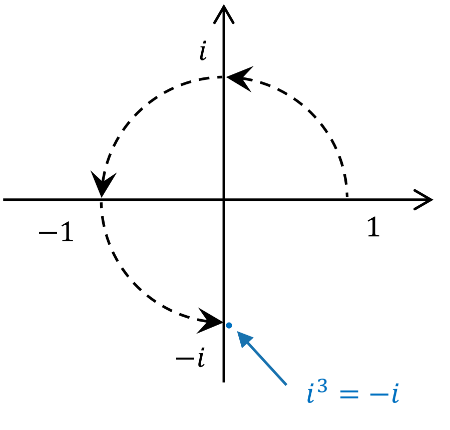
$i^2 = -1$

It represents two $90°$ turns



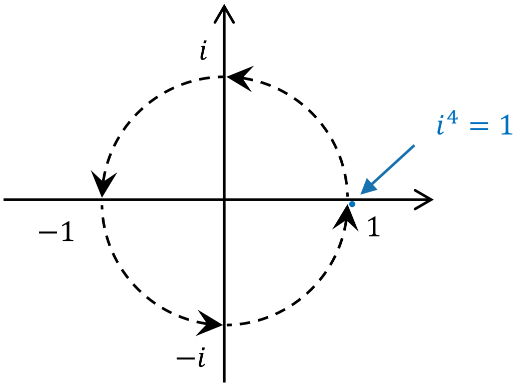
$$\begin{aligned} i^3 &= i^2 \times i \\ &= -1 \times i \\ &= -i \end{aligned}$$

It represents three $90°$ turns



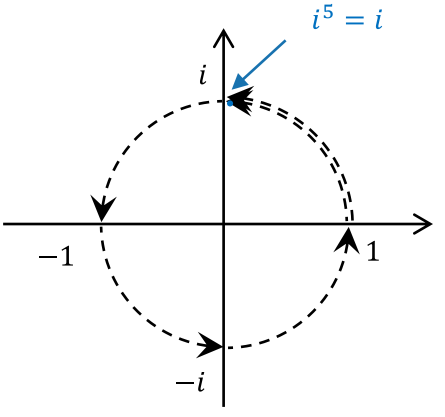
$$i^4 = (i^2)^2$$
$$= (-1)^2$$
$$= 1$$

It represents four $90°$ turns, bringing us back to 1.



$$i^5 = i^4 \times i$$
$$= 1 \times i$$
$$= i$$

It represents five $90°$ turns, ending up in the same place as only one turn.



Higher powers of i continue to cycle around the circle, a property that makes calculations with complex numbers so useful in some fields.

If the power is:

- A multiple of 4 then it is equal to 1
- One more than a multiple of 4 then it is equal to i
- Two more than a multiple of 4 and it is equal to -1
- Three more and it is equal to $-i$

We can also use negative powers of i with similar results, as the point cycles clockwise in a similar manner.

We can simplify powers of i by looking at how many revolutions have occurred, and where we end up on the unit circle.

Example 4

Simplify the following powers of i:

a i^{10} **b** i^{1000} **c** i^{99} **d** i^{-1} **e** i^{-11}

Solution

a

i^{10}

$= (i^4)^2 \times i^2$

$= 1^2 \times -1$

$= -1$

Ten lots of $90°$ is 2.5 revolutions, ending at -1.

b

i^{1000}

$= (i^4)^{250}$

$= 1^{250}$

$= 1$

1000 lots of $90°$ is 250 full revolutions, ending at 1.

c

i^{99}

$= (i^4)^{24} \times i^2 \times i$

$= 1^{24} \times -1 \times i$

$= -i$

99 lots of $90°$ is 24.75 full revolutions, ending at $-i$.

d

i^{-1}

$= (i^2)^{-1} \times i$

$= (-1)^{-1} \times i$

$= \dfrac{1}{-1} \times i$

$= -i$

1 lot of $90°$ backwards ends at $-i$.

e

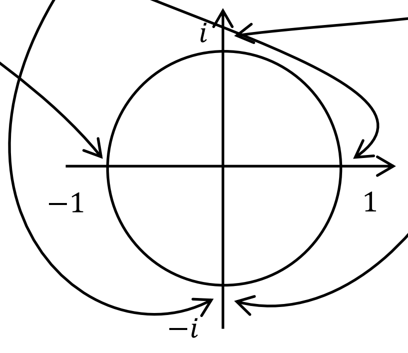
i^{-11}

$= (i^4)^{-3} \times i$

$= 1^{-3} \times i$

$= i$

11 lots of $90°$ backwards is 3.75 revolutions, ending at i.



Example 5

Simplify:

a $i^6 + i^4$ **b** $i^3(i^5 + i^8)$

Solution

a

$i^6 + i^4$

$= i^4(i^2 + 1)$

$= i^4(-1 + 1)$

$= 0$

b

$i^3(i^5 + i^8)$

$= i^8(1 + i^3)$

$= (i^4)^2(1 - i)$

$= 1 - i$

EXERCISE 2.1

BASIC

1 **a** Factorise $z^2 + 25$ **b** Hence find the square roots of -25

2 **a** Factorise $z^2 + 2z + 5$ **b** Hence solve $z^2 + 2z + 5 = 0$

c Solve $z^2 + 2z + 5 = 0$ using the quadratic formula.

3 Factorise $z^2 + 6iz + 7$

a by completing the square **b** by factorising

c Hence solve $z^2 + 6iz + 7 = 0$

4 Simplify the following powers of i:

a i^6 **b** i^{400} **c** i^{63} **d** i^{-3} **e** i^{-10}

5 Write i^9 in the form $a + ib$ where a and b are real.

6 Simplify:

a $i^5 + i^7$ **b** $i^6(i^2 + i^3)$

7 Find the square roots of -36.

MEDIUM

8 The three cube roots of 8 are the solutions to $z^3 - 8 = 0$. The cube roots include $z = 2$ plus two complex roots. Given $z^3 - 8 = (z - 2)(z^2 + 2z + 4)$ find the complex cube roots of 8.

9 Factorise $z^4 + z^2 - 12$ over:

a the rational field **b** the real field **c** the complex field

10 Solve $z^2 + 2z + 5 = 0$

a by completing the square **b** by replacing $\sqrt{b^2 - 4ac}$ with λ, where $\lambda^2 = \Delta$

11 Solve the quadratic equation $z^2 + (2 + 3i)z + (1 + 3i) = 0$, giving your answers in the form $a + bi$, where a and b are real numbers.

12 Graph the following complex numbers:

a $(-1)^{\frac{1}{3}}$ **b** $(-1)^{-\frac{2}{3}}$ **c** $(-1)^{\frac{8}{3}}$

1 **a** $z^2 + 25 = z^2 - 25i^2 = z^2 - (5i)^2 = (z+5i)(z-5i)$

b Let z be a square root of -25 $z^2 = -25$ $z^2 + 25 = 0$ $\therefore (z+5i)(z-5i) = 0$ $z = \pm 5i$

2 **a** $z^2 + 2z + 5 = z^2 + 2z + 1 + 4 = (z+1)^2 + 2^2 = (z+1)^2 - (2i)^2$

$= (z+1+2i)(z+1-2i)$

b $z^2 + 2z + 5 = 0$ $(z+1+2i)(z+1-2i) = 0$ $z = -1 \pm 2i$

c $z = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

3 **a** $z^2 + 6iz + 7 = z^2 + 6iz + (3i)^2 - (3i)^2 + 7 = (z+3i)^2 + 9 + 7 = (z+3i)^2 + 4^2$

$= (z+3i)^2 - (4i)^2 = (z+3i+4i)(z+3i-4i) = (z+7i)(z-i)$

b $z^2 + 6iz + 7 = z^2 + 6iz - 7i^2 = (z+7i)(z-i)$

c $z^2 + 6iz + 7 = 0$ $(z+7i)(z-i) = 0$ $z = -7i, i$

4 **a** $i^6 = i^4 \times i^2 = 1 \times -1 = -1$

b $i^{400} = (i^4)^{100} = 1^{100} = 1$

c $i^{63} = (i^4)^{15} \times i^2 \times i = 1^{15} \times (-1) \times i = -i$

d $i^{-3} = i^{-4} \times i = (i^4)^{-1} \times i = 1^{-1} \times i = i$

e $i^{-10} = i^{-12} \times i^2 = (i^4)^{-3} \times i^2 = 1^{-3} \times (-1) = -1$

5 $i^9 = i^8 \times i = 1 \times i = i = 0 + 1i$

6 **a** $i^5 + i^7 = i^4(i + i^3) = 1(i - i) = 0$

b $i^6(i^2 + i^3) = i^8(1+i) = (i^4)^2(1+i) = 1(1+i) = 1+i$

7 $z^2 = -36$ $z^2 = 36i^2$ $z^2 = (\pm 6i)^2$ $z = \pm 6i$ OR

Let z be a square root of -36 $\therefore z^2 = -36$ $z^2 + 36 = 0$ $\therefore (z+6i)(z-6i) = 0$ $z = \pm 6i$ OR

$z = \sqrt{-36} = \sqrt{36i^2} = \sqrt{(\pm 6i)^2} = \pm 6i$

8 Let z be a cube root of 8

$\therefore z^3 = 8$

$\therefore z^3 - 8 = 0$

$(z-2)(z^2+2z+4) = 0$

$z - 2 = 0$ or $z^2 + 2z + 4 = 0$

$z = 2$ $\qquad$ $z = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$= -1 \pm \sqrt{3}i$$

The three cube roots of 2 are 1 and $-1 \pm \sqrt{3}i$

9 **a** $z^4 + z^2 - 12 = (z^2-3)(z^2+4)$

b $z^4 + z^2 - 12 = (z^2-3)(z^2+4) = \left(z+\sqrt{3}\right)\left(z-\sqrt{3}\right)(z^2+4)$

c $z^4 + z^2 - 12 = (z^2-3)(z^2+4) = \left(z+\sqrt{3}\right)\left(z-\sqrt{3}\right)(z^2+4)$

$= \left(z+\sqrt{3}\right)\left(z-\sqrt{3}\right)(z+2i)(z-2i)$

10 **a**

$z^2 + 2z + 5 = 0$

$z^2 + 2z + 1 + 4 = 0$

$(z+1)^2 + 4 = 0$

$(z+1)^2 = -4$

$(z+1)^2 = 4i^2$

$z + 1 = \pm 2i$

$z = -1 \pm 2i$

b

$\Delta = 2^2 - 4(1)(5) = -16 = 16i^2 = (4i)^2$

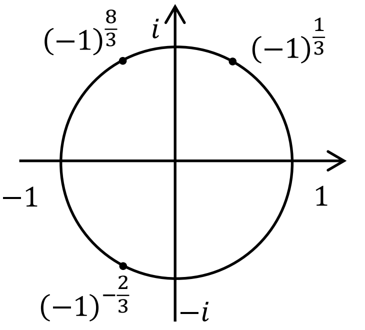
$$z = \frac{-b \pm \lambda}{2a} = \frac{-2 \pm 4i}{2 \times 1} = -1 \pm 2i$$

11 This question is easier if you notice that the sum of $2 + 3\mathrm{i}$ is one more than the product $1 + 3\mathrm{i}$, so 1 and $1 + 3\mathrm{i}$ are the numbers needed. Alternatively use the quadratic formula.
$z^2 + (2+3i)z + (1+3i) = 0 \quad (z+1)(z+1+3i) = 0 \quad \therefore z = -1$ or $-1-3i$

12 a One third of the anticlockwise rotation from 0 to -1

b Two thirds of the clockwise rotation from 0 to -1

c Eight thirds of the anticlockwise rotation from 0 to -1, so a complete revolution plus two thirds of the anticlockwise rotation.



2.2 CARTESIAN FORM

In Lesson 2 we look at the three forms of a complex number, then cover basic operations of complex numbers in Cartesian form:

- Forms of a Complex Number
- Calculations in Cartesian Form
- Addition and Subtraction
- Multiplication and Division
    - Multiplication of Complex Numbers
    - Squaring a Complex Number
    - Powers of a Complex Number
    - Conjugate of a Complex Number
    - Division of Complex Numbers
- Equal Complex Numbers
- Combining Operations

FORMS OF A COMPLEX NUMBER

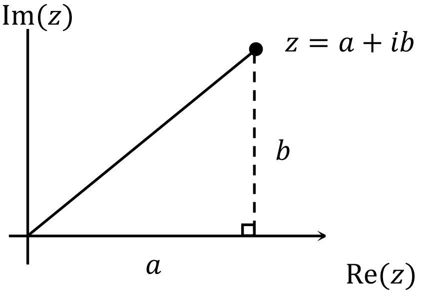
Complex Numbers can be represented in three forms, which we will look at in detail in the next lessons. Cartesian (or rectangular) form, Polar (Modulus-Argument) form and Exponential form.

CARTESIAN (RECTANGULAR) FORM

A complex number can be written in the form $z = a + ib$ where a and b are real numbers.

a is the real part also known as $\text{Re}(z)$
b is the imaginary part also known as $\text{Im}(z)$.

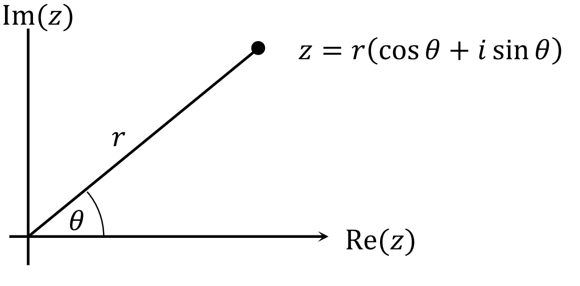
If $b = 0$ then we have a purely real number.
If $a = 0$ then we have a purely imaginary number.



For example $6 + 0i = 6$ is purely real and $0 + 6i = 6i$ is purely imaginary.

POLAR (MODULUS - ARGUMENT) FORM

We can also write the same number as $\text{z} = r(\cos\theta + i\sin\theta)$, where r is the modulus (distance from zero) and θ is the argument (the angle measured anti-clockwise from the x-axis). This is often abbreviated as $z = r\,\text{cis}\,\theta$ for convenience. We will look at this in more detail in Lesson 3.



If $\theta = 0 \text{ or } \pi$ then we have a purely real number on the real axis.

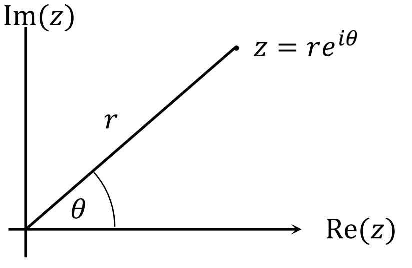
If $\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$ then we have a purely imaginary number on the imaginary axis.

For example $3(\cos 0 + i\sin 0)$ is purely real and

$2\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$ is purely imaginary.

EXPONENTIAL FORM

A form that is new to the syllabus is exponential form, which also uses the modulus and argument, but in a much simpler form. Confusingly it is also sometimes known as polar form. We will look at this in more detail in Lesson 4.



Again, if $\theta = 0 \text{ or } \pi$ then we have a purely real number, and if

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ then we have a purely imaginary number.

eg. $3e^{0i}$ is purely real and $2e^{\frac{3\pi}{2}i}$ is purely imaginary.

CALCULATIONS IN CARTESIAN FORM

For the remainder of this lesson we will concentrate on basic calculations in Cartesian form.

Cartesian form is:

- the easiest form to understand
- the best form to use for addition and subtraction
- capable of doing multiplication, powers and division (but not the easiest)
- the best form to use when translating vectors

Complex Numbers follow most but not all of the laws we have used in Years 7-10 for real numbers.

Each example will be solved algebraically (which is all you need to do in exams), but then is followed by a geometric explanation to increase your understanding of what is happening.

ADDITION AND SUBTRACTION

For addition and subtraction we simply collect like terms.

Example 1

Simplify:

a $(4+2i)+(2+3i)$

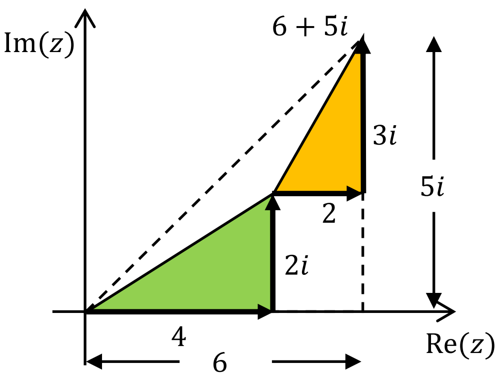
b $(4+2i)-(2+3i)$

Solution

a

$6+5i$

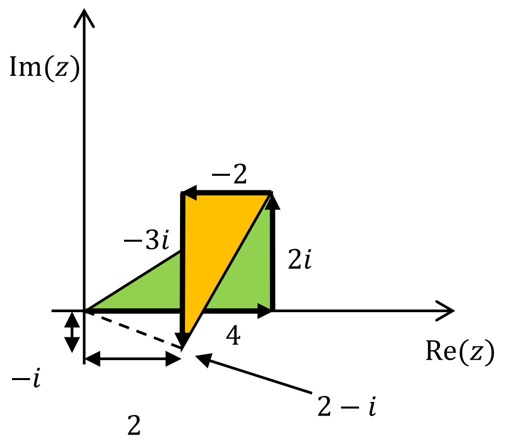
From zero go right 4, up 2, right 2 and up 3. In total that is 6 to the right and 5 up.



b

$2-i$

From zero go right 4, up 2, left 2 and down 3. In total that is 2 to the right and -1 up.



MULTIPLICATION AND DIVISION

Multiplication and division of complex numbers are easier in polar or exponential form, which we will explore in later lessons.

A complex number can be multiplied or divided by:

- a real positive number - which changes the real and imaginary parts in the same proportion - the scale changes
- a real negative number, which will reflect it to the other side of zero and may change the real and imaginary parts in the same proportion
- an imaginary number - which will rotate it by $\frac{\pi}{2}$ and may change its scale
- a complex number - which will rotate it and may change its scale

If the modulus of the imaginary number or complex number is 1 then there will be no change of scale.

Algebraically all we need to do is expand any brackets and collect the like terms.

Example 2

Simplify $(2+i)(1+2i)$

Solution

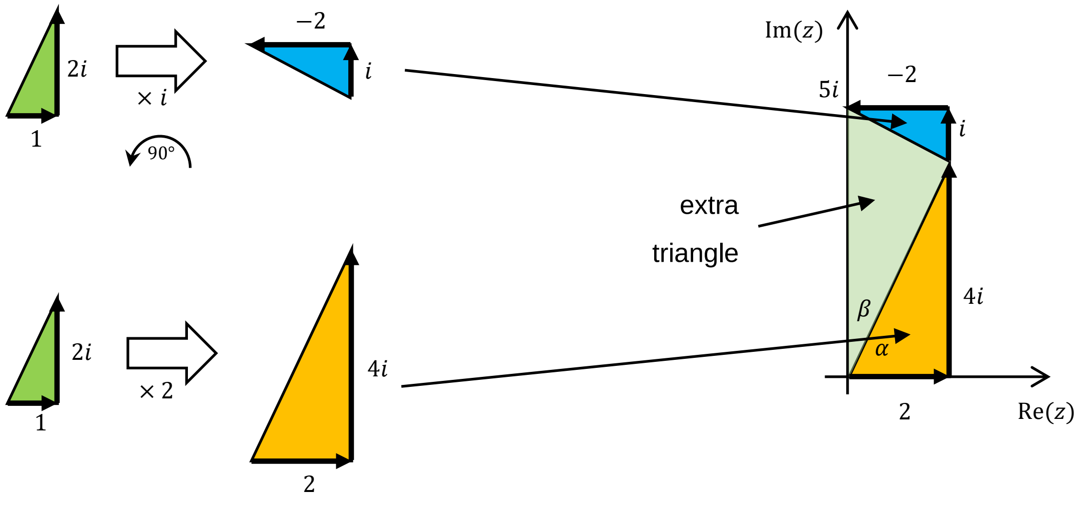
$(2+i)(1+2i)$

$= 2+4i+i+2i^2$

$= 2+4i+i-2$

$= 5i$

We can view the question as $2(1+2i)+i(1+2i)$, which is $1+2i$ scaled by 2 (since it is multiplied by 2), added with $1+2i$ rotated anti-clockwise by $90°$ (since it is multiplied by i).



Note:

- the four side lengths on the diagram $(2, 4i, i, -2)$ match the algebraic expansion top left.
- the extra triangle formed by the two triangles is a similar triangle to $2+i$. This shows that the argument of the product is equal to the sum of the arguments of the two complex numbers (marked α and β), which we will look at in a later lesson.

SQUARING A COMPLEX NUMBER

Squaring a Complex Number is a special case of multiplying two complex numbers and can be done using the expansion of a perfect square formula.

$$(a+ib)^2 = a^2 + 2iab + i^2b^2 = a^2 - b^2 + 2iab$$

Example 3

Simplify $(1+i)^2$

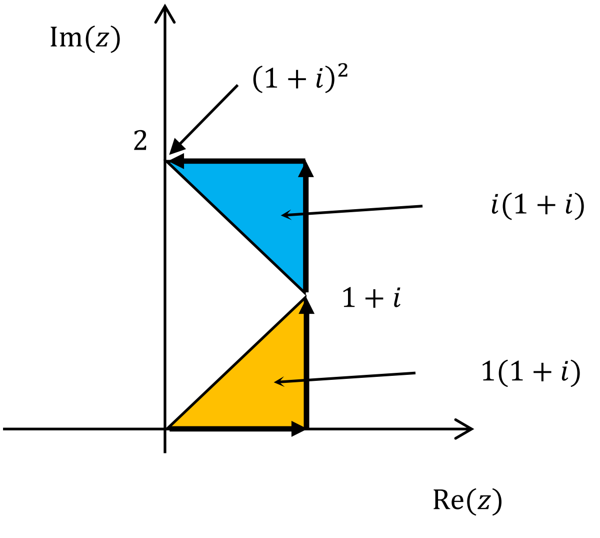
Solution

$(1+i)^2$

$= 1^2 + 2i + i^2$

$= 1 - 1 + 2i$

$= 2i$

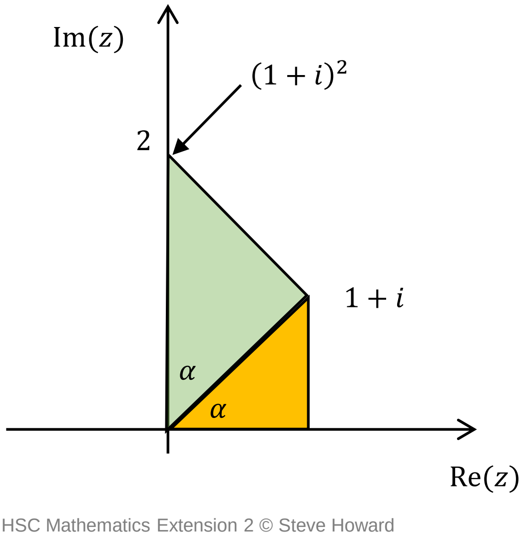


The rule above is useful for explaining some of the methods for finding the square root of a complex number that we will do later in the topic.

Note

$$\mathrm{Re}\left((a+ib)^2\right) = a^2 - b^2$$

$$\mathrm{Im}\left((a+ib)^2\right) = 2ab$$

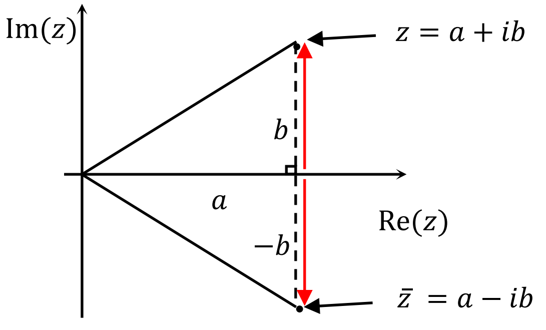


Another way to view squaring a complex number is that we can add a similar triangle with an adjoining edge. This helps us understand powers in the next example, which in turn helps us understand exponential form and de Moivre's theorem later in the topic.

CONJUGATE OF A COMPLEX NUMBER

The conjugate of a complex number is its reflection over the real axis. The conjugate is known as $\bar{z}$ which is pronounced 'z bar'.

In Cartesian form the conjugate has the same real part, but the imaginary part is of the opposite sign. The conjugate of $a + ib$ is $a - ib$. eg. the conjugate of $3 + 2i$ is $3 - 2i$. Note the similarity to surds, where the conjugate of $3 + \sqrt{2}$ is $3 - \sqrt{2}$.



Conjugates are important as:

- Multiplying by a conjugate rotates the opposite direction to multiplying by the original complex number, which we use for dividing with complex numbers
- Adding a complex number and its conjugate removes the imaginary component, while subtracting removes the real component
- If the modulus is 1 then the reciprocal of a complex number and its conjugate are equal.
- When we solve a polynomial where the coefficients are real then any complex roots occur in conjugate pairs as we see this lesson.

In Lesson 5 we will look at many important properties of conjugates.

Example 4

Find the conjugate of each complex number:

a $-3 + 2i$

b $4i$

Solution

a

$-3 - 2i$

b

$-4i$

DIVISION OF COMPLEX NUMBERS

When dividing complex numbers we must realise the denominator (it must end up with no imaginary component), which is very similar to rationalising the denominator with surds. Division can involve rotation and/or scaling, just like multiplication.

Example 5

Simplify $\dfrac{3+2i}{2i}$

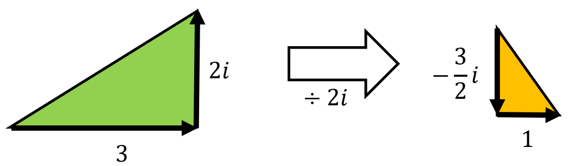
Solution

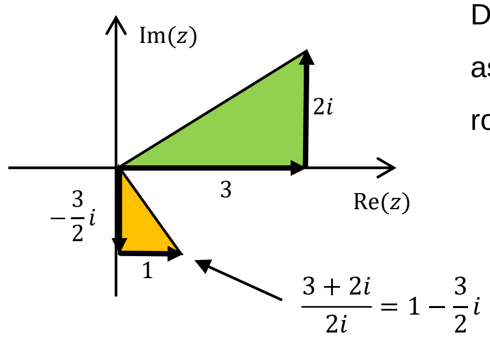
$$\frac{3+2i}{2i} \times \frac{i}{i}$$

$$= \frac{3i+2i^2}{2i^2}$$

$$= \frac{-2+3i}{-2}$$

$$= 1 - \frac{3}{2}i$$





Dividing by $2i$ is the same as dividing by 2 then rotating clockwise $90°$.

Example 6

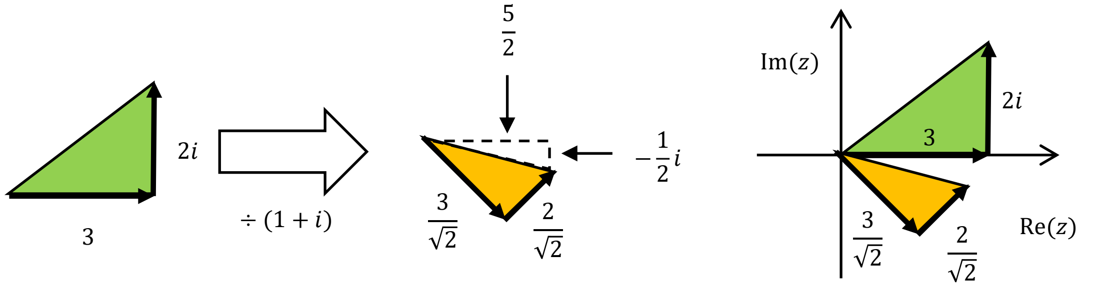
Simplify $\frac{3+2i}{1+i}$

Solution

$$\frac{3+2i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{3-3i+2i-2i^2}{1^2-i^2}$$

$$= \frac{5-i}{2}$$



In polar form, $1+i$ has modulus $\sqrt{2}$ and argument $45°$, so dividing by $1+i$ is the same as scaling by $\frac{1}{\sqrt{2}}$ then rotating clockwise $45°$. Using trigonometry we could show that there exists another triangle with horizontal and vertical sides with lengths equivalent to $\frac{5}{2}$ and $-\frac{1}{2}i$

EQUAL COMPLEX NUMBERS

An important technique in solving questions is that for two complex numbers in Cartesian form to be equal, their real parts are equal and their imaginary parts are equal.

Example 7

Given $z = a + ib$, find a and b such that $z + 3i\bar{z} = -1 + 13i$

Solution

$$\begin{aligned} \text{LHS} &= a + ib + 3i(a - ib) \\ &= a + ib + 3ai + 3b \\ &= a + 3b + (3a + b)i \end{aligned}$$

$\therefore a + 3b = -1 \qquad (1)$

$3a + b = 13 \qquad (2)$

$(2) - 3 \times (1)$:

$-8b = 16 \ \rightarrow b = -2$

sub in (2)

$3a - 2 = 13 \ \rightarrow a = 5$

$\therefore a = 5$ and $b = -2$

EXERCISE 2.2

BASIC

1 Simplify:

a $(3+2i)+(4+i)$ **b** $(3+2i)-(4+i)$

2 Simplify:

a $3(2+4i)$ **b** $-2(4-3i)$

3 Simplify:

a $i(3+i)$ **b** $-2i(3-4i)$

4 Simplify $(3+i)(4-2i)$

5 Simplify $(2-i)^2$

6 Simplify $(2+i)^3$

7 Find the conjugate of each complex number:

a $2-2i$ **b** $-6i$

8 Simplify $\dfrac{2-3i}{2i}$

9 Simplify $\dfrac{3-2i}{1-i}$

10 Given $z=3+2i$ find $\mathrm{Re}(2z+iz)$

11 Given $z=a+ib$, find a and b such that $z-2i\bar{z}=3+4i$

12 Find the real numbers a and b such that $(1+2i)(1-3i)=a+ib$

13 Let $z=1+i$ and $w=1-i$. Find, in the form $x+iy$

a $z+iw$ **b** $z^2\overline{w}$ **c** $\frac{z}{w}$

14 If $z=3+i$ and $w=-2+2i$, find:

a $z+w$ **b** $z-w$ **c** $2z$ **d** $-3w$

e iz **f** $z\times w$ **g** $\dfrac{w}{3i}$ **h** $\dfrac{z}{w}$

MEDIUM

15 Given that a and b are real numbers and $\frac{a}{1+i}-\frac{b}{2i}=2$ find the values of a and b.

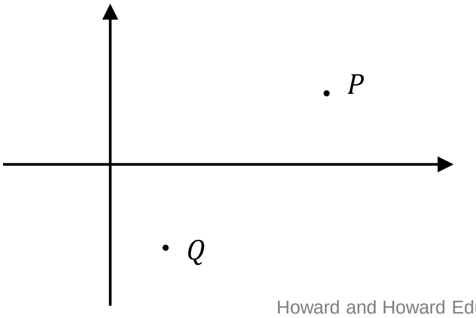
16 The points P and Q represent the complex numbers z and w respectively.

Mark the following points on the diagram.

a the point R representing $\bar{z}$

b the point S representing iw

c the point T representing $w+z$



1 **a** $(3+2i)+(4+i)=7+3i$ **b** $(3+2i)-(4+i)=-1+i$

2 **a** $3(2+4i)=6+12i$ **b** $-2(4-3i)=-8+6i$

3 **a** $i(3+i)=3i+i^2=-1+3i$ **b** $-2i(3-4i)=-6i+8i^2=-8-6i$

4 $(3+i)(4-2i)=12-6i+4i-2i^2=12-2i+2=14-2i$

5 $(2-i)^2=4-4i+i^2=4-4i-1=3-4i$

6 $(2+i)^3=(2)^3+3(2)^2(i)^1+3(2)(i)^2+i^3=8+12i-6-i=2+11i$

7 **a** $\overline{2-2i}=2+2i$ **b** $\overline{-6i}=6i$

8 $\dfrac{2-3i}{2i}\times\dfrac{i}{i}=\dfrac{2i+3}{-2}=\dfrac{-3-2i}{2}$

9 $\dfrac{3-2i}{1-i}\times\dfrac{1+i}{1+i}=\dfrac{3+3i-2i+2}{1^2+1^2}=\dfrac{5+i}{2}$

10 $\text{Re}(2z+iz)=\text{Re}\big(2(3+2i)+i(3+2i)\big)=\text{Re}(6+4i+3i-2)=\text{Re}(4+7i)=4$

11 LHS $=a+ib-2i(a-ib)=a+ib-2ai-2b=(a-2b)+(b-2a)i$

$\therefore a-2b=3 \quad (1) \quad b-2a=4 \quad (2)$

$(1)+2(2)\text{:}\ \ -3a=11 \quad \rightarrow \ a=-\dfrac{11}{3}$

$\text{sub in }(1)\text{:}\ -\dfrac{11}{3}-2b=3 \ \rightarrow \ b=-\dfrac{10}{3}$

12 $1-3i+2i-6i^2=7-i \quad \therefore a=7, b=-1$

13 **a**

$$\begin{aligned} z+iw &= 1+i+i(1-i)\\ &= 1+i+i+1\\ &= 2+2i \end{aligned}$$

b

$$\begin{aligned} z^2\overline{w} &= (1+i)^2(1+i)\\ &= (1+i)^3\\ &= 1^3+3(1)^2i+3(1)i^2\\ &= 1+3i-3-i\\ &= -2+2i \end{aligned}$$

c

$$\begin{aligned} \frac{z}{w} &= \frac{1+i}{1-i}\times\frac{1+i}{1+i}\\ &= \frac{1+2i-1}{1^2+1^2}\\ &= i \end{aligned}$$

14 **a** $z+w=(3+i)+(-2+2i)=1+3i$
b $z-w=(3+i)-(-2+2i)=5-i$
c $2z=2(3+i)=6+2i$
d $-3w=-3(-2+2i)=6-6i$
e $iz=i(3+i)=-1+3i$
f $z\times w=(3+i)(-2+2i)=-6+6i-2i-2=-8+4i$
g $\dfrac{w}{3i}=\dfrac{-2+2i}{3i}\times\dfrac{i}{i}=\dfrac{-2i-2}{-3}=\dfrac{2}{3}+\dfrac{2}{3}i$
h $\dfrac{z}{w}=\dfrac{3+i}{-2+2i}\times\dfrac{-2-2i}{-2-2i}=\dfrac{-6-6i-2i+2}{4+4}=\dfrac{-4-8i}{8}=\dfrac{-1-2i}{2}$

15

$$\frac{a}{1+i}\times\frac{1-i}{1-i}-\frac{b}{2i}\times\frac{i}{i}=2$$

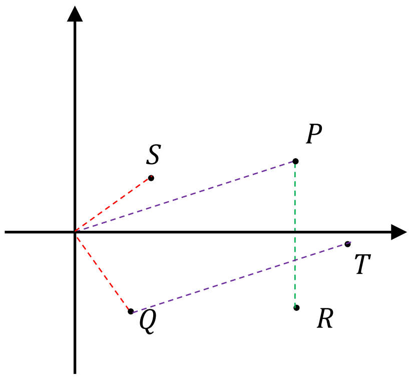
$$\frac{a-ai}{1^2+1^2}-\frac{bi}{-2}=2$$

$$\frac{a-ai+bi}{2}=2$$

$$a+(b-a)i=4$$

$$\therefore a=4, \qquad b-a=0\rightarrow b=4$$

16



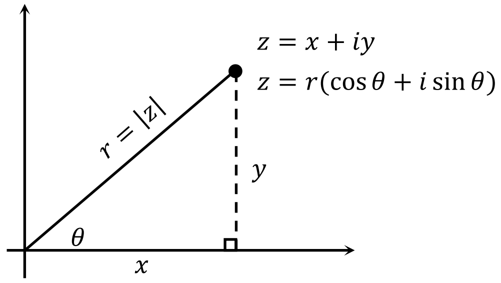
In Lesson 3 we look at the polar form of a complex number, covering:

- Modulus, Argument and Polar Form
- Multiplying Complex Numbers in Polar Form
- Division of Complex Numbers in Polar Form
- Conjugate in Polar Form
- Converting Rectangular to/from Polar Form

MODULUS, ARGUMENT AND POLAR FORM

Polar form $(r(\cos\theta + i\sin\theta)$ or $r\ \mathrm{cis}\ \theta)$ is also known as modulus-argument (mod-arg) form.

In the diagram below the complex number z can be represented as $x + iy$ in Cartesian form, or $r(\cos\theta + i\sin\theta)$ in polar form – they represent the same complex number.



The distance from zero is called the modulus, which we represent using either the pronumeral r or $|z|$.

At a glance we can see that the modulus is a hypotenuse of a right angled triangle, so we can use Pythagoras' theorem to show that $|z| = \sqrt{x^2 + y^2}$.

We call the angle the argument, labelled $\arg(z)$ and measure it anticlockwise from the x-axis.

Like on the Unit Circle a single point can be represented by many angles, each differing by 2π radians, so we define the principal argument as the angle from $-\pi$ radians exclusive to π radians inclusive, so $(-\pi, \pi]$.

Now some mathematicians prefer a principal argument to be defined for $[0, 2\pi)$, and each has its advantages. The advantage of using $(-\pi, \pi]$ is that we often have solutions that are conjugates, and so we get answers in the form $\pm\theta$ rather than $\theta, 2\pi - \theta$ which is a bit easier.

Any angle is formed by two lines. For the argument we have the real axis plus the ray from zero to the complex number. We cannot use polar form for the complex number zero, as there no line from zero to zero, so the argument is undefined. The modulus is zero.

Using some right angled trigonometry we can then easily see that $x = r\cos\theta$ and $y = r\sin\theta$, so we can convert the rectangular form of $x + iy$ into polar form $\mathrm{z} = r(\cos\theta + i\sin\theta)$ which is often abbreviated as $r \operatorname{cis}\theta$ which stands for <u>c</u>os + <u>i</u> <u>s</u>in. Purists prefer not to use cis notation, and in HSC solutions the full $\cos\theta + i\sin\theta$ is used.

In the last lesson we will use c and is as abbreviations for $\cos$ and $i\sin$ in our working.

Example 1

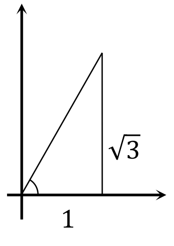
Convert the following complex numbers from Cartesian form into polar form:

a $1 + \sqrt{3}i$ **b** $-1 + i$ **c** $1 - i$

Solution

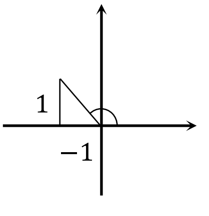
Draw a rough sketch the first few times you do it.

a



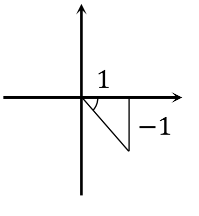
$$|z| = \sqrt{1^2 + (\sqrt{3})^2}$$
$$= 2$$
$$\arg(z) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$
$$= \frac{\pi}{3}$$
$$\therefore 1 + \sqrt{3}i = 2 \operatorname{cis}\frac{\pi}{3}$$

b



$$|z| = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$
$$\arg(z) = \pi - \tan^{-1}\left(\frac{1}{1}\right)$$
$$\therefore \arg(z) = \frac{3\pi}{4}$$
$$\therefore -1 + i = \sqrt{2} \operatorname{cis}\frac{3\pi}{4}$$

c



$$|z| = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$
$$\arg(z) = -\tan^{-1}\left(\frac{1}{1}\right)$$
$$\therefore \arg(z) = -\frac{\pi}{4}$$
$$\therefore 1 - i = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$$

Example 2

Convert the following complex numbers from polar into Cartesian form:

a $2\ \text{cis}\frac{\pi}{3}$

b $\text{cis}\left(-\frac{2\pi}{3}\right)$

Solution

a

$$
\begin{aligned}
&2\ \text{cis}\frac{\pi}{3}\\
&= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\\
&= 2 \times \left(\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right)\\
&= 1 + \sqrt{3}i
\end{aligned}
$$

b

$$
\begin{aligned}
&\text{cis}\left(-\frac{2\pi}{3}\right)\\
&= \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\\
&= -\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\\
&= -\frac{1}{2} - i \times \frac{\sqrt{3}}{2}\\
&= -\frac{1}{2} - \frac{\sqrt{3}}{2}i
\end{aligned}
$$

MULTIPLYING COMPLEX NUMBERS IN POLAR FORM

We have seen that multiplying by a complex number causes a rotation and/or a scaling, and this is much simpler when the numbers are in polar form.

We can find the product by:

- Multiplying the moduli of the factors
- Adding the arguments of the factors

$$z_1 \times z_2 = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$$

We can extend these rules to powers if we let the complex numbers be equal (call them z), then:

$$\boxed{|z^2| = |z|^2 \text{ and } \arg(z^2) = 2\arg z}$$

$$\boxed{|z^n| = |z|^n \text{ and } \arg(z^n) = n\arg z}$$

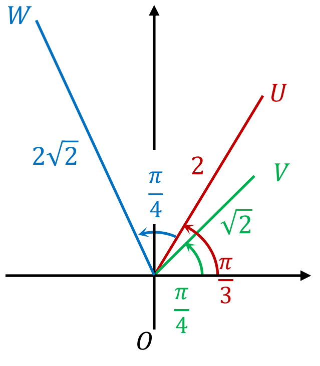
When we raise a complex number to a power we raise the modulus to the power and multiply the argument by the power. This is the basis of de Moivre's Theorem which we will look at more deeply in Lesson 8.

Example 3

If $u = 2\operatorname{cis}\frac{\pi}{3}$ and $v = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$, find w where $w = uv$.

Solution

$$\begin{aligned} w &= uv \\ &= 2\operatorname{cis}\frac{\pi}{3} \times \sqrt{2}\operatorname{cis}\frac{\pi}{4} \\ &= \left(2 \times \sqrt{2}\right)\operatorname{cis}\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\ &= 2\sqrt{2}\operatorname{cis}\frac{7\pi}{12} \end{aligned}$$



- Starting with OU which has an argument of $\frac{\pi}{3}$, we rotate it by $\frac{\pi}{4}$ (the argument of v) to find the argument of w.
- The modulus of w is the product of the moduli of u and v.

We can find the quotient by:

- Dividing the moduli of the factors
- Subtracting the arguments of the divisor (denominator) from the dividend (numerator)

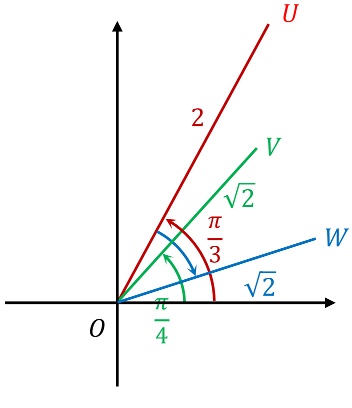
$$z_1 \div z_2 = \frac{r_1}{r_2}\Big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\Big)$$

Example 4

If $u = 2\operatorname{cis}\frac{\pi}{3}$ and $v = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$, find w where $w = \frac{u}{v}$.

Solution

$$\begin{aligned} w &= \frac{u}{v} \\ &= 2\operatorname{cis}\frac{\pi}{3} \div \sqrt{2}\operatorname{cis}\frac{\pi}{4} \\ &= \left(2 \div \sqrt{2}\right)\operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sqrt{2}\operatorname{cis}\frac{\pi}{12} \end{aligned}$$

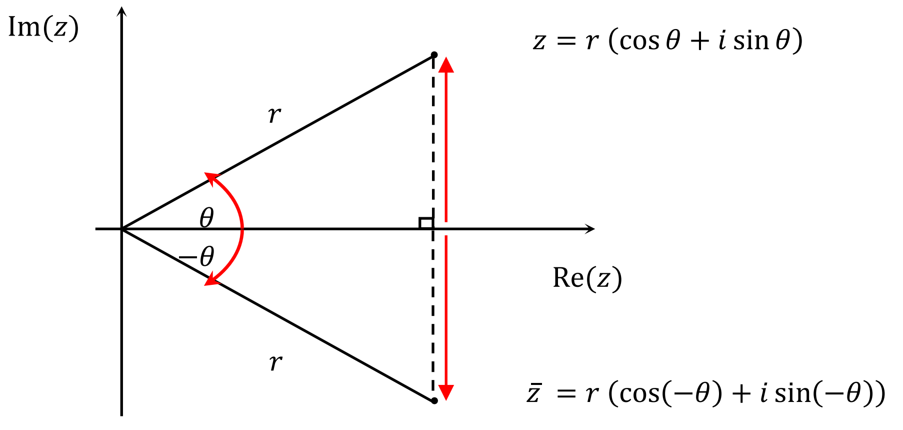


- Starting with OU which has an argument of $\frac{\pi}{3}$, we rotate it by clockwise $\frac{\pi}{4}$, the argument of v to find the argument of w.
- The modulus of w is the modulus of u divided by $\sqrt{2}$, the modulus of v.

CONJUGATE IN POLAR FORM

In polar form the conjugate of a complex number has the same modulus while the argument is the negative.

The conjugate of $z = r(\cos\theta + i\sin\theta)$ is $\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$.



Example 5

State the conjugate of each complex number, leaving your answer in polar form

a $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

b $z = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$

Solution

a

$$\bar{z} = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

b

$$\bar{z} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

EXERCISE 2.3

BASIC

1 Convert the following complex numbers from Cartesian form into polar form:

a $1+i$ **b** $-1+\sqrt{3}i$ **c** $-\sqrt{3}-i$

2 Convert the following complex numbers from polar into Cartesian form:

a $4\operatorname{cis}\frac{\pi}{6}$ **b** $\operatorname{cis}\left(-\frac{\pi}{3}\right)$

3 If $u = 4\operatorname{cis}\frac{\pi}{6}$ and $v = \operatorname{cis}\left(-\frac{\pi}{3}\right)$, find w where $w = uv$.

4 If $u = 4\operatorname{cis}\frac{\pi}{6}$ and $v = \operatorname{cis}\left(-\frac{\pi}{3}\right)$, find w where $w = \frac{u}{v}$.

5 State the conjugate of each complex number, leaving your answer in polar form

a $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ **b** $z = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$

6 Convert $\sqrt{3}+i$ into polar form in radians using a calculator (see Appendix 1)

7 Convert $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ into rectangular form using a calculator (see Appendix 1)

MEDIUM

8 Prove $z_1 \times z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$.

Hint: let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

9 Prove $z_1 \div z_2 = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$

Hint: let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

10

a Write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in the form $a+ib$ where a and b are real.

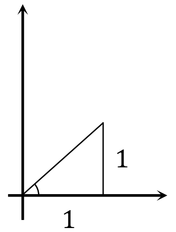
b By expressing $\sqrt{3}+i$ and $\sqrt{3}-i$ in polar form, write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in polar form.

c Hence find $\sin\frac{\pi}{3}$ in surd form

11 Multiplying a non-zero complex number by $\frac{1-i}{-1+i}$ results in a rotation about the origin. What is the angle of rotation, and in what direction?

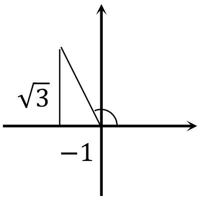
12 Given $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ find $(\bar{z})^{-1}$ in polar form.

1 a



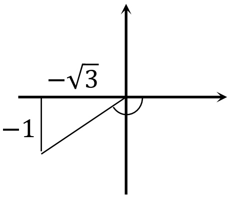
$$|z| = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$
$$\arg(z) = \tan^{-1}\left(\frac{1}{1}\right)$$
$$= \frac{\pi}{4}$$
$$\therefore 1 + i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$$

b



$$|z| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}$$
$$= 2$$
$$\arg(z) = \pi - \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$
$$\therefore \arg(z) = \frac{2\pi}{3}$$
$$\therefore -1 + \sqrt{3}i = 2\operatorname{cis}\frac{2\pi}{3}$$

c



$$|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2}$$
$$= 2$$
$$\arg(z) = -\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$\therefore \arg(z) = -\frac{5\pi}{6}$$
$$\therefore -1 - \sqrt{3}i = 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$$

2 a

$$4\operatorname{cis}\frac{\pi}{6}$$
$$= 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$= 4 \times \left(\frac{\sqrt{3}}{2} + i \times \frac{1}{2}\right)$$
$$= 2\sqrt{3} + 2i$$

b

$$\operatorname{cis}\left(-\frac{\pi}{3}\right)$$
$$= \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$$
$$= \cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)$$
$$= \frac{1}{2} - i \times \frac{\sqrt{3}}{2}$$
$$= \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

3

$$w = uv$$
$$= 4\operatorname{cis}\frac{\pi}{6} \times \operatorname{cis}\left(-\frac{\pi}{3}\right)$$
$$= (4 \times 1)\operatorname{cis}\left(\frac{\pi}{6} - \frac{\pi}{3}\right)$$
$$= 4\operatorname{cis}\left(-\frac{\pi}{6}\right)$$

4

$$w = \frac{u}{v}$$
$$= 4\operatorname{cis}\frac{\pi}{6} \div \operatorname{cis}\left(-\frac{\pi}{3}\right)$$
$$= (4 \div 1)\operatorname{cis}\left(\frac{\pi}{6} - -\frac{\pi}{3}\right)$$
$$= 4\operatorname{cis}\frac{\pi}{2}$$
$$= 4i$$

5 a

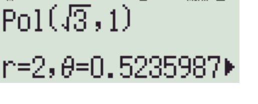
$$\overline{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)} = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

b

$$\overline{\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

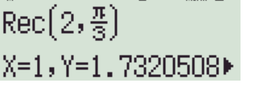
6 Keystrokes: $\boxed{\text{SHIFT}}$ $\boxed{\text{Pol}}$ $\boxed{\sqrt{\ }}$ $\boxed{3}$ $\boxed{\rightarrow}$ $\boxed{\text{SHIFT}}$ $\boxed{,}$ $\boxed{1}$ $\boxed{)}$ $\boxed{=}$

To find the argument in exact form, $\boxed{\text{SHIFT}}$ $\boxed{\pi}$ $\boxed{\div}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{Y}}$ $\boxed{=}$ gives 6, so $\frac{\pi}{6}$



7 Keystrokes: $\boxed{\text{SHIFT}}$ $\boxed{\text{Rec}}$ $\boxed{2}$ $\boxed{\text{SHIFT}}$ $\boxed{,}$ $\boxed{\text{SHIFT}}$ $\boxed{\pi}$ $\boxed{\frac{\blacksquare}{\square}}$ $\boxed{\triangledown}$ $\boxed{3}$ $\boxed{)}$ $\boxed{=}$

To find the imaginary part in exact form, $\boxed{\text{ALPHA}}$ $\boxed{\text{Y}}$ $\boxed{x^2}$ $\boxed{=}$ gives 3, so $\sqrt{3}$



8

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\begin{aligned} z_1 \times z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \\ &= r_1r_2\big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big) \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \qquad \square \end{aligned}$$

$\therefore |z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

9

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2} \\ &= \frac{r_1(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} \\ &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \end{aligned}$$

$\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \qquad \square$

10a

$$\begin{aligned} &\frac{\sqrt{3}+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i} \\ &= \frac{3 + 2\sqrt{3}i + 1}{3+1} \\ &= \frac{4 + 2\sqrt{3}i}{4} \\ &= 1 + \frac{\sqrt{3}}{2}i \end{aligned}$$

b

$$\begin{aligned} \sqrt{3}+i &= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\ \sqrt{3}-i &= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) \\ \frac{\sqrt{3}+i}{\sqrt{3}-i} &= \frac{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}{2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)} \\ &= \frac{2}{2}\left(\cos\left(\frac{\pi}{6} - -\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6} - -\frac{\pi}{6}\right)\right) \\ &= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) \end{aligned}$$

c

Equating imaginary parts of **a** and **b**

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

11

$$\frac{1-i}{-1+i} = \frac{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)}{\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)} = \cos(-\pi) + i\sin(-\pi) = -1$$

The rotation is $180°$.

12

$$\begin{aligned} (\bar{z})^{-1} &= \frac{1}{2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)} \times \frac{\left(\cos\left(-\frac{\pi}{6}\right) - i\sin\left(-\frac{\pi}{6}\right)\right)}{\left(\cos\left(-\frac{\pi}{6}\right) - i\sin\left(-\frac{\pi}{6}\right)\right)} \\ &= \frac{\left(\cos\left(-\frac{\pi}{6}\right) - i\sin\left(-\frac{\pi}{6}\right)\right)}{2} \\ &= \frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \qquad \text{since cosine is even and sine is odd} \end{aligned}$$

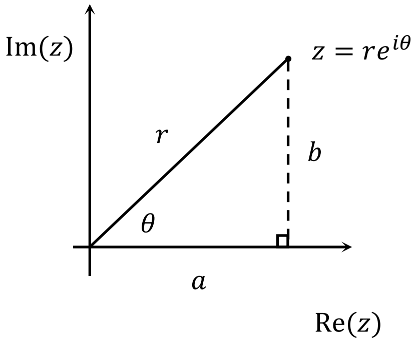
2.4 EXPONENTIAL FORM

In Lesson 4 we look at complex numbers in exponential form, covering:

- Exponential Form
- Explaining Exponential Form and Euler's Formula
- Proving Euler's Formula
- Euler's Identity
- Calculations in Exponential Form
- Compare and Convert Between Forms

EXPONENTIAL FORM

We have seen complex numbers expressed in Cartesian form $(z = x + iy)$ and polar form $\left(z = r(\cos\theta + i\sin\theta)\right)$, but there is another very useful form - exponential form, $z = re^{i\theta}$ where r is the modulus and θ is the principal argument.



Exponential form makes multiplication, division, powers and roots very simple.

EULER'S FORMULA

Joining exponential form with the polar form we can easily see that $e^{i\theta} = \cos\theta + i\sin\theta$

Euler's formula leads us to one of the most famous relationships in mathematics, Euler's Identity, which we will look at later in the lesson.

EXPLAINING EXPONENTIAL FORM AND EULER'S FORMULA

We will look at some formal algebraic proofs of Euler's formula, but first let's try and gain an understanding of how $e^{i\theta}$ works using geometry.

Now there are many potentially baffling aspects of exponential form, the two most important aspects being:

- what is e?
- what happens when we raise e to an imaginary power?

So let's have a look at these aspects first, before we look at an algebraic proof of Euler's formula.

EULER'S NUMBER

We have often come across Euler's number e, which we know is approximately 2.72, but how is it defined?

There are actually many ways to define e^x, and we will use two of them in this lesson and Appendix 2.

Our first geometric explanation is based on the power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

In Appendix 2 we look at an explanation using the definition of e as a limit:

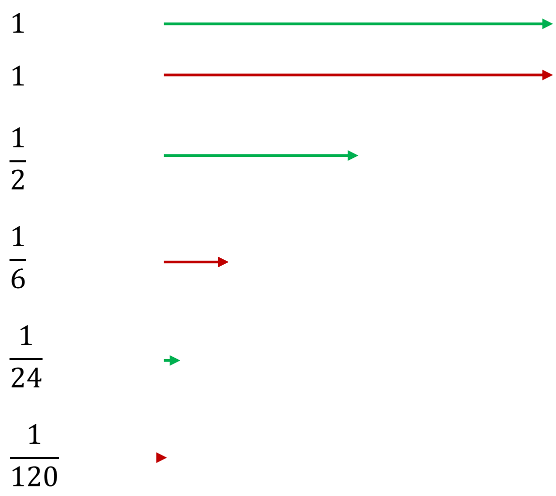
$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

Let's start by finding e^1, which is of course just e. Using the power series we can see that

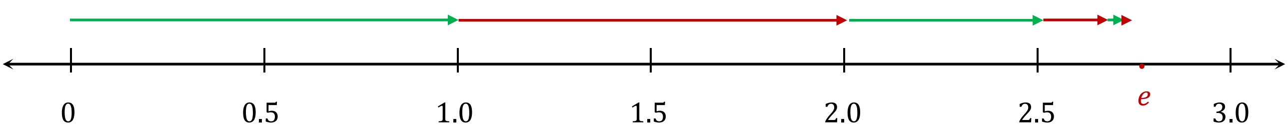
$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots$$

We can view this sum as ‘right 1’, right 1’, ‘right $\frac{1}{2}$’ etc. Representing each fraction as a vector, we can see:



We can see that the sixth term is almost negligible, and everything after that is even smaller. Arranging them in line from zero we see:



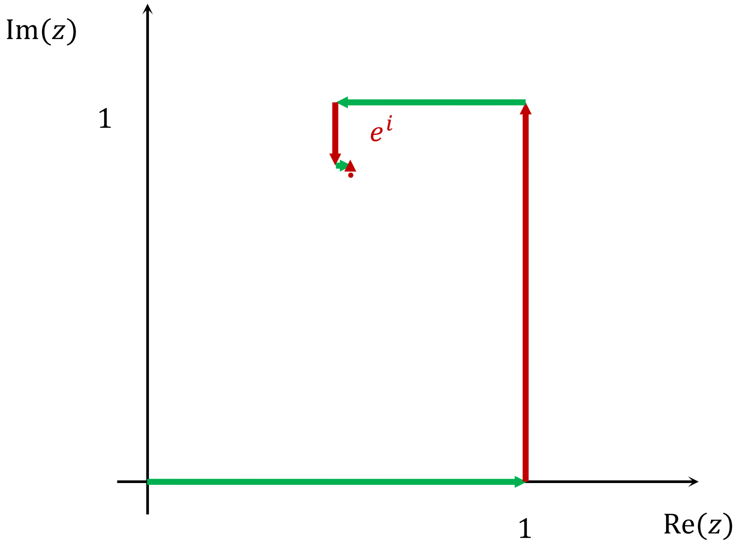
So even at a glance we can see that e is approximately 2.75. We will use the second definition in Appendix 2 to narrow it down.

Now to find e^i, using the power series we can see that

$$e^i = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \frac{i^4}{4!} + \frac{i^5}{5!} + \ldots$$

$$= 1 + i - \frac{1}{2} - \frac{i}{6} + \frac{1}{24} + \frac{i}{120} + \ldots$$

We can view this sum as ‘right 1’, ‘up 1’, ‘left $\frac{1}{2}$’, ‘down $\frac{1}{6}$’, ‘right $\frac{1}{24}$’ and ‘up $\frac{1}{120}$’ etc. Representing each fraction as a vector on the complex plane, we can see:



We can see that e^i must have a real value a bit more than $\frac{1}{2}$ and an imaginary part a bit more than $\frac{5}{6}$. We will narrow this down using the limit definition in a minute. In Appendix 2 we will narrow down the values of e and e^i for those who are interested.

PROVING EULER'S FORMULA

We now have an understanding of why Euler's formula works and what it means, so now it is time for a formal proof. We will start by assuming the power series for $e^x, \cos x$ and $\sin x$, but in Appendix 3 we look at a more thorough proof starting with the Taylor's Series.

Proof 1

Prove $e^{ix} = \cos x + i \sin x$. You are given that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad, \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad \text{and } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Solution

$$\begin{aligned} e^{ix} &= e^{0i} + ie^{0i}x + \frac{i^2 e^{0i} x^2}{2!} + \frac{i^3 e^{0i} x^3}{3!} + \frac{i^4 e^{0i} x^4}{4!} + \ldots \\ &= 1 + xi - \frac{x^2}{2!} - \frac{x^3}{3!} i + \frac{x^4}{4!} + \frac{x^5}{5!} i - \frac{x^6}{6!} - \frac{x^7}{7!} i + \ldots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \\ &= \cos x + i \sin x \quad \square \end{aligned}$$

If we let $x = \pi$ in Euler's formula, we end up with Euler's Identity, which famously finds a link between the 5 most important constants in mathematics.

$$e^{i\pi} + 1 = 0$$

The identity links:

- Euler's number e
- The imaginary unit i
- The constant π
- The number 1
- The number 0

Proof 2

Prove $e^{i\pi} + 1 = 0$

Solution

Let $x = \pi$ in Euler's formula

$\therefore e^{i\pi} = \cos\pi + i\sin\pi$

$e^{i\pi} = -1 + i(0)$

$e^{i\pi} + 1 = 0$ $\square$

Example 3

Simplify each expression and mark on a complex plane

a i^i **b** $\sqrt{e^{i\pi}}$

Solution

a

$$i^i = \left(e^{\frac{\pi}{2}i}\right)^i$$

$$= e^{\frac{\pi}{2}i^2}$$

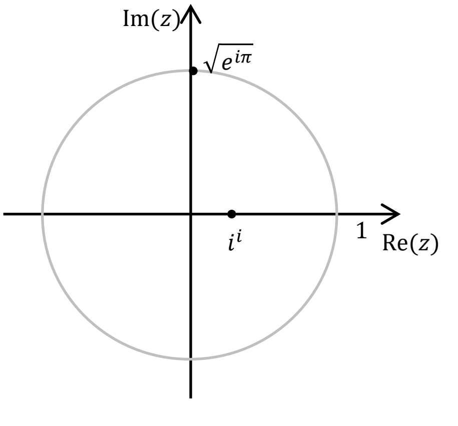
$$= e^{-\frac{\pi}{2}}$$

$$\approx 0.2$$

b

$$\sqrt{e^{i\pi}} = \sqrt{-1}$$

$$= i$$



CALCULATIONS IN EXPONENTIAL FORM

Exponential form makes many of our calculations involving multiplication, division, powers and roots a simple matter of following our index laws.

It is best to avoid using exponential form for addition and subtraction as we have no way to simplify the final answer using our index laws, unless the two complex numbers have the same argument.

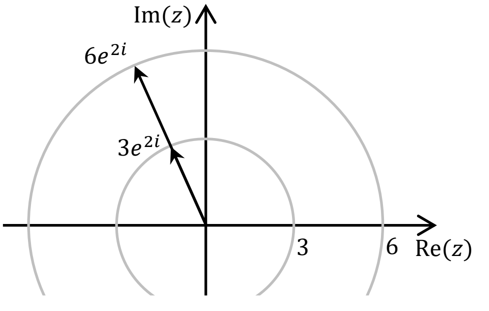
Example 4

Simplify $2(3e^{2i})$

Solution

$2\left(3e^{2i}\right) = 6e^{2i}$

Multiplying by a positive real number changes the modulus, but leaves the argument constant.



Multiplying a complex number in exponential form by a negative real number involves a simple trick, as in our final answer the modulus must be positive. We replace the minus with $e^{i\pi}$ or $e^{-i\pi}$ and use our index laws as shown below.

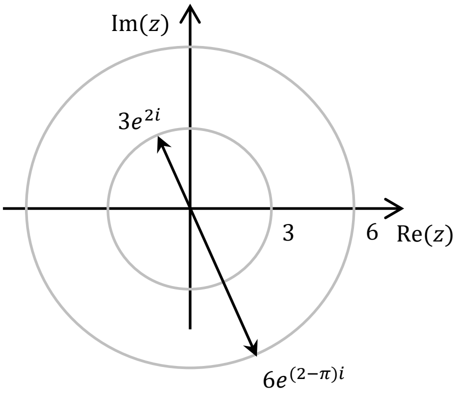
Example 5

Simplify $-2(3e^{2i})$

Solution

Simplify $-2(3e^{2i})$

Multiplying by a negative real number changes the modulus, and adds or subtracts π to the argument.



Multiplying a complex number in exponential form by an imaginary number rotates the complex number by $\frac{\pi}{2}$ - either anticlockwise (if there is a positive multiple of i) or clockwise (if there is a negative multiple of i). Remember that $i = e^{\frac{\pi}{2}i}$ and $-i = e^{-\frac{\pi}{2}i}$.

Example 6

Simplify

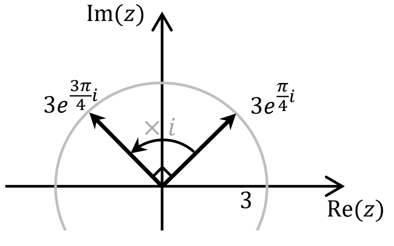
a $i(3e^{\frac{\pi}{4}i})$

b $-2i\left(3e^{\frac{\pi}{2}i}\right)$

Solution

a

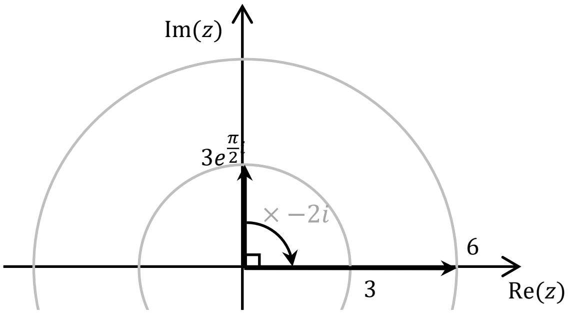
$$\begin{aligned} i\left(3e^{\frac{\pi}{4}i}\right) &= e^{\frac{\pi}{2}i} \times 3e^{\frac{\pi}{4}i} \\ &= 3e^{\left(\frac{\pi}{2}+\frac{\pi}{4}\right)i} \\ &= 3e^{\frac{3\pi}{4}i} \end{aligned}$$



Multiplying by i rotates $\frac{\pi}{2}$ anticlockwise, leaving the modulus unchanged.

b

$$\begin{aligned} -2i\left(3e^{\frac{\pi}{2}i}\right) &= -2i \times 3i \\ &= -6i^2 \\ &= 6 \end{aligned}$$



Multiplying by $-2i$ rotates $\frac{\pi}{2}$ clockwise and scales the modulus by 2.

Dividing is similar to multiplication, with the scaling of the modulus being in the opposite direction and any rotation occurring the opposite way.

In polar form the conjugate of a complex number has the same modulus while the argument is the negative.

The conjugate of $z = re^{i\theta}$ is $\bar{z} = re^{-i\theta}$.

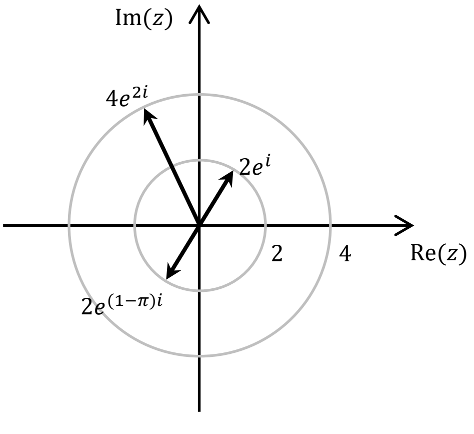
We look at square roots of complex numbers in Cartesian Form next lesson, but finding the square root of a complex number in exponential form is easy - take the square root of the modulus, halve the argument to find the one of the square roots, then add or subtract π to find the argument of the other square root.

Example 7

Find the square roots of $4e^{2i}$

Solution

The square roots are $2e^{i}$ and $2e^{(1-\pi)i}$



One square root has half the argument of the original number, and the modulus is the square root of the original. The second square root is opposite the first square root, so in this case $-2e^{i}$, but this isn't in exponential form (since the modulus cannot be negative). We need to subtract π from the argument to get $2e^{(1-\pi)i}$. If we were looking for arguments in the domain $[0,2\pi)$ then we could add π to the argument to get $2e^{(1+\pi)i}$.

COMPARE AND CONVERT BETWEEN FORMS

Both exponential and polar forms use the modulus and argument of a complex number, so we can view exponential form as a shorthand notation for polar form, plus it allows us to use our index laws for multiplication and division.

Cartesian form is the best form for addition and subtraction of complex numbers, plus their geometric equivalents of translation of vectors.

Polar form is a good way to convert between Exponential and Cartesian forms, and with de Moivre's theorem and binomial expansion allows us to find other relationships.

Converting from Cartesian form to Exponential form is the same as converting to polar form.

Example 8

Convert the following complex numbers from exponential into Cartesian form:

a $2\,e^{\frac{\pi}{3}i}$

b $e^{-\frac{2\pi}{3}i}$

Solution

a $2\,e^{\frac{\pi}{3}i}$

$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 2 \times \left(\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right)$

$= 1 + \sqrt{3}i$

b $e^{-\frac{2\pi}{3}i}$

$= \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$

$= -\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)$

$= -\frac{1}{2} - i \times \frac{\sqrt{3}}{2}$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

EXERCISE 2.4

BASIC

1 Simplify $3(4e^{3i})$

2 Simplify $-(4e^{3i})$

3 Simplify **a** $i(2e^{\frac{\pi}{3}i})$ **b** $-i\left(3e^{-\frac{\pi}{2}i}\right)$

4 Simplify $2e^{-2i} \times 3e^{i}$

5 Simplify **a** $\left(3e^{i}\right)^3$ **b** $\left(3e^{-i}\right)^2$

6 Simplify $4e^{\frac{\pi}{2}i} \div 2$

7 Simplify **a** $2e^{\frac{\pi}{3}i} \div i$ **b** $5e^{-\frac{\pi}{2}i} \div (-i)$

8 Simplify $2e^{-2i} \div 4e^{i}$

9 Find the conjugate of $z = 3e^{-\frac{\pi}{3}i}$

10 Convert the following complex numbers from polar form to exponential form:

a $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ **b** $\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$

11 Convert the following complex numbers from Cartesian form into exponential form:

a $1 + i$ **b** $-\sqrt{3} + i$

12 Convert the following complex numbers from exponential into Cartesian form:

a $e^{\frac{2\pi}{3}i}$ **b** $e^{-\frac{\pi}{4}i}$

MEDIUM

13 Prove $e^{ix} = \cos x + i\sin x$ using the Power Series

14 Prove $e^{i\pi} + 1 = 0$

15 Simplify each expression and mark on a complex plane

a i^{-i} **b** $\sqrt{e^{-i\pi}}$

16 Prove $(-1)^{\frac{1}{n}} = e^{\frac{\pi}{n}i}$

17 Find the square roots of $9e^{\frac{\pi}{3}i}$

CHALLENGING

18 Prove $e^{ix} = \cos x + i\sin x$ using the Maclaurin Series

1 $3(4e^{3i}) = 12e^{3i}$

2 $-(4e^{3i}) = e^{-i\pi} \times 4e^{3i} = 4e^{(3-\pi)i}$

3 **a** $i\left(2e^{\frac{\pi}{3}i}\right) = e^{\frac{\pi}{2}i} \times 2e^{\frac{\pi}{3}i} = 2e^{\left(\frac{\pi}{2}+\frac{\pi}{3}\right)i} = 2e^{\frac{5\pi}{6}i}$

b $-i\left(3e^{-\frac{\pi}{2}i}\right) = e^{-\frac{\pi}{2}i} \times 3e^{-\frac{\pi}{2}i} = 3e^{\left(-\frac{\pi}{2}-\frac{\pi}{2}\right)i} = 3e^{-i\pi} = 3 \times -1 = -3$

4 $2e^{-2i} \times 3e^{i} = 6e^{-i}$

5 **a** $\left(3e^{i}\right)^3 = 3^3 \times e^{i\times 3} = 27e^{3i}$ **b** $\left(3e^{-i}\right)^2 = 3^2 \times e^{-i\times 2} = 9e^{-2i}$

6 $4e^{\frac{\pi}{2}i} \div 2 = 2e^{\frac{\pi}{2}i}$

7 **a** $2e^{\frac{\pi}{3}i} \div i = 2e^{\frac{\pi}{3}i} \div e^{\frac{\pi}{2}i} = 2e^{\left(\frac{\pi}{3}-\frac{\pi}{2}\right)i} = 2e^{-\frac{\pi}{6}i}$

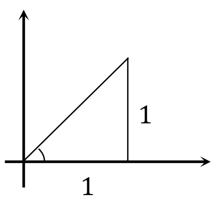
b $5e^{-\frac{\pi}{2}i} \div (-i) = -5i \div (-i) = 5$

8 $2e^{-2i} \div 4e^{i} = \dfrac{1}{2}e^{-3i}$

9 $\overline{z} = 3e^{\frac{\pi}{3}i}$

10 **a** $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{\frac{\pi}{3}i}$ **b** $\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = e^{-\frac{\pi}{4}i}$

11 **a**

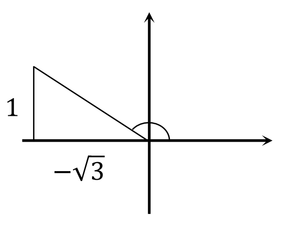


$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\therefore 1 + i = \sqrt{2}e^{\frac{\pi}{4}i}$$

b



$$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\arg(z) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\therefore \arg(z) = \frac{5\pi}{6}$$

$$\therefore -\sqrt{3} + i = 2\,e^{\frac{5\pi}{6}i}$$

12 **a** $e^{\frac{2\pi}{3}i} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = -\frac{1}{2} + i \times \frac{\sqrt{3}}{2}$

b $e^{-\frac{\pi}{4}i} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - i \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

13

$$e^{ix} = e^{0i} + ie^{0i}x + \frac{i^2e^{0i}x^2}{2!} + \frac{i^3e^{0i}x^3}{3!} + \frac{i^4e^{0i}x^4}{4!} + \ldots$$

$$= 1 + xi - \frac{x^2}{2!} - \frac{x^3}{3!}i + \frac{x^4}{4!} + \frac{x^5}{5!}i - \frac{x^6}{6!} - \frac{x^7}{7!}i + \ldots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= \cos x + i\sin x \quad \square$$

14 Let $x=\pi$ in Euler's formula → $\therefore e^{i\pi}=\cos\pi+i\sin\pi \;\rightarrow\; e^{i\pi}=-1+i(0) \;\rightarrow\; e^{i\pi}+1=0$

15 **a** $i^{-i}=\left(e^{-\frac{\pi}{2}i}\right)^{i}=e^{-\frac{\pi}{2}i^2}=e^{\frac{\pi}{2}}\approx 4.8$

b $\sqrt{e^{-i\pi}}=\sqrt{(e^{i\pi})^{-1}}=\sqrt{(-1)^{-1}}=\sqrt{-1}=i$

16

$(-1)^{\frac{1}{n}}=\left(e^{i\pi}\right)^{\frac{1}{n}}=e^{\frac{\pi}{n}i}$ Since $\arg\left(e^{\frac{\pi}{n}i}\right)=\frac{\pi}{n}$, this is equivalent to a rotation of $\frac{\pi}{n}$.

17 The square roots are $3e^{\frac{\pi}{6}i}$ and $2e^{-\frac{5\pi}{6}i}$

18

For $f(x)=e^{ix}$

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(a)}{3!}x^3+\ldots$$

$$e^{ix}=e^{0i}+ie^{0i}x+\frac{i^2e^{0i}x^2}{2!}+\frac{i^3e^{0i}x^3}{3!}+\frac{i^4e^{0i}x^4}{4!}+\ldots$$

$$=1+xi-\frac{x^2}{2!}-\frac{x^3}{3!}i+\frac{x^4}{4!}+\frac{x^5}{5!}i-\frac{x^6}{6!}-\frac{x^7}{7!}i+\ldots \qquad *$$

$$=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right) \qquad (1)$$

For $f(x)=\cos x$

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(a)}{3!}x^3+\ldots$$

$$\cos x=\cos(0)-\sin(0)\,x-\frac{\cos(0)\,x^2}{2!}+\frac{\sin(0)\,x^3}{3!}+\frac{\cos(0)\,x^4}{4!}+\ldots$$

$$=1-0x-\frac{x^2}{2!}+0x^3+\frac{x^4}{4!}+0x^5-\frac{x^6}{6!}+\ldots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \qquad (2)$$

For $f(x)=\sin x$

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(a)}{3!}x^3+\ldots$$

$$\sin x=\sin(0)+\cos(0)\,x-\frac{\sin(0)\,x^2}{2!}-\frac{\cos(0)\,x^3}{3!}+\frac{\sin(0)\,x^4}{4!}+\ldots$$

$$=0+x+0x^2-\frac{x^3}{3!}+0x^4+\frac{x^5}{5!}+0x^6+\ldots$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots \qquad (3)$$

So we have the following equations:

$$e^{ix}=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right) \qquad (1)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \qquad (2)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots \qquad (3)$$

From (1), (2) and (3) we see that $e^{ix}=\cos x+i\sin x$ $\square$

2.5 SQUARE ROOTS

In Lesson 5 we cover square roots of complex numbers:

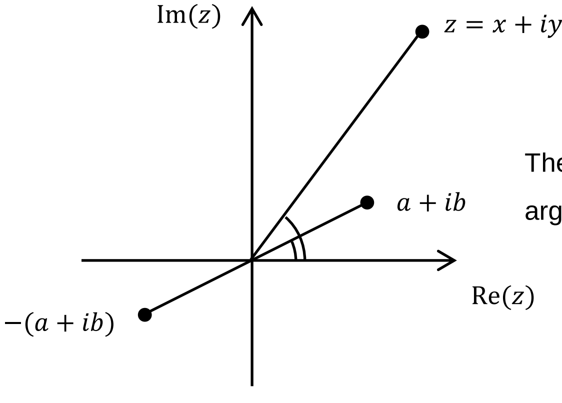
- using formal simultaneous equations
- using shortcuts for solving simultaneous equations
- using the formula

SQUARE ROOTS OF A COMPLEX NUMBER

To find the square roots of a complex number we can use the information from squaring a complex number in Lesson 2 to create simultaneous equations, or we can use a formula that works but is not in the syllabus. Use the formula if you can remember it as it is much quicker and can save a lot of fuss.

There are always two square roots of every non-zero complex number, and one is always the negative of the other, so they are in the form $\pm(a+ib)$.

If we let $a+ib$ be one of the square roots of $\mathrm{z}=x+iy$, then $x+iy=(a+ib)^2$.



The argument of this root is half the argument of the original number

From equating real and imaginary parts when squaring a complex number we found:

$$\mathrm{Re}(z)=a^2-b^2=x \quad (1)$$

$$\mathrm{Im}(z)=2ab=y \quad (2)$$

One of the square roots will always have an argument half that of the original complex number, while the second root will have an argument $\pm\pi$ that of the first root.

The modulus of both roots is the square root of the modulus of the original complex number.

In the first example we will use a formal solution of the simultaneous equations, then look at some shortcuts to solve the equations.

METHOD 1 - SIMULTANEOUS EQUATIONS

Example 1

Find the square root of $5+12i$, using simultaneous equations.

Solution

$a^2-b^2=5 \qquad (1)$

$2ab=12 \qquad (2)$

$b=\dfrac{6}{a} \qquad (2\text{ rearranged})$

$a^2-\frac{36}{a^2}=5 \qquad \big(\text{substituting into }(1)\big)$

$a^4-5a^2-36=0$

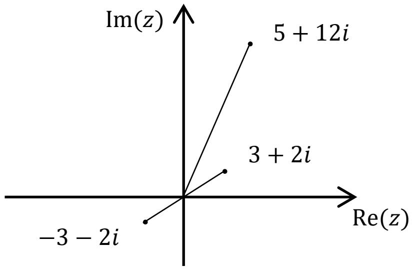
$(a^2-9)(a^2+4)=0$

$a^2-9=0$ or $a^2+4=0$

$\therefore a=\pm 3$ are the only solutions, since a must be real (not $\pm 2i$) – this is an important point

Substituting into (2) we get $a=3, b=2$ and $a=-3, b=-2$ which gives the square roots as $\pm(3+2i)$.

$3+2i$ has half the argument of $5+12i$, and its modulus is the square root of the modulus of $5+12i$. The other root is $-3-2i$.



We just solved the simultaneous equations formally, but we can solve them for simpler examples in our heads, with a little bit of writing. As a shortcut:

- Halve the imaginary part of the original number
- Find pairs of numbers that multiply to give that number
- Find which pair has a difference of their squares equal to the real part of the original number.

See the other adjustments that are needed for more difficult examples that follow.

Example 2

Find the square root of $5 + 12i$.

Solution

Half of 12 is 6. Find pairs of numbers whose product is 6:

1 and 6,   2 and 3.

Since the real part is positive then $a^2 - b^2$ is positive, so we place the number with the largest absolute value first (otherwise the difference would be negative).

In our heads: $6^2 - 1^2 \neq 5;\quad 3^2 - 2^2 = 5$ Thus 3 and 2 are the values for a and b, so $3 + 2i$ is a solution.

Therefore $-3 - 2i$ is also a solution.

The square roots of $5 + 12i$ are $\pm(3 + 2i)$.

While this can be quite quick for simple examples, it has its disadvantages. Here we will look at a range of examples - one simple then three more complicated.

Example 3

Find the square roots of $8i$

Solution

Taking the coefficient of i and halving it we get 4, so we are looking for a pair of numbers with a product of 4. We have 1 and 4, or 2 and 2 (and their negatives).

Now the real part is zero, so the difference of the squares is zero, so 2 and 2 (and their negatives) are the pairs we want.

So the square roots of $8i$ are $\pm(2 + 2i)$.

METHOD 2 - FORMULA

Now personally I never use simultaneous equations - I use the formulae below, which are easy to remember and make the task much simpler.

Equating real parts of $z = x + iy$ and $(a + ib)^2$ we have

$$\text{Re}(z) = a^2 - b^2 \qquad (1)$$

In a later lesson we learn that the modulus of a square is the square of the modulus of its root, so

$$|x + iy| = |a + ib|^2$$

$$\therefore |z| = a^2 + b^2 \qquad (2)$$

We can solve these two equations simultaneously by adding them, and rearranging we get:

$$\boxed{a = \pm\sqrt{\frac{|z| + \text{Re}(z)}{2}} \quad \text{and} \quad b = \pm\text{sgn}(y)\sqrt{\frac{|z| - \text{Re}(z)}{2}}}$$

where $\text{sgn}(y)$ is the signum function, returning the sign of y.

These are simple formula, and within the scope of many students to remember.

Example 4

Find the square root of $5 + 12i$, using Method 2

Solution

$$a = \pm\sqrt{\frac{\sqrt{5^2 + 12^2} + 5}{2}} = \pm 3$$

$$b = \pm(+)\sqrt{\frac{\sqrt{5^2 + 12^2} - 5}{2}} = \pm 2$$

$\therefore \pm(3 + 2i)$ are the solutions.

Example 6

The two square roots of $8i$ are $\pm(2+2i)$. Find the solutions of $2z^2+4z+2-i=0$

Solution

$$z=\frac{-4\pm\sqrt{4^2-4(2)(2-i)}}{2(2)}$$
$$=\frac{-4\pm\sqrt{8i}}{4}$$
$$=\frac{-4\pm(2+2i)}{4}$$
$$=-\frac{3+i}{2},\frac{-1+i}{2}$$

EXERCISE 2.5

BASIC

1 Find the square root of $3 + 4i$, using formal simultaneous equations.

2 Find the square root of $3 + 4i$, using the formula.

3 Find the square root of $3 + 4i$ using the shortcuts for simultaneous equations.

4 Find the square roots of $5 - 12i$ using the shortcuts for simultaneous equations.

5 Find the square roots of $2i$ using the shortcuts for simultaneous equations.

MEDIUM

6 Find the square roots of $2 - 2\sqrt{3}i$ using the shortcuts for simultaneous equations.

7 Find the square roots of $4 - 3i$ using the shortcuts for simultaneous equations.

8 The two square roots of $4i$ are $\pm(\sqrt{2} + \sqrt{2}i)$. Find the solutions of $z^2 + 4z + 4 - i = 0$

9 **a** Find the square roots of $-6 + 8i$

b Hence, or otherwise, solve the equation $z^2 - \sqrt{2}iz + 1 - 2i = 0$

1 $a^2 - b^2 = 3 \qquad (1)$

$2ab = 4 \qquad (2)$

$$b = \frac{2}{a} \qquad (2 \text{ rearranged})$$

$$a^2 - \frac{4}{a^2} = 3 \qquad \left(\text{substituting into } (1)\right)$$

$$a^4 - 3a^2 - 4 = 0$$

$$(a^2 - 4)(a^2 + 1) = 0$$

$a^2 - 4 = 0$ or $a^2 + 1 = 0$

$\therefore\ a = \pm 2$ are the only solutions, since a must be real (not $\pm i$)

Substituting into (2) we get $a = 2, b = 1$ and $a = -2, b = -1$ which gives the square roots as $\pm(2 + i)$.

2 Half of 4 is 2. Find pairs of numbers whose product is 2: ± 1 and ± 2 is the only pair.

Since the real part is positive then $a^2 - b^2$ is positive, so we place the number with the largest absolute value first (otherwise the difference would be negative).

In our heads: $2^2 - 1^2 3$. Thus 2 and 1 are the values for a and b, so $2 + i$ is a solution.

Therefore $-2 - i$ is also a solution.

The square roots of $3 + 4i$ are $\pm(2 + i)$.

3 Taking the coefficient of i and halving it we get -6, so we need to find a pair of numbers that have a product of -6. We have either ± 2 and ∓ 3 or ± 1 and ∓ 6.

Now we check which pair has a difference of their squares being $+5$. $3^2 - 2^2 = 9 - 4 = 5$, so the square roots are $\pm(3 - 2i)$.

4 Taking the coefficient of i and halving it we get $-\sqrt{3}$, so we need to find a pair of numbers that have a product of $\sqrt{3}$.

Trying $\pm\sqrt{3}$ and ∓ 1 we get a difference of squares of 2, which is what we want, so the square roots of $2 - 2\sqrt{3}i$ are $\pm(\sqrt{3} - i)$

5

$$a = \pm\sqrt{\frac{\sqrt{3^2 + 4^2} + 3}{2}} = \pm 2, b = \pm(+)\sqrt{\frac{\sqrt{3^2 + 4^2} - 3}{2}} = \pm 1$$

$\therefore \pm(2 + i)$ are the solutions.

6 Taking the coefficient of i and halving it we get 1, so we are looking for a pair of numbers with a product of 1. We have ± 1 and ± 1.

Now the real part is zero, so the difference of the squares is zero, so 1 and 1 (and their negatives) are the pairs we want.

So the square roots of $2i$ are $\pm(1+i)$.

7 We have a problem here – the coefficient of i isn't even so we cannot use our shortcut yet. We will double it and put it over 2 so we can use our shortcut.

$$4-3i=\frac{8-6i}{2}$$

Taking the coefficient of i in the numerator and halving it we get 3, so we need to find a pair of numbers that have a product of 3. We have ± 1 and ∓ 3. $3^2-1^2=8$, so the square roots of $8-6i$ are $\pm(3-i)$. So we have square rooted the numerator, but we must also square root the denominator.

So the square roots of $4-3i$ are $\pm\frac{3-i}{\sqrt{2}}$.

8

$$z=\frac{-4\pm\sqrt{4^2-4(1)(4-i)}}{2(1)}$$

$$=\frac{-4\pm\sqrt{4i}}{2}$$

$$=\frac{-4\pm\left(\sqrt{2}+\sqrt{2}i\right)}{2}$$

$$=\frac{-4-\sqrt{2}-\sqrt{2}i}{2},\frac{-4+\sqrt{2}+\sqrt{2}i}{2}$$

9 **a**

$|-6+8i|=\sqrt{(-6)^2+8}=10, \mathrm{Re}(-6+8i)$

$=-6$

The square roots of $-6+8i$ are

$$\pm\left(\sqrt{\frac{10-6}{2}}+\sqrt{\frac{10+6}{2}}i\right)=\pm\left(2+2\sqrt{2}i\right)$$

b

$$z=\frac{\sqrt{2}i\pm\sqrt{\left(-\sqrt{2}i\right)^2-4(1)(1-2i)}}{2(1)}$$

$$=\frac{\sqrt{2}i\pm\sqrt{-2-4+8i}}{2}$$

$$=\frac{\sqrt{2}i\pm\sqrt{-6+8i}}{2}$$

$$=\frac{\sqrt{2}i\pm\left(2+2\sqrt{2}i\right)}{2}$$

$$=-2-\sqrt{2}i, 2+3\sqrt{2}i$$

2.6 CONJUGATE THEOREMS

In Lesson 6 we cover:

- Conjugate Proofs
    - Product of Conjugates
    - Reciprocal of z and its conjugate
    - Sum of Conjugates
    - Difference of Conjugates
    - Arithmetic of Conjugates
- Conjugate Root Theorem

CONJUGATE PROOFS

We will state the rules involving conjugates here, leaving their proofs for the exercises. When doing the proofs remember that we will generally find exponential form easiest if the proof involves multiplication or division, or Cartesian form if the proof involves addition or subtraction.

PRODUCT OF CONJUGATES

The product of a complex number and its conjugate is real and equals the square of the modulus.

$$\boxed{z\bar{z} = |z|^2}$$

Example 1

If $z = 3 - 4i$ find $z\bar{z}$.

Solution

$$\begin{aligned} z\bar{z} &= |z|^2 \\ &= a^2 + b^2 \\ &= 3^2 + (-4)^2 \\ &= 25 \end{aligned}$$

RECIPROCAL OF z AND ITS CONJUGATE

Conjugates of complex numbers and division of complex numbers are closely related. If the modulus of the complex number is 1, then the conjugate and the reciprocal are the same.

$$\text{If } |z| = 1 \text{ then } \frac{1}{z} = \bar{z}$$

Example 2

If $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ find $\frac{1}{z}$

Solution

$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\therefore \frac{1}{z} = \bar{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

SUM OF CONJUGATES

The sum of a complex number and its conjugate is real and equals twice the real part.

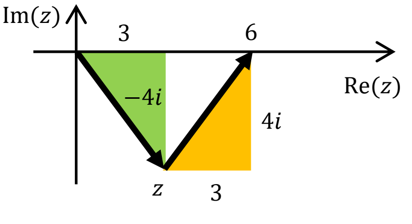
$$z + \bar{z} = 2\text{Re}(z)$$

Example 3

If $z = 3 - 4i$ find $z + \bar{z}$.

Solution

$$\begin{aligned} z + \bar{z} &= 2\text{Re}(z) \\ &= 2 \times 3 \\ &= 6 \end{aligned}$$



The imaginary parts cancel out to leave us with twice the real part.

DIFFERENCE OF CONJUGATES

The difference of a complex number and its conjugate is imaginary and equals twice the imaginary part.

$$\boxed{z - \bar{z} = 2\mathrm{Im}(z)}$$

Example 4

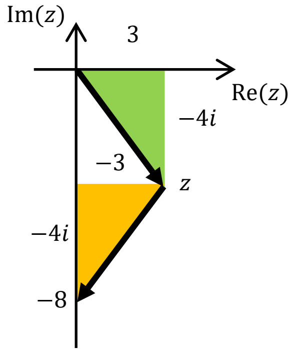
If $z = 3 - 4i$ find $z - \bar{z}$.

Solution

$z - \bar{z} = 2\mathrm{Im}(z)$

$= 2 \times (-4i)$

$= -8i$



The real parts cancel out to leave us with twice the imaginary part.

OPERATIONS WITH CONJUGATES

We will now look at the conjugate of a sum and the conjugate of a product, but these rules extend to differences and quotients, so we see that:

- the conjugate of a sum equals the sum of the conjugates $\overline{u+v} = \bar{u} + \bar{v}$
- the conjugate of a product equals the product of the conjugates $\overline{uv} = \bar{u} \times \bar{v}$
- the conjugate of a difference equals the difference of the conjugates $\overline{u-v} = \bar{u} - \bar{v}$
- the conjugate of a quotient equals the quotient of the conjugates $\overline{\left(\frac{u}{v}\right)} = \frac{\bar{u}}{\bar{v}}$

CONJUGATE OF A SUM

We can find the conjugate of the sum of multiple complex numbers by finding the conjugate of each one and adding them together (or vice versa).

$$\overline{z_1 + z_2 + \ldots + z_n} = \overline{z_1} + \overline{z_2} + \ldots + \overline{z_n}$$

CONJUGATE OF A PRODUCT

We can find the conjugate of the product of multiple complex numbers by finding the conjugate of each one and multiplying them together (or vice versa).

$$\overline{z_1 \times z_2 \times \ldots \times z_n} = \overline{z_1} \times \overline{z_2} \times \ldots \times \overline{z_n}$$

Example 5

If p and q are the roots of $2z^2 + (1+i)z + (10-3i) = 0$, find:

a $\overline{p} + \overline{q}$

b $\overline{p} \times \overline{q}$

Solution

a

$$\begin{aligned}\overline{p} + \overline{q} &= \overline{p+q} \\ &= \overline{\left(-\frac{b}{a}\right)} \\ &= \overline{-\left(\frac{1+i}{2}\right)} \\ &= -\frac{1-i}{2}\end{aligned}$$

b

$$\begin{aligned}\overline{p} \times \overline{q} &= \overline{p \times q} \\ &= \overline{\left(\frac{c}{a}\right)} \\ &= \overline{\left(\frac{10-3i}{2}\right)} \\ &= \frac{10+3i}{2}\end{aligned}$$

Conjugate Root Theorem: If z is a non-real root of a polynomial with real coefficients, then $\bar{z}$ is also a root.

Example 6

Prove that $2+i$ is a root of $z^3-3z^2+z+5=0$, and hence find the other roots.

Solution

$(2+i)^3-3(2+i)^2+(2+i)+5$

$=2^3+3(2)^2i+3(2)(i)^2+i^3-3(2^2+4i+i^2)+(2+i)+5$

$=8+12i-6-i-12-12i+3+2+i+5$

$=0$

$\therefore 2+i$ is a root.

$\therefore 2-i$ is a root.

$$\begin{aligned}\therefore 2+i+2-i+\gamma &= -\frac{b}{a}\\ &= -\frac{-3}{1}\\ &= 3\end{aligned}$$

$\therefore 4+\gamma=3$

$\therefore \gamma=-1$

$\therefore$ The roots of $z^3-3z^2+z+5=0$ are $-1, 2\pm i$.

EXERCISE 2.6

BASIC

1 If $z = 2 - 3i$ find $z\bar{z}$.

2 If $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ find $\frac{1}{z}$

3 If $z = 2 - 3i$ find $z + \bar{z}$

4 If $z = 3 - 4i$ find $z - \bar{z}$.

5 If p and q are the roots of $3z^2 + (2 - i)z + 6i = 0$, find:

a $\bar{p} + \bar{q}$

b $\bar{p} \times \bar{q}$

6 Prove that $1 + i$ is a root of $z^3 - 2z + 4 = 0$, and hence find the other roots.

MEDIUM

7 Prove $z\bar{z} = |z|^2$. Hint: let $z = re^{i\theta}$

8 Prove that if $|z| = 1$ then $\frac{1}{z} = \bar{z}$. Hint: let $z = e^{i\theta}$

9 Prove $z + \bar{z} = 2\mathrm{Re}(z)$. Hint: let $z = a + ib$

10 Prove $z - \bar{z} = 2\mathrm{Im}(z)$. Hint: let $z = a + ib$

11 Prove $\overline{z_1 + z_2 + \ldots + z_n} = \overline{z_1} + \overline{z_2} + \ldots + \overline{z_n}$. Hint: let $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$ etc

12 Prove $\overline{z_1 \times z_2 \times \ldots \times z_n} = \overline{z_1} \times \overline{z_2} \times \ldots \times \overline{z_n}$. Hint: let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ etc

13 The polynomial $P(z)$ has real coefficients, and $z = 2 + 3i$ is a root of $P(z)$. What quadratic must be a factor of $P(z)$?

14 The polynomial $P(z)$ has real coefficients, and the non-real numbers α and $-i\alpha$ are zeros of $P(z)$, where $\bar{\alpha} \neq i\alpha$. Explain why $\bar{\alpha}$ and $i\bar{\alpha}$ are also zeros of $P(z)$.

1

$$z\bar{z} = |z|^2$$
$$= a^2 + b^2$$
$$= 2^2 + (-3)^2$$
$$= 13$$

2

$$|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$\therefore \frac{1}{z} = \bar{z} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

3

$$z + \bar{z} = 2\mathrm{Re}(z)$$
$$= 2 \times 2$$
$$= 4$$

4

$$z - \bar{z} = 2\mathrm{Im}(z)$$
$$= 2 \times (-4i)$$
$$= -8i$$

5a

$$\bar{p} + \bar{q} = \overline{p + q}$$
$$= \overline{\left(-\frac{b}{a}\right)}$$
$$= -\overline{\left(\frac{2-i}{3}\right)}$$
$$= -\frac{2+i}{3}$$

b

$$\bar{p} \times \bar{q} = \overline{p \times q}$$
$$= \overline{\left(\frac{c}{a}\right)}$$
$$= \overline{\left(\frac{6i}{3}\right)}$$
$$= -2i$$

6

$$(1+i)^3 - 2(1+i) + 4$$
$$= 1^3 + 3(1)^2 i + 3(1)(i)^2 + i^3 - 2(1+i) + 4$$
$$= 1 + 3i - 3 - i - 2 - 2i + 4$$
$$= 0$$

$\therefore 1 + i$ is a root.

$\therefore 1 - i$ is a root.

$$\therefore 1 + i + 1 - i + \gamma = -\frac{b}{a}$$
$$= -\frac{0}{1}$$
$$= 0$$

$$\therefore 2 + \gamma = 0$$
$$\therefore \gamma = -2$$

$\therefore$ The roots of $z^3 - 2z + 4 = 0$ are $-2, 1 \pm i$.

7

Let $z = re^{i\theta}$

$$\therefore \bar{z} = re^{-i\theta}$$
$$z\bar{z} = re^{i\theta} \times re^{-i\theta}$$
$$= r^2 e^{i\theta - i\theta}$$
$$= r^2$$
$$= |z|^2$$

8

Let $z = e^{i\theta}$

$$\frac{1}{z} = \frac{1}{e^{i\theta}}$$
$$= e^{-i\theta}$$
$$= \bar{z} \qquad \square$$

9

$$z + \bar{z} = a + ib + a - ib$$
$$= 2a$$
$$= 2\mathrm{Re}(z) \quad \square$$

10

$$z - \bar{z} = a + ib - (a - ib)$$
$$= 2bi$$
$$= 2\mathrm{Im}(z) \quad \square$$

11

Let $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$ etc

LHS = $\overline{a_1 + ib_1 + a_2 + ib_2 + \ldots + a_n + ib_n}$

$$= \overline{(a_1 + a_2 + \ldots a_n) + i(b_1 + b_2 + \ldots + b_n)}$$
$$= (a_1 + a_2 + \ldots a_n) - i(b_1 + b_2 + \ldots + b_n)$$
$$= a_1 - ib_1 + a_2 - ib_2 + \ldots + a_n - ib_n$$
$$= \overline{z_1} + \overline{z_2} + \ldots + \overline{z_n}$$
$$= RHS \quad \square$$

12

Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ etc

LHS = $\overline{r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} \times \ldots \times r_n e^{i\theta_n}}$

$$= \overline{(r_1 r_2 \ldots r_n) e^{i\theta_1 + i\theta_2 + \ldots + i\theta_n}}$$
$$= (r_1 r_2 \ldots r_n) e^{-(i\theta_1 + i\theta_2 + \ldots + i\theta_n)}$$
$$= r_1 e^{-i\theta_1} \times r_2 e^{-i\theta_2} \times \ldots \times r_n e^{-i\theta_n}$$
$$= \overline{r_1 e^{i\theta_1}} \times \overline{r_2 e^{i\theta_2}} \times \ldots \times \overline{r_n e^{i\theta_n}}$$
$$= \overline{z_1} \times \overline{z_2} \times \ldots \times \overline{z_n}$$
$$= RHS \quad \square$$

13

$2 - 3i$ must also be a root

∴ the following quadratic must be a factor.

$$\big(z - (2 + 3i)\big)\big(z - (2 - 3i)\big) = (z - 2 - 3i)(z - 2 + 3i)$$
$$= (z - 2)^2 - (3i)^2$$
$$= z^2 - 4z + 4 + 9$$
$$= z^2 - 4z + 13$$

14

Since the coefficients are real then the conjugate of non-real zeros are also zeros.

The conjugate of α is $\bar{\alpha}$.

The conjugate of $-i\alpha$ is $\overline{-i\alpha} = \overline{-i} \times \bar{\alpha} = i\bar{\alpha}$

∴ $\bar{\alpha}$ and $i\bar{\alpha}$ are roots of $P(z)$ $\square$

2.7 COMPLEX NUMBERS AS VECTORS

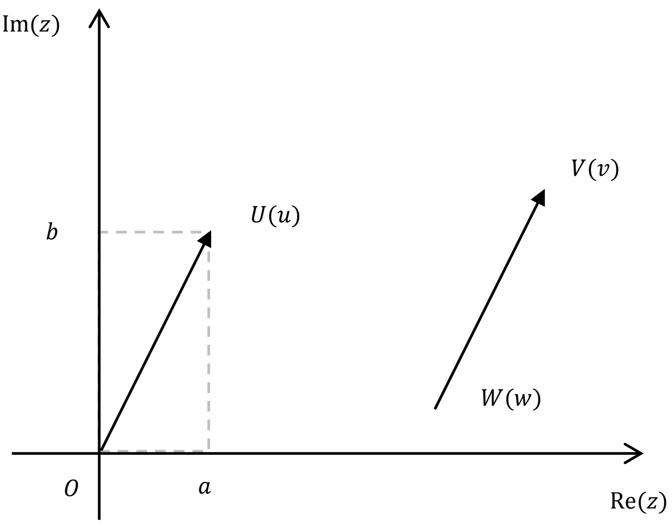
In Lesson 7 we look at complex numbers as vectors on the Complex Plane. We cover:

- Complex Numbers as Vectors
- Translations
    - Triangle Law
    - Parallelogram Law
    - Subtracting Vectors
    - Polygon Law
- Rotation and Dilation
    - Parallel Vectors
    - Perpendicular Vectors
    - Rotating Vectors
- Midpoint
- Triangle Inequality

It is important to note that two dimensional vectors (which we will cover in Vectors) and complex vectors (in Complex Numbers) behave the same with addition and subtraction but quite differently for multiplication. We can also divide with Complex vectors, as well as raising them to powers or finding their roots, which we cannot do with two dimensional vectors.

In Vectors we are only interested in the direction of a two dimensional vector as a concept, whereas in Complex Numbers we are interested in the argument of complex vectors, so will solve proofs involving rotation.

We can look at a complex number as a vector. For example, $\overrightarrow{OU}$ represents the complex number $u = a + ib$, which can be viewed as a vector starting at zero which travels a units to the right and goes up by b units. Its tail is at zero and its tip is at U.



Since $\overrightarrow{OU}$ and $\overrightarrow{WV}$ have the same magnitude and direction they are equal vectors, so $\overrightarrow{WV}$ also equals $a + ib$.

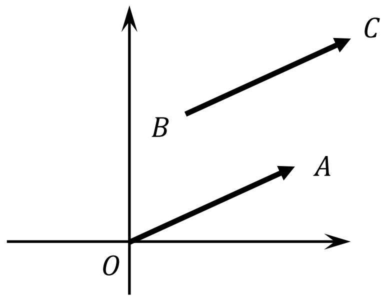
TIP MINUS TAIL

Any complex vector equals the complex number at its tip minus the complex number at its tail. When we label the vector we put the pronumeral at the tail first followed by the pronumeral at the tip.

So $\overrightarrow{WV} = v - w$.

Similarly $\overrightarrow{OU} = u - 0 = u$.

Consider the vectors $\overrightarrow{OA}$ and $\overrightarrow{BC}$ at right.

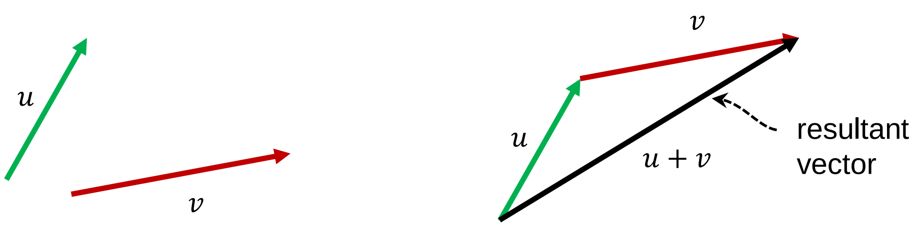


$\overrightarrow{OA}$ is a position vector as it has its tail at the origin and its tip at the position of A.

$\overrightarrow{BC}$ is a displacement (or relative) vector as it gives us the displacement of C from B.

TRIANGLE LAW - ADDING VECTORS

When we add two vectors the resultant vector can be found as the third side of the triangle formed by moving the tail of one vector to the tip of the other. The resultant vector goes from the tail of the first vector to the tip of the second as shown below.

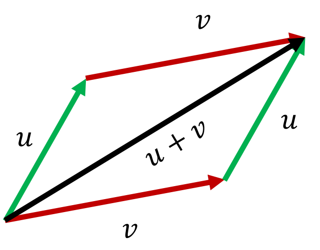


We can add the vectors in any order and achieve the same result as shown middle and right, as vectors follow the commutative law

$$u + v = v + u$$

PARALLELOGRAM LAW - ADDING VECTORS

The Triangle Law for addition leads to a few other important results, the first being the Parallelogram Law. We start by arranging the two triangles shown above into the parallelogram as shown below. The diagonal of a parallelogram (from tail to tip) with sides u and v is the sum of the vectors.

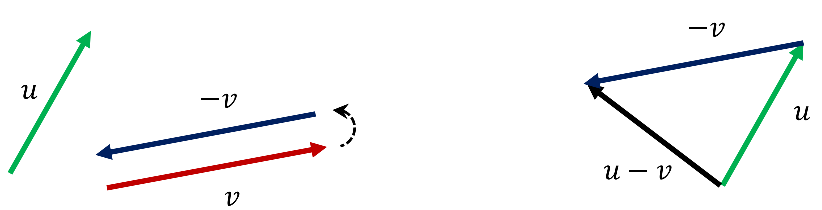


SUBTRACTING VECTORS

There are two ways to subtract vectors, each of which are useful in different circumstances.

USING THE NEGATIVE OF A VECTOR AND THE TRIANGLE LAW

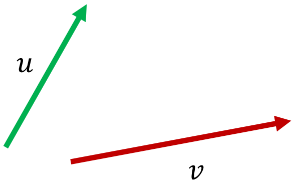
We can view $u - v$ as $u + (-v)$. We first find $-v$ by reversing v (below left), then move the tail of this new vector to the tip of u (right). The vector from tail to tip is $u - v$.

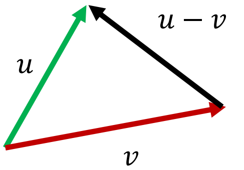


If we wanted $v - u$ then we find $-u$ and move its tail to the end of v as below. Note that $u - v$ and $v - u$ are negative vectors (same modulus with opposite direction).

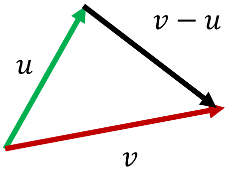
THE VECTOR BETWEEN THE TIPS

If the tails of the two vectors are at the same point (move one or both of them if not) then we can find $u - v$ as the vector from the tip of v to the tip of u. Notice this is tip minus tail.

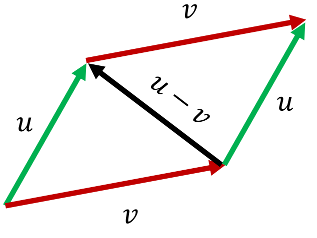


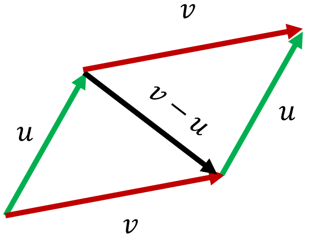


Similarly we can find $v - u$ as the vector from the tip of u to the tip of v.



Extending the two triangles above into parallelograms, we can see that the diagonal running across the parallelogram gives the difference of the two vectors. Depending on which way we draw the diagonal we get either $u - v$ or $v - u$.

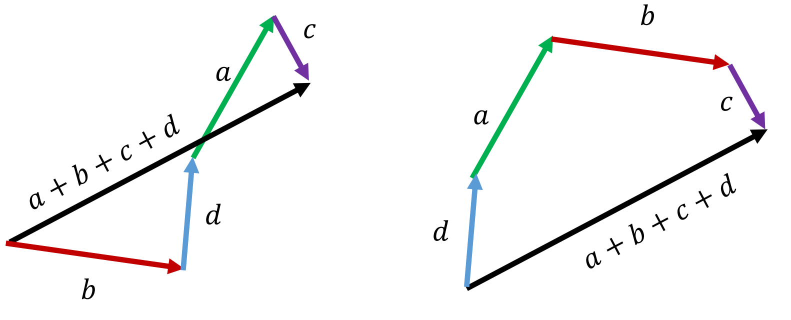




POLYGON LAW

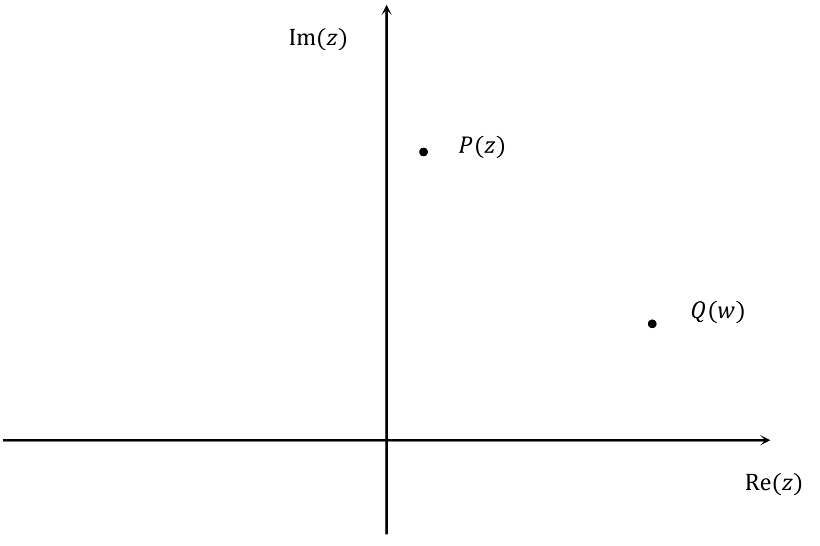
The Triangle Law for adding two vectors can also be extended to any number of vectors using the Polygon Law.

The vectors can be added together in any order and achieve the same result. The diagrams below show two more of the $4!$ ways of adding the four vectors - all give the same resultant vector.

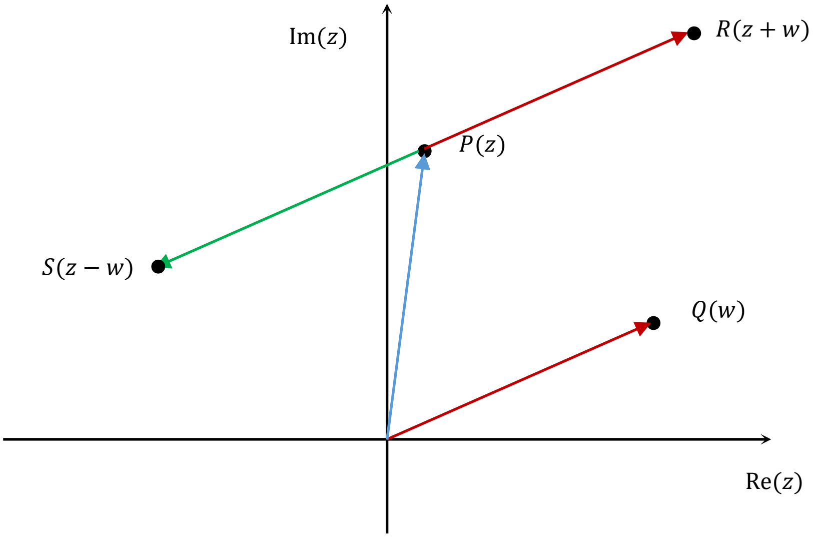


Example 1

The points P and Q on the complex plane represent the complex numbers z and w respectively. Mark the points R representing $z + w$, and S representing $z - w$.



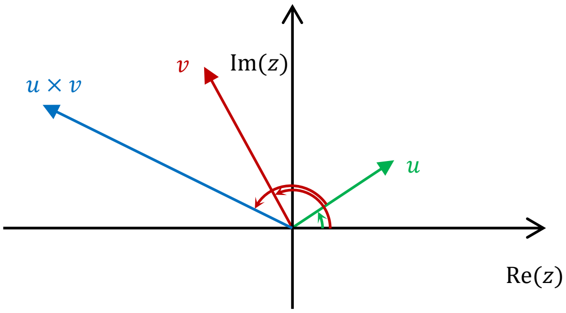
Solution



ROTATION AND DILATION

When a complex number is multiplied by another complex number, then starting with the original we:

- dilate (scale) the modulus by the modulus of the second number
- rotate the vector by the argument of the second number



The work on complex vectors is far more likely to involve multiplication than division, however division is the inverse operation of multiplication, so the effects occur in the opposite directions:

- rotate by the opposite of the argument of the second complex number
- divide by the modulus of the second complex number.

Let's have a look specifically at multiplying by a real number, an imaginary number and a complex number.

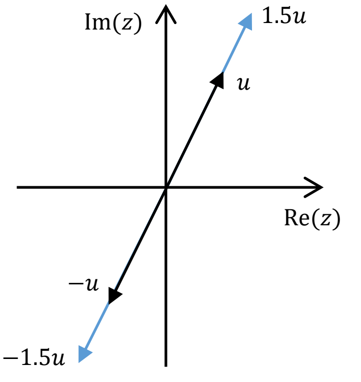
When we multiply a vector by a constant real number k:

- The modulus changes by a factor of the real number - the vector stretches or compresses (dilates)
- The direction stays the same if $k > 0$, or becomes the opposite direction if $k < 0$, so the new vector is parallel to the original.

This matches what we have just said about multiplying by a complex number, as a real number has an argument of 0 if it is positive (so no rotation) or π if it is negative (so reverses direction).

Multiplying a complex number by a positive constant leaves the argument the same (no rotation) and dilates the modulus.

Multiplying a complex number by -1 rotates the vector $180°$ and leaves the modulus the same (the vector does not change length, just rotates).

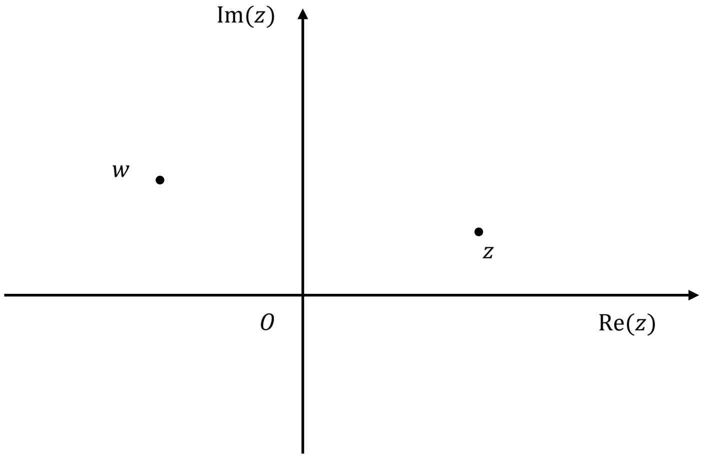


Multiplying a complex number by any other negative constant rotates the vector $180°$ and scales the modulus – the point lies on the line through the original point and zero but on the other side of zero.

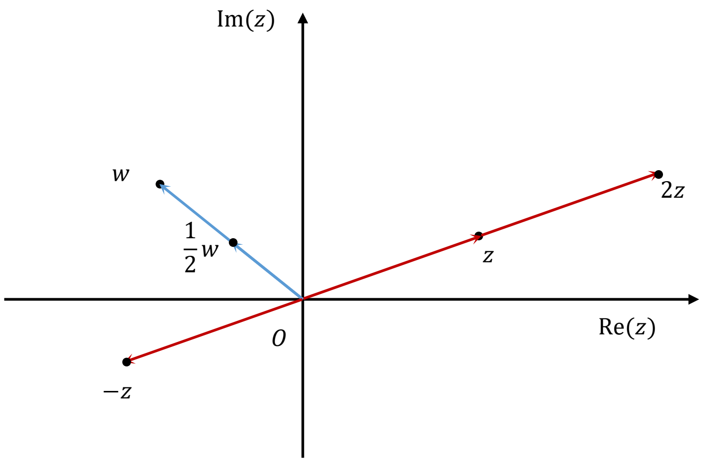
Example 2

The complex plane shows the complex numbers w and z. Mark possible positions for

a $2z$ **b** $\frac{1}{2}w$ **c** $-z$



Solution



ROTATING BY $\frac{\pi}{2}$ - PERPENDICULAR VECTORS

We have seen that an essential property of i is that when we multiply by i we rotate anticlockwise by $\frac{\pi}{2}$.

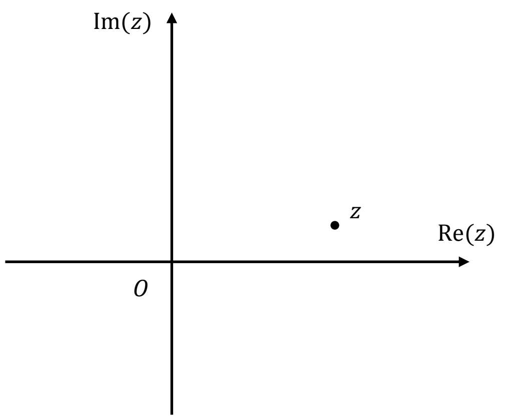
This matches what we have just seen about multiplying by a complex number, as i has an argument of $\frac{\pi}{2}$ and a modulus of 1, so we rotate by $\frac{\pi}{2}$ and scale the modulus by 1 (leave it alone).

Example 3

The complex plane shows the complex number z.

Mark possible positions for

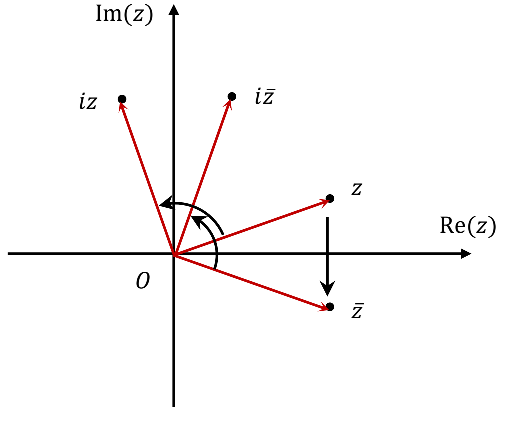
a iz **b** $i\bar{z}$



Solution

a Rotate z anticlockwise $\frac{\pi}{2}$

b Flip z over the x-axis to find the conjugate $\bar{z}$, then rotate $\bar{z}$ anticlockwise $\frac{\pi}{2}$

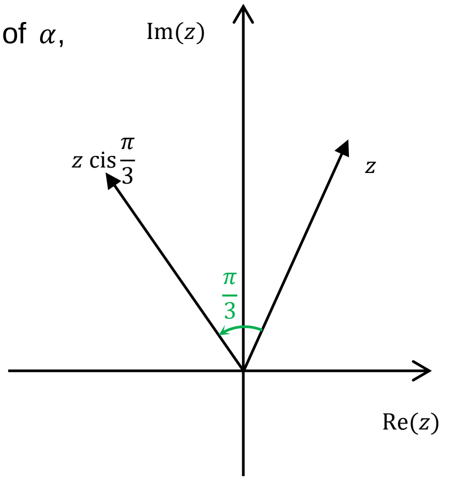


ROTATING BY α

We rotate a vector by multiplying by a second vector which has as argument equal to the angle we wish to rotate by.

If we want to rotate a vector without changing its length then we want the modulus of the multiplier to be 1 and the argument to be the angle we want the vector to rotate.

If the angle of rotation is to be α anticlockwise, we multiply by any complex number having a modulus of 1 and an argument of α, such as $e^{i\alpha}$ or $\cos\alpha + i\sin\alpha$.



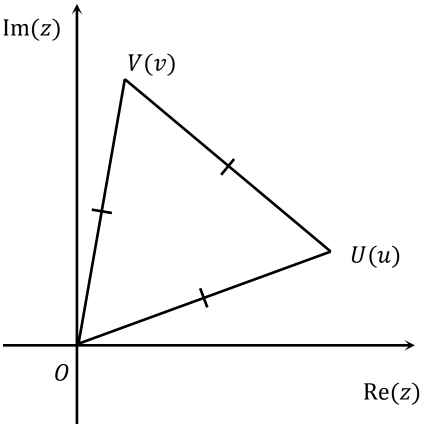
So to rotate by $\frac{\pi}{3}$ radians we would multiply by any of:

- $e^{\frac{\pi}{3}i}$
- $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$
- $\frac{1}{2} + \frac{\sqrt{3}}{2}i.$

Example 4

The complex numbers $0, u$ and v form the vertices of an equilateral triangle in the complex plane. Show that $u^2 + v^2 = uv$

Solution



$\arg(u) - \arg(v) = \frac{\pi}{3}$

Let $v = u\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

$u^2 + v^2 = u^2 + u^2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)^2$

$= u^2\left(1 + \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

$= u^2\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)$

$uv = u^2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

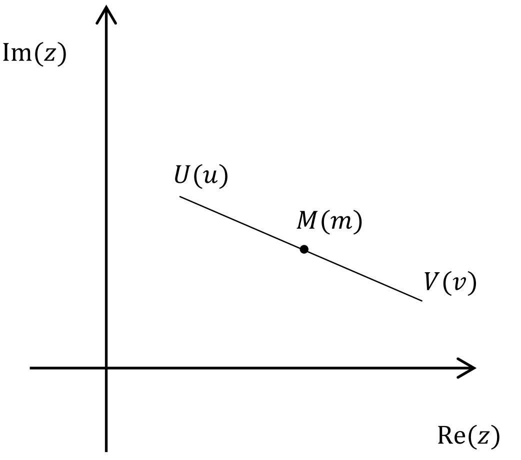
$= u^2\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)$

$\therefore u^2 + v^2 = uv$

MIDPOINT

The midpoint between two points can be found in three different ways depending on the form of the complex numbers.

Let the numbers be $u = x_1 + iy_1$ and $v = x_2 + iy_2$.



We can use the midpoint formula from the Cartesian plane:

$$\boxed{m = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2}i}$$

Given that the Complex Plane is an extension of the number line we can also find the average of the two numbers, as the midpoint is the average:

$$\boxed{m = \frac{u + v}{2}}$$

We can also find the midpoint using coinciding lines if we can show that from a common tail one vector is half the other.

$$\boxed{m - u = \frac{1}{2}(v - u)}$$

Example 5

Find the midpoint M of $\mathrm{u} = 1 + 2i$ and $v = 3 - 4i$

Solution

M is $\frac{1+3}{2} + \frac{2-4}{2}i = 2 - 2i$

TRIANGLE INEQUALITY

The Triangle Inequality is based on the sum of any two sides of a triangle being at least as big as the third side. If the sum of the two sides equals the third side then the three sides form one straight line, rather than a triangle as we know it.

So for any two complex numbers we can say that:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Example 6

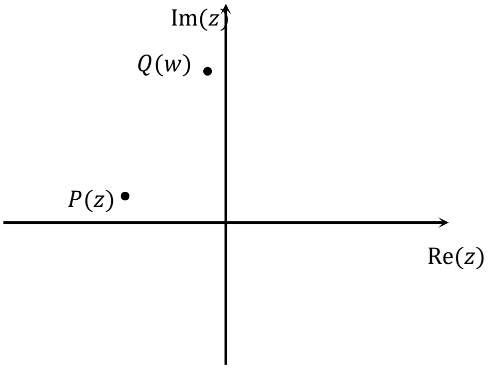
Given $|z| < \frac{1}{2}$, show that $|(1+i)z^3 + iz| < \frac{3}{4}$

Solution

$$\begin{aligned}
|(1+i)z^3 + iz| &\leq |(1+i)z^3| + |iz| \text{ (by the triangle inequality)} \\
&\leq |1+i||z|^3 + |z| \\
&< \sqrt{2}\left(\frac{1}{2}\right)^3 + \frac{1}{2} \\
&< \frac{\sqrt{2}+4}{8} \\
&< \frac{6}{8} \\
&< \frac{3}{4}
\end{aligned}$$

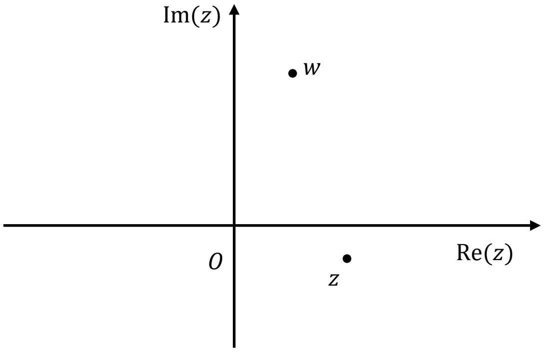
BASIC

1 The points P and Q on the complex plane represent the complex numbers z and w respectively. Mark the points R representing $z+w$, and S representing $z-w$.



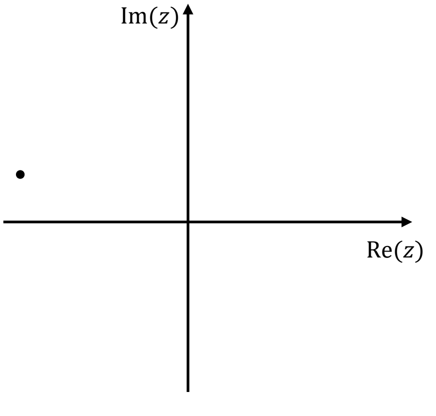
2 The complex plane shows the complex numbers w and z. Mark possible positions for

a $2z$ **b** $\frac{1}{2}w$ **c** $-z$



3 The complex plane shows the complex number z. Mark possible positions for

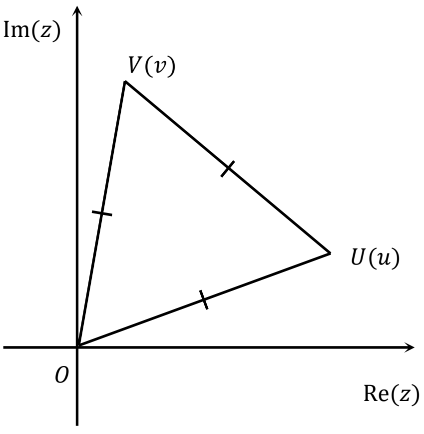
a iz **b** $i\bar{z}$



4 Find the midpoint M of $\mathrm{u} = -1 + 3i$ and $v = 5 - i$

5 The complex numbers $0, u$ and v form the vertices of an equilateral triangle in the complex plane.

It is given that $u = e^{\frac{\pi}{6}i}$. Show that $u^2 + v^2 = uv$

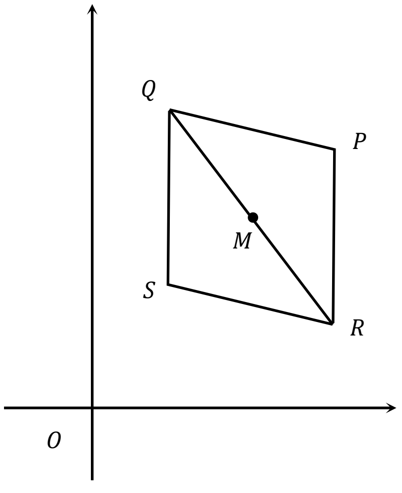


6 The point P on the Argand diagram represents the complex number z. The points Q and R represent the points ωz and $\overline{\omega} z$ respectively, where $\omega = \text{cis}\frac{\pi}{3}$.

The point M is the midpoint of QR.

a Find the complex number representing M in terms of z.

b The point S is chosen so that $PSQR$ is a parallelogram. Find the complex number represented by S.

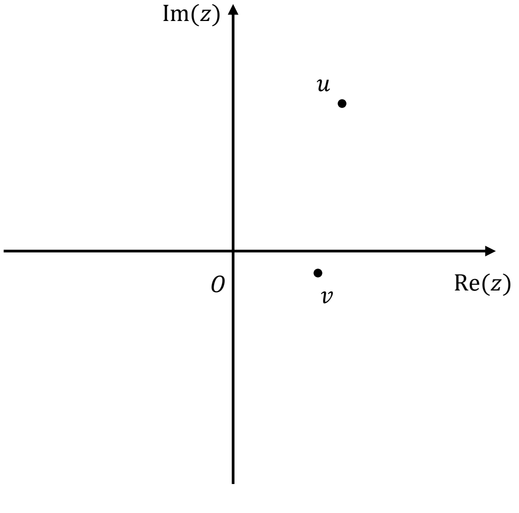


7 Given $|z| < 1$, show that $\left|(1 - i)z^2 + \sqrt{2}iz\right| < 3$

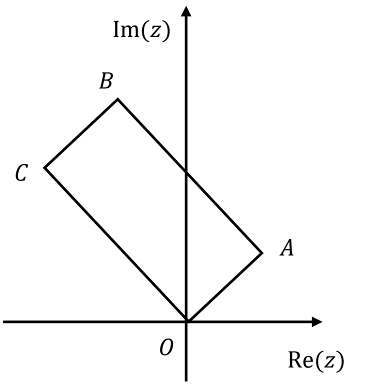
MEDIUM

8 The complex numbers u and v are indicated on the complex plane.

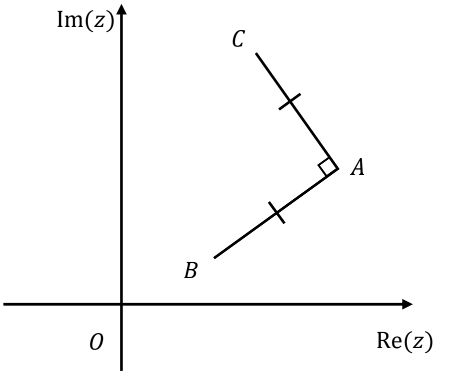
Given $u + v + w = 0$, sketch a possible location for w.



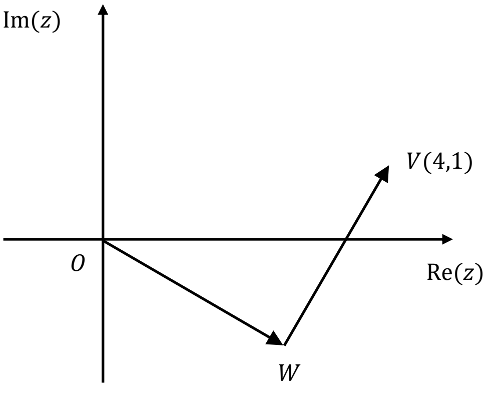
9 $OABC$ is a rectangle in a complex plane where O is the origin and point A corresponds to the complex number $1+i$. C lies in the second quadrant as shown. Given the length of the rectangle is three times its breadth and OA is one of the shorter sides, find the complex number represented by C.



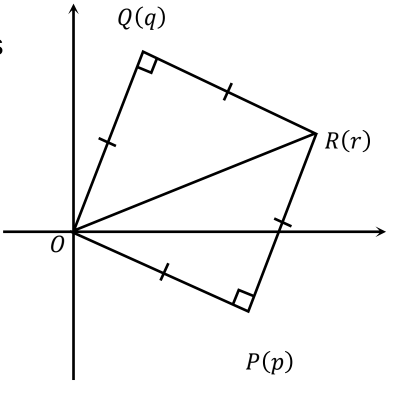
10 The point A represents the complex number a and the point B represents the complex number b. The point B is rotated clockwise about A through a right angle to take up the position C, representing the complex number c. Show that $c=(1+i)a-ib$.



11 The point V represents the complex number $4+i$. $\angle OWV=\frac{\pi}{2}$ and $|VW|=|OW|$. Find the complex number represented by the point W.

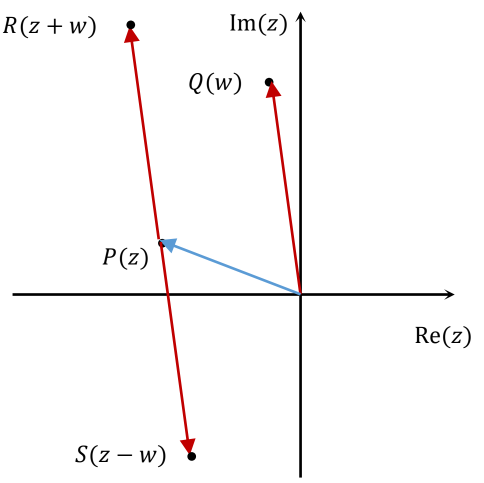


12 The points P, Q and R on the complex plane represent the complex numbers p, q and r respectively. The triangles POR and OQR are right angled isosceles with right angles at P and Q as shown. Show that $pq=\frac{r^2}{2}$



13 Prove for two complex numbers z_1, z_2 that $|z_1+z_2| \leq |z_1|+|z_2|$

1



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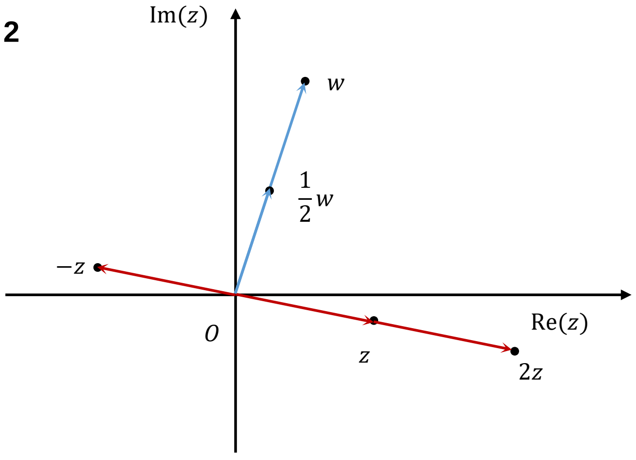

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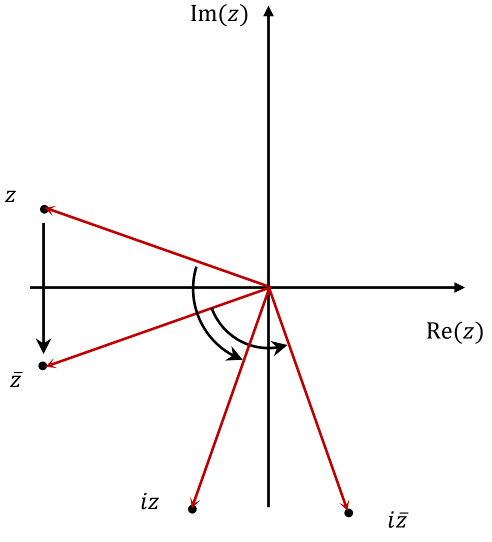


4

M is $\frac{-1+5}{2}+\frac{3-1}{2}i = 2+i$

5

$\arg(u)-\arg(v) = \frac{\pi}{3}$

Let $v = u\,e^{\frac{\pi}{3}i} = e^{\frac{\pi}{6}i} \times e^{\frac{\pi}{3}i} = e^{\frac{\pi}{2}i} = i$

$$u^2+v^2 = \left(e^{\frac{\pi}{6}i}\right)^2 + i^2$$
$$= e^{\frac{\pi}{3}i} - 1$$
$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - 1$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i - 1$$
$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$uv = e^{\frac{\pi}{6}i} \times i$$
$$= \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)i$$
$$= -\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}$$
$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$\therefore u^2+v^2 = uv$$

6

a

$$M = \frac{1}{2}(\omega z + \bar{\omega} z)$$
$$= \frac{z}{2}(\omega + \bar{\omega})$$
$$= \frac{z}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} + \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$
$$= \frac{z}{2}\left(2\cos\frac{\pi}{3}\right)$$
$$= \left(\cos\frac{\pi}{3}\right)z$$
$$= \frac{z}{2}$$

b

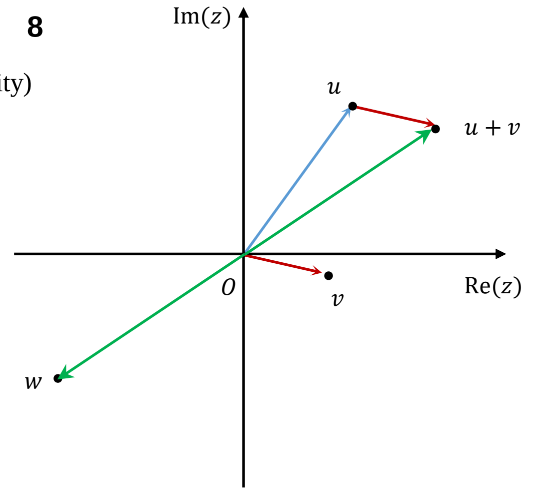
M is also the midpoint of SP

$$\frac{s+p}{2} = \frac{z}{2}$$
$$s+z = z$$
$$s = 0$$

7

$$\left|(1-i)z^2+\sqrt{2}iz\right| \le \left|(1-i)z^2\right| + \left|\sqrt{2}iz\right| \text{ (by the triangle inequality)}$$
$$\le |1-i||z|^2+\sqrt{2}|z|$$
$$\le \sqrt{2}(1)^2+\sqrt{2}$$
$$\le 2\sqrt{2}$$
$$< 3$$

8



9

$$\overrightarrow{OC} = 3i \times \overrightarrow{OA}$$
$$= 3i(1+i)$$
$$= -3+3i$$

10

$$\overrightarrow{AC} = -i\overrightarrow{AB}$$
$$c-a = -i(b-a)$$
$$c = -ib+ia+a$$
$$c = (1+i)a-ib$$

11

Let W represent $a+ib$

$$-i\overrightarrow{WV} = \overrightarrow{OW}$$
$$-i(4+i-a-ib) = a+ib$$
$$-4i+1+ai-b = a+bi$$
$$-b+1+i(a-4) = a+bi$$
$$\therefore -b+1 = a \qquad b = 1-a$$
$$a-4 = b$$
$$a-4 = 1-a$$
$$2a = 5$$
$$a = \frac{5}{2}$$
$$b = 1-\frac{5}{2} = -\frac{3}{2}$$

W represents $\frac{5-3i}{2}$

12

$$\angle POR = \angle ROQ = \frac{\pi}{4}, |r| = \sqrt{2}|p| = \sqrt{2}|q|$$

$\therefore q = \frac{1}{\sqrt{2}}\operatorname{cis}\frac{\pi}{4} \times r$ and $p = \frac{1}{\sqrt{2}}\operatorname{cis}\left(-\frac{\pi}{4}\right) \times r$

$$pq = \frac{r}{\sqrt{2}}\operatorname{cis}\frac{\pi}{4} \times \frac{r}{\sqrt{2}}\operatorname{cis}\left(-\frac{\pi}{4}\right)$$
$$= \frac{r^2}{2}\operatorname{cis}\left(\frac{\pi}{4}-\frac{\pi}{4}\right)$$
$$= \frac{r^2}{2}\operatorname{cis} 0$$
$$= \frac{r^2}{2}$$

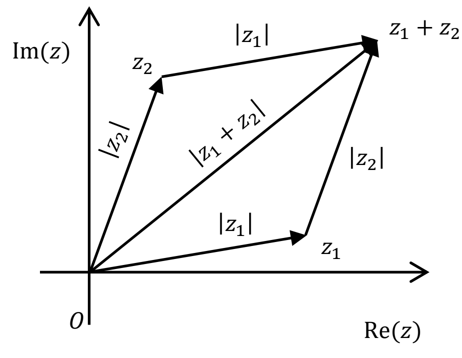
Alternatively (thanks to Chris B):

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$$
$$= \overrightarrow{OP} + \overrightarrow{OQ}$$
$(\overrightarrow{PR} = \overrightarrow{OQ}$ opposite sides of a square)
$$\therefore r = p+q$$
$$r^2 = (p+q)^2$$
$$= p^2+2pq+q^2$$
$$= p^2+2pq+(ip)^2 \qquad \text{since } q = ip$$
$$= p^2+2pq-p^2$$
$$= 2pq$$
$$\therefore pq = \frac{r^2}{2}$$

13

In the diagram at right we can see that O, z_1, z_2 and z_1+z_2 form a parallelogram.

The diagonal from zero to z_1+z_2 splits the parallelogram into two congruent triangles with lengths $|z_1|, |z_2|$ and $|z_1+z_2|$.



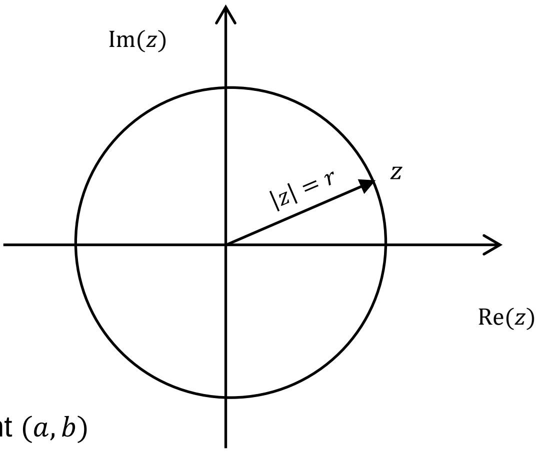
$\therefore |z_1+z_2| \le |z_1|+|z_2|$ (any side of a triangle is less than the sum of the other two sides). □

2.8 CURVES AND REGIONS

In Lesson 8 we look at locus on the Argand Diagram. We cover:

- Circles
- Perpendicular Bisector
- Rays and Sectors
- Horizontal and Vertical Lines
- Algebraic Method
- Arcs

CIRCLES



The length of a vector from the centre of a circle to any point on the circumference has a modulus equalling the radius.

$|z| = r$ is a circle of radius r centred at zero

$|z - (a + ib)| = r$ is a circle of radius r centred at the point (a, b)

Example 1

Illustrate the following on a Complex Plane:

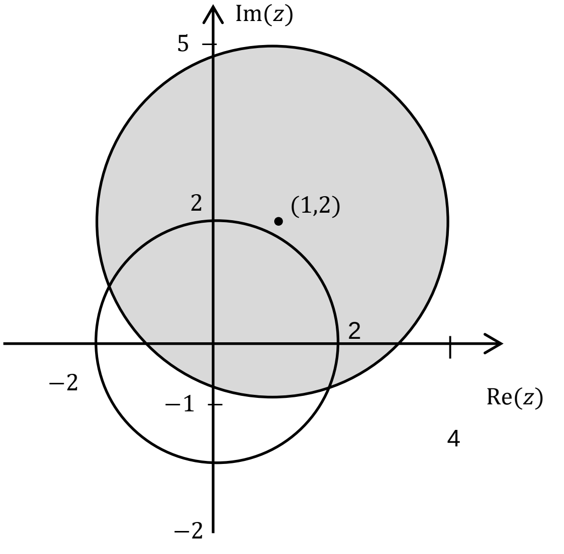
a $|z| = 2$

b $|z - 1 - 2i| \leq 3$

Solution

a This is a circle centred at zero with radius 2

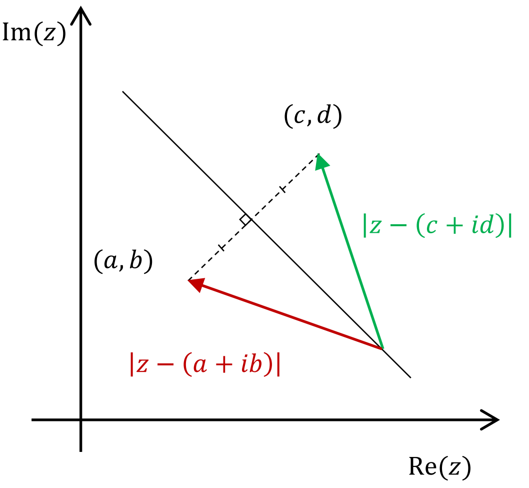
b This is the inside and circumference of a circle centred at $(1,2)$ with radius 3 units.



PERPENDICULAR BISECTOR

The length of the vectors from any point on a perpendicular bisector to the two points have the same modulus, so the perpendicular bisector of the points (a, b) and (c, d) is

$|z-(a+ib)| = |z-(c+id)|$

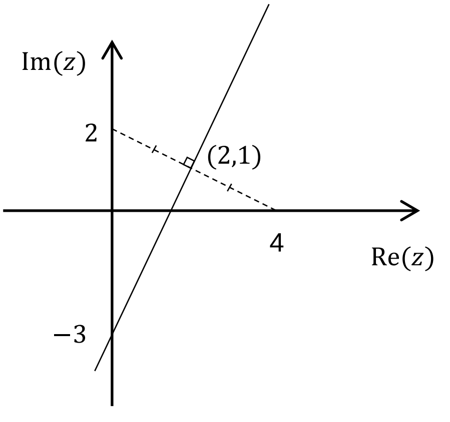


Example 2

Sketch $|z-4| = |z-2i|$

Solution

This is the perpendicular bisector of the interval joining $(4,0)$ and $(0,2)$.



RAYS AND SECTORS

A ray is a line that starts from a point and continues in one direction. The argument of any point on the ray equals the angle the ray makes with the Real axis.

A ray starting at zero at an angle of θ is $\arg(z) = \theta$.

A ray starting at (a, b) at an angle of θ is $\arg(z - (a + ib)) = \theta$.

Note that there should be an open circle on the starting point itself, as there can be no angle formed at the point itself, so the argument is undefined.

A sector with centre (a, b) and arms at angles to the x-axis of α and β is $\alpha \leq \arg\big(z - (a + ib)\big) \leq \beta$.

Again there must be an open circle at the centre, as the angle is undefined at that point.

Example 3

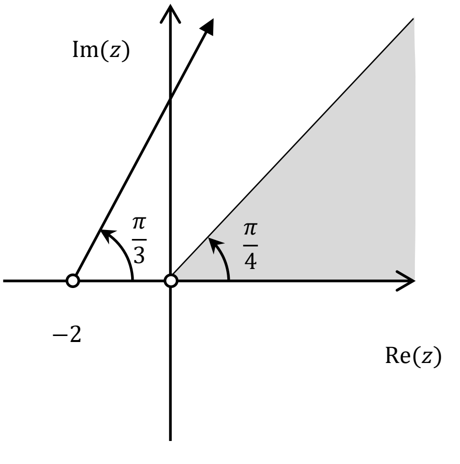
Sketch the following on a Complex Plane:

a $\arg(z + 2) = \frac{\pi}{3}$ **b** $0 \leq \arg(z) \leq \frac{\pi}{4}$

Solution

a A ray from the point $(-2,0)$ at an angle of $\frac{\pi}{3}$.

b A sector centred at zero, with rays at 0 radians and $\frac{\pi}{4}$ radians.



HORIZONTAL AND VERTICAL LINES

$\text{Re}(z) = a$ will give a vertical line at $x = a$

$\text{Im}(z) = a$ will give a horizontal line at $y = a$

Example 4

Sketch the following on a Complex Plane:

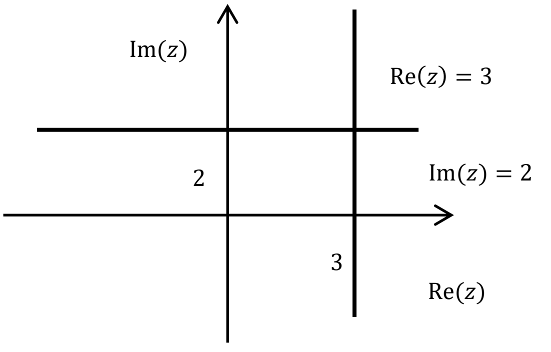
a $\text{Re}(z) = 3$

b $\text{Im}(z) = 2$

Solution

a A vertical line at $\text{Re}(z) = 3$

b A horizontal line at $\text{Im}(z) = 2$



ALGEBRAIC METHOD

Many other curves will need to be simplified algebraically until we can recognise their details. With practice most will become obvious.

Example 5

Find the curve satisfying $|z-1| = 2|z+1|$.

Solution

$$\sqrt{(x-1)^2+y^2} = 2\sqrt{(x+1)^2+y^2}$$

$$x^2-2x+1+y^2 = 4(x^2+2x+1+y^2)$$

$$x^2-2x+1+y^2 = 4x^2+8x+4+4y^2$$

$$3x^2+10x+3y^2+3=0$$

$$x^2+\frac{10}{3}x+y^2=-1$$

$$x^2+\frac{10}{3}x+\frac{25}{9}+y^2=-1+\frac{25}{9}$$

$$\left(x+\frac{5}{3}\right)^2+y^2=\frac{16}{9}$$

Which is a circle with centre $\left(-\frac{5}{3},0\right)$ with radius $\frac{4}{3}$

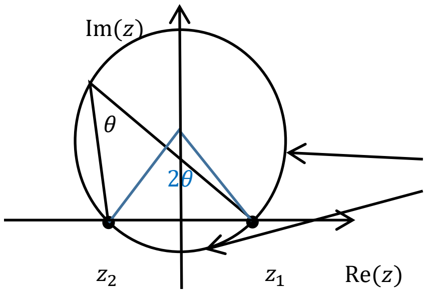
ARCS

A common textbook question from the old syllabus involved arcs, although it was not used in the HSC. Although Circle Geometry has now been dropped from the Stage 6 syllabus it is still assumed knowledge from Stage 5, so it could theoretically appear.

We have seen that when a vector is divided by another vector we subtract the arguments. This means that the following types of equation are interchangeable:

$$\arg(z - z_1) - \arg(z - z_2) = \theta \quad \text{or} \quad \arg\left(\frac{z-z_1}{z-z_2}\right) = \theta$$

These equations describe a major or minor arc of a circle if $0 < \theta < \pi$, where the two points z_1 and z_2 are on the circumference of a circle, and where the angle within the segment (measured at the circumference) is θ.



You will find either the minor or major arc, not the whole circumference!

In drawing the arc it may be of assistance to remember from Circle Geometry that the angle at the centre must be twice as large as the angle at the circumference, and that it is the apex angle of the isosceles triangle whose base vertices are z_1 and z_2.

Some other drawing hints: imagine standing at the second listed point (z_2) and facing the first point (z_1).

- If θ is positive then you want the arc on the left
- If θ is negative you want the arc on the right
- If θ is $\pm$ then you want a mirror image of the arc (so not a whole circle) either side of the chord.

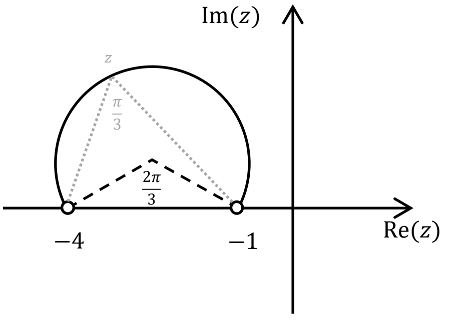
If θ is $\frac{\pi}{2}$ or less the centre of the circle will be on the same side as the arc you want and you will have a major arc. Once you have found the centre you can draw an arc from one point to the other and you have the curve.

Example 6

Sketch $\arg(z+1) - \arg(z+4) = \frac{\pi}{3}$

Solution

The angle at z (formed by the vectors from $(-1,0)$ and $(-4,0)$ as shown) is equal to $\frac{\pi}{3}$, so the locus is an arc with ends at $(-1,0)$ and $(-4,0)$ exclusive.



Standing at the second point $(-4,0)$ and facing the first point $(-1,0)$ we want the arc on the left (since $\frac{\pi}{3}$ is positive).

To find the centre of the arc, we double $\frac{\pi}{3}$ (angle at the circumference) to get $\frac{2\pi}{3}$ (angle at the centre), which is the apex angle of our isosceles triangle and gives the centre of the arc. Note: Circle Geometry rule.

We have seen that if $0 < \theta < \pi$ that $\arg\left(\frac{z-z_1}{z-z_2}\right) = \theta$ gives an arc, but if $\theta = 0, \pi$ then we get part of a straight line:

- If $\theta = 0$ then the vectors to z from z_1 and from z_2 have the same argument. This only occurs when z is on the line passing through the two points, but **outside** the two points.
- If $\theta = \pi$ then the vectors to z from z_1 and z_2 point in opposite directions. This only occurs on the line passing through the two points **between** the points.

EXERCISE 2.8

BASIC

1 Illustrate the following on a Complex Plane:

a $|z| = 3$ **b** $|z + 1 - i| \leq 1$

2 Sketch $|z + 1 - i| = |z - 1 + i|$

3 Sketch the following on a Complex Plane:

a $\arg(z) = -\frac{\pi}{4}$ **b** $0 \leq \arg(z - 1) \leq \frac{\pi}{3}$

4 Sketch the following on a Complex Plane:

a $\mathrm{Re}(z) = -2$ **b** $\mathrm{Im}(z) = 1$

MEDIUM

5 Find the locus of the point z satisfying $|z + 1| = 3|z|$.

6 Sketch $\arg(z + 1) - \arg(z - 1) = \frac{\pi}{4}$

7 Sketch $\mathrm{Im}(z^2) = 1$

8 Sketch the region on the Argand diagram where the inequalities $|z - \bar{z}| < 1$ and $|z - i| \leq 2$ hold simultaneously.

9 The point P on the Argand diagram represents the complex number z, where z satisfies $\frac{1}{z} - \frac{1}{\bar{z}} = 1$. Give a geometrical description of the locus of P as z varies.

10 Sketch

$$\arg\left(\frac{z + 1 + i}{z - 1}\right) = \pm\frac{\pi}{4}$$

CHALLENGING

11 Sketch the following on a Complex Plane:

$$\arg\left(\frac{z - 1 - i}{z + 1 - i}\right) = 0$$

12 Sketch the following on a Complex Plane:

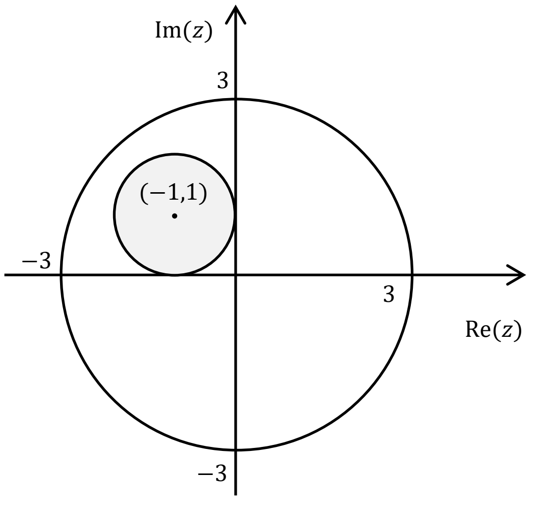
$$\arg\left(\frac{z - 2}{z + 2i}\right) = \pi$$

13 Sketch $\mathrm{Re}(z^2) = 9$
** This question involves hyperbolas with asymptotes $y = \pm x$, from the old topic of Conics so is probably beyond the scope of the current syllabus **

1

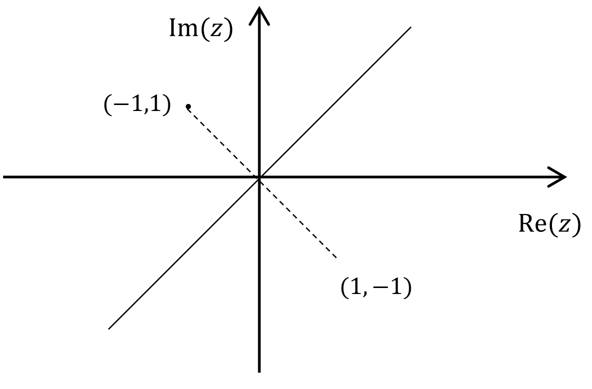
a This is a circle centred at zero with radius 3

b This is the inside and circumference of a circle centred at $(-1,1)$ with radius 1.



2

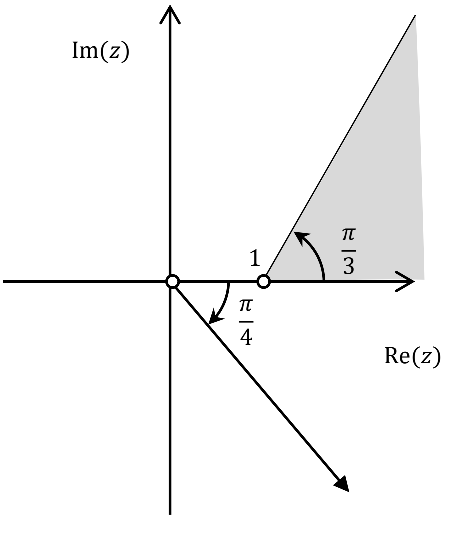
This is the perpendicular bisector of the interval joining $(-1,1)$ and $(1,-1)$.



3

a A ray from the point $(-2,0)$ at an angle of $\frac{\pi}{3}$.

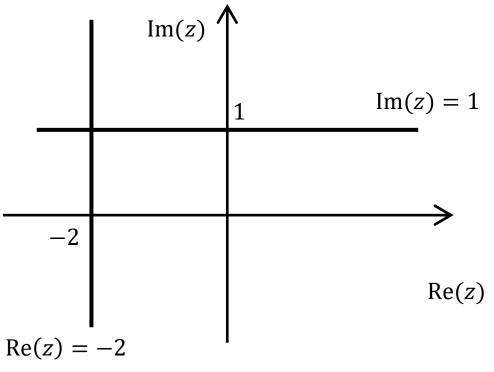
b A sector centred at zero, with rays at 0 radians and $\frac{\pi}{4}$ radians.



4

a A vertical line at $\mathrm{Re}(z) = 3$

b A horizontal line at $\mathrm{Im}(z) = 2$



5

$$\sqrt{(x+1)^2+y^2} = 3\sqrt{x^2+y^2}$$

$$x^2+2x+1+y^2 = 9(x^2+y^2)$$

$$x^2+2x+1+y^2 = 9x^2+9y^2$$

$$8x^2-2x+8y^2-1=0$$

$$x^2-\frac{1}{4}x+y^2=\frac{1}{8}$$

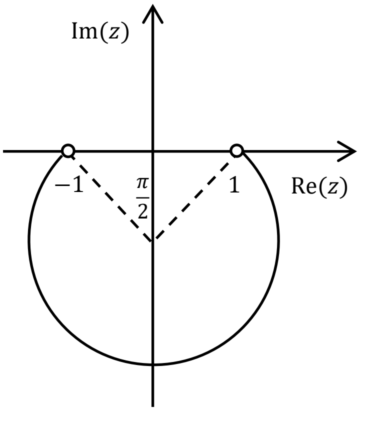
$$x^2+\frac{1}{4}x+\frac{1}{64}+y^2=\frac{1}{8}+\frac{1}{64}$$

$$\left(x+\frac{1}{8}\right)^2+y^2=\frac{9}{64}$$

Which is a circle with centre $\left(-\frac{1}{8}, 0\right)$ with radius $\frac{3}{8}$

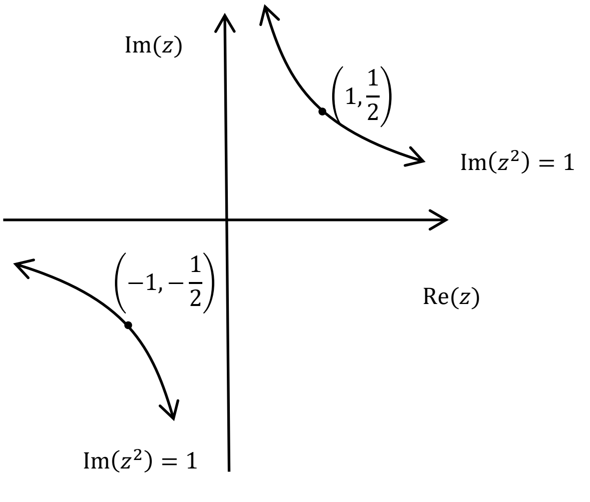
6

The angle at z (formed by the vectors from $(-1,0)$ and $(1,0)$ is equal to $\frac{\pi}{4}$, so an arc with ends at $(-1,0)$ and $(1,0)$. Standing at $(-4,0)$ and facing $(-1,0)$ we want the arc on the left (since the angle is positive). We double the $\frac{\pi}{4}$ angle at the circumference to get $\frac{\pi}{2}$, which is the apex angle of our isosceles triangle. Since our angle is less than $\frac{\pi}{2}$ put the centre to the left (below) the chord, and draw our arc.



7

A hyperbola with asymptotes the x and y axes.



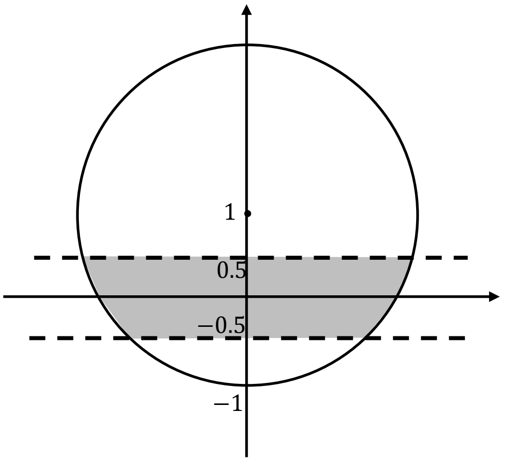
8

$z - \bar{z} = x + iy - (x - iy) = 2iy$

$\therefore |z - \bar{z}| < 1$ is the same as $|2iy| <$, so $|y| < \frac{1}{2}$,

or the region between the lines $y = \pm\frac{1}{2}$.

$|z - i| \leq 2$ is the circumference of a circle centred at (0.1) with radius 2, plus everything inside the circle.



9

$$\frac{1}{z+1} + \frac{1}{\bar{z}+1} = 1$$

$\bar{z} + 1 + z + 1 = (z+1)(\bar{z}+1)$

$x - iy + 1 + x + iy + 1 = (x + 1 + iy)(x + 1 - iy)$

$2x + 2 = (x+1)^2 + y^2$

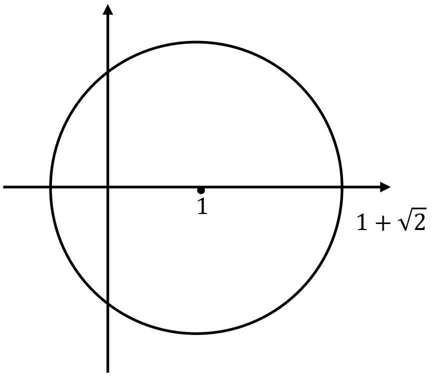
$2x + 2 = x^2 + 2x + 1 + y^2$

$x^2 - 2x + y^2 - 1 = 0$

$x^2 - 2x + 1 + y^2 = 1 + 1$

$(x-1)^2 + y^2 = \left(\sqrt{2}\right)^2$

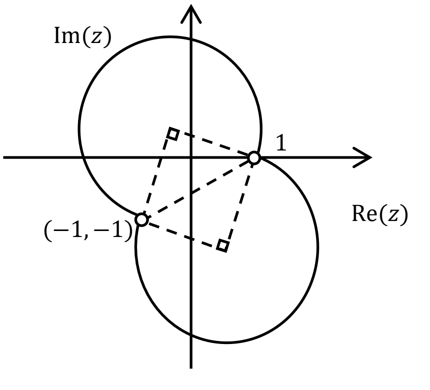
So the locus of P is a circle of radius $\sqrt{2}$ with centre (1,0)



10

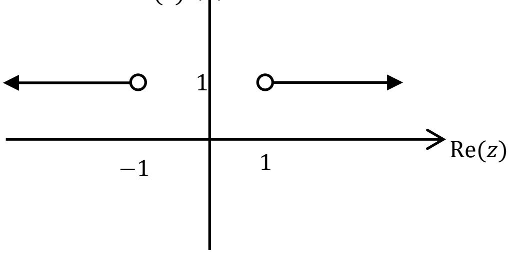
The angle at z (formed by the vectors from $(-1,-1)$ and $(1,0)$ is equal to $\pm\frac{\pi}{4}$, so an arc with ends at $(-1,-1)$ and $(1,0)$.

We want isosceles triangles drawn to both sides since the angle can be positive or negative, with apex angles of $2 \times \frac{\pi}{4} = \frac{\pi}{2}$ and can then draw both arcs to form a snowman.



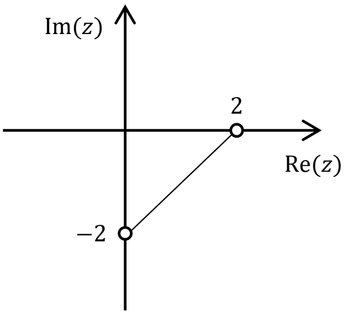
11

The angle at z formed by the vectors from $(1,1)$ and (-1,1) is equal to 0. The line passing through (1,0) and (0,1), but outside those points.



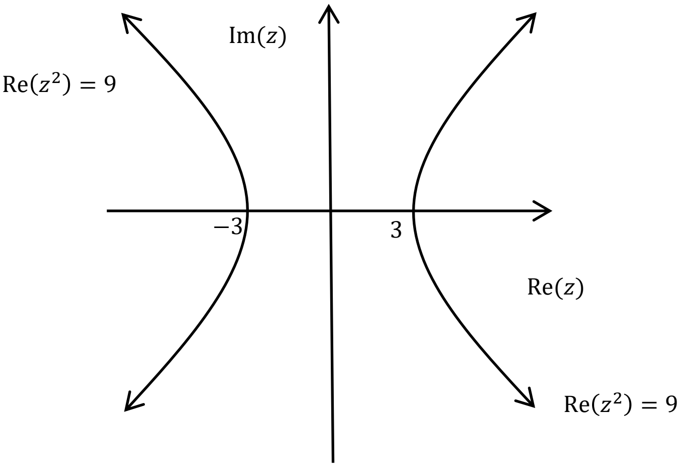
12

The angle at z formed by the vectors from $(2,0)$ and $(0.-2)$ is equal to π. The line between $(2,0)$ and $(0,-2)$ not including the points.



13

A hyperbola with asymptotes $y = \pm x$ and x-intercepts ± 2.



In Lesson 9 we look at de Moivre's Theorem. We cover:

- De Moivre's Theorem
- Trigonometrical Applications of De Moivre's Theorem

DE MOIVRE'S THEOREM

De Moivre's Theorem states that if we raise a complex number to a power then we raise the modulus to that power and multiply the argument by the power.

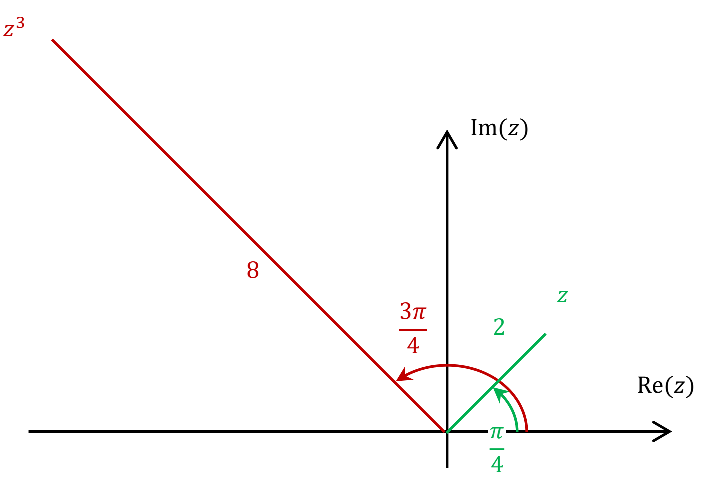
$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

De Moivre's theorem is often used in calculations as an intermediate step, such as simplifying powers of complex numbers in Cartesian form, and in combination with binomial expansion to find relationships between trigonometric ratios.

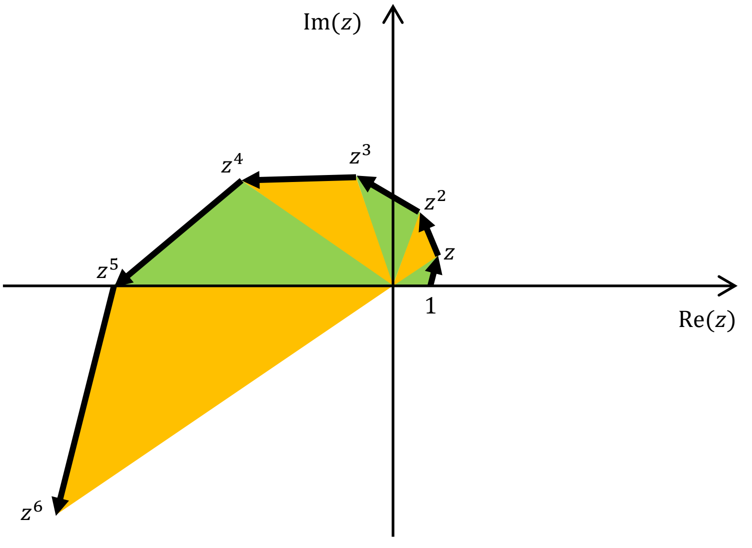
We mostly use polar form, but in exponential form we get the simple result $\left(e^{i\theta}\right)^n = e^{in\theta}$, although it is often not as useful for further steps.

If a question does not use Cartesian form, keep in mind that doing the calculation via exponential form may be easier than de Moivre's theorem.

In the diagram the complex number $z = 2\operatorname{cis}\frac{\pi}{4}$ has been raised to the power of 3 to become z^3, so the argument is multiplied by 3 to become $\frac{3\pi}{4}$ and the modulus is cubed to become 8.



To help our understanding of what happens with de Moivre's theorem, let's have a look at what happens to a complex number as we raise it to successively higher powers. If we take $1.5\,\text{cis}\,\frac{\pi}{5}$ and raise it to increasing powers of n, we can see that we end up with an increasing anti-clockwise spiral (since $|z| > 1$ and $\arg z > 0$), created by the successive similar triangles as we saw earlier in the topic.



Example 1

Let $z = 1 + i$

a Express z in polar form

b Express z^5 in polar form

c Hence express z^5 in the form $x + iy$

Solution

a

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\arg(1+i) = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$$

$$1+i = \sqrt{2}\,\mathrm{cis}\frac{\pi}{4}$$

b

$$\therefore (1+i)^5$$

$$= \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^5$$

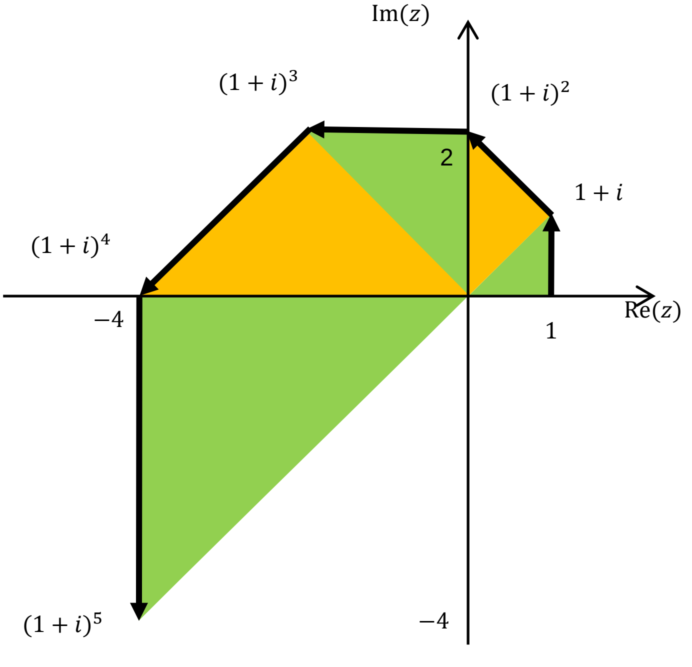
$$= \left(\sqrt{2}\right)^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$$= 4\sqrt{2}\left(-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

c

$$= -4\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= -4 - 4i$$



Every power of $1 + i$ has a modulus $\sqrt{2}$ times as long as the one before, and an argument $\frac{\pi}{4}$ more, so we get an increasing anti-clockwise spiral.

We can also use de Moivre's theorem to simplify more complicated power expressions.

Example 2

Simplify $\left(1+\sqrt{3}i\right)^4(1-i)^2$

Solution

$1+\sqrt{3}i = 2\operatorname{cis}\frac{\pi}{3} \qquad 1-i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

$\therefore \left(1+\sqrt{3}i\right)^4(1-i)^2 = \left(2\operatorname{cis}\frac{\pi}{3}\right)^4\left(\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^2$

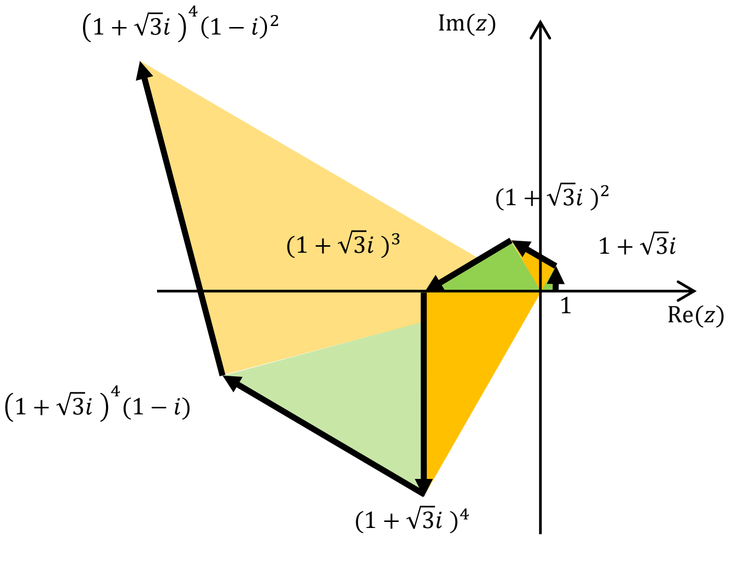
$= 2^4 \times \sqrt{2}^2 \times \operatorname{cis}\frac{4\pi}{3} \times \operatorname{cis}\left(-\frac{2\pi}{4}\right)$

$= 32\operatorname{cis}\frac{5\pi}{6}$

$= 32\left(-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 32\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$= -16\sqrt{3} + 16i$



Each $1+\sqrt{3}i$ increases the modulus and the argument so we have an increasing anti-clockwise spiral, then each $1-i$ increases the modulus and decreases the argument so the spiral continues to increase but clockwise.

Example 3

a Using de Moivre's theorem, or otherwise, show that for every positive integer n,

$$(1+i)^n+(1-i)^n = 2\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}.$$

b Hence, or otherwise, show that for every positive integer n divisible by 4,

$$\binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\binom{n}{6}+\ldots+\binom{n}{n}=(-1)^{\frac{n}{4}}\left(\sqrt{2}\right)^n$$

Solution

a $1+i=\sqrt{2}\text{cis}\frac{\pi}{4}$ and $1-i=\sqrt{2}\text{ cis}\left(-\frac{\pi}{4}\right)$

$\therefore (1+i)^n+(1-i)^n$

$$=\left(\sqrt{2}\text{ cis}\frac{\pi}{4}\right)^n+\left(\sqrt{2}\text{ cis}\left(-\frac{\pi}{4}\right)\right)^n$$

$$=2^{\frac{n}{2}}\text{ cis}\frac{n\pi}{4}+2^{\frac{n}{2}}\text{ cis}\left(-\frac{n\pi}{4}\right)$$

$$=2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}+\cos\left(-\frac{n\pi}{4}\right)+i\sin\left(-\frac{n\pi}{4}\right)\right)$$

$$=2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}+\cos\left(\frac{n\pi}{4}\right)-i\sin\left(\frac{n\pi}{4}\right)\right)$$

$$=2^{\frac{n}{2}}\left(2\cos\frac{n\pi}{4}\right)$$

$=2\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}$ as required

b

$(1+i)^n+(1-i)^n$

$$=\binom{n}{0}+\binom{n}{1}i+\binom{n}{2}i^2+\binom{n}{3}i^3+\ldots+\binom{n}{n}i^n$$

$$+\binom{n}{0}-\binom{n}{1}i+\binom{n}{2}i^2-\binom{n}{3}i^3+\ldots+\binom{n}{n}i^n$$

$$=2\left(\binom{n}{0}+\binom{n}{2}i^2+\binom{n}{4}i^4+\ldots+\binom{n}{n}i^n\right)$$

$$=2\left(\binom{n}{0}-\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}\right)$$

$$\therefore 2\left(\binom{n}{0}-\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}\right)=2\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}$$

$$\binom{n}{0}-\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}=\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}$$

but since n is a multiple of 4, $\frac{n\pi}{4}$ is equal to 0 or π, and thus $\cos\frac{n\pi}{4}$ is equal to 1 or -1, which we can say mathematically as $(-1)^{\frac{n}{4}}$.

$\therefore \binom{n}{0}-\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}=(-1)^{\frac{n}{4}}\left(\sqrt{2}\right)^n$ as required

Combining de Moivre's theorem and binomial expansion allows us to find links between powers of trig ratios. For brevity in the examples below we will let $c = \cos\theta$ and $is = i\sin\theta$.

Example 4

a Expand $(\cos\theta + i\sin\theta)^5$

b Hence find an expression for $\cos 5\theta$ in terms of $\cos\theta$

Solution

a

$(\cos\theta + i\sin\theta)^5$

$= \binom{5}{0}c^5 + \binom{5}{1}c^4(is) + \binom{5}{2}c^3(is)^2 + \binom{5}{3}c^2(is)^3 + \binom{5}{4}c(is)^4 + \binom{5}{5}(is)^5$

$= c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$

$= (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5)$

b

From de Moivre's Theorem we also know that $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$.

Equating real parts with part a we get:

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2 \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \qquad \square\end{aligned}$$

EXERCISE 2.9

BASIC

1 Let $z = 1 - i$

a Express z in polar form **b** Express z^5 in polar form

c Hence express z^5 in the form $x + iy$

2 Simplify $\left(1-\sqrt{3}i\right)^4(1+i)^2$

MEDIUM

3 **a** Expand $(\cos\theta + i\sin\theta)^5$

b Hence show that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

4 For what values of n is $(1-i)^n$ purely imaginary?

5 Prove $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ for non-negative integers by induction.

6 Prove $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ for negative integers.

7 **a** Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^4$

b Use de Moivre's theorem and your result from part (a) to prove that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

c Hence, or otherwise, find the smallest positive solution of

$$8\cos^4\theta - 8\cos^2\theta = -1$$

CHALLENGING

8 **a** Using de Moivre's theorem, or otherwise, show that for every positive integer n,

$$(1+i)^n - (1-i)^n = 2\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}i.$$

b Hence, or otherwise, show that for every positive integer n divisible by 4,

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1} = 0$$

9 **a** Show that $(1+i\tan\theta)^n - (1-i\tan\theta)^n = \dfrac{2\sin n\theta}{\cos^n\theta}i$

where $\cos\theta \neq 0$ and n is a positive integer.

b Hence show that if z is a purely imaginary number, the roots of

$(1+z)^4 - (1-z)^4 = 0$ are $z = 0, \pm i$.

1a

$$|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$
$$\arg(1-i) = \tan^{-1}\frac{-1}{1} = -\frac{\pi}{4}$$
$$1+i = \sqrt{2}\text{ cis}\left(-\frac{\pi}{4}\right)$$

b

$$\therefore (1-i)^5 = \left(\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\right)^5$$
$$= \left(\sqrt{2}\right)^5\left(\cos\left(-\frac{5\pi}{4}\right) + i\sin\left(-\frac{5\pi}{4}\right)\right)$$
$$= 4\sqrt{2}\left(-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

c

$$= 4\sqrt{2}\left(-\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}\right)$$
$$= -4+4i$$

2

$$1-\sqrt{3}i = 2\text{ cis}\left(-\tfrac{\pi}{3}\right) \qquad 1-i = \sqrt{2}\text{ cis}\left(\tfrac{\pi}{4}\right)$$
$$\therefore \left(1+\sqrt{3}i\right)^4(1-i)^2 = \left(2\text{ cis}\left(-\tfrac{\pi}{3}\right)\right)^4\left(\sqrt{2}\text{ cis}\left(\tfrac{\pi}{4}\right)\right)^2$$
$$= 2^4\times\sqrt{2}^2\times\text{cis}\left(-\frac{4\pi}{3}\right)\times\text{cis}\left(\frac{2\pi}{4}\right)$$
$$= 32\text{ cis}\left(-\frac{5\pi}{6}\right)$$
$$= 32\left(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$
$$= 32\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$
$$= -16\sqrt{3}-16i$$

3a

$$(\cos\theta + i\sin\theta)^5 = \text{c}^5 + 5\text{c}^4(is) + 10\text{c}^3(is)^2 + 10\text{c}^2(is)^3 + 5\text{c}(is)^4 + (is)^5$$
$$= \text{c}^5 + 5\text{c}^4\text{s}i - 10\text{c}^3\text{s}^2 - 10\text{c}^2\text{s}^3 i + 5\text{cs}^4 + \text{s}^5 i$$

b

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Equating imaginary parts

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$
$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \text{ as required}$$

4

$$(1-i)^n = \left(\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)\right)^n$$
$$= 2^{\frac{n}{2}}\text{cis}\left(-\frac{n\pi}{4}\right)$$

The real part is $2^{\frac{n}{2}}\cos\left(-\frac{n\pi}{4}\right)$, which we let equal 0.

$$\cos\left(-\frac{n\pi}{4}\right) = 0$$
$$\therefore \cos\left(\frac{n\pi}{4}\right) = 0 \quad \text{since cosine is even}$$

$\therefore \frac{n\pi}{4} = \frac{(2k+1)\pi}{2}$ where k is an integer

$n = 4k+2$ so n is a multiple of 4

$\therefore (1+i)^n$ is purely imaginary when n is two more than a multiple of 4.

5

Let $P(n)$ represent the proposition.

$P(0)$ is true since $\text{LHS} = [r(\cos\theta + i\sin\theta)]^0 = 1$

$$\text{RHS} = r^0(\cos 0\theta + i\sin 0\theta) = 1(1 + 0i) = 1$$

$P(k)$: let $k \geq 0$ be an arbitrary integer for which $P(k)$ is true, that is

$$[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$$

RTP $P(k+1)$: $\quad [r(\cos\theta + i\sin\theta)]^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$

$$\begin{aligned}
\text{LHS} &= [r(\cos\theta + i\sin\theta)]^k \times r(\cos\theta + i\sin\theta) \\
&= r^k(\cos k\theta + i\sin k\theta) \times r(\cos\theta + i\sin\theta) \\
&= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\
&= r^{k+1}[\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta] \\
&= r^{k+1}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i\,(\sin k\theta\cos\theta + \cos k\theta\sin\theta)] \\
&= r^{k+1}[\cos(k\theta + \theta) + i\sin(k\theta + \theta)] \\
&= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for all $n \geq 0$ by induction.

6

Let $n = -k$ where k is a positive integer

$$\begin{aligned}
[r(\cos\theta + i\sin\theta)]^n &= [r(\cos\theta + i\sin\theta)]^{-k} \\
&= r^{-k} \times \left((\cos\theta + i\sin\theta)^{-1}\right)^k \\
&= r^{-k}(\overline{\cos\theta + i\sin\theta})^k \quad \text{since } |\cos\theta + i\sin\theta| = 1 \\
&= r^{-k}(\cos(-\theta) + i\sin(-\theta))^k \\
&= r^{-k}(\cos(-k\theta) + i\sin(-k\theta)) \quad \text{since } k \geq 0 \\
&= r^n(\cos n\theta + i\sin n\theta)
\end{aligned}$$

$\therefore$ de Moivre’s Theorem is true for negative integers

7a

Let $\mathrm{c} = \cos\theta \;\; is = i\sin\theta$ and $z = c + is$

$z^4 = (c + is)^4$

$\quad = \mathrm{c}^4 + 4\mathrm{c}^3(is) + 6\mathrm{c}^2(is)^2 + 4\mathrm{c}(is)^3 + (is)^4$

$\quad = \mathrm{c}^4 + i4\mathrm{c}^3\mathrm{s} - 6\mathrm{c}^2\mathrm{s}^2 - i4\mathrm{cs}^3 + \mathrm{s}^4$

b

$z^4 = \cos 4\theta + i\sin 4\theta$

Equating real parts with (a)

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\\ &= \cos^4\theta - 6\cos^2\theta\,(1-\cos^2\theta) + (1-\cos^2\theta)^2\\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta\\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

c

$$\begin{aligned}8\cos^4\theta - 8\cos^2\theta &= -1\\ 8\cos^4\theta - 8\cos^2\theta + 1 &= 0\\ \cos 4\theta &= 0\\ 4\theta &= \frac{(2k+1)\pi}{2} \quad \text{for } k = 0,1,\ldots\\ \theta &= \frac{(2k+1)\pi}{8}\\ &= \frac{\pi}{8}, \frac{3\pi}{8}\ldots\end{aligned}$$

The smallest positive solution is $\frac{\pi}{8}$

8

a $1 + i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$ and $1 - i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

$\therefore (1+i)^n - (1-i)^n$

$$= \left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^n - \left(\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^n$$

$$= 2^{\frac{n}{2}}\operatorname{cis}\frac{n\pi}{4} - 2^{\frac{n}{2}}\operatorname{cis}\left(-\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} - \cos\left(-\frac{n\pi}{4}\right) - i\sin\left(-\frac{n\pi}{4}\right)\right)$$

$$= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} - \cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right)$$

$$= 2^{\frac{n}{2}}\left(2\sin\frac{n\pi}{4}\right)i$$

$= 2\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}\, i$ as required

b

$(1+i)^n - (1-i)^n$

$$= \binom{n}{0} + \binom{n}{1}i + \binom{n}{2}i^2 + \binom{n}{3}i^3 + \ldots + \binom{n}{n}i^n$$

$$-\binom{n}{0} + \binom{n}{1}i - \binom{n}{2}i^2 + \binom{n}{3}i^3 + \ldots - \binom{n}{n}i^n$$

$$= 2\left(\binom{n}{1}i + \binom{n}{3}i^3 + \binom{n}{5}i^5 + \ldots + \binom{n}{n-1}i^{n-1}\right)$$

$$= 2\left(\binom{n}{1} - \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1}\right)i$$

$$\therefore 2\left(\binom{n}{1} - \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1}\right) = 2\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$$

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1} = \left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$$

but since n is a multiple of 4, $\frac{n\pi}{4}$ is equal to 0 or π,

and thus $\sin\frac{n\pi}{4}$ is equal to 0.

$\therefore \binom{n}{1} - \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1} = 0$ as required

9a

$(1 + i\tan\theta)^n - (1 - i\tan\theta)^n$

$= \left(\dfrac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n - \left(\dfrac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$

$= \left(\dfrac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n - \left(\dfrac{\cos(-\theta) + i\sin(-\theta)}{\cos\theta}\right)^n$

$= \dfrac{\cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta)}{\cos^{\mathrm{n}}\theta}$

$= \dfrac{\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta}{\cos^{\mathrm{n}}\theta}$

$= \dfrac{2\sin n\theta}{\cos^n\theta} i$

b

$(1 + z)^4 - (1 - z)^4 = 0$

Let $z = i\tan\theta$

$\therefore \left((1 + i\tan\theta)^4 - (1 - i\tan\theta)^4\right) = 0$

$\therefore \dfrac{2\sin 4\theta}{\cos^4\theta} = 0$ from (a)

$\sin 4\theta = 0$

$4\theta = k\pi \quad$ for $k = -1,0,1,2$

$\theta = \dfrac{k\pi}{4}$

$= -\dfrac{\pi}{4}, 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}$

$\therefore z = i\tan\left(-\dfrac{\pi}{4}\right), i\tan(0), i\tan\left(\dfrac{\pi}{4}\right), \tan\left(\dfrac{\pi}{2}\right)$

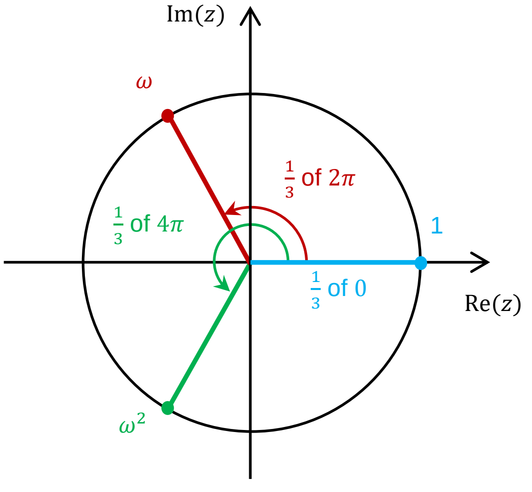
$= 0, \pm i \quad$ since $\tan\dfrac{\pi}{2}$ is undefined

2.10 COMPLEX ROOTS

In Lesson 10 we look at roots of Complex Numbers. We look at

- Complex Roots of Unity
- Complex Roots of -1
- Roots of Other Complex Numbers

The complex n^{th} roots of the number 1 are called the complex roots of unity. They are the solutions to $z^n = 1$ (or $z^n - 1 = 0$). The complex roots of 1 are spread equally around the unit circle, starting at the number 1 itself. The cube roots of 1 are shown below.



Note that the argument at 1 is not just 0, but also the multiples of 2π. Since we are after the three cube roots we are interested in the first three multiples of 2π, so $0, 2\pi$ and 4π.

We can think of complex roots of unity as:

- the cube root of e^{0i} which is $\text{cis}\left(\frac{0\pi}{3}\right) = \text{cis}\, 0 = 1$
- the cube root of $e^{2\pi i}$ which is $\text{cis}\left(\frac{2\pi}{3}\right)$
- the cube root of $e^{4\pi i}$ which is $\text{cis}\left(\frac{4\pi}{3}\right)$

Since the multiples of 2π are equally spaced, the arguments of the three roots are also equally spaced.

We normally use arguments from $-\pi$ to π, but for complex roots it is often easier to use arguments from 0 to 2π, which we simplify if the answer is purely real or imaginary.

When solving questions like this in exams the most successful technique is to start by sketching the roots, then simply write down the roots using the modulus and argument.

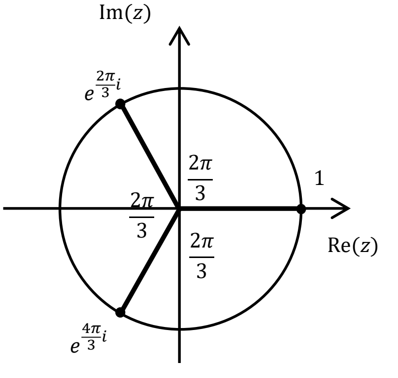
Exponential form is easiest if you only have to find the roots and nothing else. In many questions there will be a second part to a question which requires the roots to be in polar form.

Example 1

Find the cube roots of 1 by sketching the roots first.

Solution

1 is a cube root of 1, and the other roots must be spaced by $\frac{2\pi}{3}$ as shown. The cube roots of 1 are $1, e^{\frac{2\pi}{3}i}$ and $e^{\frac{4\pi}{3}i}$.



The third root could be written as $e^{-\frac{2\pi}{3}i}$ instead.

Using the same diagram but using polar form we can say the three cube roots are $1, \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$ and $\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$. We could then convert to Cartesian form if required.

Example 2

Find the cube roots of 1 using de Moivre's theorem.

Solution

Let a root be $z = r(\cos\theta + i\sin\theta)$

$$\therefore z^3 = 1$$

$$r^3(\cos\theta + i\sin\theta)^3 = \cos 2k\pi + i\sin 2k\pi \quad \text{for } k = 0,1,2$$

$$r^3(\cos 3\theta + i\sin 3\theta) = \cos 2k\pi + i\sin 2k\pi$$

$$\therefore r^3 = 1 \quad \text{and } 3\theta = 2k\pi$$

$$\therefore r = 1 \quad \text{and } \theta = \frac{2k\pi}{3}$$

$$\therefore z = \cos 0 + i\sin 0, \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right), \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$

$$= 1, \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right), \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$

SOME POINTS TO NOTE BEFORE WE CONTINUE

Complex roots of unity are solutions to a polynomial with real coefficients, so using the conjugate root theorem the conjugate of every non-real root will also be a root, as we can see from the sketches we have done where any root off the real axis is matched by one directly above or below on the unit circle.

All complex roots of 1 are powers of the non-real root with the smallest argument, so if ω is the non-real root with the smallest argument, then the other roots are $\omega^2, \omega^3, \omega^4 \ldots \omega^{n-1}$. So if we let $\omega = e^{\frac{2\pi}{3}i}$ then we see that $\omega^2 = e^{\frac{4\pi}{3}i}$ and $\omega^3 = e^{\frac{6\pi}{3}i} = e^{2\pi i} = e^{0i} = 1$

This rule only works for complex roots of 1. Combining the two points above we see relationships like $\omega^2 = \overline{\omega}$ for cube roots of unity.

Exponential form is the easiest form to use when finding complex roots and should always be used if the question doesn't specify a form, but we can find the n^{th} complex roots using any of the three forms of a complex number.

- For polar form use the same method as we used in Example 1 - find as many arguments as you need for the original number and divide each of these arguments by n. Alternatively use de Moivre's theorem like Example 2.
- For Cartesian form we normally need to find the modulus and argument then use exponential form (or polar form), converting back to Cartesian form if needed. Alternatively the following formula may be handy to memorise for some proofs:

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1})$$

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots + y^{n-1}) \text{ only if } n \text{ is odd}$$

We will prove both of these results in Mathematical Induction.

So if ω is a complex nth root of 1, ie a complex solution of $z^n - 1 = 0$ (which is $z^n - 1^n = 0$) then using the top rule with $x = z$ and $y = 1$ we can say:

$(\omega - 1)(\omega^{n-1} + \omega^{n-2} + \omega^{n-3} + \ldots + \omega^2 + \omega + 1) = 0$

$\therefore\ 1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^{n-1} = 0$ since ω is not real

This means that the sum of the n^th roots is equal to zero, an important result that we could also prove using the polygon rule for adding vectors.

We can also prove that the sum of the nth roots of any complex number is zero using the sum of the roots one at a time

$$\sum \alpha = -\frac{b}{a}$$

$$= \frac{0}{1}$$

$$= 0$$

Example 3

Prove that if ω is a non-real cube root of unity, then $(1 + \omega - \omega^2)^7 = -128\omega^2$

Solution

$1 + \omega + \omega^2 = 0$

$\therefore (1 + \omega - w^2)^7$

$= (1 + \omega + \omega^2 - 2\omega^2)^7$

$= (-2\omega^2)^7$

$= -128\omega^{14}$

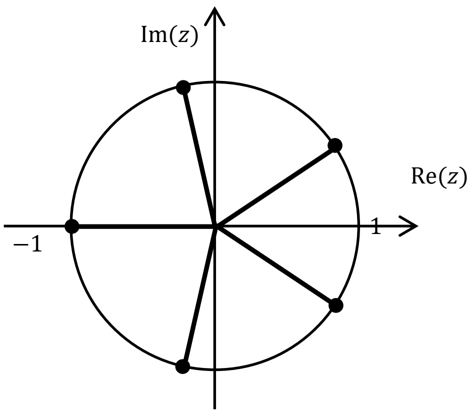
$= -128(\omega^3)^4\omega^2$

$= -128\omega^2$

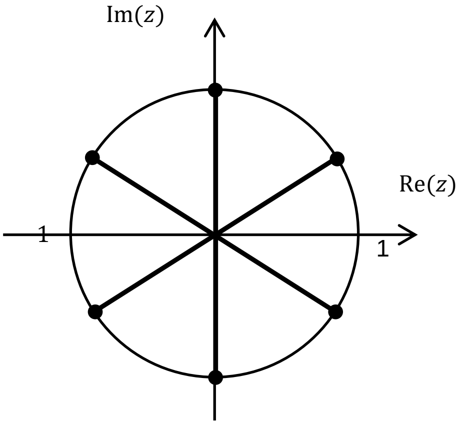
COMPLEX ROOTS OF -1

The complex roots of -1 are also spread equally around the circle, but finding the first root can be tricky if n is even.

- If n is odd then the roots include -1
- If n is even then there are no purely real roots, and the two roots either side of -1 are spaced equally either side (one is still the conjugate of the other).



Spacing of
Odd Roots



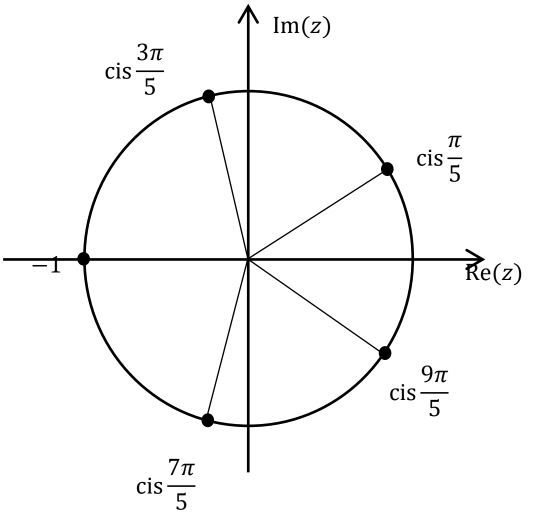
Spacing of
Even Roots

Example 4

Find the fifth roots of -1 in polar form.

Solution

-1 is one of the roots, and the others are spread evenly around the unit circle, so there is $\frac{2\pi}{5}$ between each.



$\therefore$ the roots are

$$\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}, -1,$$

$$\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}, \cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$$

ROOTS OF OTHER COMPLEX NUMBERS

New to the course is finding the complex roots of complex numbers, rather than the roots of real numbers only from the old course.

- To find the first root, take the n^{th} root of the modulus and divide the argument by n. The first root is $\sqrt[n]{r}e^{\frac{\theta}{n}i}$.
- The roots will be equally spaced around a circle, so they have the same modulus as the first root and their arguments differ by $\frac{2\pi}{n}$. The other roots are $\sqrt[n]{r}e^{\frac{2\pi+\theta}{n}i}, \sqrt[n]{r}e^{\frac{4\pi+\theta}{n}i}, \sqrt[n]{r}e^{\frac{6\pi+\theta}{n}i}, \ldots$

As a shortcut the roots are:

- $\sqrt[n]{r}e^{\frac{2k\pi+\theta}{n}i}$ for $k = 0,1, \ldots n-1$, or
- $\sqrt[n]{r}\left(\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right)$ for $k = 0,1, \ldots n-1$

Example 5

Find the fifth roots of i

Solution

For i $r = 1, \theta = \frac{\pi}{2}$

The first root has modulus $\sqrt[5]{1} = 1$ and argument $\frac{\pi}{2} \div 5 = \frac{\pi}{10}$.

The other roots have arguments which differ by $\frac{2\pi}{5} = \frac{4\pi}{10}$

$\therefore$ the fifth roots of i are $e^{\frac{\pi}{10}i}, e^{\frac{5\pi}{10}i}, e^{\frac{9\pi}{10}i}, e^{\frac{13\pi}{10}i}, e^{\frac{17\pi}{10}i}$ which simplify to $e^{\frac{\pi}{10}i}, i, e^{\frac{9\pi}{10}i}, e^{\frac{13\pi}{10}i}, e^{\frac{17\pi}{10}i}$

Example 6

Find the cube roots of $-4\sqrt{2}-4\sqrt{2}i$, leaving answers in exponential form.

Solution

For $-4\sqrt{2}-4\sqrt{2}i$, $r=\sqrt{\left(-4\sqrt{2}\right)^2+\left(-4\sqrt{2}\right)^2}=8$ and $\theta=\frac{5\pi}{4}$

The first root has modulus $\sqrt[3]{8}=2$ and argument $\frac{5\pi}{4}\div 3=\frac{5\pi}{12}$.

The other roots have arguments which differ by $\frac{2\pi}{3}=\frac{8\pi}{12}$

$\therefore$ the cube roots of i are $2e^{\frac{5\pi}{12}i}, 2e^{\frac{13\pi}{12}i}, 2e^{\frac{21\pi}{12}i}$

EXERCISE 2.10

BASIC

1 Find the fifth roots of 1 by sketching the roots first.

2 Find the fifth roots of 1 using de Moivre's theorem.

3 Prove that if ω is a non-real cube root of unity, then $(1-\omega+\omega^2)^4 = 16\omega$

4 Find the cube roots of -1 in polar form.

5 Find the cube roots of i

6 Find the eighth roots of 1.

MEDIUM

7 Find the fourth roots of $8\sqrt{2}-8\sqrt{2}i$, leaving answers in exponential form.

8 Find the cube roots of 8 in Cartesian form.

9 Find the remainder when $P(x)$ is divided by $x+i$ if $P(x) = (x^2+1)Q(x)+4x-2$

10 If w is a non-real cube root of unity prove

$$\frac{1}{1+w} - \frac{1}{1+w^2} = -(1+2w)$$

11 If ω is a non-real cube root of unity, prove that $(a-b)(a-\omega b)(a-\omega^2 b) = a^3-b^3$

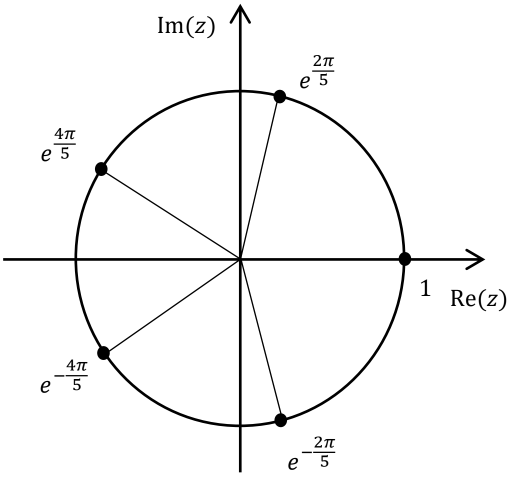
12 Given $z^5-1 = (z-1)(z^4+z^3+z^2+z+1)$ let w be a solution to $z^5-1=0$ where $w \neq -1$.

a Prove that $1+w^2+w^4 = -(w+w^3)$

b Hence show that $\cos\frac{2\pi}{5} - \cos\frac{\pi}{5} = -\frac{1}{2}$

1

1 is one of the roots, and the others are spread evenly around the unit circle, so there is $\frac{2\pi}{5}$ between each.



$\therefore$ the roots are $e^{-\frac{4\pi}{5}}, e^{-\frac{2\pi}{5}}, 1, e^{\frac{2\pi}{5}}, e^{\frac{4\pi}{5}}$

2

Let a root be $z = r(\cos\theta + i\sin\theta)$

$$\therefore z^5 = 1$$

$$r^5(\cos\theta + i\sin\theta)^5 = \cos 2k\pi + i\sin 2k\pi \quad \text{for } k = -2, -1, 0, 1, 2$$

$$r^5(\cos 5\theta + i\sin 5\theta) = \cos 2k\pi + i\sin 2k\pi$$

$$\therefore r^5 = 1 \quad \text{and } 5\theta = 2k\pi$$

$$\therefore r = 1 \quad \text{and } \theta = \frac{2k\pi}{5}$$

$$\therefore z = \text{cis}\left(-\frac{4\pi}{5}\right), \text{cis}\left(-\frac{2\pi}{5}\right), 1, \text{cis}\left(\frac{2\pi}{5}\right), \text{cis}\left(\frac{4\pi}{5}\right)$$

3

$1 + \omega + \omega^2 = 0$

$\therefore (1 - \omega + w^2)^4$

$= (1 + \omega + \omega^2 - 2\omega)^4$

$= (-2\omega)^4$

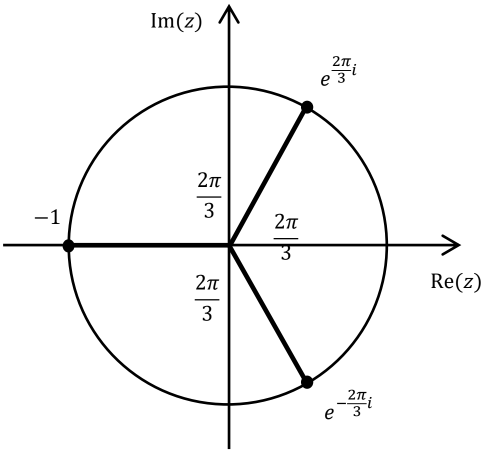
$= 16\omega^4$

$= 16\omega^3\omega$

$= 16\omega$

4

-1 is a cube root of -1, and the other roots must be spaced by $\frac{2\pi}{3}$ as shown. The cube roots of -1 are $-1, e^{\frac{2\pi}{3}i}$ and $e^{-\frac{2\pi}{3}i}$.



The third root could be written as $e^{\frac{5\pi}{3}i}$ instead.

5

For i $r = 1, \theta = \frac{\pi}{2}$

The first root has modulus $\sqrt[3]{1} = 1$ and argument $\frac{\pi}{2} \div 3 = \frac{\pi}{6}$.

The other roots have arguments which differ by $\frac{2\pi}{3} = \frac{4\pi}{6}$

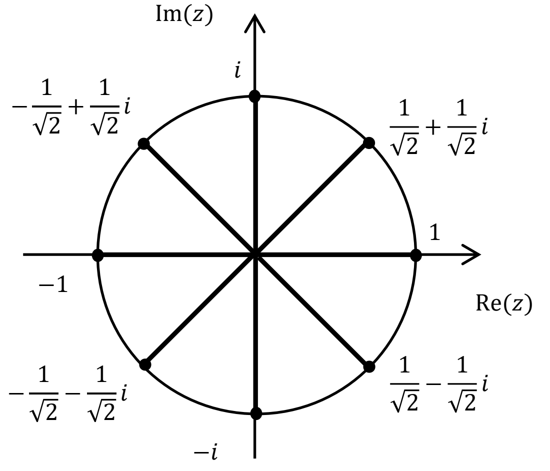
$\therefore$ the fifth roots of i are $e^{\frac{\pi}{6}i}, e^{\frac{5\pi}{6}i}, e^{\frac{9\pi}{6}i}$ which simplify to $e^{\frac{\pi}{6}i}, e^{\frac{5\pi}{6}i}, -i$.

6

1 is an eighth root of 1, and the other roots must be spaced by $\frac{2\pi}{8} = \frac{\pi}{4}$ as shown. The eighth roots of 1 are $\pm 1, \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, \pm i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$.

Alternatively in exponential form the roots are

$1, e^{\pm\frac{\pi}{4}i}, \pm i, e^{\pm\frac{3\pi}{4}i}$



7

For $4 - i$, $r = \sqrt{(8\sqrt{2})^2 + \left(-8\sqrt{2}\right)^2} = 16$ and $\theta = -\frac{\pi}{4}$

The first root has modulus $\sqrt[4]{16} = 2$ and argument $-\frac{\pi}{4} \div 4 = -\frac{\pi}{16}$.

The other roots have arguments which differ by $\frac{2\pi}{4} = \frac{8\pi}{16}$

$\therefore$ the cube roots of i are $2e^{-\frac{\pi}{16}i}, 2e^{\frac{7\pi}{16}i}, 2e^{\frac{15\pi}{16}i}, 2e^{\frac{23\pi}{16}i}$

8

Let z be a cube root of 8

$$\therefore z^3 = 8$$

$$\therefore z^3 - 8 = 0$$

$$(z-2)(z^2 + 2z + 4) = 0$$

$$z - 2 = 0 \quad \text{or} \quad z^2 + 2z + 4 = 0$$

$$z = 2 \qquad\qquad z = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$= -1 \pm \sqrt{3}i$$

The three cube roots of 8 are 2 and $-1 \pm \sqrt{3}i$

9

$$P(-i) = \left((-i)^2 + 1\right)Q(-i) + 4(-i) - 2$$

$$= (-1 + 1)Q(i) - 4i - 2$$

$$= -4i - 2$$

10

$$\frac{1}{1+w}-\frac{1}{1+w^2}$$

$$=\frac{1+w^2-1-w}{1+w^2+1+w}$$

$$=\frac{w^2-w}{1+w+w^2+1}$$

$$=\frac{1+w+w^2-1-2w}{0+1}$$

$$=-(1+2w) \quad \square$$

11

$$(a-b)(a-\omega b)(a-\omega^2 b)$$

$$=(a-b)(a^2-ab\omega^2-ab\omega+b^2\omega^3)$$

$$=(a-b)(a^2-ab(1+\omega+\omega^2-1)+b^2)$$

$$=(a-b)(a^2-ab(0-1)+b^2)$$

$$=(a-b)(a^2+ab+b^2)$$

$$=a^3-b^3 \quad \square$$

12a

$$w^5-1=0$$

$$\therefore (w-1)(w^4+w^3+w^2+w+1)=0$$

$$\therefore w^4+w^3+w^2+w+1=0 \qquad \text{since } w\neq -1$$

$$\therefore 1+w^2+w^4=-(w+w^3)$$

b

Let $z=r\operatorname{cis}\theta$

$$z^5-1=0$$

$$z^5=1$$

$$\therefore r^5\operatorname{cis}^5\theta=\operatorname{cis}2k\pi$$

$$r^5(\cos 5\theta+i\sin 5\theta)=\cos(2k\pi)+i\sin(2k\pi)$$

$$\therefore r=1 \text{ and } 5\theta=(2k\pi)$$

$$\theta=\frac{2k\pi}{5}$$

$$=0,\pm\frac{2\pi}{5},\pm\frac{4\pi}{5}$$

Let $w=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$

$$\therefore 1+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^2+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^4=-\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^3\right)$$

$$1+\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}=-\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}\right)$$

$$1-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}=-\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}-\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}\right)$$

$$1-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}=-\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}+\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$$

$$\therefore 1-2\cos\frac{\pi}{5}+2\cos\frac{2\pi}{5}=0$$

$$\therefore \cos\frac{2\pi}{5}-\cos\frac{\pi}{5}=-\frac{1}{2}$$

CONVERTING RECTANGULAR TO POLAR/EXPONENTIAL FORM

In Complex Numbers we often have to convert between rectangular form ($z = X + iY$) and polar or exponential forms ($z = r \operatorname{cis} \theta$ or $z = re^{i\theta}$), where $r = \sqrt{X^2 + Y^2}$ and $\theta = \tan^{-1} \frac{Y}{X}$.

It is always better to know how to do this properly as we have done in the examples, but if a student has a mental blank in a test, or wants to check their answer, here is how to do it on a CASIO fx82 AUPlusII:

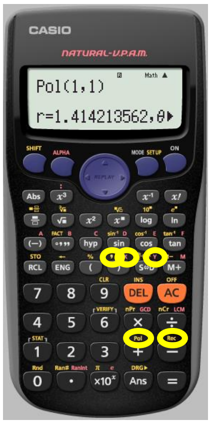
- Set your calculator to DEG or RAD as needed.
- Press $\boxed{\text{SHIFT}}$ $\boxed{\text{Pol}}$
- Enter the X value, then $\boxed{\text{SHIFT}}$ $\boxed{,}$ Y value $\boxed{)}$ $\boxed{=}$
- The calculator will display r and θ on the screen as decimals - you may need to use the right arrow to see the value of θ.
- ProTip: the calculator stores the r value in both Ans and memory X, and the θ value in memory Y, so you can perform further calculations on these values if needed. For example if you press x^2 straight away the calculator will square r and in the example on the right tell you 2, so working backwards this means $r = \sqrt{2}$.

Example 1

Convert $1 + i$ into polar form

Solution

Keystrokes: $\boxed{\text{SHIFT}}$ $\boxed{\text{Pol}}$ $\boxed{1}$ $\boxed{\text{SHIFT}}$ $\boxed{,}$ $\boxed{1}$ $\boxed{)}$ $\boxed{=}$



CONVERTING FROM POLAR/EXPONENTIAL TO RECTANGULAR FORM

To convert from polar or exponential form to rectangular form on a CASIO fx82 AUPlusII:

- Set your calculator to DEG or RAD as needed.
- Press $\boxed{\text{SHIFT}}$ $\boxed{\text{Rec}}$
- Enter the r value, then $\boxed{\text{SHIFT}}$ $\boxed{,}$ θ value $\boxed{)}$ $\boxed{=}$
- The calculator will display X and Y on the screen as decimals - you may need to use the right arrow to see the value of θ.
- ProTip: the calculator stores the X value in both Ans and memory X, and the Y value in memory Y, so you can perform further calculations on these values if needed. For example if you press x^2 straight away the calculator will square X and in the example on the right tell you 3, so working backwards this means $X = \sqrt{3}$.

Example 2

Convert $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ into rectangular form

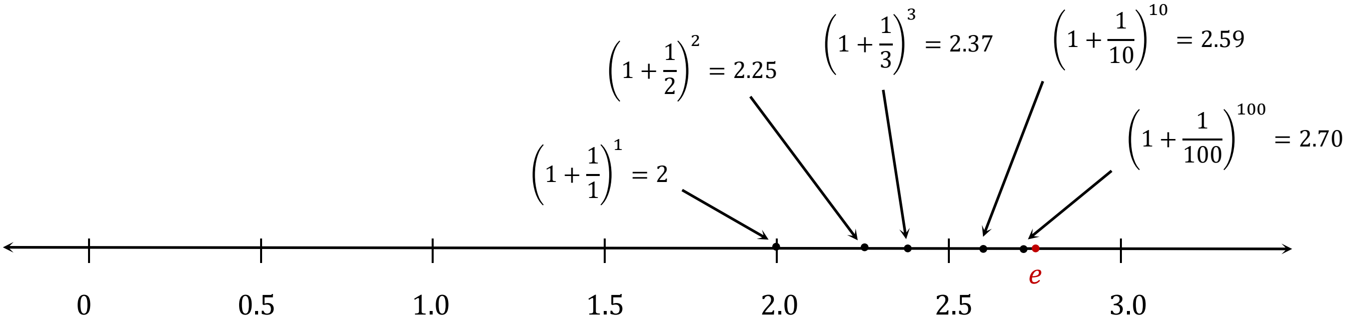
Solution

Keystrokes:

$\boxed{\text{SHIFT}}$ $\boxed{\text{Rec}}$ $\boxed{2}$ $\boxed{\text{SHIFT}}$ $\boxed{,}$ $\boxed{\text{SHIFT}}$ $\boxed{\pi}$ $\boxed{\frac{\blacksquare}{\square}}$ $\boxed{\triangledown}$ $\boxed{6}$ $\boxed{)}$ $\boxed{=}$

This would be the same steps for converting $2e^{\frac{\pi}{6}i}$ into rectangular form.

Now let's start narrowing down the values of e and e^i using the limit definition for e^x. We start by looking at $\left(1+\frac{1}{n}\right)^n$ for some values of n.



As we try larger and larger values we find that the result moves from 2 to 2.25 to 2.37, gradually approaching $2.718\ldots$

Since $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, so we see that $e = 2.718\ldots$

So what happens if x is imaginary? Does it make e^x imaginary? Well, not quite.

Consider e^i - if we have a look at our definition of e^x, we can see

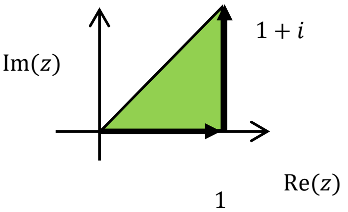
$$e^i = \lim_{n\to\infty}\left(1+\frac{i}{n}\right)^n$$

Now looking at the expression in the brackets, $1+\frac{i}{n}$, we see that it is a complex number, and $\left(1+\frac{i}{n}\right)^n$ is a complex number multiplied by itself n times, which is also complex. So e^i is complex rather than imaginary.

Let's follow similar steps to what we have just done with e^1 to find where e^i is on the complex plane.

Letting $n = 1$

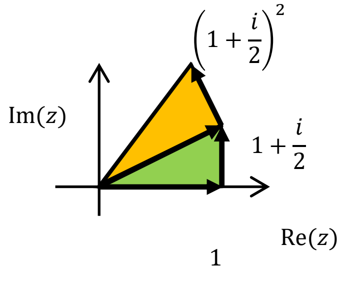
$$\left(1+\frac{i}{1}\right)^1$$
$$= 1 + i$$



1 is multiplied by $1 + i$.
$\arg z = 0.79$
$|z| = 1.41$

Letting $n = 2$

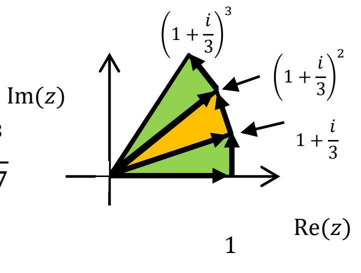
$$\left(1+\frac{i}{2}\right)^2$$
$$= 1 + 2\left(\frac{i}{2}\right) + \frac{i^2}{4}$$
$$= \frac{3}{4} + i$$



1 is multiplied by $1 + \frac{i}{2}$ twice.
$\arg z = 0.93$
$|z| = 1.25$

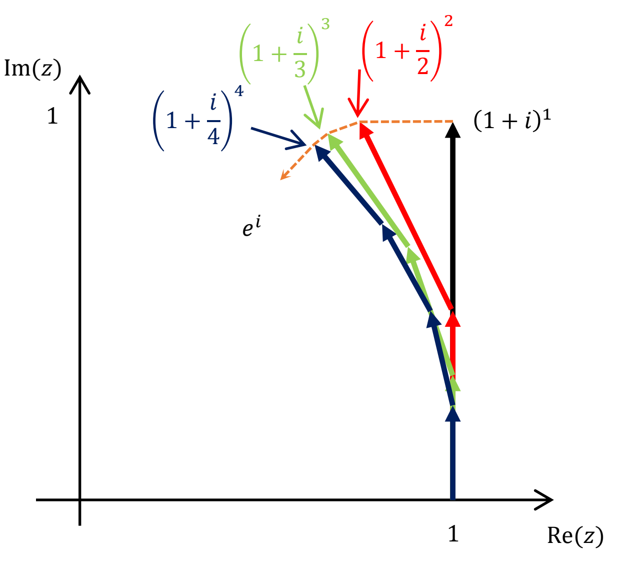
Letting $n = 3$

$$\left(1+\frac{i}{3}\right)^3$$
$$= 1 + 3\left(\frac{i}{3}\right) + 3\left(\frac{i}{3}\right)^2 + \frac{i^3}{27}$$
$$= \frac{2}{3} + \frac{26}{27}i$$



1 is multiplied by $1 + \frac{i}{3}$ three times.
$\arg z = 0.97$
$|z| = 1.17$

Looking at the pattern as n increases from 1 to 4 we see:

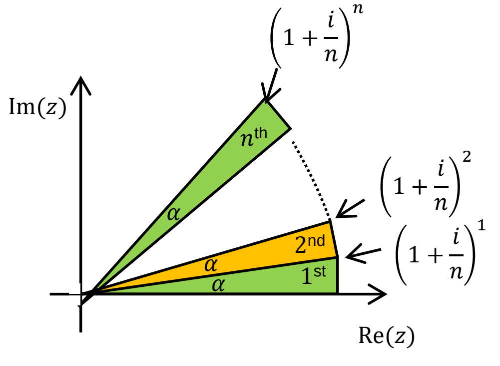


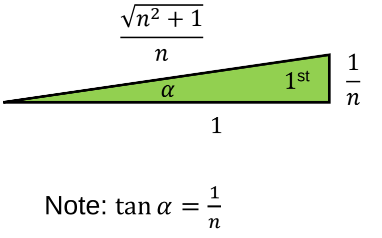
As n increases we see that:

- The path to $\left(1+\frac{i}{n}\right)^n$ bends to the left and down, moving a smaller distance each time
- The modulus of $\left(1+\frac{i}{n}\right)^n$ is decreasing a little each time
- The argument of $\left(1+\frac{i}{n}\right)^n$ is increasing a little each time

So the question is how far does the modulus decrease and how far does the argument increase to get to e^i?

Consider $\left(1+\frac{1}{n}\right)^n$. We have n similar triangles where the hypotenuse of one is the adjacent side of the next (below left). The first triangle (shown below right) would have sides of length 1 and $\frac{1}{n}$, and using Pythagoras we can see the hypotenuse is $\frac{\sqrt{n^2+1}}{n}$.





As $n \to \infty$ the opposite sides of the triangles approach an arc of the unit circle.

We can see that for $\left(1+\frac{1}{n}\right)^n$ we keep adding α to find the argument and multiplying by $\frac{\sqrt{n^2+1}}{n}$ to get the modulus:

$$\arg\left(\left(1+\frac{i}{n}\right)^n\right) = n \times \alpha \qquad \left|\left(1+\frac{i}{n}\right)^n\right| = \left(\frac{\sqrt{n^2+1}}{n}\right)^n$$

Since $e^i = \lim_{n\to\infty}\left(1+\frac{i}{n}\right)^n$ we find the argument and modulus of e^i by finding the limit of the argument and modulus of $\left(1+\frac{i}{n}\right)^n$:

$$\begin{aligned}
\therefore \left|e^i\right| &= \lim_{n\to\infty}\left|\left(1+\frac{i}{n}\right)^n\right| \\
&= \lim_{n\to\infty}\left(\frac{\sqrt{n^2+1}}{n}\right)^n \\
&= \lim_{n\to\infty}\left(\frac{\sqrt{n^2}}{n}\right)^n \qquad \text{since } n \text{ is large 1 is negligible } * \\
&= \lim_{n\to\infty} 1^n \\
&= 1
\end{aligned}$$

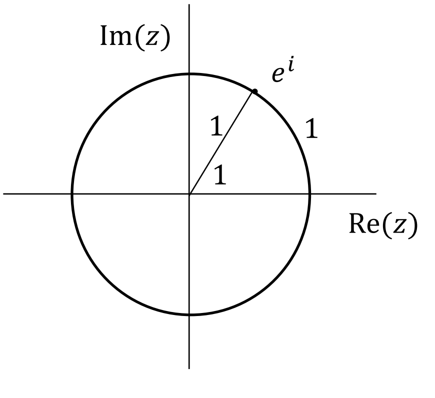
Another way of viewing this step, is that as $n \to \infty$ the hypotenuse approaches the same length as the adjacent side.

$$\begin{aligned}
\therefore \arg(e^i) &= \lim_{n\to\infty}\arg\left(\left(1+\frac{i}{n}\right)^n\right) \\
&= \lim_{n\to\infty}(n\times\alpha) \\
&= \lim_{n\to\infty}(n\times\tan\alpha) \qquad \text{since } \alpha \text{ is small} \\
&= \lim_{n\to\infty}\left(n\times\frac{1}{n}\right) \\
&= \lim_{n\to\infty} 1 \\
&= 1
\end{aligned}$$

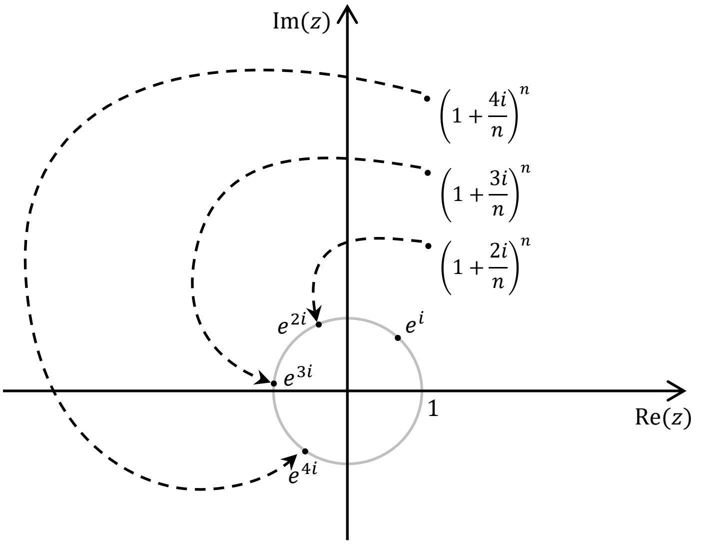
e^i has modulus 1 and argument 1 radian

$\therefore e^i = \cos 1 + i\sin 1$

So e^i is on the unit circle with an argument of 1 radian.



We could extend this process to $e^{i\theta}$. The diagram below shows the path taken by $\left(1+\frac{i\theta}{n}\right)^n$ for $\theta = 2, 3$ and 4 as n moves from 1 tending to infinity, giving us the position of e^{2i}, e^{3i} and e^{4i} respectively.



We now have an understanding of why Euler's formula works and what it means, so now it is time for a formal proof. This might be the basis for part of an investigative task.

We will start from the Taylor Series, but for a simpler proof you might get students to start with the power series.

We will do the proof with the variable x, as stated in the syllabus, as it is easier to follow the differentiation, but it is best to show students the rule in terms of θ, as a reminder that the variable represents an argument or rotation, rather than a horizontal distance.

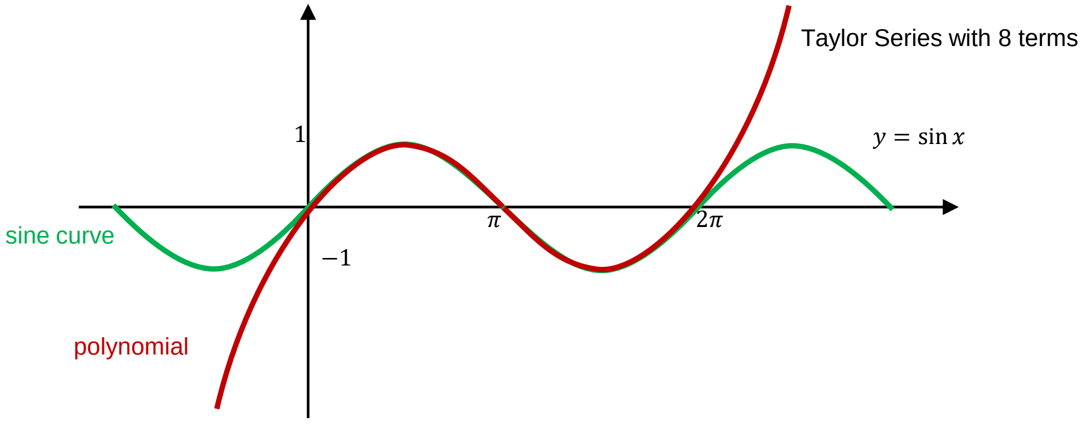
BACKGROUND - TAYLOR SERIES

A Taylor Series is used to approximate any differentiable non-polynomial function as a polynomial, around some point $x = a$. It is

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

The Cartesian plane below (not a complex plane) shows how the first 8 terms of a Taylor Series gives us a polynomial that approximates $y = \sin x$ about $x = \pi$. So below we have graphed $y = \sin x$ and the polynomial

$$y = \sin\pi + \cos\pi\,(x-\pi) - \frac{\sin\pi\,(x-\pi)^2}{2} - \frac{\cos\pi\,(x-\pi)^3}{6} + \ldots + \frac{\cos\pi(x-\pi)^7}{5040}$$



We can see that it is quite accurate around $x = \pi$, and as we take more and more terms the approximation becomes more accurate through more of the domain.

Now as $n \to \infty$ the Taylor Series is no longer an approximation, but equals the function. The point about which we base the approximation no longer matters, so let's make it zero for easier calculations.

A Taylor Series where $n \to \infty$ and where $a = 0$ is a Maclaurin Series.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \ldots$$

Proof 1

Prove $e^{ix} = \cos x + i \sin x$ using the Maclaurin Series

Solution

Now we will find the Maclaurin Series for $e^{i\theta}$, $\cos\theta$ and $\sin\theta$.

For $f(x) = e^{ix}$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \ldots$$

$$e^{ix} = e^{0i} + ie^{0i}x + \frac{i^2e^{0i}x^2}{2!} + \frac{i^3e^{0i}x^3}{3!} + \frac{i^4e^{0i}x^4}{4!} + \ldots$$

$$= 1 + xi - \frac{x^2}{2!} - \frac{x^3}{3!}i + \frac{x^4}{4!} + \frac{x^5}{5!}i - \frac{x^6}{6!} - \frac{x^7}{7!}i + \ldots \qquad *$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \qquad (1)$$

* This is a power series.

For $f(x) = \cos x$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \ldots$$

continues over page

$$\cos x = \cos(0) - \sin(0)\, x - \frac{\cos(0)\, x^2}{2!} + \frac{\sin(0)\, x^3}{3!} + \frac{\cos(0)\, x^4}{4!} + \ldots$$

$$= 1 - 0x - \frac{x^2}{2!} + 0x^3 + \frac{x^4}{4!} + 0x^5 - \frac{x^6}{6!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad (2)$$

For $f(x) = \sin x$

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(a)}{3!} x^3 + \ldots$$

$$\sin x = \sin(0) + \cos(0)\, x - \frac{\sin(0)\, x^2}{2!} - \frac{\cos(0)\, x^3}{3!} + \frac{\sin(0)\, x^4}{4!} + \ldots$$

$$= 0 + x + 0x^2 - \frac{x^3}{3!} + 0x^4 + \frac{x^5}{5!} + 0x^6 + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad (3)$$

So we have the following equations:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \qquad (1)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad (2)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad (3)$$

From $(1), (2)$ and (3) we see that $e^{ix} = \cos x + i \sin x$ □

Chapter 3
Mathematical Induction

MEX-P2: Further Proof by Mathematical Induction.

Mathematical Induction is a great topic for Extension 2 students as
- it tests your ability to use mathematical language and plain language to reason and communicate
- it promotes clear, simple and logical thought processes
- it forces you to master your algebra skills - you need to be quick and accurate, and have a deep understanding of how the rules work.
- the topic has a wide variety of applications so questions do not need to be repetitive.

LESSONS
Further Mathematical Induction is covered in 2 lessons. In the Appendix you will find two lessons on the Extension 1 content if you have not already done them - do them before the Extension 2 lessons.

3.1 Further Algebraic Induction Proofs
3.2 Other Induction Proofs
Appendix 1: Extension 1 Mathematical Induction
  3.0A Introduction to Mathematical Induction and Proofs Involving Series
  3.0B Proofs Involving Divisibility, False Proofs and Inappropriate Situations

REVISION QUESTIONS
In '1000 Extension 2 Revision Questions', the revision book that goes with this textbook you will find the following questions matching this chapter:
- Revision Exercise 3
  40 graded questions on this topic only
- Revision Exercises 7 (Basic), 8 (Medium) and 9 (Challenging)
  Another 40 questions mixed through other topics for when you finish the course.

Don't forget to do any questions from the exercises in this textbook you haven't done.

3.1 FURTHER ALGEBRAIC INDUCTION PROOFS

In Lesson 1 we look at the first of two lessons covering the types of Mathematical Induction not included in Extension 1. In this first lesson we will look at algebraic proofs which are quite similar to those we covered in Extension 1, covering:

- Induction in Extension 2
- The Initial Value and Step not equal to 1
- Sigma Notation
- Divisibility Proofs in Extension 2
- Inequality Proofs by Induction

THE INITIAL VALUE AND STEP NOT EQUAL TO 1

In Extension 1 the results we prove always apply to all natural numbers, so 1, 2, 3 etc. We always test $n = 1$ for the base case and in the inductive step prove that $P(k) \Rightarrow P(k+1)$.

In Extension 2

- The base case we test can be greater than 1 (not 0 for some unknown reason . . . sigh)
- The numbers we are proving the result for may jump by 2s or any other number.

For example, to prove a result is true for positive even numbers means that we test $n = 2$ as the base case, then check that $P(k) \Rightarrow P(k+2)$ in the inductive step.

Example 1

Prove that $n^2 + 2n$ is a multiple of 8 if n is an even positive integer

Solution

Let $P(n)$ represent the proposition.

$P(2)$ is true since $2^2 + 2(2) = 8$

If $P(k)$ is true for some arbitrary even $k \geq 2$ then $k^2 + 2k = 8m$ for integral m

RTP $P(k+2)$ $\quad (k+2)^2 + 2(k+2) = 8p$ for integral p

$$\begin{aligned}
\text{LHS} &= (k+2)^2 + 2(k+2) \\
&= k^2 + 4k + 4 + 2k + 4 \\
&= k^2 + 2k + 4(k+2) \\
&= 8m + 8\left(\frac{k}{2} + 1\right) \quad \text{from } P(k) \\
&= 8\left(m + \frac{k}{2} + 1\right) \\
&= 8p \quad \text{for integral } p \text{ since } m \text{ is integral and } k \text{ is even} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for even $n \geq 2$ by induction $\square$

SIGMA NOTATION

Sigma notation is not officially part of the Extension 1 course, even though you may have used it in topics like Polynomials. So in Extension 1 the sum of a series had to be written out in full, while in Extension 2 we can use sigma notation. Presumably the questions will also be harder!

Example 2

Prove that $\sum_{r=1}^{n}(2r+2^r)=n(n+1)+2^{n+1}-2$ for $n\geq 1$.

Solution

Let $P(n)$ represent the proposition.

$P(1)$ is true since $2(1)+2^1=4; 1(1+1)+2^{1+1}-2=4$

If $P(k)$ is true for some arbitrary $k\geq 1$ then $\sum_{r=1}^{k}(2r+2^r)=k(k+1)+2^{k+1}-2$

RTP $P(k+1)$ $\sum_{r=1}^{k+1}(2r+2^r)=(k+1)(k+2)+2^{k+2}-2$

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k}(2r+2^r)+2(k+1)+2^{k+1} \\
&= k(k+1)+2^{k+1}-2+2(k+1)+2^{k+1} \qquad \text{from } P(k) \\
&= (k+1)(k+2)+2\times 2^{k+1}-2 \\
&= (k+1)(k+2)+2^{k+2}-2 \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\geq 1$ by induction $\square$

DIVISIBILITY PROOFS IN EXTENSION 2

As an educated guess, we can expect to see divisibility expressions with only one index in Extension 1, while those with two or more indices or other more complicated expressions will be in Extension 2. The only real difference is that the algebraic manipulation in the inductive step is harder, particularly rearranging the LHS of $P(k+1)$ in terms of $P(k)$.

Example 3

Prove that $3^{2n+4} - 2^{2n}$ is divisible by 5 for any positive integer n.

Solution

Let $P(n)$ represent the proposition.

$P(1)$ is true since $3^{2(1)+4} - 2^{2(1)} = 725 = 5 \times 145$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $3^{2k+4} - 2^{2k} = 5m$ for integral m

RTP $P(k+1)$ $\quad$ $3^{2k+6} - 2^{2k+2} = 5p$ for integral p

$$\begin{aligned}
\text{LHS} &= 3^{2k+6} - 2^{2k+2} \\
&= 9(3^{2k+4}) - 4(2^{2k}) \\
&= 9\left(3^{2k+4} - 2^{2k}\right) + 5(2^{2k}) \\
&= 9(5m) + 5(2^{2k}) && \text{from } P(k) \\
&= 5\left(9m + 2^{2k}\right) \\
&= 5p && \text{for integral } p \text{ since } m \text{ and } k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

INEQUALITY PROOFS BY INDUCTION

We now look at induction proofs where we bring our skills from direct inequality proofs into the inductive step. As with any inequality proof, obtaining a deep understanding of inequalities is the key.

Example 4

Prove by induction that $2^n > n^2$ for positive integers $n > 4$.

Solution

Let $P(n)$ represent the proposition.

$P(5)$ is true since $2^5 = 32;\ 5^2 = 25$

If $P(k)$ is true for some arbitrary $k > 4$ then $2^k > k^2$

RTP $P(k+1)$ $\qquad$ $2^{k+1} > (k+1)^2$

$$\begin{aligned}
\text{LHS} &= 2 \cdot 2^k \\
&> 2k^2 \qquad \text{from } P(k)
\end{aligned}$$

$\therefore 2^{k+1} > k^2 + k^2 \qquad (1)$

$$\begin{aligned}
\text{Now } & k^2 - 2k - 1 \\
&= k^2 - 2k + 1 - 2 \\
&= (k-1)^2 - 2 \\
&> 0 \qquad \text{for } k > 4
\end{aligned}$$

$\therefore k^2 > 2k + 1 \quad \text{for } k > 4 \qquad (2)$

From (1) and (2)

$$\begin{aligned}
2^{k+1} &> k^2 + 2k + 1 \\
&= (k+1)^2 \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n > 4$ by induction $\quad\square$

EXERCISE 3.1

BASIC

1 Prove by induction that $2^n + 1$ is divisible by 3 for all odd integers.

2 Prove by induction that the square of an even number is even.

MEDIUM

3 Prove by induction that the product of n even integers is even for $n \geq 2$.

4 Prove by induction that $\sum_{r=1}^{n} 4r + 4^r = 2n(n+1) + \frac{4^{n+1} - 4}{3}$ for $n \geq 1$.

5 Prove by induction that $3^{2n} - 4^n$ is divisible by 5 if n is a positive odd number.

6 Prove by induction that $4^n + 5^n$ is divisible by 9 if n is a positive odd number.

7 Prove by induction that $x^n - y^n$ is divisible by $x - y$, $(x \neq y)$ for integral x, y with n a positive integer.

8 Prove by induction that $4^{n+1} + 6^n$ is divisible by 10 when n is even

9 Prove by induction that $6n + 6 < 2^n \text{ for } n \geq 6$

CHALLENGING

10 Prove by induction that $n^2 < 4^n$ for n a positive integer.

11 Prove by induction that $12^n > 7^n + 5^n$ *for* $n \geq 2$

12 Prove by induction for positive integers n that $1! \times 3! \times 5! \times \ldots \times (2n-1)! \geq (n!)^n$

13 Prove by induction for $n \geq 2$ that $1^3 + 2^3 + \ldots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \ldots + n^3$

1 Let $P(n)$ represent the proposition.

$P(1)$ is true since $2^1 + 1 = 3$

If $P(k)$ is true for some arbitrary odd $k \geq 1$ then $2^k + 1 = 3m$ for integral m

RTP $P(k+2)$ $\quad 2^{k+2} + 1 = 3p$ for integral p

$$\begin{aligned}
\text{LHS} &= 2^{k+1} + 1 \\
&= 2^2(2^k + 1) - 3 \\
&= 4(3m) - 3 \qquad \text{from } P(k) \\
&= 3(4m - 1) \\
&= 3p \qquad \text{for integral } p \text{ since } m \text{ is integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd $n \geq 1$ by induction $\square$

2 Let $P(n)$ represent the proposition.

$P(2)$ is true since $2^2 = 4$ which is even

If $P(k)$ is true for some arbitrary even $k \geq 2$ then $k^2 = 2m$ for integral m

RTP $P(k+2)$ $\quad (k+2)^2 = 2p$ for integral p

$$\begin{aligned}
\text{LHS} &= k^2 + 4k + 4 \\
&= 2m + 4k + 4 \qquad \text{from } P(k) \\
&= 2(m + 2k + 2) \\
&= 2p \qquad \text{for integral } p \text{ since } m \text{ and } k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd $n \geq 1$ by induction $\square$

3 Let $P(n)$ represent the proposition, and the even numbers be $2j_1, 2j_2, \ldots, 2j_n$ for integral $j_1, j_2, \ldots j_n$.

$P(2)$ is true since $(2j_1)(2j_2) = 4j_1j_2 = 2(2j_1j_2)$ which is even since j_1, j_2 are integral.

If $P(k)$ is true for some arbitrary $k \geq 2$ then $(2\mathrm{j}_1)(2j_2)\ldots(2j_k) = 2m$ for integral m

RTP $P(k+1)$ $\quad (2\mathrm{j}_1)(2j_2)\ldots(2j_k)(2j_{k+1}) = 2p$ for integral p

$$\begin{aligned}
\text{LHS} &= (2\mathrm{j}_1)(2j_2)\ldots(2j_k)(2j_{k+1}) \\
&= (2m)(2j_{k+1}) \qquad \text{from } P(k) \\
&= 4mj_{k+1} \\
&= 2(2mj_{k+1}) \\
&= 2p \qquad \text{for integral } p \text{ since } m \text{ and } j_{k+1} \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction $\square$

4 Let $P(n)$ represent the proposition.

$P(1)$ is true since $4(1)+4^1=8; 2(1)(1+1)+\frac{4^{1+1}-4}{3}=8$

If $P(k)$ is true for some arbitrary $k\geq 1$ then $\displaystyle\sum_{r=1}^{k} 4r+4^r=2k(k+1)+\frac{4^{k+1}-4}{3}$

RTP $P(k+1)$ $\displaystyle\sum_{r=1}^{k+1} 4r+4^r=2(k+1)(k+2)+\frac{4^{k+2}-4}{3}$

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k} 4r+4^r+4(k+1)+4^{k+1} \\
&= 2k(k+1)+\frac{4^{k+1}-4}{3}+4(k+1)+4^{k+1} \qquad \text{from } P(k) \\
&= 2k(k+1)+4(k+1)+\frac{4^{k+1}-4}{3}+4^{k+1} \\
&= 2(k+1)\big((k+1)+1\big)+\frac{4^{k+1}-4+3\times 4^{k+1}}{3} \\
&= 2(k+1)(k+2)+\frac{4\times 4^{k+1}-4}{3} \\
&= 2(k+1)(k+2)+\frac{4^{k+2}-4}{3} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\geq 1$ by induction □

5 Let $P(n)$ represent the proposition.

$P(1)$ is true since $3^{2(1)}-4^1=5$

If $P(k)$ is true for some arbitrary odd $k\geq 1$ then $3^{2k}-4^k=5m$ for integral m

RTP $P(k+2)$ $3^{2(k+2)}-4^{k+2}=5p$ for integral p

$$\begin{aligned}
\text{LHS} &= 3^{2k+4}-4^{(k+2)} \\
&= 81\left(3^{2k}\right)-16\left(4^k\right) \\
&= 81\left(3^{2k}-4^k\right)+65\left(4^k\right) \\
&= 9(5m)+65\left(4^k\right) \quad \text{from } P(k) \\
&= 5\left(9m+13\times 4^k\right) \\
&= 5p \qquad \text{for integral } p \text{ since } m \text{ and } k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k)\Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd $n\geq 1$ by induction □

6 Let $P(n)$ represent the proposition.

$P(1)$ is true since $4^1+5^1=9$

If $P(k)$ is true for some arbitrary odd $k\geq 1$ then $4^k+5^k=9m$ for integral m

RTP $P(k+2)$ $4^{k+2}+5^{k+2}=9p$ for integral p

LHS $= 16 \cdot 4^k + 25 \cdot 5^k$

$= 16(4^k + 5^k) + 9 \cdot 5^k$

$= 16(9m) + 9 \cdot 5^k$ from $P(k)$

$= 9(16m + 5^k)$

$= 9p$ for integral p since m and k are integral

$=$ RHS

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd $n \geq 1$ by induction □

7 Let $P(n)$ represent the proposition.

$P(1)$ is true since $x^1 - y^1 = x - y$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $x^k - y^k = m(x-y)$ for integral m

RTP $P(k+1)$ $x^{k+1} - y^{k+1} = p(x-y)$ for integral p

LHS $= x^{k+1} - y^{k+1}$

$= x \cdot x^k - y \cdot y^k$

$= x(x^k - y^k) - (y - x)y^k$

$= x(m(x-y)) + (x-y)y^k$ from $P(k)$

$= (x-y)(mx + y^k)$

$= p(x-y)$ for integral p since m, x, y and k are integral

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

8 Let $P(n)$ represent the proposition.

$P(2)$ is true since $4^{2+1} + 6^2 = 100 = 10(10)$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $4^{k+1} + 6^k = 10m$ for integral m

RTP $P(k+2)$ $4^{k+3} + 6^{k+2} = 10p$ for integral p

LHS $= 16(4^{k+1}) + 36(6^k)$
$= 16(4^{k+1} + 6^k) + 20(6^k)$
$= 16(10m) + 20(6^k)$ from $P(k)$
$= 10(16m + 2 \times 6^k)$
$= 10p$ for integral p since m and k are integral
$=$ RHS
$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for even $n \geq 2$ by induction □

9 Let $P(n)$ represent the proposition.

$P(6)$ is true since $\text{LHS} = 6(6) + 6 = 42; \text{RHS} = 2^6 = 64$

If $P(k)$ is true for some arbitrary $k \geq 6$ then $6k + 6 < 2^k$

RTP $P(k+1)$ $\quad 6k + 12 < 2^{k+1}$

$$\begin{aligned}\text{LHS} &= 6k + 6 + 6\\ &< 2^k + 6 \quad \text{from } P(k)\\ &< 2^k + 2^k \quad \text{for } k \geq 6\\ &= 2^{k+1}\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 6$ by induction $\square$

10 Let $P(n)$ represent the proposition.

$P(1)$ is true since $1^2 < 4^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $k^2 < 4^k$

RTP $P(k+1)$ $\quad (k+1)^2 < 4^{k+1}$

$$\begin{aligned}\text{LHS} &= (k+1)^2\\ &= k^2 + 2k + 1\\ &< 4^k + 2k + 1 \quad \text{from } P(k)\\ &< 4^k + 3 \times 4^k \quad \text{since } 2k + 1 < 3\left(4^k\right) \text{ for } k \geq 1\\ &= 4\left(4^k\right)\\ &= 4^{k+1}\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

11 Let $P(n)$ represent the proposition.

$P(2)$ is true since $\text{LHS} = 12^2 = 144; \text{RHS} = 7^2 + 5^2 = 74$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $12^k > 7^k + 5^k$

RTP $P(k+1)$ $\quad 12^{k+1} > 7^{k+1} + 5^{k+1}$

$$\begin{aligned}\text{LHS} &= 12(12^k)\\ &> 12\left(7^k + 5^k\right) \quad \text{from } P(k)\\ &= 12(7^k) + 12(5^k)\\ &> 7\left(7^k\right) + 5\left(5^k\right)\\ &= 7^{k+1} + 5^{k+1}\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction $\square$

12 Let $P(n)$ represent the proposition.

$P(1)$ is true since $1! \geq \left((1)!\right)^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $1! \times 3! \times 5! \times \ldots \times (2k-1)! \geq (k!)^k$

RTP $P(k+1)$ $\quad 1! \times 3! \times 5! \times \ldots \times (2k-1)! \times (2k+1)! \geq ((k+1)!)^{k+1}$

$$\begin{aligned}
\text{LHS} &= 1! \times 3! \times 5! \times \ldots \times (2k-1)! \times (2k+1)! \\
&\geq (k!)^k \times (2k+1)! \quad \text{from } P(k) \\
&= (k!)^k \cdot \left(\underbrace{(2k+1)\cdot(2k)\cdot(2k-1)\ldots(k+2)}_{k \text{ terms}}(k+1)k!\right) \\
&\geq (k!)^k \cdot \left((k+1)^k (k+1)!\right) \\
&= \left((k+1)k!\right)^k \cdot (k+1)! \\
&= \left((k+1)!\right)^k \cdot (k+1)! \\
&= \left((k+1)!\right)^{k+1} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

13 Let $P(n)$ represent the proposition.

$P(2)$ is true since $1^3 < \frac{2^4}{4} < 1^3 + 2^3 \rightarrow 1 < 4 < 9$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $1^3 + 2^3 + \ldots + (k-1)^3 < \frac{k^4}{4} < 1^3 + 2^3 + \ldots + k^3$

RTP $P(k+1)$ $\quad 1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{(k+1)^4}{4} < 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$

$$1^3 + 2^3 + \ldots + (k-1)^3 < \frac{k^4}{4} < 1^3 + 2^3 + \ldots + k^3 \quad \text{from } P(k)$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 < 1^3 + 2^3 + \ldots + k^3 + k^3$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 < 1^3 + 2^3 + \ldots + k^3 + k^3$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 + \frac{3}{2}k^2 + k + \frac{1}{4} < 1^3 + 2^3 + \ldots + k^3 + k^3 + \frac{3}{2}k^2 + k + \frac{1}{4}$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{4} < 1^3 + 2^3 + \ldots + k^3 + k^3 + 3k^2 + 3k + 1$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{(k+1)^4}{4} < 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

3.2 OTHER INDUCTION PROOFS

In Lesson 2 we look at unusual applications of mathematical induction, some of which have appeared in the Extension 2 HSC in recent years. We cover:

- Choosing a Method in Unusual Questions
- Calculus
- Probability
- Geometry
- First Order Recursive Formula

CHOOSING A METHOD IN UNUSUAL QUESTIONS

In all of the questions to follow, the trick in solving them involves finding how the relation being true for $n = k$ leads to it being true for $n = k + 1$. This means we need to find an instance of the smaller problem in the larger problem. In the first five examples we will use algebraic and calculus skills to prove $P(k) \Rightarrow P(k + 1)$, while in the geometry questions we will grow or change the previous diagram.

CALCULUS

We will start by looking at using induction to prove calculus results. We need to keep in mind that although these results are actually true for all real n, induction only proves that they are true for the positive integers.

Example 1

Prove that for any positive integer n, $\frac{d}{dx}(x^n) = nx^{n-1}$

Solution

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{d}{dx}(x^1) = 1$ and $1x^{1-1} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$

RTP $P(k+1)$ $\qquad\qquad \frac{d}{dx}\left(x^{k+1}\right) = (k+1)x^k$

$$\begin{aligned}
\text{LHS } &= \frac{d}{dx}\left(x \cdot x^k\right) \\
&= x \cdot \frac{d}{dx}\left(x^k\right) + x^k \cdot \frac{d}{dx}(x) \quad \text{by the product rule} \\
&= x \cdot kx^{k-1} + x^k \cdot 1 \qquad \text{from } P(k) \\
&= kx^k + x^k \\
&= (k+1)x^k \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

Example 2

Prove that for any positive integer n,

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

Solution

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\text{LHS} = (x+a)^1 = x + a$ and

$$\text{RHS} = \sum_{r=0}^{1} {}^1C_r\, x^{1-r} a^r = {}^1C_0 x^1 a^0 + {}^1C_1 x^0 a^1 = x + a$$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $(x+a)^k = \sum_{r=0}^{k} {}^kC_r\, x^{k-r} a^r$

RTP $P(k+1)$ $(x+a)^{k+1} = \sum_{r=0}^{k+1} {}^{k+1}C_r\, x^{k-r+1} a^r$

$$\begin{aligned}
\text{LHS} &= (x+a)(x+a)^k \\
&= (x+a)\sum_{r=0}^{k} {}^kC_r\, x^{k-r} a^r \qquad \text{from } P(k) \\
&= \sum_{r=0}^{k} {}^kC_r\, x^{k-r+1} a^r + \sum_{r=0}^{k} {}^kC_r\, x^{k-r} a^{r+1} \\
&= \sum_{r=0}^{k} {}^kC_r\, x^{k-r+1} a^r + \sum_{r=1}^{k+1} {}^kC_{r-1}\, x^{k-r+1} a^r \\
&= x^{k+1} + \sum_{r=1}^{k} \left({}^kC_r + {}^kC_{r-1}\right) x^{k-r+1} a^r + a^{k+1} \\
&= x^{k+1} + \sum_{r=1}^{k} {}^{k+1}C_r x^{k-r+1} a^r + a^{k+1} \qquad \text{since } {}^{k+1}C_r = {}^kC_r + {}^kC_{r-1} \\
&= \sum_{r=0}^{k+1} {}^{k+1}C_r\, x^{k-r+1} a^r \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

Example 3

Prove that the sum of the exterior angles of an n-sided plane convex polygon is $360°$

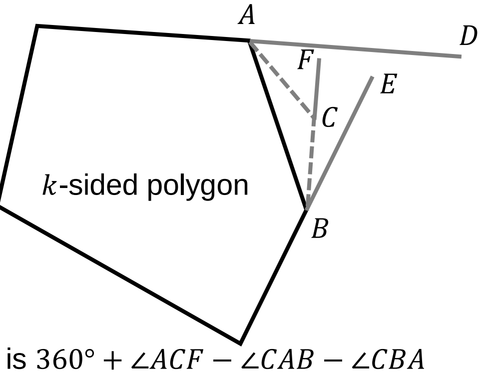
Solution

Let $P(n)$ represent the proposition.

$P(3)$ is true since the sum of the interior and exterior angle at each of the three vertices is $180°$, totalling $540°$. Subtracting the interior angle sum of a triangle, $180°$, we are left with an exterior angle sum of $360°$.

If $P(k)$ is true for some arbitrary $k \geq 3$, then the sum of the exterior angles of a k-sided plane convex polygon is $360°$

RTP $P(k+1)$ The sum of the exterior angles of a $(k+1)$-sided plane convex polygon is $360°$



Consider the $(k+1)$ - sided polygon at right, formed by adding ΔABC to one side of the k-sided polygon.

The exterior angle sum of the new polygon will be equal to that of the k-sided polygon:

- minus $\angle CAB$ from exterior angle $\angle DAB$
- minus $\angle CBA$ from exterior angle $\angle EBA$
- plus the new exterior angle $\angle ACF$

$\therefore$ the sum of the exterior angles of a $(k+1)$-sided polygon is $360° + \angle ACF - \angle CAB - \angle CBA$ from $P(k)$.
Now $\angle ACF = \angle CAB + \angle CBA$ (exterior angle of ΔABC).
The sum of the exterior angles of a $k+1$-sided plane convex polygon is $360° + \angle ACF - \angle AFC$ $= 360°$
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 3$ by induction □

FIRST ORDER RECURSIVE FORMULA

A first order recursive formula creates a sequence where each term is determined by the previous term. There always seem to be two formulas in these questions, but you are given the recursive formula (the one linking T_n and T_{n-1}) and must prove the formula linking T_n and n. Reading the questions carefully you will see this is what they ask.

The syllabus limits us to first order recursive formula, so the Fibonacci Sequence cannot be tested as it relies on the two previous terms so is a second order recursive formula. It is it still useful and interesting.

Example 4

A sequence b_n is defined by $b_0 = 5$ and $b_n = 4 + b_{n-1}$ for $n \geq 1$.
Prove that $b_n = 5 + 4n$ for $n \geq 1$.

Solution

b_1 is true since $\text{LHS} = 4 + b_0 = 4 + 5 = 9$ and $\text{RHS} = 5 + 4(1) = 9$.

If $P(k)$ is true for some arbitrary $k \geq 1$ then $b_k = 5 + 4k$

RTP $P(k+1)$ $\quad b_{k+1} = 5 + 4(k+1)$

$$\begin{aligned}
\text{LHS} &= b_{k+1} \\
&= 4 + b_k && \text{from the recursive formula} \\
&= 4 + 5 + 4k && \text{from } P(k) \\
&= 5 + 4(k+1) \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\quad \square$

EXERCISE 3.2

BASIC

1 Prove that for any positive integer n, $\frac{d}{dx}(x^n) = nx^{n-1}$

2 Prove that for any positive integer n, $(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$

3 Prove that the sum of the interior angles of an n-sided plane convex polygon is $180(n-2)°$

4 Given $a_1 = 2$ and $a_n = 5a_{n-1}$ for $n \geq 2$, prove that $a_n = 2 \times 5^{n-1}$ for $n \geq 1$.

5 Given $a_0 = A$ and $a_n = (1+r)a_{n-1}$, show that $a_n = A(1+r)^n$ for $n \geq 0$

MEDIUM

6 Prove that for any positive integer n, $\frac{d}{dx}(\cos^n x) = -n\cos^{n-1} x \sin x$

7 Prove that for any positive integer $n \geq 1$ that $\binom{n}{1} = n$.

You may assume $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ and $\binom{n}{0} = 1$

8 A plane is divided into regions by one or more intersecting circles. Prove that it is possible to colour the regions with only two colours, such that no two regions sharing an edge are the same colour.

9 Given $a_1 = 2$ and $a_n = \frac{a_{n-1}}{n}$ for $n \geq 2$, prove that $a_n = \frac{2}{n!}$ for $n \geq 1$

CHALLENGING

10 Chessboard Problem Prove that it is possible to cover a $2^n \times 2^n$ grid with L tiles consisting of 3 squares if 1 square is removed.

11 Tower of Hanoi You have three pegs and a collection of disks of different sizes. Initially all of the disks are stacked on top of each other according to size on the first peg

- the largest disk being on the bottom and the smallest on top, as shown above. A move in this game consists of moving a disk from one peg to another, subject to the condition that a larger disk may never rest on a smaller one. The objective of the game is to find a number of permissible moves that will transfer all of the disks from the first peg to the third peg, making sure that the disks are assembled on the third peg according to size. The second peg is used as an intermediate peg. Prove that it takes $2^n - 1$ moves to move n disks from the first peg to the third peg.

12 Postage Stamp Problem Prove any integer amount of postage in cents $n \geq 12$ can be paid for exactly using only 4 cent and 5 cent stamps.

Question 1-3

See the lesson for solutions

4 Let $P(n)$ represent the proposition.

$P(1)$ is true since $a_1 = 2$ and $a_1 = 2 \times 5^{1-1} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = 2 \times 5^{k-1}$

RTP $P(k+1)$ $\quad a_{k+1} = 2 \times 5^k$

$\text{LHS} = a_{k+1}$

$= 5a_k$ $\quad$ from the recursive formula

$= 5\left(2 \times 5^{k-1}\right)$ $\quad$ from $P(k)$

$= 2 \times 5^k$

$= \text{RHS}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

5 Let $P(n)$ represent the proposition.

$P(0)$ is true since $a_0 = A$ and $a_0 = A(1+r)^0 = A$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = A(1+r)^k$

RTP $P(k+1)$ $\quad a_{k+1} = A(1+r)^{k+1}$

$\text{LHS} = a_{k+1}$

$= (1+r)a_k$ $\quad$ from the recursive formula

$= (1+r) \times A(1+r)^k$ $\quad$ from $P(k)$

$= A(1+r)^{k+1}$

$= \text{RHS}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 0$ by induction $\square$

6 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{d}{dx}(\cos^1 x) = -\sin x$ and $-1\cos^{1-1} x \sin x = -\sin x$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{d}{dx}\left(\cos^k x\right) = -k\cos^{k-1} x \sin x$

RTP $P(k+1)$ $\quad \frac{d}{dx}\left(\cos^{k+1} x\right) = -(k+1)\cos^k x \sin x$

$$\text{LHS } = \frac{d}{dx}\left(\cos x \cos^k x\right)$$

$$= \cos x \frac{d}{dx}\left(\cos^k x\right) + \cos^k x \times \frac{d}{dx}(\cos x)$$

$$= \cos x \left(-k\cos^{k-1} x \sin x\right) + \cos^k x\,(-\sin x) \qquad \text{from } P(k)$$

$$= -k\cos^k x \sin x - \cos^k x \sin x$$

$$= -(k+1)\cos^k x \sin x$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

7 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\binom{1}{1} = \frac{1!}{0!1!} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\binom{k}{1} = k$

RTP $P(k+1)$ $\quad \binom{k+1}{1} = k+1$

$$\text{LHS} = \binom{k+1}{1}$$

$$= \binom{k}{1} + \binom{k}{0} \qquad \text{since} \binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

$$= k+1 \qquad \text{from } P(k)$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

8 Let $P(n)$ represent the proposition.

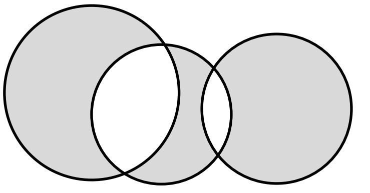
$P(1)$ is true since with one circle we can colour the inside of the circle with one colour and the outside the other.

If $P(k)$ is true for some arbitrary, then with k circles we can colour the resulting regions with only two colours so no two regions sharing an edge are the same colour.

RTP $P(k+1)$ We can colour the regions that result from $k+1$ circles with only two colours so no two regions sharing an edge are the same colour.

Consider the k circles on the plane, with no two regions sharing a border being the same colour as shown (top right).

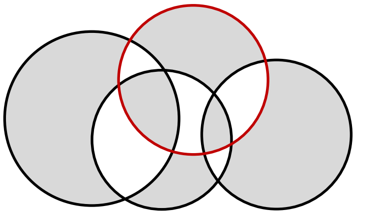
k circles



Add a circle, here shown in red, cutting some regions in two, and possibly leaving others unaffected.

For all regions on the inside of the circle swap their colour, as shown (bottom right).

$k+1$ circles



No two regions sharing an edge are the same colour.

$\therefore$ We can colour the regions that result from $k+1$ circles with only two colours so no two regions sharing an edge are the same colour.

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

9 Let $P(n)$ represent the proposition.

$P(1)$ is true since $a_1 = 2$ and $a_1 = \frac{2}{1!} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = \frac{2}{k!}$

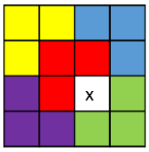
RTP $P(k+1)$ $\quad a_{k+1} = \frac{2}{(k+1)!}$

$$\begin{aligned} \text{LHS} &= a_{k+1} \\ &= \frac{a_k}{k+1} \qquad \text{from the recursive formula} \\ &= \frac{2}{k!} \div (k+1) \qquad \text{from } P(k) \\ &= \frac{2}{(k+1)!} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

10



Here we have a $2^2 \times 2^2$ grid with 1 square removed, with the remainder covered in 5 L tiles.

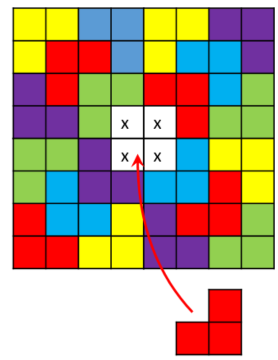
Let $P(n)$ represent the proposition.

$P(1)$ is true since a $2^1 \times 2^1$ grid with one square removed can be covered by one L tile.

If $P(k)$ is true for some arbitrary k then it is possible to completely cover a $2^k \times 2^k$ grid with L tiles consisting of 3 squares if 1 square is removed.

RTP $P(k+1)$ It is possible to completely cover a $2^{k+1} \times 2^{k+1}$ grid with L tiles consisting of 3 squares if 1 square is removed.

Arrange 4 of the $2^k \times 2^k$ grids in a square, removing the square from each grid that is in the centre of the new square, which is a $2^{k+1} \times 2^{k+1}$ grid, as shown at right.



Place one extra tile in the centre, so only 1 square has been removed from the $2^{k+1} \times 2^{k+1}$ grid and the rest is covered.

$\therefore$ It is possible to completely cover a $2^{k+1} \times 2^{k+1}$ grid with L tiles consisting of 3 squares if 1 square is removed.
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

11

Let $P(n)$ represent the proposition.

$P(1)$ is true since the one disk can be moved to the third peg in 1 move, and $2^1 - 1 = 1$

If $P(k)$ is true for some arbitrary k then it takes $2^k - 1$ moves to move k disks from the first peg to the third peg.

RTP $P(k+1)$ It takes $2^{k+1} - 1$ moves to move k disks from the first peg to the third peg.

Move the first k disks to the second peg instead of the third peg, in $2^k - 1$ moves, from $P(k)$.
Move the bottom disk to the third peg in one move.
Move the first k disks again until they are on the third peg on top of the largest disk, in $2^k - 1$ moves, again from $P(k)$.

$$\begin{aligned} \text{Total moves} &= (2^k - 1) + 1 + (2^k - 1) \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \text{as required} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

12 Let $P(n)$ represent the proposition.

$P(12)$ is true since we can use three 4 cent stamps.

If $P(k)$ is true for some arbitrary $k \geq 12$ then we can pay for it exactly using only 4 cent and 5 cent stamps.

RTP $P(k+1)$
We can pay for $k+1$ cents of postage exactly using only 4 cent and 5 cent stamps.

Case 1: We have used at least one 4 cent stamp to make k cents postage
Remove one 4 cents stamp and replace it with a 5 cent stamp, increasing the postage by 1 cent from k to $k+1$ cents.

Case 2: There are no 4 cent stamps used to make k cents postage.
There must be at least three 5 cent stamps here, and if we remove three 5 cent stamps and replace them with four 4 cent stamps the postage increases by 1 cent from k to $k+1$ cents.

$\therefore$ We can pay for $k+1$ cents postage.
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 12$ by induction $\square$

APPENDIX 1: EXTENSION 1 MATHEMATICAL INDUCTION

In Appendix 1 we will cover the work on Mathematical Induction from the Extension 1 course, for those who find themselves up to Extension 2 Induction without having done Extension 1 Induction yet.

3.0A INTRODUCTION TO MATHEMATICAL INDUCTION

In Lesson 3.0A we look at the theory of mathematical induction, and examples involving sums.

We will cover:

- An Introduction to Mathematical Induction
    - Base Case
    - Inductive Step
- Analogies
    - Falling Dominos
    - Climbing a Ladder
    - Burning Down a Skyscraper
    - How does the Analogy Match the Proof?
- Looking at the Two Steps in Depth
    - Inductive Step & If-then Statements
    - Base Case
- Setting out a Mathematical Induction Proof
- Miscellaneous Notes on Induction
- Proving results with Sums

THE PRINCIPLE OF MATHEMATICAL INDUCTION

Mathematical Induction is a great technique that is used when we need to prove a proposition for an infinite number of positive integers.

It consists of two proofs:

- Base Case: We prove that it is true for $n = 1$
- Inductive Step: We prove that if it is true for $n = k$ (the Inductive Hypothesis) then it is true for $n = k + 1$.

The two proofs together create an automated loop that proves a proposition for every positive integer.

Let's look at an example before we study the technique more deeply. We will prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$, with the proof set out in a slightly different way to that which many of us are used to.

Example 1

Prove $1+2+3+\ldots+n=\frac{n(n+1)}{2}$ for $n \geq 1$ by induction.

Solution

Let $P(n)$ represent the proposition.

Definition

$P(1)$ is true since $1=\frac{1(1+1)}{2}$

Base Case

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$1+2+3+\ldots+k=\frac{k(k+1)}{2}$$

Induction Hypothesis

RTP $P(k+1)$:

$$1+2+3+\ldots+k+(k+1)=\frac{(k+1)(k+2)}{2}$$

RTP

$$\begin{aligned}
\text{LHS} &= \frac{k(k+1)}{2}+(k+1) && \text{from } P(k) \\
&= \frac{k(k+1)}{2}+\frac{2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$P(k) \Rightarrow P(k+1)$

Inductive Step

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

Finishing Statement

Now we have seen an example, let's have a look at some analogies to explain what we have just done.

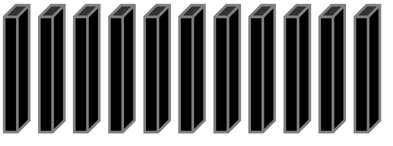
ANALOGIES FOR MATHEMATICAL INDUCTION

The new courses have a greater emphasis on understanding how the Base Case and Inductive Step interact. A lot of mathematicians can use the technique successfully without understanding how it works, and with the changes in the syllabus this will not be good enough.

Let's start by looking at some analogies that help us gain a deeper understanding of how proof by induction works.

FALLING DOMINOS

By far the most popular analogy is falling dominos. We can think of Induction proofs like knocking over dominos standing in a line - what conditions will cause all the dominos to fall?



If the first domino falls over, **and,**
if when any domino falls **then** the next domino falls,
then all the dominos will fall.

When we combine the two steps then the first domino knocks over the second domino, the second knocks over the third, the third knocks over the fourth, with the process repeating infinitely, and so all dominoes fall.

What happens if:

- The first domino doesn't fall (no one pushes it over).
- Any domino that falls doesn't knock over the next domino (the dominos are too far apart say).

You need to realise the significance of both steps - if either is missing then at most only one domino will fall.

The only reservation with the domino analogy is that it almost presumes that every proposition will be true, because as everyone knows, if you line up some dominos they must fall!

We can adjust the analogy so that it fits more involved induction proofs in Extension 2, such as:

- we could start by pushing over the second or third domino rather than the first, knocking over all dominos after that
- we could arrange the dominos in two offset rows - so that when we push over the first or second domino, only the odd or even dominos fall respectively

CLIMBING A LADDER

Another popular analogy involves climbing an infinitely tall ladder. How do you climb every rung?

If you step from the ground onto the first rung, **and if** you step from any rung to the one above **then** you will climb every rung.

What happens if:

- You can't step onto the first rung (it is too high off the ground)
- You can't step from any rung to the next (the rungs are too far apart).

Again you need to realise the significance of both steps - if either is missing then at most only one rung will be climbed.

When we combine the two steps then we step from the ground to the first rung, then the first rung to the second rung, and continue infinitely.

There a couple of reservations with the ladder analogy:

- ladders can be climbed or descended, whereas mathematical induction only works in one direction.
- ladders are meant to be climbed, which again almost presumes that every proposition will be true.

It is harder to adjust the ladder analogy to fit more detailed induction proofs in Extension 2 and still make sense, but it can be done:

- we could somehow start climbing from the second or third rung rather than the first
- we could climb two rungs at a time, so we only step on the odd or even rungs

BURNING DOWN A SKYSCRAPER

So now let's look at an analogy that you won't intuitively think is always true.

Dastardly Dan the Dodgy Demolition Dude has taken the contract to demolish an infinitely tall skyscraper. He didn't understand the concept of infinity when he was a student, and has just realised he is about to lose lots of money … an infinite amount! Luckily he was listening when his class was taught about mathematical induction.

So instead of demolishing it properly he wants to see if he can burn it to the ground with one match - think of the money he will save! He determines that:

If he sets fire to the first* floor, **and if** every floor once alight causes the floor above to burn **then** the skyscraper will burn to the ground.

* Call the lowest floor the first floor, not the ground floor, as in Extension 1 the Base Case is always $n = 1$.

What happens if:
- He can't set fire to the first floor (it might have sprinklers)
- Any floor catching fire doesn't cause the one above to catch fire (the exteriors might be fire resistant).

Again you need to realise the significance of both steps - if either is missing then at most only one floor will burn.

Because most buildings don't burn down, or only partially burn, you will have a much better intuitive understanding of the importance of both steps.

When we combine the two steps then the first floor being on fire causes the second floor to catch alight, then the second floor causes the third floor to catch alight etc.

There are of course many reservations about the burning building analogy, but more social than mathematical! We must assume that fire can only spread up not down.

We can adjust the analogy so that it fits more detailed induction proofs in Extension 2, such as:
- the fire could be started on a higher floor rather than the first floor, and only burn the floors from there up
- every second floor could have a wider balcony causing the fire to skip a floor, only burning down the even (or odd) floors

HOW DOES THE ANALOGY MATCH THE PROOF?

Now let’s match the steps of the domino analogy with the steps of an induction proof. We could do this with any of the analogies.

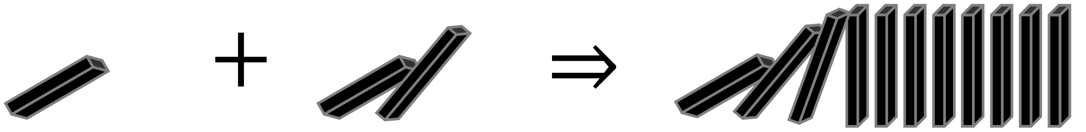
The Base Case is like checking that the first domino falls over.

The Inductive Step is like checking that if any domino falls it will knock over the next domino. This involves three steps:

- We assume* that a domino falls over - this is the inductive hypothesis.
- Then we see what the next domino falling over would look like - this is the RTP.
- We prove that the next domino will fall.

* The word ‘assume’ can cause misunderstanding - I prefer to use ‘if’.

When we combine the Base Case and the Inductive Step we can see that the proposition must be true for any positive number. This is like proving that all dominoes must be knocked over



1^{st} k^{th} $k+1^{\text{th}}$ 1^{st} 2^{nd} 3^{rd} 4^{th} …

Base Case   Inductive Step   The whole row falls

If the proposition is true for the Base Case **and, if** when the proposition is true for $n = k$ **then** it must be true for $n = k + 1$, **then** the proposition must be true for all $n \geq 1$.

The first couple of times you write out an induction proof it can be a good idea to write down what each step is down the side, as we will see in the next example - only do it for the first one or two proofs though.

Example 2

Prove $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for $n \geq 1$

Solution

Let $P(n)$ represent the proposition.	Definition
$P(1)$ is true since $1 = 1^2$	Base Case
If $P(k)$ is true for some arbitrary $k \geq 1$ then $1 + 3 + 5 + \ldots + (2k - 1) = k^2$	Induction Hypothesis
RTP $P(k+1)$ $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$	RTP
$\text{LHS} = k^2 + 2k + 1 \quad \text{from } P(k)$ $= (k + 1)^2$ $= \text{RHS}$ $\therefore P(k) \Rightarrow P(k+1)$	$P(k) \Rightarrow P(k+1)$
$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$	Finishing Statement

Inductive Step

Example 3

Prove $1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1 \text{ for } n \geq 1$

Solution

Let $P(n)$ represent the proposition.

$P(1)$ is true since $1 = 2^1 - 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $1 + 2 + 2^2 + 2^3 + \ldots + 2^{k-1} = 2^k - 1$

RTP $P(k+1)$ $\quad 1 + 2 + 2^2 + 2^3 + \ldots + 2^{k-1} + 2^k = 2^{k+1} - 1$

$$\begin{aligned}\text{LHS} &= 2^k - 1 + 2^k \quad \text{from } P(k)\\ &= 2 \times 2^k - 1\\ &= 2^{k+1} - 1\\ &= \text{RHS}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

Example 4

Prove $1+4+9+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ for $n\geq 1$

Solution

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\text{LHS}=1^2=1$; $\text{RHS}=\frac{1(1+1)(2(1)+1)}{6}=1$

If $P(k)$ is true for some arbitrary $k\geq 1$ then $1^2+4+9+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$

RTP $P(k+1)$ $\qquad 1^2+4+9+\ldots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$

$$\begin{aligned}\text{LHS}&=\frac{k(k+1)(2k+1)}{6}+(k+1)^2 \qquad \text{from } P(k)\\&=(k+1)\left[\frac{2k^2+k}{6}+k+1\right]\\&=(k+1)\left(\frac{2k^2+k+6k+6}{6}\right)\\&=\frac{(k+1)(2k^2+7k+6)}{6}\\&=\frac{(k+1)(k+2)(2k+3)}{6}\\&=\text{RHS}\end{aligned}$$

Look at RTP and take out common factor rather than expanding

Look at RTP and use it to factorise rather than trying to do it from scratch.

$\therefore P(k)\Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n\geq 1$ by induction $\square$

EXERCISE 3.0A

BASIC

1 Prove $2+4+6+\ldots+2n=n(n+1)$ for $n\geq 1$ by induction.

Mark definition, base case etc down the right hand side.

2 Prove $3+6+9+\ldots+3n=\frac{3n(n+1)}{2}$ for $n\geq 1$

MEDIUM

3 Prove $1+4+4^2+\ldots+4^{n-1}=\frac{4^n-1}{3}$ for $n\geq 1$

4 Prove $1^2+4^2+7^2+\ldots+(3n-2)^2=\frac{n(6n^2-3n-1)}{2}$ for $n\geq 1$

5 Prove $1\times 4+2\times 5+3\times 6+\ldots+n(n+3)=\frac{n(n+1)(n+5)}{3}$ for $n\geq 1$

6 Prove $\frac{1}{1(1+1)}+\frac{1}{2(2+1)}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for $n\geq 1$

7 Prove $2\times 1!+5\times 2!+10\times 3!+\ldots+(n^2+1)\left[n!\right]=n\left[(n+1)!\right]$ for $n\geq 1$

8 Prove $2\times 2+3\times 2^2+4\times 2^3+\ldots+(n+1)\times 2^n=n\times 2^{n+1}$ for $n\geq 1$

9 Prove $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=n+1$ for $n\geq 1$

1

Let $P(n)$ represent the proposition. — Definition

$P(1)$ is true since $2 = 1(1+1)$ — Base Case

If $P(k)$ is true for some arbitrary $k \geq 1$ then — Induction Hypothesis

$$2 + 4 + 6 + \ldots + 2k = k(k+1)$$

RTP $P(k+1)$: — RTP
$2 + 4 + 6 + \ldots + 2k + 2(k+1) = (k+1)(k+2)$

LHS $= k(k+1) + 2(k+1)$ from $P(k)$
$= (k+1)(k+2)$
$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$ — $P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$ — Finishing Statement

Inductive Step

2

Let $P(n)$ represent the proposition.

$P(1)$ is true since $3 = \frac{3(1)(1+1)}{2}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $3 + 6 + 9 + \ldots + 3k = \frac{3k(k+1)}{2}$

RTP $P(k+1)$ $3 + 6 + 9 + \ldots + 3k + 3(k+1) = \frac{3(k+1)(k+2)}{2}$

$$\text{LHS} = \frac{3k(k+1)}{2} + 3(k+1) \qquad \text{from } P(k)$$

$$= 3(k+1)\left[\frac{k}{2} + 1\right]$$

$$= \frac{3(k+1)(k+2)}{2}$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

3
Let $P(n)$ represent the proposition.

$P(1)$ is true since $1 = \frac{4^1-1}{3}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $1 + 4 + 4^2 + \ldots + 4^{k-1} = \frac{4^k-1}{3}$

RTP $P(k+1)$ $\quad 1 + 4 + 4^2 + \ldots + 4^{k-1} + 4^k = \frac{4^{k+1}-1}{3}$

$$\begin{aligned}
\text{LHS} &= \frac{4^k - 1}{3} + 4^k \quad \text{from } P(k) \\
&= \frac{4^k - 1}{3} + 3 \cdot \frac{4^k}{3} \\
&= \frac{4 \cdot 4^k - 1}{3} \\
&= \frac{4^{k+1} - 1}{3} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

4
Let $P(n)$ represent the proposition.

$P(1)$ is true since $\text{LHS} = 1^2 = 1$; $\text{RHS} = \frac{1(6(1)^2-3(1)-1)}{2} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $1^2 + 4^2 + 7^2 + \ldots + (3k-2)^2 = \frac{k(6k^2-3k-1)}{2}$

RTP $P(k+1)$ $\quad 1^2 + 4^2 + 7^2 + \ldots + (3k-2)^2 + (3k+1)^2 = \frac{(k+1)(6(k+1)^2-3(k+1)-1)}{2} = \frac{(k+1)(6k^2+9k+2)}{2}$

$$\begin{aligned}
\text{LHS} &= \frac{k(6k^2 - 3k - 1)}{2} + (3k+1)^2 \quad \text{from } P(k) \\
&= \frac{6k^3 - 3k^2 - k + 18k^2 + 12k + 2}{2} \\
&= \frac{6k^3 + 15k^2 + 11k + 2}{2} \\
&= \frac{6k^3 + 9k^2 + 2k + 6k^2 + 9k + 2}{2} \\
&= \frac{(k+1)(6k^2 + 9k + 2)}{2} \\
&= \frac{(k+1)(6(k+1)^2 - 3(k+1) - 1)}{2} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

5

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\text{LHS} = 1 \times 4 = 4$; $\text{RHS} = \frac{1(1+1)(1+5)}{3} = 4$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $1 \times 4 + 2 \times 5 + 3 \times 6 + \ldots + k(k+3) = \frac{k(k+1)(k+5)}{3}$

RTP $P(k+1)$ $\quad 1 \times 4 + 2 \times 5 + 3 \times 6 + \ldots + k(k+3) + (k+1)(k+4) = \frac{(k+1)(k+2)(k+6)}{3}$

$$\begin{aligned}
\text{LHS} &= \frac{k(k+1)(k+5)}{3} + (k+1)(k+4) \qquad \text{from } P(k) \\
&= (k+1)\left[\frac{k^2+5k}{3} + k + 4\right] \\
&= (k+1)\left[\frac{k^2+5k+3k+12}{3}\right] \\
&= \frac{k+1}{3}(k^2+8k+12) \\
&= \frac{(k+1)(k+2)(k+6)}{3} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

6

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{1}{1(1+1)} = \frac{1}{1+1}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

RTP $P(k+1)$ $\quad \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

$$\begin{aligned}
\text{LHS} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{from } P(k) \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

7

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\text{LHS} = 2 \times 1! = 2$; $\text{RHS} = 1(1+1)! = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (k^2+1)\left[k!\right] = k\left[(k+1)!\right]$$

RTP $P(k+1)$

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (k^2+1)(k!) + \left[(k+1)^2+1\right]\left[(k+1)!\right] = (k+1)\left[(k+2)!\right]$$

$\text{LHS} = k\left[(k+1)!\right] + \left[(k+1)^2+1\right]\left[(k+1)!\right]$ from $P(k)$

$= \left[(k+1)!\right]\left[k + k^2 + 2k + 1 + 1\right]$

$= \left[(k+1)!\right]\left[k^2 + 3k + 2\right]$

$= \left[(k+1)!\right](k+1)(k+2)$

$= (k+1)\left[(k+2)!\right]$

$= \text{RHS}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

8

Let $P(n)$ represent the proposition.

$P(1)$ is true since $\text{LHS} = 2 \times 2 = 4$; $\text{RHS} = 2 \times 2^{1+1} = 4$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \ldots + (k+1) \times 2^k = k \times 2^{k+1}$$

RTP $P(k+1)$

$$2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \ldots + (k+1) \times 2^k + (k+2) \times 2^{k+1} = (k+1) \times 2^{k+2}$$

$\text{LHS} = k \times 2^{k+1} + (k+2) \times 2^{k+1} \qquad \text{from } P(k)$

$= 2^{k+1}\left[k + k + 2\right]$

$= 2^{k+1} \times 2(k+1)$

$= (k+1) \times 2^{k+2}$

$= \text{RHS}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

9

Let $P(n)$ represent the proposition.

$P(1)$ is true since $1 = \frac{1+1}{2(1)}$

If $P(k)$ is true for some arbitrary $k \geq 1$ then

$$\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\ldots\left(1 + \frac{1}{k}\right) = k + 1$$

RTP $P(k+1)$ $\quad \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\ldots\left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k+1}\right) = k + 2$

$\text{LHS} = (k+1) \times \left(1 + \frac{1}{k+1}\right) \quad \text{from } P(k)$

$= k + 1 + 1$

$= k + 2$

$= \text{RHS}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

In Lesson 3.0B we look at divisibility proofs and some miscellaneous aspects related to mathematical induction. We will cover:

- Simple divisibility proofs by induction
- False proof by induction
- Cases where proof by induction is not appropriate

DIVISIBILITY PROOFS

Example 1

Prove $4^n - 1$ is divisible by 3 for $n \geq 1$

Solution

Let $P(n)$ represent the proposition.

$P(1)$ is true since $4^1 - 1 = 3$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $4^k - 1 = 3m$ for integral m

RTP $P(k+1)$ $\quad$ $4^{k+1} - 1 = 3p$ for integral p

$$\begin{aligned} \text{LHS} &= 4\left(4^k\right) - 1 \\ &= 4\left(4^k - 1\right) + 3 \\ &= 4(3m) + 3 && \text{from } P(k) \\ &= 3(4m + 1) \\ &= 3p && \text{since } 4, m \text{ and } 1 \text{ are integral} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\quad \square$

Now we will look at an example involving two indices that might be beyond the Extension 1 syllabus - best to be able to do it anyway!

Example 2

Prove $5^n - 2^n$ is divisible by 3 for $n \geq 1$

Solution

Let $P(n)$ represent the proposition.

$P(1)$ is true since $5^1 - 2^1 = 3 = 3 \times 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $5^k - 2^k = 3m$ for integral m

RTP $P(k+1)$ $5^{k+1} - 2^{k+1} = 3p$ for integral p

$$\begin{aligned}
\text{LHS} &= 5\left(5^k\right) - 2\left(2^k\right) \\
&= 5\left(5^k - 2^k\right) + 3\left(2^k\right) \\
&= 5(3m) + 3\left(2^k\right) && \text{from } P(k) \\
&= 3\left(5m + 2^k\right) \\
&= 3p && \text{since } 5, m, 2 \text{ and } k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

WHEN CAN WE USE MATHEMATICAL INDUCTION?

Mathematical Induction can only be used to prove propositions that meet all of the following criteria.

1. The variable must be a positive integer
2. The rule for any integer must be able to be found from the rule for a smaller integer - we need to find an instance of $P(k)$ in $P(k+1)$ for the induction step to work.
3. There must be a lowest value that we can test

We cannot use induction to prove results for all positive real numbers, as the step-wise nature of the induction proof misses an infinite number of real numbers between each pair of integers. So even in Extension 2 where we will prove that $\frac{d}{dx}(x^n) = nx^{n-1}$, a result that is true for all real values of n except $n = 0$, the induction proof will only prove that it is true for integral values of n.

If we look through the Reference Sheet, the majority of rules cannot be proved by induction as they involve real variables rather than integers. So looking at the top of the first page:

- $A = P(1+r)^n$ is appropriate as the variable n is an integer (P and r being constants within a given question), and $P(1+r)^n$ can be found from $P(1+r)^{n-1}$.
- $\ell = \frac{\theta}{360} \times 2\pi r$ is inappropriate as the variable θ is real (r is a constant within a given circle). We could use induction to prove $\ell = \frac{\theta}{360} \times 2\pi r$ for all integral angles, just not for all angles.

Example 3

Which of the following rules from the Reference Sheet can be proved by induction for all possible values?

a) $\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

b) ${}^nP_r = \frac{n!}{(n-r)!}$

c) $\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$

Solution

a) No, as n is a real variable

b) Yes, as n can only take integral values and ${}^nP_r$ can be found from ${}^{n-1}P_r$

c) No, as n is a real variable - we could prove it just for the integral values though.

CAN WE PROVE A FALSE RESULT BY MATHEMATICAL INDUCTION?

If Mathematical Induction is used properly it is impossible to prove a false result - you will find that either you cannot prove a base case or the inductive step fails.

We can however mistakenly prove a false result if we do any of the following:

- Only test one or more base values without the inductive step
- Only do the inductive step and not test a base case
- Perform both steps but have a logic error or algebraic mistake

Example 4

What is the error in this false proof?

Prove $n^2 - 3n + 2 = 0$ for all $n \geq 1$

FALSE PROOF

Let $P(n)$ represent the proposition.

$P(1)$ is true since $(1)^2 - 3(1) + 2 = 0$

$P(2)$ is true since $(2)^2 - 3(2) + 2 = 0$

…

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

Solution

There is no Inductive Step. We have just tested two cases and extrapolated from there. If we had tested $n = 3$ we would see it is false.

Example 5

Prove all odd numbers are even.

Let $P(n)$ represent the proposition.

If $P(k)$ is true for some arbitrary $k \geq 1$, then

$$2k + 1 = 2m \text{ for integral } m$$

RTP $P(k+1)$ $\quad 2k + 3 = 2p$ for integral p

FALSE PROOF

$$\begin{aligned} \text{LHS} &= 2k + 1 + 2 \\ &= 2m + 2 \qquad \text{from } P(k) \\ &= 2(m+1) \\ &= 2p \qquad \text{for integral } p \text{ since m and 1 are integral} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

Solution

There is no Base Case. We are saying that it will be true everywhere if it is true anywhere, but we haven't shown that it is true anywhere.

BASIC

1 Prove $5^n - 1$ is divisible by 4 for $n \geq 1$

2 Which of the following rules from the Reference Sheet can be proved by induction for all possible values?

a) $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$ b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ c) $S_n = \dfrac{n}{2}(a+l)$

3 What is the error in this false proof?

Prove $6 - 11n + 6n^2 - n^3 = 0$ for all $n \geq 1$

Let $P(n)$ represent the proposition.

$P(1)$ is true since $6 - 11(1) + 6(1)^2 - (1)^3 = 0$

$P(2)$ is true since $6 - 11(2) + 6(2)^2 - (2)^3 = 0$

$P(2)$ is true since $6 - 11(3) + 6(3)^2 - (3)^3 = 0$

…

$\therefore P(n)$ is true for $n \geq 1$ by induction □

FALSE PROOF

4 What is the error in this false proof?

Prove that the sum of the first n even numbers is odd.

Let $P(n)$ represent the proposition.

If $P(k)$ is true for some arbitrary $k \geq 1$, then $2 + 4 + 6 + \ldots + 2k = 2m + 1$
RTP $P(k+1)$
$2 + 4 + 6 + \ldots + 2k + 2(k+1) = 2q + 1$ for integral q
$\text{LHS} = 2m + 1 + 2(k+1)$ from $P(k)$
$= 2(m + k + 1) + 1$
$= 2q + 1$ since m, k and 1 are integral
$= \text{RHS}$
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

FALSE PROOF

So all the sum of the first n even numbers is odd, eg $2 + 4$ is odd!

5 Prove $4^{3n} + 8$ is divisible by 9 for $n \geq 1$

6 Prove $n^3 - n$ is divisible by 3 for $n \geq 1$

7 What is the error in this false proof?

Every positive integer n is much less than 1, 000, 000.

Let $P(n)$ represent the proposition.

FALSE PROOF

$P(1)$ is true, since 1 is much less than a million

If $P(k)$ is true for some arbitrary $k \geq 1$, then k is much less than a million.

RTP $P(k+1)$ that $k+1$ is much less than a million.

Since k is much less than a million from $P(k)$, then adding 1 to it will still create a number much less than a million, so $k+1$ is much less than a million.
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

So all positive integers are less than one million!

CHALLENGING

8 Prove $7^n - 2^n$ is divisible by 5 for $n \geq 1$

9 What is the error in this false proof?

Prove that any set of n points lie on one line.

Let $P(n)$ represent the proposition.

FALSE PROOF

$P(1)$ is true, since one point must lie on one line.

If $P(k)$ is true for some arbitrary $k \geq 1$, then any k points must lie on one line.

RTP $P(k+1)$ that any $k+1$ points lie on one line.

If we add one point to $P(k)$ then there will be a group of $k+1$ points. From $P(k)$ the first k points are on one line, and the last k points must also be on one line, and since at least one point belongs to both groups then all the $k+1$ points are on one line.
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

1 Let $P(n)$ represent the proposition.

$P(1)$ is true since $5^1 - 1 = 4$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $5^k - 1 = 4m$ for integral m

RTP $P(k+1)$ $\quad 5^{k+1} - 1 = 4p$ for integral p

$$\begin{aligned}
\text{LHS} &= 5\left(5^k\right) - 1 \\
&= 5\left(5^k - 1\right) + 4 \\
&= 5(4m) + 4 && \text{from } P(k) \\
&= 4(5m + 1) \\
&= 4p && \text{since } 4, m \text{ and } 1 \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\quad \square$

2 a) Yes, as n can only take integral values and $\binom{n}{r}$ can be found from $\binom{n-1}{r}$

b) No, as u and v are functions, so we cannot use induction.

c) Hmmm sort of. If a and l are fixed then yes, but that is not what this formula is about, as l for $n-1$ and for n are different last terms. We would need to be able to relate the last terms, which would need the common difference d. The other version of S_n is a definite yes, while this is a maybe depending on assumptions.

3 There is no Inductive Step. We have just tested three cases and extrapolated from there. If we had tested $n = 4$ we would see it is false.

4 There is no Base Case. We are saying that it will be true everywhere if it is true anywhere, but we haven't shown that it is true anywhere.

5 Let $P(n)$ represent the proposition.

$P(1)$ is true since $4^{3(1)} + 8 = 72 = 9 \times 8$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $4^{3k} + 8 = 9m$ for integral m

RTP $P(k+1)$ $\quad 4^{3k+3} + 8 = 9p$ for integral p.

$$\begin{aligned}
\text{LHS} &= 4^{3k+3} + 8 \\
&= 4^3 \times 4^{3k} + 8 \\
&= 4^3\left(4^{3k} + 8\right) - 63 \times 8 \\
&= 64(9m) - 9 \times 56 \\
&= 9(64m - 56) \\
&= 9p \quad \text{since } 64, m \text{ and } 56 \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\quad \square$

6 Let $P(n)$ represent the proposition.

$P(1)$ is true since $1^3 - 1 = 0 = 3 \times 0$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $k^3 - k = 3m$ for integral m

RTP $P(k+1)$ $(k+1)^3 - (k+1) = 3p$ for integral p

$$\begin{aligned}
\text{LHS} &= k^3 + 3k^2 + 3k + 1 - k - 1 \\
&= k^3 - k + 3k^2 + 3k \\
&= 3m + 3k^2 + 3k \quad \text{from } P(k) \\
&= 3(m + k^2 + k) \\
&= 3p \quad \text{since } m \text{ and } k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

7 Here we have a logic error created by the inexact language. We have no definition for 'much less', and so at some point $P(k)$ won't imply the truth of $P(k+1)$. For instance, if we say a number is only much less than a million of the difference between them is 1000, then $P(999\,000)$ is true but $P(999\,001)$ is false.

8 Let $P(n)$ represent the proposition.

$P(1)$ is true since $7^1 - 2^1 = 5$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $7^k - 2^k = 5m$ for integral m

RTP $P(k+1)$ $7^{k+1} - 2^{k+1} = 5p$ for integral p

$$\begin{aligned}
\text{LHS} &= 7\left(7^k\right) - 2\left(2^k\right) \\
&= 7\left(7^k - 2^k\right) + 5\left(2^k\right) \\
&= 7(5m) + 5\left(2^k\right) \quad \text{from } P(k) \\
&= 5\left(7m + 2^k\right) \\
&= 5p \quad \text{since } 5, m, 2 \text{ and } k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

9 There is a logic error, as we would need to show that there were two points that belong to both groups. It first breaks down for $P(2) \Rightarrow P(3)$, but also breaks down everywhere after that.

Chapter 4
Integration

Integration and Complex Numbers were the two most important topics, in terms of marks, in HSCs from the old course and we can probably expect this to continue. Integration is a great Extension 2 topic, as you are often given the question with no hints as to how you should approach it, so need to develop a deep understanding of the dozens of different techniques.

Lessons

Integration is covered in 10 lessons.

4.1 Standard Integrals & Completing the Square

4.2 The Reverse Chain Rule and U Substitutions

4.3 Splitting the Numerator and Partial Fractions by Inspection

4.4 Partial Fractions

4.5 Other Substitutions

4.6 Trigonometric Functions I - Powers of Trig Functions and Product to Sum identities

4.7 Trigonometric Functions II - t-results, trig substitutions and rationalising the numerator

4.8 Integration by Parts

4.9 Recurrence Relationships

4.10 Definite Integrals

Appendix 1: Tabular Integration by Parts

Revision Questions

In '1000 Revision Questions', the revision book that goes with this textbook you will find the following questions matching this chapter:

- Revision Exercise 4

  100 graded questions on this topic only
- Revision Exercises 7 (Basic), 8 (Medium) and 9 (Challenging)

  Another 100 questions mixed through other topics for when you finish the course.

Don't forget to do any questions from the exercises in this textbook you haven't done.

In Lesson 1 we introduce Extension 2 Integration and look at the standard integrals, then look at simple questions that can be solved by first completing the square.

We will cover:

- Integration in Extension 2
- Understanding Integration
- Standard Integrals
    - From the Reference Sheet
    - Other Useful Standard Integrals
    - Mnemonics (memory aids)
- Completing the Square

INTEGRATION IN EXTENSION 2

Extension 2 Integration involves a massive number of different techniques, some you have seen before in Advanced and Extension 1, and others which will be new. As well as the new techniques you will learn, Integration in Extension 2 is also harder as you will often be given a bare question with no hints as to how it should be solved - you generally won't be given the substitution if one is needed, or any other hint if a different technique is needed.

In the old course Integration questions were mainly in the easy to medium range (for students who had practised enough) with a few harder questions, although we may see more harder integrations in the new course with the change of content. Concentrate on the small differences between integrands that require different approaches.

There are online integral calculators that can help you if you are doing a question without worked solutions. Search for 'integral calculator' and end up at sites like integral-calculator.com - type in the integrand and it will give you the final solution plus working if you select it. Just keep in mind that it may use different techniques to what you would like, and the final answer may be equivalent to what you want but in a different form.

At the end of this document is a summary of the topic, including many standard integrals that you would benefit from memorising.

UNDERSTANDING INTEGRATION

One of the results of having so many techniques to learn is we often focus on how to do each type without really understanding what they do and how they do it.

The trick to successfully being able to solve the greatest variety of integration questions is to deeply understand that

Integration is the inverse operation of Differentiation

This might seem to be an obvious statement, but until we continually remind ourselves that Integration is all about finding the function that can be differentiated to get the integrand, we will only be following processes without understanding.

STANDARD INTEGRALS

Standard Integrals are common primitive functions, which form the basis of all our integration. They are the anti-derivatives from work we have looked at in Advanced and Extension 1. So for instance:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \Rightarrow \qquad \therefore \int \cos x \, dx = \sin x + c$$

The Reference Sheet has a list of some of the standard integrals, and that will certainly cover most questions in the HSC, but it is important to look at some other standard integrals listed in the next pages, as they can make answering more obscure questions a lot simpler.

LIST OF STANDARD INTEGRALS ON THE CURRENT REFERENCE SHEET

Here is the list of standard integrals from the reference sheet. You should memorise them as you use them in the course, only using the reference sheet as a back up if you forget.

$$\int f'(x)\,[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

$$\text{where } n \neq 1$$

$$\int f'(x) \sin f(x)\, dx = -\cos f(x) + c$$

$$\int f'(x) \cos f(x)\, dx = \sin f(x) + c$$

$$\int f'(x) \sec^2 f(x)\, dx = \tan f(x) + c$$

$$\int f'(x) e^{f(x)}\, dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$$

$$\int f'(x)\, a^{f(x)} dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

OTHER USEFUL STANDARD INTEGRALS

There are other standard integrals that you might occasionally find useful.

$$\int f'(x) \operatorname{cosec}^2 f(x)\;\, dx = -\cot f(x) + c$$

$$\int f'(x) \sec f(x) \tan f(x)\, dx = \sec f(x) + c$$

$$\int f'(x) \operatorname{cosec} f(x) \cot f(x)\, dx = -\operatorname{cosec} f(x) + c$$

$$\int f'(x) \sec f(x)\;\, dx = \ln|\sec f(x) + \tan f(x)| + c$$

$$\int f'(x) \operatorname{cosec} f(x)\, dx = -\ln|\operatorname{cosec} f(x) + \cot f(x)| + c$$

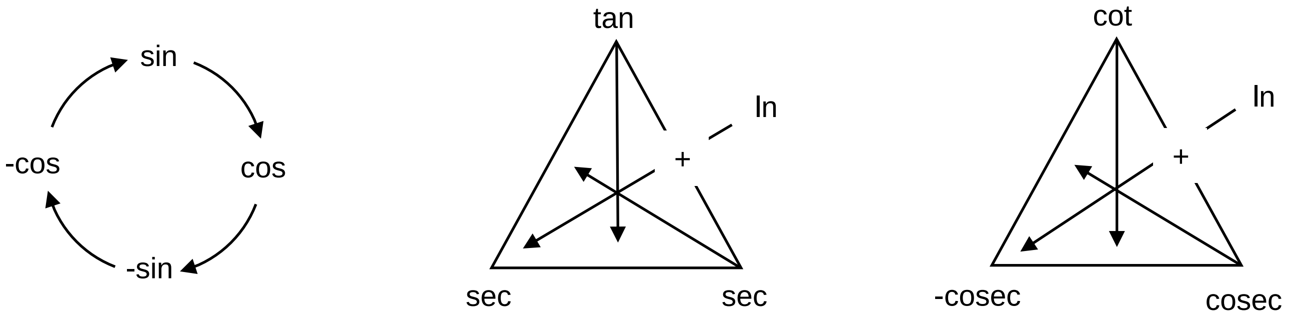
$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}} dx = \ln\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + c$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}} dx = \ln\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + c$$

Now that is a lot of rules to remember, and the trigonometry rules in particular can be quite confusing, so let’s look at some mnemonics that help us remember some of them.

MNEMONICS FOR DIFFERENTIATING AND INTEGRATING WITH TRIGONOMETRY

The mnemonics below can be a handy way to remember the rules for differentiating or integrating with trig functions. For all diagrams the arrows represent differentiation, so go against the arrows to find the primitives.



Sine and Cosine (Left)

To differentiate we follow the arrows clockwise, or to integrate go against the arrows (anticlockwise for the anti-derivatives).

Tangent and Secant (Middle)

To find the derivative of $\tan x$ we follow the arrow from 'tan' - it ends half way between 'sec' and 'sec' - taking the product of these the derivative is $\sec^2 x$. Similarly the derivative of $\sec x$ is $\sec x \tan x$. A more obscure rule also shown is that the derivative of $\ln|\tan x + \sec x|$ is $\sec x$, which we will sometimes use when integrating $\sec x$.

Cotangent and Cosecant (Right)

To find the derivative of $\cot x$ we follow the arrow from 'cot' - it ends half way between '-cosec' and '-cosec' - so the derivative is $-\text{cosec}^2 x$. The derivative of $\text{cosec}\, x$ is $-\text{cosec}\, x \cot x$, and of $\ln|\cot x + \text{cosec}\, x|$ is $-\text{cosec}\, x$.

At the end of this chapter is a summary of the standard integrals and mnemonics.

COMPLETING THE SQUARE

Some integrands need to be manipulated into a different form before we can use the standard integrals. We will start by looking at some requiring completing the square.

Example 1

Find $\displaystyle\int \frac{dx}{x^2 + 10x + 29}$

Solution

$$\int \frac{dx}{x^2 + 10x + 29}$$

$$= \int \frac{dx}{x^2 + 10x + 25 + 4}$$

$$= \int \frac{dx}{(x + 5)^2 + 2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x + 5}{2}\right) + c$$

Example 2

Find $\displaystyle\int \frac{1}{\sqrt{16 - 6x - x^2}}\,dx$

Solution

$$\int \frac{1}{\sqrt{16 - 6x - x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2 + 6x - 16)}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2 + 6x + 9 - 25)}}\,dx$$

$$= \int \frac{1}{\sqrt{5^2 - (x + 3)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x + 3}{5}\right) + c$$

Example 3

Find $\int \frac{1}{\sqrt{x^2 - 6x + 5}} dx$, given $\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}} dx = \ln\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + c$

Solution

$$\int \frac{1}{\sqrt{x^2 - 6x + 5}} dx$$

$$= \int \frac{1}{\sqrt{x^2 - 6x + 9 - 4}} dx$$

$$= \int \frac{1}{\sqrt{(x-3)^2 - 2^2}} dx$$

$$= \ln\left|x - 3 + \sqrt{x^2 - 6x + 5}\right| + c$$

Example 4

Find $\int \frac{e^x}{e^{2x} + 8e^x + 25} dx$

Solution

$$\int \frac{e^x}{e^{2x} + 8e^x + 25} dx$$

$$= \int \frac{e^x}{(e^x + 4)^2 + 3^2} dx$$

$$= \frac{1}{3} \tan^{-1}\left(\frac{e^x + 4}{3}\right) + c$$

EXERCISE 4.1

BASIC

Find

1 $$\int \frac{dx}{x^2 + 6x + 25}$$

2 $$\int \frac{1}{\sqrt{20 - 8x - x^2}}\,dx$$

3 $$\int \frac{1}{\sqrt{x^2 - 8x + 7}}\,dx$$

4 $$\int \frac{1}{x^2 - 3x + 3}\,dx$$

MEDIUM

5 $$\int \frac{1}{\sqrt{4x^2 - 16}}\,dx$$

6 $$\int \frac{2}{4x^2 - 4x + 17}\,dx$$

7 $$\int \frac{2x}{x^4 + 2x^2 + 5}\,dx$$

CHALLENGING

8 $$\int \frac{\cos x}{\sin^2 x + 2\sin x + 5}\,dx$$

9 $$\int \frac{e^x}{\sqrt{e^{2x} + 2e^x - 3}}\,dx$$

10 $$\int \frac{\cos x - \sin x}{2 + \sin 2x}\,dx$$

1

$$\int \frac{dx}{x^2+6x+25}$$

$$= \int \frac{dx}{x^2+6x+9+16}$$

$$= \int \frac{dx}{(x+3)^2+4^2}$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x+3}{4}\right)+c$$

2

$$\int \frac{dx}{\sqrt{20-8x-x^2}}\,dx$$

$$= \int \frac{dx}{\sqrt{-(x^2+8x-20)}}$$

$$= \int \frac{dx}{\sqrt{-\left((x+4)^2-36\right)}}$$

$$= \int \frac{dx}{\sqrt{6^2-(x+4)^2}}$$

$$= \sin^{-1}\left(\frac{x+4}{6}\right)+c$$

3

$$\int \frac{dx}{\sqrt{x^2-8x+7}}$$

$$= \int \frac{dx}{\sqrt{x^2-8x+16-9}}$$

$$= \int \frac{1}{\sqrt{(x-4)^2-3^2}}\,dx$$

$$= \ln\left|x-4+\sqrt{x^2-8x+7}\right|+c$$

4

$$\int \frac{1}{x^2-3x+3}\,dx$$

$$= \int \frac{1}{x^2-3x+\frac{9}{4}+\frac{3}{4}}\,dx$$

$$= \int \frac{1}{\left(x-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{3}{2}}{\sqrt{3}/2}\right)+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-3}{\sqrt{3}}\right)+c$$

5

$$\int \frac{dx}{\sqrt{4x^2-16}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2-2^2}}$$

$$= \frac{1}{2}\ln\left|x+\sqrt{x^2-4}\right|+c$$

6

$$\int \frac{2}{4x^2-4x+17}\,dx$$

$$= \int \frac{2}{(2x-1)^2+4^2}\,dx$$

$$= 2\times\frac{1}{4}\tan^{-1}\left(\frac{2x-1}{4}\right)+c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2x-1}{4}\right)+c$$

7

$$\int \frac{2x}{x^4+2x^2+5}\,dx$$

$$= \int \frac{2x}{(x^2+1)^2+2^2}\,dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{2}\right)+c$$

8

$$\int \frac{\cos x}{\sin^2 x+2\sin x+5}\,dx$$

$$= \int \frac{\cos x}{(\sin x+1)^2+2^2}\,dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\sin x+1}{2}\right)+c$$

9

$$\int \frac{e^x}{\sqrt{e^{2x}+2e^x-3}}\,dx$$

$$= \int \frac{e^x}{\sqrt{(e^x+1)^2-2^2}}\,dx$$

$$= \ln\left|e^x+1+\sqrt{e^{2x}+2e^x-3}\right|+c$$

10

$$\int \frac{\cos x-\sin x}{2+\sin 2x}\,dx$$

$$= \int \frac{\cos x-\sin x}{2+2\sin x\cos x}\,dx$$

$$= \int \frac{\cos x-\sin x}{\sin^2 x+2\sin x\cos x+\cos^2 x+1}\,dx$$

$$= \int \frac{\cos x-\sin x}{(\sin x+\cos x)^2+1}\,dx$$

$$= \tan^{-1}(\sin x+\cos x)+c$$

In Lesson 2 we look at integrals where we need to use the Reverse Chain Rule or a u substitution before we can use the standard integrals.

We will cover:

- The Reverse Chain Rule and u Substitutions

THE REVERSE CHAIN RULE & U SUBSTITUTIONS

Many integration questions in Extension 1 involve u substitutions, and in Extension 1 you will always be given the substitution. You will also often be told you have to use the u substitution. In Extension 2 we could get exactly the same question, but not be told how to solve it - we either need to work out what substitution to use, or use the Reverse Chain Rule which is often quicker. The Reverse Chain Rule is actually the first Standard Integral on the reference sheet.

Mastering the Reverse Chain Rule also makes Integration by Parts and Recurrence Relationships much easier when we get to them later in the chapter.

Consider $\int x(2x^2+3)^3\,dx$. We have the compound function $(2x^2+3)^3$ multiplied by the function x. If we were to use a u substitution then we would let u be equal to the inner function of the compound function, so $u = 2x^2+3$, and we could then solve it like in Extension 1.

With practice it is easier to solve questions like this using the Reverse Chain Rule. Consider first the compound function $y=(2x^2+3)^4$, where the inner function is $f(x)=2x^2+3$ and the outer function is raising this to the power of 4. Differentiating the compound function we would get:

$$
\begin{aligned}
y &= (2x^2+3)^4 \\
\frac{dy}{dx} &= 4(2x^2+3)^3(4x) \\
&= 16x(2x^2+3)^3
\end{aligned}
$$

So to summarise

$$
\begin{array}{c}
(2x^2+3)^4 \\
\text{differentiate} \ \downarrow \quad \uparrow \ \text{integrate} \\
16x(2x^2+3)^3
\end{array}
$$

This means that if we get an integrand that is a multiple of $16x(2x^2+3)^3$ then the primitive will be the same multiple of $(2x^2+3)^4$. So going to the original question we can say:

$$\int x(2x^2+3)^3\,dx$$

$$=\frac{1}{16}\int 16x(2x^2+3)^3\,dx$$

$$=\frac{(2x^2+3)^4}{16}+c$$

Now that explains what is happening for this one question, but it needs a little tweak before we get a method that will work to easily solve many integration questions. Let's redo the same question, but this time we will rearrange the integrand into the form $f'(x)g\big(f(x)\big)$, so the first function $f'(x)$ is the derivative of the inner function of the compound function $g\big(f(x)\big)$. We will often need to multiply by a constant (the fudge factor) to make the integral equivalent. Once the integrand is in that form we can take the primitive of the outer function, leaving the inner function the same as shown below, then simplifying the solution.

$$\int x(2x^2+3)^3\,dx$$

$$=\frac{1}{4}\int 4x(2x^2+3)^3\,dx$$

Take the primitive of the outer function, leaving the inner function the same

$$=\frac{1}{4}\times\frac{(2x^2+3)^4}{4}+c$$

$$=\frac{(2x^2+3)^4}{16}+c$$

We will now look at finding integrals where we can use either the Reverse Chain Rule of u substitutions - each question will be solved using both methods.

It is helpful to think of the functions we use in Integration in five groups: Algebraic, Trig, Inverse Trig, Exponential and Logarithmic. We will use a mix of the five groups as the inner and outer functions.

Example 1

Find $\int x^3(x^4-2)^5\,dx$

Solution

$$\int x^3(x^4-2)^5$$

$$= \frac{1}{4}\int 4x^3(x^4-2)^5\,dx$$

$$= \frac{1}{4}\times\frac{(x^4-2)^6}{6}+c$$

$$= \frac{(x^4-2)^6}{24}+c$$

$$\int x^3(x^4-2)^5$$

$$= \int x^3\times u^5\times\frac{du}{4x^3}$$

$$= \frac{1}{4}\int u^5\,du$$

$$= \frac{1}{4}\times\frac{u^6}{6}+c$$

$$= \frac{(x^4-2)^6}{24}+c$$

$$u = x^4-2$$
$$\frac{du}{dx} = 4x^3$$
$$dx = \frac{du}{4x^3}$$

Example 2

Find $\int \frac{x}{\sqrt{2x^2-1}}\,dx$

Solution

$$\int \frac{x}{\sqrt{2x^2-1}}\,dx$$

$$= \frac{1}{4}\int 4x(2x^2-1)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{4}\times 2(2x^2-1)^{\frac{1}{2}}+c$$

$$= \frac{\sqrt{2x^2-1}}{2}+c$$

$$\int \frac{x}{\sqrt{2x^2-1}}\,dx$$

$$= \int \frac{x}{u}\times\frac{u\,du}{2x}$$

$$= \frac{1}{2}\int du$$

$$= \frac{1}{2}u+c$$

$$= \frac{\sqrt{2x^2-1}}{2}+c$$

$$u^2 = 2x^2-1$$
$$2u\,du = 4x\,dx$$
$$dx = \frac{u\,du}{2x}$$

Example 3

Find $\int e^{3x}\sqrt{e^{3x}+1}\,dx$

Solution

$$\int e^{3x}\sqrt{e^{3x}+1}\,dx$$

$$= \frac{1}{3}\int 3e^{3x}(e^{3x}+1)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\times\frac{2}{3}(e^{3x}+1)^{\frac{3}{2}}+c$$

$$= \frac{2\sqrt{(e^{3x}+1)^3}}{9}+c$$

$$\int e^{3x}\sqrt{e^{3x}+1}\,dx$$

$$= \int e^{3x}\times u\times\frac{2u\,du}{3e^{3x}}$$

$$= \frac{2}{3}\int u^2\,du$$

$$= \frac{2}{3}\times\frac{u^3}{3}+c$$

$$= \frac{2\sqrt{(e^{3x}+1)^3}}{9}+c$$

$$\boxed{\begin{aligned} u^2 &= e^{3x}+1 \\ 2u\,du &= 3e^{3x}dx \\ dx &= \frac{2u\,du}{3e^{3x}} \end{aligned}}$$

Example 4

Find $\int x\sin x^2\,dx$

Solution

$$\int x\sin x^2\,dx$$

$$= \frac{1}{2}\int 2x\sin x^2\,dx$$

$$= -\frac{1}{2}\cos x^2+c$$

$$\int x\sin x^2\,dx$$

$$= \int x\times\sin u\times\frac{du}{2x}$$

$$= \frac{1}{2}\int\sin u\,du$$

$$= -\frac{1}{2}\cos u+c$$

$$= -\frac{1}{2}\cos x^2+c$$

$$\boxed{\begin{aligned} u &= x^2 \\ \frac{du}{dx} &= 2x \\ dx &= \frac{du}{2x} \end{aligned}}$$

Example 5

Find $\int \sqrt{\cos 2x} \sin 2x \, dx$

Solution

$$\int \sqrt{\cos 2x} \sin 2x \, dx$$

$$= -\frac{1}{2}\int (-2\sin 2x)(\cos 2x)^{\frac{1}{2}} \, dx$$

$$= -\frac{1}{2} \times \frac{2}{3}(\cos 2x)^{\frac{3}{2}} + c$$

$$= -\frac{\sqrt{\cos^3 2x}}{3} + c$$

$$\int \sqrt{\cos 2x} \sin 2x \, dx$$

$$= \int u \times \sin 2x \times \left(-\frac{u \, du}{\sin 2x}\right)$$

$$= -\int u^2 \, du$$

$$= -\frac{u^3}{3} + c$$

$$= -\frac{\sqrt{\cos^3 2x}}{3} + c$$

$$\boxed{\begin{aligned} u^2 &= \cos 2x \\ 2u \, du &= -2\sin 2x \, dx \\ dx &= -\frac{u \, du}{\sin 2x} \end{aligned}}$$

Example 6

Find $\int e^x \sin e^x \, dx$

Solution

$$\int e^x \sin e^x \, dx$$

$$= -\cos e^x + c$$

$$\int e^x \sin e^x \, dx$$

$$= \int e^x \sin u \times \frac{du}{e^x}$$

$$= \int \sin u \, du$$

$$= -\cos u + c$$

$$= -\cos e^x + c$$

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x dx \\ dx &= \frac{du}{e^x} \end{aligned}}$$

Example 7

Find $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx$

Solution

$$\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx$$

$$= \int \frac{1}{\sqrt{1-x^2}} \times e^{\sin^{-1} x}\, dx$$

$$= e^{\sin^{-1} x} + c$$

$$\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx$$

$$= \int \frac{e^u}{\sqrt{1-x^2}} \times \sqrt{1-x^2}\, du$$

$$= \int e^u\, du$$

$$= e^u + c$$

$$= e^{\sin^{-1} x} + c$$

$$\boxed{\begin{aligned} u &= \sin^{-1} x \\ du &= \frac{1}{\sqrt{1-x^2}} dx \\ dx &= \sqrt{1-x^2}\, du \end{aligned}}$$

Example 8

Find $\int \frac{\sin(\ln|x|)}{x} dx$

Solution

$$\int \frac{\sin(\ln|x|)}{x} dx$$

$$= \int \frac{1}{x} \sin(\ln|x|)\, dx$$

$$= -\cos(\ln|x|) + c$$

$$\int \frac{\sin(\ln|x|)}{x} dx$$

$$= \int \frac{\sin u}{x} \times x\, du$$

$$= \int \sin u\, du$$

$$= -\cos u + c$$

$$= -\cos(\ln|x|) + c$$

$$\boxed{\begin{aligned} u &= \ln|x| \\ du &= \frac{1}{x} dx \\ dx &= x\, du \end{aligned}}$$

EXERCISE 4.2

BASIC

Find

1 $\displaystyle\int x^2(x^3+4)^5\,dx$

2 $\displaystyle\int \frac{x^2}{\sqrt{2x^3-1}}\,dx$

3 $\displaystyle\int e^{2x}\sqrt{e^{2x}-1}\,dx$

4 $\displaystyle\int x\cos x^2\,dx$

5 $\displaystyle\int \tan^3 x\sec^2 x\,dx$

6 $\displaystyle\int \frac{\cos x}{\sin^5 x}\,dx$

MEDIUM

7 $\displaystyle\int \sin^{\frac{3}{2}} 2x\cos 2x\,dx$

8 $\displaystyle\int \sqrt{\sin 2x}\cos 2x\,dx$

9 $\displaystyle\int e^x\cos e^x\,dx$

10 $\displaystyle\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}}\,dx$

11 $\displaystyle\int \frac{\cos(\ln|x|)}{x}\,dx$

12 $\displaystyle\int (3x^2+2x)\sqrt{x^3+x^2}\,dx$

13 $\displaystyle\int x^2e^{x^3}\,dx$

14 $\displaystyle\int \frac{x^3+x^2}{3x^4+4x^3}\,dx$

15 $\displaystyle\int \frac{\sin(\tan^{-1} x)}{1+x^2}\,dx$

16 $\displaystyle\int (e^{t^2}+16)te^{t^2}\,dt$

17 $\displaystyle\int \frac{\operatorname{cosec} x\cot x}{1+\operatorname{cosec}^2 x}\,dx$

CHALLENGING

18 $\displaystyle\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$

19 $\displaystyle\int \frac{x\sin\left(\sqrt{2x^2-1}\right)}{\sqrt{2x^2-1}}\,dx$

20 $\displaystyle\int \frac{e^{\sqrt{\sin x}}}{\sec x\sqrt{\sin x}}\,dx$

1

$$\int x^2(x^3+4)^5\,dx$$

$$= \frac{1}{3}\int 3x^2(x^3+4)^5\,dx$$

$$= \frac{1}{3}\times\frac{(x^3+4)^6}{6}+c$$

$$= \frac{(x^3+4)^6}{18}+c$$

$$\int x^2(x^3+4)^5\,dx$$

$$= \int x^2\times u^5\times\frac{du}{3x^2}$$

$$= \frac{1}{3}\int u^5\,du$$

$$= \frac{1}{3}\times\frac{u^6}{6}+c$$

$$= \frac{(x^3+4)^6}{18}+c$$

$$u = x^3+4$$

$$\frac{du}{dx} = 3x^2$$

$$dx = \frac{du}{3x^2}$$

2

$$\int \frac{x^2}{\sqrt{2x^3-1}}\,dx$$

$$= \frac{1}{6}\int 6x^2(2x^3-1)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{6}\times 2(2x^3-1)^{\frac{1}{2}}+c$$

$$= \frac{\sqrt{2x^3-1}}{3}+c$$

$$\int \frac{x^2}{\sqrt{2x^3-1}}\,dx$$

$$= \int \frac{x^2}{u}\times\frac{u\,du}{3x^2}$$

$$= \frac{1}{3}\int du$$

$$= \frac{1}{3}u+c$$

$$= \frac{\sqrt{2x^3-1}}{3}+c$$

$$u^2 = 2x^3-1$$

$$2u\,du = 6x^2\,dx$$

$$dx = \frac{u\,du}{3x^2}$$

3

$$\int e^{2x}\sqrt{e^{2x}-1}\,dx$$

$$= \frac{1}{2}\int 2e^{2x}(e^{2x}-1)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\times\frac{2}{3}(e^{2x}-1)^{\frac{3}{2}}+c$$

$$= \frac{\sqrt{(e^{2x}-1)^3}}{3}+c$$

$$\int e^{2x}\sqrt{e^{2x}-1}\,dx$$

$$= \int e^{2x}\times u\times\frac{u\,du}{e^{2x}}$$

$$= \int u^2\,du$$

$$= \frac{u^3}{3}+c$$

$$= \frac{\sqrt{(e^{2x}-1)^3}}{3}+c$$

$$u^2 = e^{2x}-1$$

$$2u\,du = 2e^{2x}dx$$

$$dx = \frac{u\,du}{e^{2x}}$$

4

$$\int x\cos x^2\,dx$$

$$= \frac{1}{2}\int 2x\cos x^2\,dx$$

$$= \frac{1}{2}\sin x^2 + c$$

$$\int x\cos x^2\,dx$$

$$= \int x \times \cos u \times \frac{du}{2x}$$

$$= \frac{1}{2}\int \cos u\,du$$

$$= \frac{1}{2}\sin u + c$$

$$= \frac{1}{2}\sin x^2 + c$$

$$\boxed{\begin{aligned} u &= x^2 \\ \frac{du}{dx} &= 2x \\ dx &= \frac{du}{2x} \end{aligned}}$$

5

$$\int \tan^3 x\sec^2 x\,dx$$

$$= \int \sec^2 x\,(\tan x)^3\,dx$$

$$= \frac{\tan^4 x}{4} + c$$

$$\int \tan^3 x\sec^2 x\,dx$$

$$= \int u^3 \times \sec^2 x \times \frac{du}{\sec^2 x}$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + c$$

$$= \frac{\tan^4 x}{4} + c$$

$$\boxed{\begin{aligned} u &= \tan x \\ du &= \sec^2 x\,dx \\ dx &= \frac{du}{\sec^2 x} \end{aligned}}$$

6

$$\int \frac{\cos x}{\sin^5 x}\,dx$$

$$= \int \cos x\,(\sin x)^{-5}\,dx$$

$$= \frac{(\sin x)^{-4}}{-4} + c$$

$$= -\frac{1}{4\sin^4 x} + c$$

$$\int \frac{\cos x}{\sin^5 x}\,dx$$

$$= \int \frac{\cos x}{u^5} \times \frac{du}{\cos x}$$

$$= \int u^{-5}\,du$$

$$= \frac{u^{-4}}{-4} + c$$

$$= -\frac{1}{4\sin^4 x} + c$$

$$\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x\,dx \\ dx &= \frac{du}{\cos x} \end{aligned}}$$

7

$$\int \sin^{\frac{3}{2}} 2x \cos 2x \, dx$$
$$= \frac{1}{2}\int 2\cos 2x \,(\sin 2x)^{\frac{3}{2}}\, dx$$
$$= \frac{1}{2} \times \frac{2}{5} (\sin 2x)^{\frac{5}{2}} + c$$
$$= \frac{\sin^{\frac{5}{2}} 2x}{5} + c$$

$$\int \sin^{\frac{3}{2}} 2x \cos 2x \, dx$$
$$= \int u^{\frac{3}{2}} \times \cos 2x \times \frac{du}{2\cos 2x}$$
$$= \frac{1}{2}\int u^{\frac{3}{2}}\, du$$
$$= \frac{1}{2} \times \frac{2}{5} u^{\frac{5}{2}} + c$$
$$= \frac{\sin^{\frac{5}{2}} 2x}{5} + c$$

$$u = \sin 2x$$
$$du = 2\cos 2x \, dx$$
$$dx = \frac{du}{2\cos 2x}$$

8

$$\int \sqrt{\sin 2x} \cos 2x \, dx$$
$$= \frac{1}{2}\int (2\cos 2x)(\sin 2x)^{\frac{1}{2}}\, dx$$
$$= \frac{1}{2} \times \frac{2}{3} (\sin 2x)^{\frac{3}{2}} + c$$
$$= \frac{\sqrt{\sin^3 2x}}{3} + c$$

$$\int \sqrt{\sin 2x} \cos 2x \, dx$$
$$= \int u \times \cos 2x \times \frac{u\, du}{\cos 2x}$$
$$= \int u^2 \, du$$
$$= \frac{u^3}{3} + c$$
$$= \frac{\sqrt{\sin^3 2x}}{3} + c$$

$$u^2 = \sin 2x$$
$$2u\, du = 2\cos 2x \, dx$$
$$dx = \frac{u\, du}{\cos 2x}$$

9

$$\int e^x \cos e^x \, dx$$
$$= \sin e^x + c$$

$$\int e^x \cos e^x \, dx$$
$$= \int e^x \cos u \times \frac{du}{e^x}$$
$$= \int \cos u \, du$$
$$= \sin u + c$$
$$= \sin e^x + c$$

$$u = e^x$$
$$du = e^x dx$$
$$dx = \frac{du}{e^x}$$

10

$$\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}}\,dx$$
$$= -\int \frac{-1}{\sqrt{1-x^2}} \times e^{\cos^{-1} x}\,dx$$
$$= -e^{\cos^{-1} x} + c$$

$$\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}}\,dx$$
$$= \int \frac{e^u}{\sqrt{1-x^2}} \times \left(-\sqrt{1-x^2}\,du\right)$$
$$= -\int e^u\,du$$
$$= -e^u + c$$
$$= -e^{\cos^{-1} x} + c$$

$$\boxed{\begin{aligned} u &= \cos^{-1} x \\ du &= \frac{-1}{\sqrt{1-x^2}}\,dx \\ dx &= -\sqrt{1-x^2}\,du \end{aligned}}$$

11

$$\int \frac{\cos(\ln|x|)}{x}\,dx$$
$$= \int \frac{1}{x}\cos(\ln|x|)\,dx$$
$$= \sin(\ln|x|) + c$$

$$\int \frac{\cos(\ln|x|)}{x}\,dx$$
$$= \int \frac{\cos u}{x} \times x\,du$$
$$= \int \cos u\,du$$
$$= \sin u + c$$
$$= \sin(\ln|x|) + c$$

$$\boxed{\begin{aligned} u &= \ln|x| \\ du &= \frac{1}{x}\,dx \\ dx &= x\,du \end{aligned}}$$

12

$$\int (3x^2+2x)\sqrt{x^3+x^2}\,dx$$
$$= \int (3x^2+2x)(x^3+x^2)^{\frac{1}{2}}\,dx$$
$$= \frac{2(x^3+x^2)^{\frac{3}{2}}}{3} + c$$
$$= \frac{2\sqrt{(x^3+x^2)^3}}{3} + c$$

$$\int (3x^2+2x)\sqrt{x^3+x^2}\,dx$$
$$= \int (3x^2+2x) \times u \times \frac{2u\,du}{3x^2+2x}$$
$$= 2\int u^2\,du$$
$$= 2 \times \frac{u^3}{3} + c$$
$$= \frac{2(x^3+x^2)^{\frac{3}{2}}}{3} + c$$
$$= \frac{2\sqrt{(x^3+x^2)^2}}{3} + c$$

$$\boxed{\begin{aligned} u^2 &= x^3+x^2 \\ 2u\,du &= (3x^2+2x)dx \\ dx &= \frac{2u\,du}{3x^2+2x} \end{aligned}}$$

13

$$\int x^2 e^{x^3}\,dx$$
$$= \frac{1}{3}\int 3x^2 e^{x^3}\,dx$$
$$= \frac{1}{3}e^{x^3} + c$$

$$\int x^2 e^{x^3}\,dx$$
$$= \int x^2 \times e^u \times \frac{du}{3x^2}$$
$$= \frac{1}{3}\int e^u\,du$$
$$= \frac{1}{3}e^u + c$$
$$= \frac{1}{3}e^{x^3} + c$$

$$\boxed{\begin{aligned} u &= x^3 \\ du &= 3x^2 dx \\ dx &= \frac{du}{3x^2} \end{aligned}}$$

14

$$\int \frac{x^3+x^2}{3x^4+4x^3}\,dx$$

$$= \frac{1}{12}\int \frac{12x^3+12x^2}{3x^4+4x^3}\,dx$$

$$= \frac{1}{12}\ln|3x^4+4x^3| + c$$

$$\int \frac{x^3+x^2}{3x^4+4x^3}\,dx$$

$$= \int \frac{x^3+x^2}{u} \times \frac{du}{12x^3+12x^2}$$

$$= \frac{1}{12}\int \frac{1}{u}\,du$$

$$= \frac{1}{12}\ln|u| + c$$

$$= \frac{1}{12}\ln|3x^4+4x^3| + c$$

$$\boxed{\begin{aligned} u &= 3x^4+4x^3 \\ du &= (12x^3+12x^2)dx \\ dx &= \frac{du}{12x^3+12x^2} \end{aligned}}$$

15

$$\int \frac{\sin(\tan^{-1}x)}{1+x^2}\,dx$$

$$= \int \frac{1}{1+x^2}\sin(\tan^{-1}x)\,dx$$

$$= -\cos(\tan^{-1}x) + c$$

$$= -\frac{1}{\sqrt{1+x^2}} + c$$

$$\int \frac{\sin(\tan^{-1}x)}{1+x^2}\,dx$$

$$= \int \frac{\sin u}{1+x^2} \times (1+x^2)du$$

$$= \int \sin u\,du$$

$$= -\cos u + c$$

$$= -\cos(\tan^{-1}x) + c$$

$$= -\frac{1}{\sqrt{1+x^2}} + c$$

$$\boxed{\begin{aligned} u &= \tan^{-1}x \\ du &= \frac{1}{1+x^2}dx \\ dx &= (1+x^2)du \end{aligned}}$$

16

$$\int (e^{t^2}+16)te^{t^2}\,dt$$

$$= \frac{1}{2}\int 2te^{t^2}\left(e^{t^2}+16\right)dt$$

$$= \frac{1}{2} \times \frac{\left(e^{t^2}+16\right)^2}{2} + c$$

$$= \frac{1}{4}\left(e^{t^2}+16\right)^2 + c$$

$$\int (e^{t^2}+16)te^{t^2}\,dt$$

$$= \int u \times te^{t^2} \times \frac{du}{2te^{t^2}}$$

$$= \frac{1}{2}\int u\,du$$

$$= \frac{1}{2} \times \frac{u^2}{2} + c$$

$$= \frac{1}{4}\left(e^{t^2}+16\right)^2 + c$$

$$\boxed{\begin{aligned} u &= e^{t^2}+16 \\ du &= 2te^{t^2}dt \\ dt &= \frac{du}{2te^{t^2}} \end{aligned}}$$

17

$$\int \frac{\operatorname{cosec}x\cot x}{1+\operatorname{cosec}^2 x}\,dx$$

$$= -\int \frac{-\operatorname{cosec}x\cot x}{1+(\operatorname{cosec}x)^2}\,dx$$

$$= -\tan^{-1}(\operatorname{cosec}x) + c$$

$$\boxed{\begin{aligned} u &= \operatorname{cosec}x \\ du &= -\operatorname{cosec}x\cot x\,dx \\ dx &= -\frac{du}{\operatorname{cosec}x\cot x} \end{aligned}}$$

$$\int \frac{\operatorname{cosec}x\cot x}{1+\operatorname{cosec}^2 x}\,dx$$

$$= \int \frac{\operatorname{cosec}x\cot x}{1+u^2} \times \left(-\frac{du}{\operatorname{cosec}x\cot x}\right)$$

$$= -\int \frac{1}{1+u^2}\,du$$

$$= -\tan^{-1}u + c$$

$$= -\tan^{-1}(\operatorname{cosec}x) + c$$

18

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$$

$$= \int \frac{\cos^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$

$$= \int \frac{1-\sin^2 x}{\sqrt{\sin x}} \times \cos x\,dx$$

$$= \int \cos x\left((\sin x)^{-\frac{1}{2}} - (\sin x)^{\frac{3}{2}}\right)dx$$

$$= 2(\sin x)^{\frac{1}{2}} - \frac{2}{5}(\sin x)^{\frac{5}{2}} + c$$

$$= 2\sqrt{\sin x} - \frac{2\sqrt{\sin^5 x}}{5} + c$$

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$$

$$= \int \frac{\cos^3 x}{u} \times \frac{2u\,du}{\cos x}$$

$$= 2\int \cos^2 x\,du$$

$$= 2\int (1-\sin^2 x)\,du$$

$$= 2\int (1-u^4)\,du$$

$$= 2u - \frac{2u^5}{5} + c$$

$$= 2\sqrt{\sin x} - \frac{2\sqrt{\sin^5 x}}{5} + c$$

$$\boxed{\begin{aligned} u^2 &= \sin x \\ 2u\,du &= \cos x\,dx \\ dx &= \frac{2u\,du}{\cos x} \end{aligned}}$$

19

$$\int \frac{x\sin\left(\sqrt{2x^2-1}\right)}{\sqrt{2x^2-1}}\,dx$$

$$= \frac{1}{2}\int \left(\frac{1}{2}(2x^2-1)^{-\frac{1}{2}} \times 4x\right)\sin\left(\sqrt{2x^2-1}\right)dx$$

$$= -\frac{1}{2}\cos\sqrt{2x^2-1} + c$$

$$\int \frac{x\sin\left(\sqrt{2x^2-1}\right)}{\sqrt{2x^2-1}}\,dx$$

$$= \int \frac{x\sin u}{u} \times \frac{u\,du}{2x}$$

$$= \frac{1}{2}\int \sin u\,du$$

$$= -\frac{1}{2}\cos u + c$$

$$= -\frac{1}{2}\cos\sqrt{2x^2-1} + c$$

$$\boxed{\begin{aligned} u &= (2x^2-1)^{\frac{1}{2}} \\ du &= \frac{1}{2}(2x^2-1)^{-\frac{1}{2}}(4x)dx \\ dx &= \frac{u\,du}{2x} \end{aligned}}$$

20

$$\int \frac{e^{\sqrt{\sin x}}}{\sec x\sqrt{\sin x}}\,dx$$

$$= 2\int \frac{\cos x}{2\sqrt{\sin x}} e^{\sqrt{\sin x}}\,dx$$

$$= 2e^{\sqrt{\sin x}} + c$$

$$\int \frac{e^{\sqrt{\sin x}}}{\sec x\sqrt{\sin x}}\,dx$$

$$= \int \frac{e^u}{\sec x\sqrt{2x^2-1}} \times \frac{2u\,du}{\cos x}$$

$$= 2\int e^u\,du$$

$$= 2e^u + c$$

$$= 2e^{\sqrt{\sin x}} + c$$

$$\boxed{\begin{aligned} u^2 &= \sin x \\ 2u\,du &= \cos x\,du \\ dx &= \frac{2u\,du}{\cos x} \end{aligned}}$$

In Lesson 3 we spend the first of two lessons looking at integrals involving rational fractions, which must be broken into simpler fractions before we can use the standard integrals. In this lesson we will look at simple rational functions which we can break up by splitting the numerator, or partial fractions by inspection.

We will cover:

- Splitting the numerator, where the degree of the numerator
    - is greater than or equal to that of the denominator
    - is one less than that of the denominator
- Partial Fractions by Inspection

SPLITTING THE NUMERATOR

Rational functions are fractions where the numerator and denominator are both polynomials. Examples include

$$\frac{x^2+2x+3}{x-1} \qquad \frac{x+2}{x+1} \qquad \frac{4x}{x^2+4x-6}$$

Splitting the numerator is a method of rewriting the numerator in terms of another polynomial - the denominator and the derivative of the denominator are useful in different situations. You may have used this method for dividing polynomials in Year 11, where it is a much better alternative to long division. We can also use this method for similar functions - functions that would be rational functions after making a u substitution.

The polynomial we choose depends on the relative degrees of the numerator and denominator. Looking at the three examples above, we see:

1. on the left and in the middle, the degree of the numerator is greater than or equal to that of the denominator
2. on the right the numerator has a degree one less than that o the denominator

In all cases we need to have a rational function where the numerator is less than the denominator before we can use any of the standard integrals.

Case 1: The degree of the numerator is greater than or equal to that of the denominator

If the numerator has degree that is greater than or equal to the degree of the denominator, then we rewrite the numerator as multiples of the denominator, leaving us with the sum of a polynomial and a rational fraction.

Example 1

Find $\int \frac{x^2 + 2x + 3}{x - 1} dx$

Solution

$$\int \frac{x^2 + 2x + 3}{x - 1} dx$$

$$= \int \frac{x(x - 1) + 3(x - 1) + 6}{x - 1} dx$$

$$= \int \left(x + 3 + \frac{6}{x - 1}\right) dx$$

$$= \frac{x^2}{2} + 3x + 6 \ln|x - 1| + c$$

Example 2

Find $\int \frac{x + 2}{x + 1} dx$

Solution

$$\int \frac{x + 2}{x + 1} dx$$

$$= \int \frac{x + 1 + 1}{x + 1} dx$$

$$= \int \left(1 + \frac{1}{x + 1}\right) dx$$

$$= x + \ln|x + 1| + c$$

Example 3

Find $\int \frac{x^2+5}{x^2+1}\,dx$

Solution

$$\int \frac{x^2+5}{x^2+1}\,dx$$

$$= \int \frac{x^2+1+4}{x^2+1}\,dx$$

$$= \int \left(1 + \frac{4}{x^2+1^2}\right)dx$$

$$= x + 4\tan^{-1} x + c$$

The same techniques work for functions that would be rational functions after a u substitution.

Example 4

Find $\int \frac{e^{2x}+e^x}{e^{2x}-e^x}\,dx$

Solution

$$\int \frac{e^{2x}+e^x}{e^{2x}-e^x}\,dx$$

$$= \int \frac{e^{2x}-e^x+2e^x}{e^{2x}-e^x}\,dx$$

$$= \int \left(1 + \frac{2e^x}{e^{2x}-e^x}\right)dx$$

$$= \int \left(1 + \frac{2e^{-x}}{1-e^{-x}}\right)dx$$

$$= \int \left(1 + 2 \times \frac{e^{-x}}{1-e^{-x}}\right)dx$$

$$= x + 2\ln|1-e^{-x}| + c \quad \longleftarrow$$

Getting to this step in an exam would probably be enough

$$= x + 2\ln\left|\frac{e^x-1}{e^x}\right| + c$$

$$= x + 2\ln|e^x-1| - 2\ln|e^x| + c$$

$$= 2\ln|e^x-1| - x + c$$

Case 2: The degree of the numerator is one less than that of the denominator

If the numerator has degree that is one less than the degree of the denominator, then we rewrite the numerator as multiples of the derivative of the denominator, leaving us with the sum of a polynomial and a rational fraction.

Example 5

Find $\int \frac{4x}{x^2 + 4x + 13} dx$

Solution

$$\int \frac{4x}{x^2 + 4x + 13} dx$$

$$= \int \frac{2(2x + 4) - 8}{x^2 + 4x + 4 + 9} dx$$

$$= 2\int \frac{2x + 4}{x^2 + 4x + 13} dx - 8\int \frac{1}{(x + 2)^2 + 3^2} dx$$

$$= 2\ln|x^2 + 4x + 13| - \frac{8}{3}\tan^{-1}\left(\frac{x + 2}{3}\right) + c$$

PARTIAL FRACTIONS BY INSPECTION

Partial Fractions involves breaking a rational function into the sum or difference of two or more simpler fractions. It only works for rational fractions whose denominator can be factorised (think cross multiplying to realise why).

We can create partial fractions by inspection for easier examples, or next lesson we will look at three more formal methods for harder examples.

Example 6

Find $\int \frac{3}{x(x+1)} dx$

Solution

$$\int \frac{3}{x(x+1)} dx$$
$$= 3\int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$
$$= 3\ln|x| - 3\ln|x+1| + c$$
$$= 3\ln\left|\frac{x}{x+1}\right| + c$$

We will try putting 1 over each factor, and subtracting the two fractions, which will get rid of the x in the numerator when we cross multiply. We then need to place a fudge factor out the front, so that when it multiplies by the new fractions we get an answer equivalent to the original question. In this case $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}$ so we need a fudge factor of 3 out the front.

Example 7

Find $\int \frac{1}{x(x^2+1)} dx$

Solution

$$\int \frac{1}{x(x^2+1)} dx$$
$$\int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$$
$$= \int \left(\frac{1}{x} - \frac{1}{2} \times \frac{2x}{x^2+1}\right) dx$$
$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + c$$

We will try putting 1 over each factor, and subtracting the two fractions, but cross multiplying we would be left with an x^2 we don't want and be subtracting an x. We can get around these problems if we make the second denominator x.
Cross multiplying in our heads we see that no fudge factor is needed.

EXERCISE 4.3

BASIC

Find:

1 $\displaystyle\int \frac{x^2-2x-2}{x+1}\,dx$

2 $\displaystyle\int \frac{2x-3}{x+1}\,dx$

3 $\displaystyle\int \frac{2x^2+19}{x^2+9}\,dx$

4 $\displaystyle\int \frac{e^{-x}}{e^{-x}-1}\,dx$

5 $\displaystyle\int \frac{2x}{x^2+4x+20}\,dx$

6 $\displaystyle\int \frac{3x+2}{x^2+6x+10}\,dx$

7 $\displaystyle\int \frac{2}{x(x-1)}\,dx$

8 $\displaystyle\int \frac{2}{x(x^2+4)}\,dx$

MEDIUM

9 $\displaystyle\int \frac{e^{2x}+e^x}{e^{2x}+1}\,dx$

10 $\displaystyle\int \frac{x^4+1}{x^2+2}\,dx$

11 $\displaystyle\int \frac{x^3}{x^2+2}\,dx$

CHALLENGING

12 $\displaystyle\int \frac{2\cos x\,(\sin x+1)}{(\sin x+3)(\sin x-1)}\,dx$

13 $\displaystyle\int \frac{\cos x}{\sin^2 x-3\sin x+2}\,dx$

14 $\displaystyle\int \frac{8x}{1+e^2 x}\,dx$

1

$$\int \frac{x^2 - 2x - 2}{x + 1}\,dx$$

$$= \int \frac{x(x+1) - 3(x+1) + 1}{x+1}\,dx$$

$$= \int \left(x - 3 + \frac{1}{x+1}\right)dx$$

$$= \frac{x^2}{2} - 3x + \ln|x+1| + c$$

2

$$\int \frac{2x - 3}{x + 1}\,dx$$

$$= \int \frac{2(x+1) - 5}{x+1}\,dx$$

$$= \int \left(2 - \frac{5}{x+1}\right)dx$$

$$= 2x - 5\ln|x+1| + c$$

3

$$\int \frac{2x^2 + 19}{x^2 + 9}\,dx$$

$$= \int \frac{2(x^2+9) + 1}{x^2+9}\,dx$$

$$= \int \left(2 + \frac{1}{x^2 + 3^2}\right)dx$$

$$= 2x + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$$

4

$$\int \frac{e^x + 1}{e^x - 1}\,dx$$

$$= \int \frac{e^x - 1 + 2}{e^x - 1}\,dx$$

$$= \int \left(1 + \frac{2}{e^x - 1}\right)dx$$

$$= \int \left(1 + 2\frac{e^{-x}}{1 - e^{-x}}\right)dx$$

$$= x + 2\ln\left|1 - e^{-x}\right| + c$$

5

$$\int \frac{2x}{x^2 + 4x + 20}\,dx$$

$$= \int \frac{2x + 4 - 4}{x^2 + 4x + 4 + 16}\,dx$$

$$= \int \frac{2x+4}{x^2 + 4x + 20}\,dx - 4\int \frac{1}{(x+2)^2 + 4^2}\,dx$$

$$= \ln|x^2 + 4x + 20| - \tan^{-1}\left(\frac{x+2}{4}\right) + c$$

6

$$\int \frac{3x + 2}{x^2 + 6x + 10}\,dx$$

$$= \int \frac{\frac{3}{2}(2x+6) - 7}{x^2 + 6x + 9 + 1}\,dx$$

$$= \frac{3}{2}\int \frac{2x+6}{x^2 + 6x + 10}\,dx - 7\int \frac{1}{(x+3)^2 + 1^2}\,dx$$

$$= \frac{3}{2}\ln|x^2 + 6x + 10| - 7\tan^{-1}(x+3) + c$$

7

$$\int \frac{2}{x(x-1)}\,dx$$

$$= 2\int \left(\frac{1}{x-1} - \frac{1}{x}\right)dx$$

$$= 2\ln\left|x - 1\right| - 2\ln\left|x\right| + c$$

8

$$\int \frac{2}{x(x^2+4)}\,dx$$

$$= \frac{1}{2}\int \left(\frac{1}{x} - \frac{x}{x^2+4}\right)dx$$

$$= \frac{1}{2}\int \left(\frac{1}{x} - \frac{1}{2}\times\frac{2x}{x^2+4}\right)dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2 + 4| + c$$

9

$$\int \frac{e^{2x}+e^x}{e^{2x}+1}\,dx$$

$$= \int \left(\frac{1}{2}\times\frac{2e^{2x}}{e^{2x}+1}+\frac{e^x}{(e^x)^2+1}\right)dx$$

$$= \frac{1}{2}\ln\left|e^{2x}+1\right| + \tan^{-1}(e^x) + c$$

10

$$\int \frac{x^4+1}{x^2+2}\,dx$$

$$= \int \frac{x^2(x^2+2)-2(x^2+2)+5}{x^2+2}\,dx$$

$$= \int \left(x^2-2+\frac{5}{x^2+2}\right)dx$$

$$= x^3-2x+\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$$

11

$$\int \frac{x^3}{x^2+2}\,dx$$

$$= \int \frac{x(x^2+2)-2x}{x^2+2}\,dx$$

$$= \int \left(x-\frac{2x}{x^2+2}\right)dx$$

$$= \frac{x^2}{2}-\ln\left|x^2+2\right|+c$$

12

$$\int \frac{2\cos x\,(\sin x+1)}{(\sin x+3)(\sin x-1)}\,dx$$

$$= \int \frac{2\sin x\cos x+2\cos x}{\sin^2 x+2\sin x-3}\,dx$$

$$= \ln\left|\sin^2 x+2\sin x-3\right|+c$$

13

$$\int \frac{\cos x}{\sin^2 x-3\sin x+2}\,dx$$

$$= \int \frac{\cos x}{(\sin x-2)(\sin x-1)}\,dx$$

$$= \int \left(\frac{\cos x}{\sin x-2}-\frac{\cos x}{\sin x-1}\right)dx$$

$$= \ln\left|\sin x-2\right|-\ln\left|\sin x-1\right|+c$$

$$= \ln\left|\frac{\sin x-2}{\sin x-1}\right|+c$$

14

$$\int \frac{8x}{1+e^2x}\,dx$$

$$= \int \frac{\frac{8}{e^2}(1+e^2x)-\frac{8}{e^2}}{1+e^2x}\,dx$$

$$= \int \left(\frac{8}{e^2}-\frac{8}{e^4}\times\frac{e^2}{1+e^2x}\right)dx$$

$$= \frac{8x}{e^2}-\frac{8\ln\left|1+e^2x\right|}{e^4}+c$$

4.4 PARTIAL FRACTIONS

In Lesson 4 we look at splitting rational functions into simple fractions, using more formal methods than we used last lesson. We look at three different methods of splitting a fraction into partial fractions, which can be simply integrated.

We will cover:

- Background Information
- Partial fractions by Equating coefficients
- Partial fractions by Elimination by substitution
- Partial fractions by The Cover Up Method
- Integrating the partial fractions using Standard Integrals

PARTIAL FRACTIONS

Questions involving partial fractions require us to take a rational function and convert it into two or more simple fractions (the partial fractions) that we can easily integrate using standard integrals.

Last lesson we looked at simple examples where we could split the function up by inspection, while in this lesson we look at more formal methods needed for harder examples. We will work through examples, with important notes beside each solution.

There are three methods we will investigate. All students should learn the first method, as it will work for all questions, even if a little more slowly than the other methods. More capable students should also learn the second and third methods as they each allow more efficient solutions to some types of questions. In exams you can choose whichever method you like.

Most exam questions will tell you the original integrand (a rational function) and the form of the partial fractions to use. Use the hints in the following examples so that you can answer those questions where the form of the partial fractions are not given.

Method 1 - Equating coefficients

The first method works for all questions but takes longer than the other two methods. We start by cross multiplying the partial fractions on the RHS, then equating the coefficients of x^2, x and the constant with the LHS. In more complicated questions the other two methods need to equate one or more of the coefficients anyway, so make sure you can do this method.

Example 1 - Equating Coefficients

i Express $\dfrac{2x-7}{(x+1)(x-2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x-2}$ by equating coefficients.

ii Hence find $\displaystyle\int \frac{2x-7}{(x+1)(x-2)}\,dx$

Solution

i $\dfrac{2x-7}{(x+1)(x-2)} = \dfrac{a}{x+1}+\dfrac{b}{x-2}$

1 Cross multiply on the RHS

$$\frac{2x-7}{(x+1)(x-2)} = \frac{a(x-2)+b(x+1)}{(x+1)(x-2)}$$

$$2x-7 = (a+b)x - 2a + b$$

2 Equate the numerators & simplify the RHS

$$\therefore 2 = a + b \qquad (1)$$

$$-7 = -2a + b \qquad (2)$$

3 Equate the coefficients of the LHS and RHS to create simultaneous equations

$$(1)-(2) \quad 9 = 3a \quad \rightarrow \quad a = 3$$

$$\text{sub in } (1) \quad 2 = 3 + b \quad \rightarrow \quad b = -1$$

4 Solve the simultaneous equations by inspection

$$\therefore \frac{2x-7}{(x+1)(x-2)} = \frac{3}{x+1} - \frac{1}{x-2}$$

ii $\displaystyle \therefore \int \frac{2x-7}{(x+1)(x-2)}\,dx = \int \left(\frac{3}{x+1} - \frac{1}{x-2}\right) dx$

$$= 3\ln|x+1| - \ln|x-2| + c$$

Hints for the previous example:

- Look at the original fraction and each of the partial fractions - the degree of the numerator is less than that of the denominator in each fraction, which we need for our standard integrals.
- For the unknown numerators we always use a function with a degree one less than the denominator - in this case both denominators were linear functions, so we used a constant for the numerator.

Method 2 - Elimination by Substitution

The second method works quickly when the original integrand has single linear factors, but for more involved questions we need to equate one or more of the coefficients. We start by cross multiplying the partial fractions on the RHS and equating numerators (don't simplify though), just like for Method 1, then we solve for the variables by substituting values for x that allow us to find one variable by eliminating the others. If we used this method for Example 1 it would have saved us one line of working.

Hints for the following example:

- When the original denominator has a double factor the simplest form of the partial fractions involves a constant over a single power of that factor and a constant over the factor squared. Alternatively we could expand the perfect square and use a linear numerator:
$$\frac{2x-3}{x^2(x-1)} = \frac{ax+b}{x^2} + \frac{c}{x-1}$$
- When there is a repeated factor then the substitution method cannot find all variables - equating coefficients will be needed for at least one variable. This will also occur for the Cover Up Method we will use next, which is an abbreviated form of the elimination by substitution method.

Example 2 – Elimination by Substitution

i Express $\dfrac{2x-3}{x^2(x-1)}$ in the form $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x-1}$ by elimination by substitution.

ii Hence evaluate $\displaystyle\int_2^3 \frac{2x-3}{x^2(x-1)}\,dx$

Solution

1 Cross multiply on the RHS and equate the numerators.

2 Substitute values of x that will make each set of brackets equal zero, then solve for the variables.

$$\textbf{i}\ \frac{2x-3}{x^2(x-1)}=\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x-1}$$

$$(2x-3)=ax(x-1)+b(x-1)+cx^2 \quad \mathbf{1}$$

Let $x=0\ \therefore 2(0)-3=a(0)+b(0-1)+c(0)\ \rightarrow\ -3=-b\ \rightarrow\ b=3$ **2**

Let $x=1\ \therefore 2(1)-3=a(0)+b(0)+c(1)\ \rightarrow\ -1=c\ \rightarrow c=-1$

equating coefficients of x^2: $0=a+c\ \rightarrow\ 0=a-1\ \rightarrow\ a=1$ **3**

$$\therefore \frac{2x-3}{x^2(x-1)}=\frac{1}{x}+\frac{3}{x^2}-\frac{1}{x-1}$$

3 No substitution would allow us to find the value of a, so we have to equate coefficients – since a is part of the coefficient of x^2 and x we could use either one.

$$\textbf{ii}\ \therefore \int_2^3 \frac{2x-3}{x^2(x-1)}\,dx=\int_2^3\left(\frac{1}{x}+\frac{3}{x^2}-\frac{1}{x-1}\right)dx$$

$$=\left[\ln|x|-\frac{3}{x}-\ln|x-1|\right]_2^3$$

$$=\left(\ln 3-1-\ln 2\right)-\left(\ln 2-\frac{3}{2}-0\right)$$

$$=\ln 3-2\ln 2+\frac{1}{2}$$

$$=\ln\left(\frac{3}{4}\right)+\frac{1}{2}$$

Method 3 - Cover Up Method

The third method is a variation of elimination by substitution, quickly finding variables in simple questions with a minimum of working. It will not find all variables in harder examples, requiring us to also equate coefficients. HSC questions are chosen so that this method cannot be used by itself, as it reduces these questions to mindless technique without any understanding

To find a variable on the RHS we do the following:

- Ignore (cover up) everything on the RHS except the single variable.
- Find the zero of the denominator under the variable.
- On the LHS ignore (cover up) the factor matching the denominator for the variable, then substitute the zero into the LHS and evaluate.

Let's redo Example 1, then try a harder example.

Example 3 - Cover Up Method

Express $\dfrac{2x-7}{(x+1)(x-2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x-2}$ using the cover up method

Solution

$$a=\frac{2(-1)-7}{-1-2}=3$$

1 To find a, we look at the denominator underneath it on the RHS - the zero of $x+1$ is $x=-1$, which we substitute into the LHS.

$$b=\frac{2(2)-7}{2+1}=-1$$

2 To find b, we look at the denominator underneath it on the RHS - the zero of $x-2$ is $x=2$, which we substitute into the LHS.

$$\therefore \frac{2x-7}{(x+1)(x-2)}=\frac{3}{x+1}-\frac{1}{x-2}$$

Let's redo Example 2 using the cover up method, and see why the combination of the cover up method then equating coefficients is generally the best way to approach these questions. We will use this combination as the default method in the solutions to the exercises. With practice we can work out the coefficients in our heads as below, without writing them out.

Example 4 - Cover Up Method then Equating Coefficients

Express $\dfrac{2x-3}{x^2(x-1)}$ in the form $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x-1}$ using the cover up method.

Solution

$$b=\frac{2(0)-3}{0-1}=3$$

$$c=\frac{2(1)-3}{1^2}=-1$$

equating coefficients of x^2: $0=a+c \quad \to \quad 0=a-1 \quad \to \quad a=1$

$$\therefore \frac{2x-3}{x^2(x-1)}=\frac{1}{x}+\frac{3}{x^2}-\frac{1}{x-1}$$

Although the variables in the partial fractions must be real numbers, we can use imaginary numbers as tools to help us determine them more quickly using either the substitution method or the cover up method. Particularly useful when one factor is an irreducible quadratic.

Example 5 - Cover Up Method with Imaginary Numbers

Express $\dfrac{5x+10}{(x-2)(x^2+1)}$ in the form $\dfrac{a}{x-2}+\dfrac{bx+c}{x^2+1}$ and hence evaluate $\displaystyle\int_3^4 \frac{5x+10}{(x-2)(x^2+1)}\,dx$

Solution

$$a=\frac{5(2)+10}{2^2+1}=4$$

Let $x=i$ $\quad bi+c=\dfrac{5i+10}{i-2}\times\dfrac{i+2}{i+2}=\dfrac{-5+10i+10i+20}{i^2-2^2}$

$$bi+c=-3-4i$$

$\therefore c=-3, b=-4$

$$\therefore \frac{x+2}{(x-2)(x^2+1)}=\frac{4}{x-2}-\frac{4x+3}{x^2+1}$$

$$\therefore \int_3^4 \frac{5x+10}{(x-2)(x^2+1)}\,dx=\int_3^4\left(\frac{4}{x-2}-\frac{4x+3}{x^2+1}\right)dx$$

$$=\int_3^4\left(\frac{4}{x-2}-2\times\frac{2x}{x^2+1}-3\times\frac{1}{1+x^2}\right)dx$$

$$=\Big[4\ln|x-2|-2\ln|x^2+1|-3\tan^{-1}x\Big]_3^4$$

$$=(4\ln 2-2\ln 17-3\tan^{-1}4)-(0-2\ln 10-3\tan^{-1}3)$$

$$=4\ln 2-2\ln 17+2\ln 10-3\tan^{-1}4+3\tan^{-1}3$$

BASIC

1 **i** Express $\dfrac{2x+1}{(x+1)(x+2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x+2}$ by equating coefficients.

ii Hence find $\displaystyle\int \frac{2x+1}{(x+1)(x+2)}\,dx$

2 **i** Express $\dfrac{2x+3}{x^2(x+1)}$ in the form $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x+1}$ by elimination by substitution.

ii Hence evaluate $\displaystyle\int_2^3 \frac{2x+3}{x^2(x+1)}\,dx$

3 Express $\dfrac{2x+1}{(x+1)(x+2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x+2}$ using the cover up method

4 Express $\dfrac{2x+3}{x^2(x+1)}$ in the form $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x+1}$ using the cover up method.

MEDIUM

5 Express $\dfrac{5x-5}{(x+2)(x^2+1)}$ in the form $\dfrac{a}{x+2}+\dfrac{bx+c}{x^2+1}$ and hence evaluate $\displaystyle\int_3^4 \frac{5x-5}{(x+2)(x^2+1)}\,dx$

6 It can be shown that $\dfrac{2}{x^3+x^2-x+1}=\dfrac{1}{x+1}-\dfrac{x}{x^2+1}+\dfrac{1}{x^2+1}$ (Do NOT prove this.)

Use this result to evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1}\,dx$

7 It can be shown that $\dfrac{8(1-x)}{(2-x^2)(2-2x+x^2)}=\dfrac{4-2x}{2-2x+x^2}-\dfrac{2x}{2-x^2}$ (Do NOT prove this.)

Use this result to evaluate $\displaystyle\int_0^1 \frac{8(1-x)}{(2-x^2)(2-2x+x^2)}\,dx$

8 Evaluate $\displaystyle\int_2^5 \frac{x-6}{x^2+3x-4}\,dx$

9 **i** Given that $\dfrac{16x-43}{(x-3)^2(x+2)}$ can be written as $\dfrac{16x-43}{(x-3)^2(x+2)}=\dfrac{a}{(x-3)^2}+\dfrac{b}{x-3}+\dfrac{c}{x+2}$ where a, b and c are real numbers, find a, b and c.

ii Hence find $\displaystyle\int \frac{16x-43}{(x-3)^2(x+2)}\,dx$

10 Find $\int \frac{3x^2+8}{x(x^2+4)}dx$

11 Find $\int \frac{2x^3-x^2-8x-2}{x(x-2)}dx$

12 Use partial fractions to show that $\frac{3!}{x(x+1)(x+2)(x+3)} = \frac{1}{x} - \frac{3}{x+1} + \frac{3}{x+2} - \frac{1}{x+3}$

13 It is given that $x^4+4 = (x^2+2x+2)(x^2-2x+2)$.

i Find A and B so that $\frac{16}{x^4+4} = \frac{A+2x}{x^2+2x+2} + \frac{B-2x}{x^2-2x+2}$

ii Hence, or otherwise, show that for any real number m,

$$\int_0^m \frac{16}{x^4+4}dx = \ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right) + 2\tan^{-1}(m+1) + 2\tan^{-1}(m-1)\,.$$

iii Find the limiting value as $m \to \infty$ of $\int_0^m \frac{16}{x^4+4}dx$

1

i $\dfrac{2x+1}{(x+1)(x+2)} = \dfrac{a}{x+1} + \dfrac{b}{x+2}$

$$\frac{2x+1}{(x+1)(x+2)} = \frac{a(x+2)+b(x+1)}{(x+1)(x+2)}$$

$$2x+1 = (a+b)x + 2a + b$$

$$\therefore 2 = a + b \qquad (1)$$

$$1 = 2a + b \qquad (2)$$

$$(2)-(1) \quad -1 = a \quad \to \quad a = -1$$

$$\text{sub in } (1) \quad 2 = -1 + b \quad \to \quad b = 3$$

$$\therefore \frac{2x+1}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{3}{x+2}$$

ii $\therefore \displaystyle\int \frac{2x+1}{(x+1)(x+2)}\,dx = \int \left(-\frac{1}{x+1} + \frac{3}{x+2}\right)dx$

$$= -\ln|x+1| + 3\ln|x+2| + c$$

2

i $\dfrac{2x+3}{x^2(x+1)} = \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x+1}$

$$(2x-3) = ax(x+1) + b(x+1) + cx^2$$

Let $x = 0 \ \therefore 2(0) + 3 = a(0) + b(0+1) + c(0) \ \to \ 3 = b \ \to \ b = 3$

Let $x = -1 \ \therefore 2(-1) + 3 = a(0) + b(0) + c(1) \ \to \ 1 = c \ \to c = 1$

equating coefficients of x^2: $0 = a + c \ \to \ 0 = a + 1 \ \to \ a = -1$

$$\therefore \frac{2x+3}{x^2(x+1)} = -\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}$$

ii $\therefore \displaystyle\int_2^3 \frac{2x+3}{x^2(x+1)}\,dx = \int_2^3 \left(-\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}\right)dx$

$$= \left[-\ln|x| - \frac{3}{x} + \ln|x+1|\right]_2^3$$

$$= \left(-\ln 3 - 1 + \ln 4\right) - \left(-\ln 2 - \frac{3}{2} - \ln 3\right)$$

$$= \ln 4 + \ln 2 - 2\ln 3 + \frac{1}{2}$$

$$= \ln\left(\frac{8}{9}\right) + \frac{1}{2}$$

3

$$a = \frac{2(-1)+1}{-1+2} = -1$$

$$b = \frac{2(-2)+1}{(-2)+1} = 3$$

$$\therefore \frac{2x+1}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{3}{x+2}$$

4

$$b = \frac{2(0)+3}{0+1} = 3$$

$$c = \frac{2(-1)+3}{(-1)^2} = 1$$

equating coefficients of x^2: $0 = a + c$

$\rightarrow \; 0 = a + 1 \;\rightarrow\; a = -1$

$$\therefore \frac{2x+3}{x^2(x+1)} = -\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}$$

5

$$a = \frac{5(-2)-5}{(-2)^2+1} = -3$$

Let $x = i$ $\quad bi + c = \frac{5i-5}{i+2} \times \frac{i-2}{i-2} = \frac{-5-10i-5i+10}{i^2-2^2}$

$$bi + c = -1 + 3i$$

$\therefore c = -1, b = 3$

$$\therefore \frac{5x-5}{(x+2)(x^2+1)} = -\frac{3}{x+2} + \frac{3x-1}{x^2+1}$$

$$\therefore \int_3^4 \frac{5x-5}{(x+2)(x^2+1)}\,dx = \int_3^4 \left(-\frac{3}{x+2} + \frac{3x-1}{x^2+1}\right) dx$$

$$= \int_3^4 \left(-\frac{3}{x+2} + \frac{3}{2} \times \frac{2x}{x^2+1} - \frac{1}{1+x^2}\right) dx$$

$$= \left[-3\ln|x+2| + \frac{3}{2}\ln|x^2+1| - \tan^{-1} x\right]_3^4$$

$$= \left(-3\ln 6 + \frac{3}{2}\ln 17 - \tan^{-1} 4\right) - \left(-3\ln 5 + \frac{3}{2}\ln 10 - \tan^{-1} 3\right)$$

$$= \frac{3}{2}\ln\frac{17}{10} + 3\ln\frac{5}{6} + \tan^{-1} 3 - \tan^{-1} 4$$

6

$$\int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1}\,dx$$

$$= \int_{\frac{1}{2}}^{2} \left(\frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right) dx$$

$$= \left[\ln\left|x+1\right| - \frac{1}{2}\ln\left|x^2+1\right| + \tan^{-1} x\right]_{\frac{1}{2}}^{2}$$

$$= \left(\ln 3 - \frac{1}{2}\ln 5 + \tan^{-1} 2\right) - \left(\ln\frac{3}{2} - \frac{1}{2}\ln\frac{5}{4} + \tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$= \ln 3 - \ln\frac{3}{2} - \frac{1}{2}\ln 5 + \frac{1}{2}\ln\frac{5}{4} + \tan^{-1} 2 - \tan^{-1}\frac{1}{2}$$

$$= \ln 2 - \frac{1}{2}\ln 4 + \tan^{-1} 2 - \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1} 2 - \tan^{-1}\frac{1}{2}$$

7

$$\int_0^1 \frac{8(1-x)}{(2-x^2)(2-2x+x^2)}\,dx$$

$$= \int_0^1 \left(\frac{4-2x}{2-2x+x^2} - \frac{2x}{2-x^2}\right) dx$$

$$= \int_0^1 \left(-\frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1^2} - \frac{2x}{2-x^2}\right) dx$$

$$= \left[-\ln|x^2-2x+2| + 2\tan^{-1}(x-1) + \ln|2-x^2|\right]_0^1$$

$$= (0+0+0) - \left(-\ln 2 + 2\left(-\frac{\pi}{4}\right) + \ln 2\right)$$

$$= \frac{\pi}{2}$$

8

$$\frac{x-6}{x^2+3x-4} = \frac{x-6}{(x+4)(x-1)} = \frac{a}{x+4} + \frac{b}{x-1}$$

$$a = \frac{(-4)-6}{(-4)-1} = 2$$

$$b = \frac{(1)-6}{(1)+4} = -1$$

$$\therefore \int_2^5 \frac{x-6}{x^2+3x-4}\,dx$$

$$= \int_2^5 \left(\frac{2}{x+4} - \frac{1}{x-1}\right) dx$$

$$= \Big[2\ln|x+4| - \ln|x-1|\Big]_2^5$$

$$= (2\ln 9 - \ln 4) - (2\ln 6 - 0)$$

$$= \ln 81 - \ln 4 - \ln 36$$

$$= \ln\frac{81}{144}$$

$$= \ln\frac{9}{16}$$

9

i $\dfrac{16x-43}{(x-3)^2(x+2)} = \dfrac{a}{(x-3)^2} + \dfrac{b}{x-3} + \dfrac{c}{x+2}$

$$a = \frac{16(3)-43}{(3)+2} = 1$$

$$c = \frac{16(-2)-43}{\big((-2)-3\big)^2} = -3$$

equating coefficients of x^2: $b + c = 0 \;\; \rightarrow \;\; b = 3$

ii $\displaystyle\int \frac{16x-43}{(x-3)^2(x+2)}\,dx$

$$= \int \left(\frac{1}{(x-3)^2} + \frac{3}{x-3} - \frac{3}{x+2}\right) dx$$

$$= -\frac{1}{x-3} + 3\ln|x-3| - 3\ln|x+2| + c$$

$$= 3\ln\left|\frac{x-3}{x+2}\right| - \frac{1}{x-3} + c$$

10 $$\frac{3x^2+8}{x(x^2+4)}=\frac{a}{x}+\frac{bx+c}{x^2+4}$$

$$a=\frac{3(0)+8}{(0)^2+4}=2$$

$$\begin{aligned}b(2i)+c&=\frac{3(2i)^2+8}{2i}\\&=-\frac{4}{2i}\times\frac{i}{i}\\&=2i\end{aligned}$$

$\therefore b=1, c=0$

$$\begin{aligned}\int\frac{3x^2+8}{x(x^2+4)}dx&=\int\left(\frac{2}{x}+\frac{x}{x^2+4}\right)dx\\&=2\ln|x|+\frac{1}{2}\ln|x^2+4|+c\end{aligned}$$

11 $$\int\frac{2x^3-x^2-8x-2}{x(x-2)}dx$$

$$=\int\frac{2x(x^2-2x)+3(x^2-2x)-2x-2}{x(x-2)}dx$$

$$=\int\left(2x+3-\frac{2x+2}{x(x-2)}\right)dx \qquad (*)$$

Let $\frac{2x+2}{x(x-2)}=\frac{a}{x}+\frac{b}{x-2}$

$$a=\frac{2(0)+2}{(0)-2}=-1$$

$$b=\frac{2(2)+2}{(2)}=3$$

sub in $(*)$

$$\int\frac{2x^3-x^2-8x-2}{x(x-2)}dx$$

$$=\int\left(2x+3+\frac{1}{x}-\frac{3}{x-2}\right)dx$$

$$=x^2+3x+\ln|x|-3\ln|x-2|+c$$

12

Let $\dfrac{3!}{x(x+1)(x+2)(x+3)} = \dfrac{a}{x} + \dfrac{b}{x+1} + \dfrac{c}{x+2} + \dfrac{d}{x+3}$

$a = \dfrac{3!}{(0+1)(0+2)(0+3)} = 1$

$b = \dfrac{3!}{(-1)(-1+2)(-1+3)} = -3$

$c = \dfrac{3!}{(-2)(-2+1)(-2+3)} = 3$

$d = \dfrac{3!}{(-3)(-3+1)(-3+2)} = -1$

$\therefore \dfrac{3!}{x(x+1)(x+2)(x+3)} = \dfrac{1}{x} - \dfrac{3}{x+1} + \dfrac{3}{x+2} - \dfrac{1}{x+3}$

13

i RHS $= \dfrac{\big((A+2x)(x^2-2x+2)+(B-2x)(x^2+2x+2)\big)}{(x^4+4)}$

equating coefficients of x $\;-2A+4x+2B-4x=0 \;\; \therefore A=B$

equating the constant terms $\;2A+2B=16 \;\; \therefore A=B=4$

ii $\displaystyle\int_0^m \frac{16}{x^4+4}\,dx = \int_0^m \left(\frac{4+2x}{x^2+2x+2} + \frac{4-2x}{x^2-2x+2}\right)dx$

$\displaystyle= \int_0^m \left(\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1}\right)\,dx$

$\displaystyle= \Big[\ln(x^2+2x+2) + 2\tan^{-1}(x+1) - \ln(x^2-2x+2) + 2\tan^{-1}(x-1)\Big]_0^m$

$= (\ln(m^2+2m+2) + 2\tan^{-1}(m+1) - \ln(m^2-2m+2) + 2\tan^{-1}(m-1))$

$-(\ln 2 + 2\tan^{-1}1 - \ln 2 + 2\tan^{-1}1)$

$= \ln\left(\dfrac{m^2+2m+2}{m^2-2m+2}\right) + 2\tan^{-1}(m+1) + 2\tan^{-1}(m-1)$

iii as $m \to \infty$

$\ln\left(\dfrac{m^2+2m+2}{m^2-2m+2}\right) + 2\tan^{-1}(m+1) + 2\tan^{-1}(m-1) \to$

$\ln(1) + 2\left(\dfrac{\pi}{2}\right) + 2\left(\dfrac{\pi}{2}\right) = 2\pi$

$\displaystyle\therefore \int_0^m \frac{16}{x^4+4}\,dx \to 2\pi$

In Lesson 5 we look at other algebraic substitutions, which cannot be solved using the Reverse Chain Rule. These questions could also occur in Extension 1, but the substitution would be given there, so the only difference is being able to recognise the best substitution to make.

We will cover substitutions that simplify:

- Compound Functions
- Other Square Roots

SIMPLIFYING INTEGRANDS

In general, the simpler the integrand is the easier it is to integrate it. A common difficulty arises when the integrand includes the square root of the variable, say x, or some function involving x. In these situations we can use a u or u^2 substitution to simplify the integrand so that we can integrate it more easily.

When a square root is involved, it is easiest to let u^2 equal the radicand as the algebra is easier. We assume that u is restricted to positive values only, without the need to define it as such - this matches official HSC solutions.

Later in the chapter we will look at trig substitutions, which are only needed in a small number of cases.

In each of the examples to follow we will see that letting u^2 equal the radicand simplifies the integrand, replacing any fractional powers with integral powers that are much easier to manipulate.

Example 1

Find $\int x^3\sqrt{x^2-1}\,dx$

Solution

$$\int x^3\sqrt{x^2-1}\,dx$$

$$\boxed{\begin{aligned} u^2 &= x^2-1 \\ 2udu &= 2xdx \\ dx &= \frac{udu}{x} \end{aligned}}$$

$$= \int x^3 u \times \frac{udu}{x}$$

$$= \int x^2u^2du$$

$$= \int (u^2+1)u^2du$$

$$= \int (u^4+u^2)du$$

$$= \frac{u^5}{5} + \frac{u^3}{3} + c$$

$$= \frac{\sqrt{(x^2-1)^5}}{5} + \frac{\sqrt{(x^2-1)^3}}{3} + c$$

Example 2

Find $\int \frac{\sqrt{x}}{1+x} dx$

Solution

$$\int \frac{\sqrt{x}}{1+x} dx$$

$$\boxed{\begin{aligned} u^2 &= x \\ 2u\,du &= dx \end{aligned}}$$

$$= \int \frac{u}{1+u^2} \times 2u \; du$$

$$= 2\int \frac{u^2}{1+u^2} du$$

$$= 2\int \frac{1+u^2-1}{1+u^2} du$$

$$= 2\int \left(1 - \frac{1}{1+u^2}\right) du$$

$$= 2u - 2\tan^{-1} u + c$$

$$= 2\sqrt{x} - 2\tan^{-1} \sqrt{x} + c$$

Example 3

Find $\int \frac{x^7}{\sqrt{1-x^4}}\,dx$

Solution

$$\int \frac{x^7}{\sqrt{1-x^4}}\,dx$$

$$\boxed{\begin{aligned} u^2 &= 1-x^4 \\ 2u\,du &= -4x^3dx \\ dx &= -\frac{u\,du}{2x^3} \end{aligned}}$$

$$= \int \frac{x^7}{u} \times \left(-\frac{u\,du}{2x^3}\right)$$

$$= -\frac{1}{2}\int x^4\,du$$

$$= -\frac{1}{2}\int (1-u^2)\,du$$

$$= -\frac{1}{2}\left(u - \frac{u^3}{3}\right) + c$$

$$= \frac{\sqrt{(1-x^4)^3}}{6} - \frac{\sqrt{1-x^4}}{2} + c$$

There are some substitutions that are much more difficult to see - the only way to be able to solve questions like these is lots of practice! Notice in the example below at one point we rationalise the numerator, not the denominator, so that we can split the fraction.

Example 4

Find $\int x\sqrt{\dfrac{1-x^2}{1+x^2}}\,dx$

Solution

$$\int x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \\ dx &= \frac{du}{2x} \end{aligned}}$$

$$= \int x\sqrt{\frac{1-u}{1+u}} \times \frac{du}{2x}$$

$$= \frac{1}{2}\int \sqrt{\frac{1-u}{1+u}} \times \sqrt{\frac{1-u}{1-u}}\,du$$

$$= \frac{1}{2}\int \frac{1-u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du - \frac{1}{2}\int \frac{u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du + \frac{1}{4}\int (1-u^2)^{-\frac{1}{2}}(-2u)\,du$$

$$= \frac{1}{2}\sin^{-1} u + \frac{1}{2}\sqrt{1-u^2} + c$$

$$= \frac{\sin^{-1} x^2}{2} + \frac{\sqrt{1-x^4}}{2} + c$$

EXERCISE 4.5

BASIC

Find the following indefinite integrals:

1 $\int x^5\sqrt{x^2+1}\,dx$

2 $\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx$

3 $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

4 $\int \frac{1}{2+\sqrt{x}}\,dx$

MEDIUM

5 $\int \frac{x^3}{(x^2+1)^2}\,dx$

6 $\int x^5\sqrt{2-x^3}\,dx$

7 $\int x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$

CHALLENGING

8 $\int \frac{\sec x \tan x}{\sec x + \sec^2 x}\,dx$

9 Let $I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$

i Use the substitution $u = 4 - x$ to show that $I = \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$

ii Hence, find the value of I.

1

$$\int x^5\sqrt{x^2+1}\,dx$$

$$\boxed{\begin{aligned} u^2 &= x^2+1 \\ 2udu &= 2xdx \\ dx &= \frac{udu}{x}\end{aligned}}$$

$$= \int x^4 \times x \times u \times \frac{udu}{x}$$

$$= \int (u^2-1)^2u^2du$$

$$= \int (u^4-2u^2+1)u^2du$$

$$= \int (u^6-2u^4+u^2)\,du$$

$$= \frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3}+c$$

$$= \frac{\sqrt{(x^2+1)^7}}{7}-\frac{2\sqrt{(x^2+1)^5}}{5}+\frac{\sqrt{(x^2+1)^3}}{3}+c$$

2

$$\int \frac{\sqrt{x}}{1-\sqrt{x}}dx$$

$$\boxed{\begin{aligned} u^2 &= x \\ 2u\,du &= dx\end{aligned}}$$

$$= \int \frac{u}{1-u} \times 2u\ \ du$$

$$= 2\int \frac{u^2}{1-u}du$$

$$= 2\int \frac{-u(1-u)-(1-u)+1}{1-u}du$$

$$= 2\int \left(-u-1+\frac{1}{1-u}\right)du$$

$$= -u^2-2u-2\ln|1-u|+c$$

$$= -x-2\sqrt{x}-2\ln\left|1-\sqrt{x}\right|+c$$

3

$$\int \frac{x^3}{\sqrt{1+x^2}}dx$$

$$\boxed{\begin{aligned} u^2 &= 1+x^2 \\ 2u\,du &= 2xdx \\ dx &= \frac{u\,du}{x}\end{aligned}}$$

$$= \int \frac{x^2\times x}{u}\times\left(\frac{u\,du}{x}\right)$$

$$= \int (u^2-1)\,du$$

$$= \frac{u^3}{3}-u+c$$

$$= \frac{\sqrt{(1+x^2)^3}}{3}-\sqrt{1+x^2}+c$$

4

$$\int \frac{1}{2+\sqrt{x}}dx$$

$$\boxed{\begin{aligned} u^2 &= x \\ 2u\,du &= \ dx\end{aligned}}$$

$$= \int \frac{1}{2+u}\times 2u\,du$$

$$= 2\int \frac{u}{u+2}du$$

$$= 2\int \frac{u+2-2}{u+2}du$$

$$= 2\int \left(1-\frac{2}{u+2}\right)du$$

$$= 2u-4\ln|u+2|+c$$

$$= 2\sqrt{x}-4\ln\left|2+\sqrt{x}\right|+c$$

5

$$\int \frac{x^3}{(x^2+1)^2}\,dx$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \\ dx &= \frac{du}{2x} \end{aligned}}$$

$$= \int \frac{x^2 \times x}{u^2} \times \frac{du}{2x}$$

$$= \frac{1}{2}\int \frac{u-1}{u^2}\,du$$

$$= \frac{1}{2}\int \left(\frac{1}{u} - u^{-2}\right) du$$

$$= \frac{1}{2}\ln|u| + \frac{1}{2u} + c$$

$$= \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)} + c$$

6

$$\int x^5\sqrt{2-x^3}\,dx$$

$$\boxed{\begin{aligned} u^2 &= 2-x^3 \\ 2u\,du &= -3x^2 dx \\ dx &= -\frac{2u}{3x^2}\,du \end{aligned}}$$

$$= \int x^5 \times u \times \left(-\frac{2u}{3x^2}\right) du$$

$$= -\frac{2}{3}\int x^3u^2\,du$$

$$= -\frac{2}{3}\int (2-u^2)u^2\,du$$

$$= -\frac{2}{3}\int (2u^2-u^4)\,du$$

$$= -\frac{2}{3}\left(\frac{2u^3}{3} - \frac{u^5}{5}\right) + c$$

$$= -\frac{4\sqrt{(2-x^3)^3}}{9} + \frac{2\sqrt{(2-x^3)^5}}{15} + c$$

7

$$\int x\sqrt{\frac{1+x^2}{1-x^2}}\,dx$$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \\ dx &= \frac{du}{2x} \end{aligned}}$$

$$= \int x\sqrt{\frac{1+u}{1-u}} \times \frac{du}{2x}$$

$$= \frac{1}{2}\int \sqrt{\frac{1+u}{1-u}} \times \sqrt{\frac{1+u}{1+u}}\,du$$

$$= \frac{1}{2}\int \frac{1+u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du + \frac{1}{2}\int \frac{u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du - \frac{1}{4}\int (-2u)(1-u^2)^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\sin^{-1}u - \frac{1}{2}\sqrt{1-u^2} + c$$

$$= \frac{\sin^{-1}x^2}{2} - \frac{\sqrt{1-x^4}}{2} + c$$

8

$$\int \frac{\sec x\tan x}{\sec x + \sec^2 x}\,dx$$

$$\boxed{\begin{aligned} u &= \sec x \\ du &= \sec x\tan x\,dx \\ dx &= \frac{du}{\sec x\tan x} \end{aligned}}$$

$$= \int \frac{\sec x\tan x}{u+u^2} \times \frac{du}{\sec x\tan x}$$

$$= \int \frac{1}{u(1+u)}\,du$$

$$= \int \left(\frac{1}{u} - \frac{1}{1+u}\right) du$$

$$= \ln|u| - \ln|1+u| + c$$

$$= \ln\left|\frac{u}{1+u}\right| + c$$

$$= \ln\left|\frac{\sec x}{1+\sec x}\right| + c$$

$$= \ln\left|\frac{1}{\cos x + 1}\right| + c$$

$$= -\ln|\cos x + 1| + c$$

9

$$I=\int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$$

$$=-\int_3^1 \frac{\cos^2\left(\frac{\pi}{8}(4-u)\right)}{(4-u)u}\,du$$

$$=\int_1^3 \frac{\cos^2\left(\frac{\pi}{2}-\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

$$=\int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

ii

$$2I=\int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx+\int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

$$2I=\int_1^3 \frac{1}{x(4-x)}\,dx$$

$$2I=\frac{1}{4}\int_1^3 \left(\frac{1}{x}+\frac{1}{4-x}\right)dx$$

$$I=\frac{1}{8}\Big[\ln x-\ln|4-x|\Big]_1^3$$

$$=\frac{1}{8}(\ln 3-\ln 1-\ln 1+\ln 3)$$

$$=\frac{\ln 3}{4}$$

In Lesson 6 we look at integrals where the integrand is a power of one or more trig functions.

We will cover:

- Integrands involving powers of Sine and/or Cosine
- Integrands involving powers of Tangent and/or Secant
- Integrands involving powers of Cotangent and/or Cosecant (briefly)
- Integrands involving Product to Sum Identities

POWERS OF TRIGONOMETRIC FUNCTIONS

We have seen some trigonometric and inverse trigonometric results already in this chapter, and in this lesson we will focus on powers of trig functions. Powers of trigonometric integrals requires the use of the Pythagorean Identities and Standard Integrals.

When we have trig functions raised to a power, singly or in pairs, we have many different approaches depending on the function(s) and the powers. The only way to learn is to practise!

POWERS OF SINE AND COSINE

To integrate powers of sine and cosine we will use the following rules and the reverse chain rule:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \sin x\, dx = -\cos x + c$$

$$\int \cos x\, dx = \sin x + c$$

Memorise these bottom two identities.
Notice that the only difference is the sign.

Example 1

Find $\int \sin^3 x\ dx$

Hint for first step:
$\sin^2 x = 1 - \cos^2 x$

Solution

$$\int \sin^3 x\ dx$$

$$= \int \sin^2 x \sin x\ dx$$

$$= \int (1 - \cos^2 x)\sin x\, dx$$

$$= \int \sin x\, dx - \int \sin x \cos^2 x\, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + c$$

POWERS OF TANGENT AND SECANT

Integration involving $\tan x$ and $\sec x$ can be more involved, as the two functions are linked by many different rules, which can cause confusion. Not all integrands involving tangent and/or secant can be integrated using the simple techniques below - some rely on integration by parts as we will see later in the chapter. This complexity makes integration using tangent and secant great for Extension 2. We need to solve the integrals using the following Pythagorean Identity and standard integrals.

$$\tan^2 x + 1 = \sec^2 x$$
$$\int \tan x \, dx = -\ln|\cos x| + c$$
$$\int \sec x \, dx = \ln|\sec x + \tan x| + c$$
$$\int \sec^2 x \, dx = \tan x + c$$
$$\int \sec x \tan x \, dx = \sec x + c$$

Example 2

Find $\int \tan^4 x \, dx$

Hint for first step:
$\tan^2 x = \sec^2 x - 1$

Solution

$$\int \tan^4 x \, dx$$

$$= \int \tan^2 x \, (\sec^2 x - 1) \, dx$$

$$= \int \sec^2 x \, (\tan x)^2 dx - \int \tan^2 x \, dx$$

$$= \int \sec^2 x \, (\tan x)^2 dx - \int (\sec^2 x - 1) \, dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x + c$$

Example 3

Prove $\int \sec x \, dx = \ln|\tan x + \sec x| + c$

We could differentiate the RHS and prove we get $\sec x$, but let's do it using integration

Hint for first step:
$\int \sec x \, dx = \int \sec x \times \frac{\tan x + \sec x}{\sec x + \tan x} dx$
If you are not required to prove it then just use the mnemonic.

Solution

$$\int \sec x \, dx$$

$$= \int \sec x \times \frac{\tan x + \sec x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$$

$$= \ln|\tan x + \sec x| + c$$

Example 4

Find $\int \tan^3 x \sec x \, dx$

Solution

$$\int \tan^3 x \sec x \, dx$$

$$= \int \tan^2 x \,.\, \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1) \tan x \sec x \, dx$$

$$= \int (\tan x \sec x)(\sec x)^2 \, dx - \int \tan x \sec x \, dx$$

$$= \frac{\sec^3 x}{3} - \sec x + c$$

POWERS OF COTANGENT AND COSECANT

There are many similarities between the pair of functions $\cot x$ and $\operatorname{cosec} x$ and the pair of functions $\tan x$ and $\sec x$ that we have just looked at.

There are no separate examples for this pair – have a look at the previous slides for the techniques to be used, replacing $\sec x$ with $\operatorname{cosec} x$ and $\tan x$ with $\cot x$, remembering that there will be a minus sign involved in each of them too.

PRODUCT TO SUM IDENTITIES

The Product to Sum Identities are another source of integrals, which were not in the old course. If you have the option, use double angle results or other trig identities as the solution will be easier.

Example 5

Find $\int (\sin 2x + \sin 4x)^2 \, dx$

Solution

$$\int (\sin 2x + \sin 4x)^2 \, dx$$

$$= \int (\sin^2 2x + 2\sin 4x \sin 2x + \sin^2 4x) \, dx$$

$$= \int \left(\frac{1}{2}(1 - \cos 4x) + 2\left(\frac{1}{2}[\cos(4x - 2x) - \cos(4x + 2x)]\right) + \frac{1}{2}(1 - \cos 8x)\right) dx$$

$$= \int \left(1 - \frac{1}{2}\cos 4x + \cos 2x - \cos 6x - \frac{1}{2}\cos 8x\right) dx$$

$$= x - \frac{\sin 8x}{16} - \frac{\sin 4x}{8} - \frac{\sin 6x}{6} + \frac{\sin 2x}{2} + c$$

EXERCISE 4.6

BASIC

Find the following indefinite integrals:

1 $\int \cos^3 x \; dx$

2 $\int \sec^4 x \, dx$

3 Prove $\int \operatorname{cosec} x \, dx = -\ln|\cot x + \operatorname{cosec} x| + c$

4 $\int \tan x \sec^3 x \, dx$

5 $\int (\cos 2x + \cos 4x)^2 \, dx$

6 $\int \cos^4 x \, dx$

7 $\int \cot^3 x \operatorname{cosec}^2 x \, dx$

MEDIUM

8 $\int \operatorname{cosec}^4 x \, dx$

9 $\int \sin 3x \sin 2x \, dx$

10 $\int \frac{\cos 2x}{\cos x} dx$

11 $\int \frac{\tan x}{\cos^2 x} dx$

12 $\int (\cos^4 x - \sin^4 x) \, dx$

13 $\int \sin mx \sin nx \, dx$ for positive integral m, n and $m \neq n$

CHALLENGING

14 $\int \frac{(\cos x + \sin x)^3}{\sin 2x + 1} dx$

15 Prove $\int \sin x \sin 2x \, dx = \frac{2\sin^3 x}{3} + c$

i Using the double angle results **ii** Using the Product to Sum results.

16 $\int \frac{\sqrt{1 + \sin x}}{\sec x} dx$

17 $\int \sqrt{1 + \sin x} \, dx$

1

$$\int \cos^3 x \ dx$$

$$= \int \cos^2 x \cos x \ dx$$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int \cos x \, dx - \int \cos x \sin^2 x \, dx$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

2

$$\int \sec^4 x \, dx$$

$$= \int \sec^2 x \sec^2 x \, dx$$

$$= \int \sec^2 x \, (\tan^2 x + 1) \, dx$$

$$= \int \sec^2 x \, (\tan x)^2 dx + \int \sec^2 x \, dx$$

$$= \frac{\tan^3 x}{3} + \tan x + c$$

3

$$\int \operatorname{cosec} x \, dx$$

$$= \int \operatorname{cosec} x \times \frac{\cot x + \operatorname{cosec} x}{\operatorname{cosec} x + \cot x} dx$$

$$= \int \frac{\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x} dx$$

$$= -\int \frac{-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x} dx$$

$$= -\ln|\cot x + \operatorname{cosec} x| + c$$

4

$$\int \tan x \sec^3 x \, dx$$

$$= \int \tan x \sec x \, (\sec x)^2 \, dx$$

$$= \frac{\sec^3 x}{3} + c$$

5

$$\int (\cos 2x + \cos 4x)^2 \, dx$$

$$= \int (\cos^2 2x + 2 \cos 2x \cos 4x + \sin^2 4x) \, dx$$

$$= \int \left(\frac{1}{2}(1 + \cos 4x) + 2\left(\frac{1}{2}[\cos(4x + 2x) + \cos(4x - 2x)]\right) + \frac{1}{2}(1 + \cos 8x)\right) dx$$

$$= \int \left(1 + \frac{1}{2}\cos 4x + \cos 6x + \cos 2x + \frac{1}{2}\cos 8x\right) dx$$

$$= x + \frac{\sin 8x}{16} + \frac{\sin 6x}{6} + \frac{\sin 4x}{8} + \frac{\sin 2x}{2} + c$$

6

$$\int \cos^4 x\,dx$$

$$= \int (\cos^2 x)^2$$

$$= \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right) dx$$

$$= \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}\left(\frac{1}{4}\sin 4x + x\right)\right) + c$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + \frac{x}{2} + c$$

$$= \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + \frac{3x}{8} + c$$

7

$$\int \cot^3 x \operatorname{cosec}^2 x\,dx$$

$$= -\int (-\operatorname{cosec}^2 x)(\cot x)^3\,dx$$

$$= \frac{\cot^4 x}{4} + c$$

8

$$\int \operatorname{cosec}^4 x\,dx$$

$$= \int \operatorname{cosec}^2 x \times \operatorname{cosec}^2 x\,dx$$

$$= \int \operatorname{cosec}^2 x\,(1+\cot^2 x)\,dx$$

$$= \int (\operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cot^2 x)\,dx$$

$$= -\cot x - \frac{\cot^3 x}{3} + c$$

9

$$\int \sin 3x \sin 2x\,dx$$

$$= \int \frac{1}{2}\Big[\cos(3x-2x) - \cos(3x+2x)\Big]\,dx$$

$$= \frac{1}{2}\int \Big[\cos x - \cos 5x\Big]\,dx$$

$$= \frac{\sin x}{2} - \frac{\sin 5x}{10} + c$$

10

$$\int \frac{\cos 2x}{\cos x}\,dx$$

$$= \int \frac{2\cos^2 x - 1}{\cos x}\,dx$$

$$= \int (2\cos x - \sec x)\,dx$$

$$= 2\sin x - \ln|\tan x + \sec x| + c$$

11

$$\int \frac{\tan x}{\cos^2 x}\,dx$$

$$= \int \tan x \sec^2 x\,dx$$

$$= \int \sec^2 x\,(\tan x)^1\,dx$$

$$= \frac{\tan^2 x}{2} + c$$

12

$$\int (\cos^4 x - \sin^4 x)\,dx$$

$$= \int (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)\,dx$$

$$= \int (\cos^2 x - \sin^2 x)\,dx$$

$$= \int \cos 2x\,dx$$

$$= \frac{1}{2}\sin 2x + c$$

13 $\int \sin mx \sin nx \, dx$

$= \int \frac{1}{2}\left[\cos(mx - nx) - \cos(mx + nx)\right] dx$

$= \int \frac{1}{2}\left[\cos(m - n)x - \cos(m + n)x\right] dx$

$= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c$

14 $\int \frac{(\cos x + \sin x)^3}{\sin 2x + 1} dx$

$= \int \frac{(\cos x + \sin x)^3}{2\sin x \cos x + \cos^2 x + \sin^2 x} dx$

$= \int \frac{(\cos x + \sin x)^3}{(\cos x + \sin x)^2} dx$

$= \int (\cos x + \sin x)\, dx$

$= \sin x - \cos x + c$

15

i $\int \sin x \sin 2x \, dx$

$= \int 2\sin^2 x \cos x \, dx$

$= 2\int (\sin x)^2 \times \frac{d}{dx}(\sin x)\, dx$

$= \frac{2\sin^3 x}{3} + c$

ii $\int \sin x \sin 2x \, dx$

$= \int \frac{1}{2}\left[\cos(2x - x) - \cos(2x + x)\right] dx$

$= \frac{1}{2}\int \left[\cos x - \cos 3x\right] dx$

$= \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + c$

$= \frac{1}{2}\sin x - \frac{1}{6}(\sin 2x \cos x + \cos 2x \sin x) + c$

$= \frac{1}{2}\sin x - \frac{1}{6}(2\sin x \cos^2 x + \sin x - 2\sin^3 x) + c$

$= \frac{1}{2}\sin x - \frac{1}{3}\sin x\,(1 - \sin^2 x) - \frac{1}{6}\sin x + \frac{1}{3}\sin^3 x + c$

$= \frac{1}{2}\sin x - \frac{1}{3}\sin x + \frac{1}{3}\sin^3 x - \frac{1}{6}\sin x + \frac{1}{3}\sin^3 x + c$

$= \frac{2\sin^3 x}{3} + c$

16 $\int \frac{\sqrt{1 + \sin x}}{\sec x} dx$

$= \int \cos x\,(1 + \sin x)^{\frac{1}{2}} dx$

$= \frac{2}{3}(1 + \sin x)^{\frac{3}{2}} + c$

$= \frac{2}{3}\sqrt{(1 + \sin x)^3} + c$

17 $\int \sqrt{1 + \sin x}\, dx$

$= \int \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + \sin 2\left(\frac{x}{2}\right)}\, dx$

$= \int \sqrt{\cos^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2} + \sin^2\frac{x}{2}}\, dx$

$= \int \sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\, dx$

$= \int \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) dx$

$= 2\sin\frac{x}{2} - 2\cos\frac{x}{2} + c$

4.7 TRIGONOMETRIC FUNCTIONS II

In Lesson 7 we look at more trigonometric integrands, this time looking at methods we keep in reserve for when easier methods will not work.

We will cover:

- Integrating with t-formulae
- Integrating with trigonometric substitutions

INTEGRATING WITH T-FORMULAE

In the HSC for the old syllabus we were always told when to use the t-formulae, and it is likely that this will continue as it is, otherwise it takes much practice to recognise when to use t-formulae.

T-formulae are one of our methods of last resort, as many integrands involving trig functions can be solved more easily using other methods.

Once you have substituted for t, the integrand can be rearranged into one of the standard integrals and solved. If you are using an indefinite integral, remember to rewrite your answer in terms of the original variable.

Example 1

Find $\int \frac{dx}{1+\cos x}$ using $t = \tan\frac{x}{2}$

Solution

$$\int \frac{dx}{1+\cos x}$$

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{1+t^2+1-t^2}\,dt$$

$$= 2\int \frac{1}{2}\,dt$$

$$= t + c$$

$$= \tan\frac{x}{2} + c$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ \frac{dt}{dx} &= \frac{1}{2}\sec^2\frac{x}{2} \\ &= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) \\ &= \frac{1}{2}(1+t^2) \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

The official HSC solutions show all steps in finding dx, as shown in the example on the left. It is unclear whether writing just the steps below would suffice instead.

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

Note that the RHS on the final line is almost the same as the t-result for $\sin\theta$ which is on the Reference Sheet.

Example 2

Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int \frac{1}{4 + 5\cos x}\,dx$

Solution

$$\int \frac{1}{4 + 5\cos x}\,dx$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1 + t^2} \end{aligned}}$$

$$= \int \frac{1}{4 + 5\left(\frac{1 - t^2}{1 + t^2}\right)} \times \frac{2dt}{1 + t^2}$$

$$= 2\int \frac{1}{4 + 4t^2 + 5 - 5t^2}\,dt$$

$$= 2\int \frac{1}{9 - t^2}\,dt$$

$$= \frac{1}{3}\int \left(\frac{1}{3 - t} + \frac{1}{3 + t}\right)dt$$

$$= \frac{1}{3}(-\ln|3 - t| + \ln|3 + t|) + c$$

$$= \frac{1}{3}\ln\left|\frac{3 + t}{3 - t}\right| + c$$

$$= \frac{1}{3}\ln\left|\frac{3 + \tan\frac{x}{2}}{3 - \tan\frac{x}{2}}\right| + c$$

TRIG SUBSTITUTIONS

Another method we can use when nothing else works is to use a trig substitution - we will normally be told in an exam when to use a trig substitution, as they are rarely needed. Often we can use a u^2 substitution instead and save a lot of work, as trig substitutions are often quite long.

Trig substitutions can be used whenever the integrand involves a square root (including in the denominator) where the radicand is the sum or difference of two squares. This might involve completing the square to become obvious.

In the following examples we will see how the trig substitution gives us an expression that we can then simplify using a Pythagorean Identity.

Example 3

Using the substitution $x = \sec\theta$, or otherwise, evaluate $\int x^3\sqrt{x^2-1}\,dx$

Solution

$$\int x^3\sqrt{x^2-1}\,dx$$

$$\boxed{\begin{aligned} x &= \sec\theta \\ dx &= \sec\theta\tan\theta\,d\theta \end{aligned}}$$

$$= \int \sec^3\theta\sqrt{\sec^2\theta-1}\,\sec\theta\tan\theta\,d\theta$$

$$= \int \sec^3\theta\sqrt{\tan^2\theta}\,\sec\theta\tan\theta\,d\theta$$

$$= \int \sec^4\theta\tan^2\theta\,d\theta$$

$$= \int \sec^2\theta\sec^2\theta\tan^2\theta\,d\theta$$

$$= \int \sec^2\theta\,(\tan^2\theta+1)\tan^2\theta\,d\theta$$

$$= \int \sec^2\theta\tan^4\theta\,d\theta + \int \sec^2\theta\tan^2\theta\,d\theta$$

$$= \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3} + c$$

$$\boxed{\begin{aligned} \tan^2\theta &= \sec^2\theta - 1 \\ &= x^2 - 1 \\ \tan\theta &= \sqrt{x^2-1} \end{aligned}}$$

$$= \frac{\sqrt{(x^2-1)^5}}{5} + \frac{\sqrt{(x^2-1)^3}}{3} + c$$

The trig substitution chosen is $x = \sec\theta$, as then the radicand becomes the LHS of the Pythagorean $\sec^2\theta - 1 = \tan^2\theta$, which the simplifies to $\tan\theta$, simplifying the integrand.

Example 4

Using the substitution $x = \tan\theta$, or otherwise, evaluate $\displaystyle\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$

Solution

$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$$

$$\boxed{\begin{aligned} x &= 2\tan u \\ dx &= 2\sec^2 u\,du \end{aligned}}$$

$$= \int \frac{1}{4\tan^2 u\sqrt{4+4\tan^2 u}} \times 2\sec^2 u\,du$$

$$= \frac{1}{2}\int \frac{\sec^2 u}{\tan^2 u\sqrt{4\sec^2 u}}\,du$$

$$= \frac{1}{2}\int \frac{\sec^2 u}{\tan^2 u \times 2\sec u}\,du$$

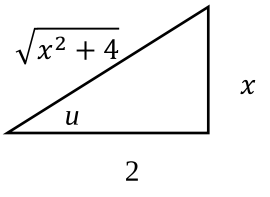
$$= \frac{1}{4}\int \frac{\sec u}{\tan^2 u}\,du$$

$$= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\,du$$

$$= \frac{1}{4}\int (\sin u)^{-2} \times \cos u\,du$$

$$= \frac{1}{4}\int \frac{d}{du}\left(-(\sin u)^{-1}\right)du$$

$$= -\frac{1}{4\sin u} + c$$



$$= -\frac{1}{4\left(\frac{x}{\sqrt{x^2+4}}\right)} + c$$

$$= -\frac{\sqrt{x^2+4}}{4x} + c$$

The trig substitution chosen is $x = 2\tan u$, as then the radicand becomes the LHS of the Pythagorean $4(1+\tan^2 u) = 4\sec^2 u$, which then simplifies to $2\sec u$, simplifying the integrand.

Integrals in the form $\displaystyle\int \frac{1}{x\sqrt{x^n - c}}\,dx$ $n = 1,2,3\ldots$ can be solved using $u^2 = \text{radicand}$, avoiding trig substitutions. For example in the integral $\displaystyle\int \frac{1}{x\sqrt{x-4}}\,dx$ we would let $u^2 = x - 4$

EXERCISE 4.7

BASIC

1 Find $\int \frac{dx}{1+\sin x}$ using $t = \tan\frac{x}{2}$

2 Using the substitution $t = \tan\frac{x}{2}$, or otherwise, find $\int \frac{5}{12+13\cos x}\,dx$

3 Using the substitution $x = \sin\theta$, or otherwise, find $\int x^3\sqrt{1-x^2}\,dx$

4 Using the substitution $x = 2\sec\theta$, or otherwise, find $\int \frac{1}{x\sqrt{x^2-4}}\,dx$

MEDIUM

5 Using the substitution $t = \tan\frac{\theta}{2}$ find $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$

6 Using the substitution $x = 4\tan^2 u$, or otherwise, find $\int \sqrt{x+4}\,dx$

7 Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx$

8 Using the substitution $x = \sin^2\theta$, or otherwise, evaluate $\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx$

9 Use the substitution $t = \tan\frac{\theta}{2}$ to show that $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta} = \frac{1}{2}\log_e 3$

CHALLENGING

Find the following indefinite integrals using a suitable trig substitution:

10 $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

11 $\int \frac{3+\cos x}{2-\cos x}\,dx$

12 $\int \sqrt{2x-x^2}\,dx$

1

$$\int \frac{dx}{1+\sin x}$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{1+t^2+2t}\,dt$$

$$= 2\int \frac{1}{(t+1)^2}\,dt$$

$$= 2\int (t+1)^{-2}\,dt$$

$$= -\frac{2}{t+1} + c$$

$$= -\frac{2}{\tan\frac{x}{2}+1} + c$$

2

$$\int \frac{5}{12+13\cos x}\,dx$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int \frac{5}{12+13\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$

$$= 10\int \frac{1}{12+12t^2+13-13t^2}\,dt$$

$$= 10\int \frac{1}{25-t^2}\,dt$$

$$= \int \left(\frac{1}{5-t}+\frac{1}{5+t}\right)dt$$

$$= -\ln|5-t| + \ln|5+t| + c$$

$$= \ln\left|\frac{5+\tan\frac{x}{2}}{5-\tan\frac{x}{2}}\right| + c$$

3

$$\int x^3\sqrt{1-x^2}\,dx$$

$$\boxed{\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\,d\theta \end{aligned}}$$

$$= \int \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$$

$$= \int \sin^3\theta\sqrt{\cos^2\theta}\cos\theta\,d\theta$$

$$= \int \sin^3\theta\cos^2\theta\,d\theta$$

$$= \int \sin\theta\sin^2\theta\cos^2\theta\,d\theta$$

$$= \int \sin\theta\,(1-\cos^2\theta)\cos^2\theta\,d\theta$$

$$= -\int (-\sin\theta)(\cos^2\theta-\cos^4\theta)\,d\theta$$

$$= \frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3} + c$$

$$\boxed{\begin{aligned} \cos\theta &= \sqrt{1-\sin^2\theta} \\ &= \sqrt{1-x^2} \end{aligned}}$$

$$= \frac{\sqrt{(1-x^2)^5}}{5} - \frac{\sqrt{(1-x^2)^3}}{3} + c$$

4

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx$$

$$\boxed{\begin{aligned} x &= 2\sec u \\ dx &= 2\sec u \tan u\,du \end{aligned}}$$

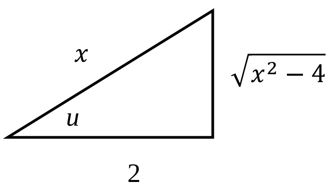
$$= \int \frac{1}{2\sec u\sqrt{4\sec^2 u - 4}} \times 2\sec u \tan u\,du$$

$$= \int \frac{\tan u}{\sqrt{4\tan^2 u}}\,du$$

$$= \int \frac{\tan u}{2\tan u}\,du$$

$$= \frac{1}{2}\int du$$

$$= \frac{u}{2} + c$$



$$= \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2-4}}{2} + c$$

Alternatively:

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx$$

$$\boxed{\begin{aligned} u^2 &= x^2 - 4 \\ 2u\,du &= 2x\,dx \\ dx &= \frac{u\,du}{x} \end{aligned}}$$

$$= \int \frac{1}{xu} \times \left(u\frac{du}{x}\right)$$

$$= \int \frac{1}{x^2}\,du$$

$$= \int \frac{1}{u^2+4}\,du$$

$$= \frac{1}{2}\tan^{-1}\frac{u}{2} + c$$

$$= \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2-4}}{2} + c$$

Hint: Integrands in the form $\frac{1}{x\sqrt{x^2-c}}$ can be solved using a u^2 substitution.

5

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$$

$$\boxed{\begin{aligned} t &= \tan\frac{\theta}{2} \\ d\theta &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int_0^1 \frac{1}{2-\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{1}{2+2t^2-1+t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{1+3t^2}\,dt$$

$$= \frac{2}{\sqrt{3}}\int_0^1 \frac{\sqrt{3}}{1+3t^2}\,dt$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}t)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}\right)$$

$$= \frac{2\sqrt{3}\pi}{9}$$

6

$$\int \sqrt{x+4}\,dx$$

$$\boxed{\begin{aligned} x &= 4\tan^2 u \\ dx &= 8\tan u \sec^2 u\,du \end{aligned}}$$

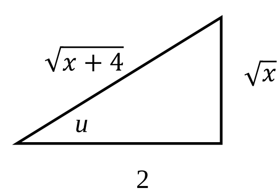
$$= \int \sqrt{4\tan^2 u + 4} \times 8\tan u \sec^2 u\,du$$

$$= 8\int \tan u \sec^2 u\sqrt{4\sec^2 u}\,du$$

$$= 16\int \tan u \sec^3 u\,du$$

$$= 16\int \tan u \sec u\,(\sec u)^2\,du$$

$$= \frac{16\sec^3 u}{3} + c$$



$$= \frac{2\sqrt{(x+4)^3}}{3} + c$$

7

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2}\end{aligned}}$$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx$$
$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) + 5} \times \frac{2dt}{1+t^2}$$
$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{6t - 4 + 4t^2 + 5 + 5t^2}\,dt$$
$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{9t^2 + 6t + 1}\,dt$$
$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{(3t+1)^2}\,dt$$
$$= 2\left[\frac{(3t+1)^{-1}}{-1\times 3}\right]_{\frac{1}{\sqrt{3}}}^{1}$$
$$= -\frac{2}{3}\left[\frac{1}{3t+1}\right]_{\frac{1}{\sqrt{3}}}^{1}$$
$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{1}{\sqrt{3}+1}\right)$$
$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}\right)$$
$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{\sqrt{3}-1}{2}\right)$$
$$= -\frac{2}{3}\frac{\left(3 - 2\sqrt{3}\right)}{4}$$
$$= \frac{2\sqrt{3} - 3}{6}$$

8

$$\boxed{\begin{aligned} x &= \sin^2\theta \\ dx &= 2\sin\theta\cos\theta\,d\theta\end{aligned}}$$

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx$$
$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \times 2\sin\theta\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\,d\theta$$
$$= 2\int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} (1 - \cos 2\theta)\,d\theta$$
$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$
$$= \left(\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0 - 0)\right)$$
$$= \frac{\pi}{4} - \frac{1}{2}$$

9

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta}$$
$$= \int_1^{\sqrt{3}} \frac{1+t^2}{2t} \times \frac{2dt}{1+t^2}$$
$$= \int_1^{\sqrt{3}} \frac{dt}{t}$$
$$= \Big[\ln t\Big]_1^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \ln 1$$
$$= \ln 3^{\frac{1}{2}}$$
$$= \frac{1}{2}\ln 3$$

10

$$\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta\end{aligned}}$$

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx$$
$$= \int \frac{4\sin^2\theta}{\sqrt{4 - 4\sin^2\theta}} \times 2\cos\theta\,d\theta$$
$$= \int \frac{4\sin^2\theta}{2\cos\theta} \times 2\cos\theta\,d\theta$$
$$= 4\int \sin^2\theta\,d\theta$$
$$= 4\int \frac{1}{2}(1 - \cos 2\theta)\,d\theta$$
$$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + c$$
$$= 2\theta - 2\sin\theta\cos\theta + c$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + c$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + c$$

11

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2}\end{aligned}}$$

$$\int \frac{3 + \cos x}{2 - \cos x}\,dx$$
$$= \int \frac{3 + \frac{1-t^2}{1+t^2}}{2 - \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$
$$= 2\int \frac{3 + 3t^2 + 1 - t^2}{2 + 2t^2 - 1 + t^2} \times \frac{dt}{1+t^2}$$
$$= 2\int \frac{2t^2 + 4}{(3t^2+1)(1+t^2)}\,dt$$
$$= 2\int \left(\frac{5}{3t^2+1} - \frac{1}{1+t^2}\right)dt$$
$$= \frac{10}{\sqrt{3}}\int \frac{\sqrt{3}}{3t^2+1}\,dt - 2\int \frac{1}{1+t^2}\,dt$$
$$= \frac{10}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}t\right) - 2\tan^{-1}t + c$$
$$= \frac{10}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{x}{2}\right) - x + c$$

12 $\int \sqrt{2x - x^2}\, dx$

$$= \int \sqrt{-(x^2 - 2x + 1 - 1)}\, dx$$

$$= \int \sqrt{1 - (x-1)^2}\, dx$$

$$\boxed{\begin{aligned} x - 1 &= \sin\theta \\ dx &= \cos\theta\, d\theta \end{aligned}}$$

$$= \int \sqrt{1 - \sin^2\theta} \times \cos\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int (1 + \cos 2\theta) d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + c$$

$$= \frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4} + c$$

$$= \frac{\sin^{-1}(x-1)}{2} + \frac{(x-1)\sqrt{1-(x-1)^2}}{2} + c$$

$$= \frac{\sin^{-1}(1-x)}{2} + \frac{(x-1)\sqrt{2x-x^2}}{2} + c$$

4.8 INTEGRATION BY PARTS

In Lesson 8 we look at Integration by Parts (IBP), a method that we use to integrate an integrand that involves two different types of functions - it is sometimes used for two similar functions as we will see. Although it will work on the simpler integrals we have already solved, we reserve it for more complicated integrals.

We will cover:

- How Integration by Parts works
- Integration by Parts using DETAIL
- Integrations by Parts using the Reverse Chain Rule

HOW INTEGRATION BY PARTS WORKS

Integration by Parts comes about by taking the Product Rule from Differentiation, integrating both sides and rearranging. Again note that we are using integration as the inverse operation to remove differentiation:

$$\frac{d}{dx}(u \times v) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\therefore \int \frac{d}{dx}(u \times v)dx = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$u \times v = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$\boxed{\int u\frac{dv}{dx}dx \ = uv - \int v\frac{du}{dx}dx}$$

In words:

1. split the integrand into two functions
2. find the primitive of one of the functions and put it in both parts of the RHS
3. the other function stays the same in the square brackets and is differentiated for the integral.

This makes more sense when you see it used in the examples to follow.

Notes:

- In some questions you will need to integrate by parts twice to get the answer
- In Appendix 1 we will look at an alternative method - Tabular Integration by Parts.

INTEGRATION BY PARTS USING DETAIL

Once you split the original expression into the correct functions for u and $\frac{dv}{dx}$ then this work is straightforward, however if you choose the wrong functions then the work becomes either much harder, or impossible and you will have to start again.

The trick to Integration by Parts lies in correctly splitting the integrand into one function that is easy to integrate and a second function that is simpler when it is differentiated.

To split the integrand and make our solution work as easily as possible we have two choices:

- **DETAIL *** - This rule of thumb helps us to choose which function is most easily integrated (which we make $\frac{dv}{dx}$) when we have two different types of functions. It works in the vast majority of questions in the HSC or from textbooks.
- **Reverse Chain Rule** - If one of the functions is a compound function we find $\frac{dv}{dx}$ as the product of the composite function and the derivative of the inner function. This can then be integrated using the Reverse Chain Rule.

* Alternatives to **DETAIL** are **LIATE** or **ILATE**, which focus on choosing the function for u rather than the function for $\frac{dv}{dx}$. In other countries **LIPET** is also used for choosing u, where P stands for Polynomials rather than Algebraic. Note that since we will probably never have an integrand with a logarithmic and an inverse trig function it doesn't matter whether LIATE or ILATE is used. Since Exponential and Trigonometric functions can both be integrated we can swap the order of those terms and still achieve success.

I used to use LIATE, but now prefer DETAIL as it helps us focus our attention on $\frac{dv}{dx}$, which is the pressure point for harder questions in Integration by Parts and Recurrence Relationships next lesson.

DETAIL

D – Let $\dfrac{dv}{dx}$ equal the first function from this list to appear in the integrand

E – Exponential

T – Trigonometric

Easiest to Integrate, so these are the best functions for $\frac{dv}{dx}$

A – Algebraic

Easy to Integrate, so still good for $\frac{dv}{dx}$

I – Inverse Trig

L – Logarithmic

Hard to Integrate, so no good for $\frac{dv}{dx}$

For example $\displaystyle\int x^2 \ln x\, dx$ has x^2 (an **A**lgebraic term) and $\ln x$ (a **L**ogarithmic term). Since **A** comes before **L** in DETAIL we let $\dfrac{dv}{dx} = x^2$ while u equals the rest of the integrand, in this case $u = \ln x$.

DETAIL focuses on the functions that are best to integrate, but it is also interesting to consider how well each function works as u. We want $\dfrac{du}{dx}$ to be much simpler than u.

- Logarithmic functions differentiate to become simple algebraic functions, so are the best choice for u if available.
- Inverse Trig functions differentiate to become algebraic functions, so are also a good choice for u.
- Algebraic functions differentiate easily so are a good choice for u.
- Trig functions often cycle from sine to cosine, so while we can use them for u they will require us to integrate twice (see Example 2)
- Exponential functions don't get simpler when they are differentiated, plus we need to know extra rules, so are no good as u.

If we only have one function in the integrand we often let $\frac{dv}{dx} = 1$ (as this is easy to integrate) and let u equal the function.

In the exercises we will do one of the integrals twice, once using DETAIL and the other going against it

- when we follow DETAIL the answer is straightforward
- when we go against DETAIL we will still get the correct answer but it will be a harder solution.

Example 1

Find $\int x \ln x \, dx$

This example is also done using Tabular IBP in Appendix 1

Solution

$$\int x \ln x \, dx$$

$$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \times \frac{1}{x} \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

$$\boxed{\begin{array}{ll} u = \ln x & \dfrac{dv}{dx} = x \\ \dfrac{du}{dx} = \dfrac{1}{x} & v = \dfrac{x^2}{2} \end{array}}$$

Using DETAIL $\frac{dv}{dx} = x$

At the start it can help to draw these lines to see which pairs we need to multiply

Example 2 - Using IBP twice

Evaluate $\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$

This example is also done using Tabular IBP in Appendix 1

Solution

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$$= \Big[e^x \sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$= \Big[e^x \sin x\Big]_0^{\frac{\pi}{2}} - \left(\Big[e^x \cos x\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx\right)$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \Big[e^x \sin x\Big]_0^{\frac{\pi}{2}} - \Big[e^x \cos x\Big]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \frac{1}{2}\left(\left(e^{\frac{\pi}{2}} - 0\right) - \left(0 - 1\right)\right)$$

$$= \frac{e^{\frac{\pi}{2}} + 1}{2}$$

$$\boxed{\begin{array}{ll} u = \sin x & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = \cos x & v = e^x \end{array}}$$

Using DETAIL $\frac{dv}{dx} = e^x$

$$\boxed{\begin{array}{ll} u = \cos x & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = -\sin x & v = e^x \end{array}}$$

We have to use Integration by Parts a second time. Since the u and v don't form part of our actual solution we can use them again,
Using DETAIL $\frac{dv}{dx} = e^x$

INTEGRATION BY PARTS WITH THE REVERSE CHAIN RULE

The vast majority of questions you come across will be solved using DETAIL, but there are more obscure questions where DETAIL will not work since both functions are of the same type, commonly with one of the functions being a compound function. Here we will use the Reverse Chain Rule. While we could often simplify the integrand first using u substitution it would cause more confusion (you would need a different variable as you couldn't use u twice) and make for a longer solution. This method is also good for Recurrence Relationships where we are not given the rule.

Let $\frac{dv}{dx}$ be the product of the composite function and the derivative of the inner function.

In Example 3 we will look at $\int x^3 e^{x^2}\, dx$

- e^{x^2} is the compound function.
- The derivative of the inner function, x^2, is $2x$
- Let $\frac{dv}{dx} = 2xe^{x^2}$
- That leaves us with $\frac{x^2}{2}$ from the integrand, which gives us u.

Example 3 - Reverse Chain Rule

Evaluate $\int x^3 e^{x^2}\, dx$

Solution

$$\int x^3 e^{x^2}\, dx$$

$u = \frac{x^2}{2}$	$\frac{dv}{dx} = 2xe^{x^2}$
$\frac{du}{dx} = x$	$v = e^{x^2}$

Using RCR
$\frac{dv}{dx} = 2xe^{x^2}$

$$= \frac{x^2 e^{x^2}}{2} - \int xe^{x^2}\, dx$$

$$= \frac{x^2 e^{x^2}}{2} - \frac{1}{2}\int xe^{x^2}\, dx$$

$$= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + c$$

SIMILAR FUNCTIONS

We can also use IBP with the Reverse Chain Rule to integrate integrands with two similar functions, although quite often it is a hidden example of reversing the quotient rule as we will see in the next example. Again a u substitution would create a more involved and longer solution.

Example 4

Find $\displaystyle\int \frac{\ln x}{(1+\ln x)^2}\,dx$

i Using IBP

ii By using the quotient rule to find a function whose derivative equals the integrand

Solution

i $\displaystyle\int \frac{\ln x}{(1+\ln x)^2}\,dx$

$$u = x\ln x \qquad \frac{dv}{dx} = \frac{1}{x}(1+\ln x)^{-2}$$

$$\frac{du}{dx} = \ln x + 1 \qquad v = -(1+\ln x)^{-1} = -\frac{1}{1+\ln x}$$

Using RCR

$$\frac{dv}{dx} = \frac{1}{x}(1+\ln x)^{-2}$$

$$= \int \frac{x\ln x}{x(1+\ln x)^2}\,dx$$

$$= -\frac{x\ln x}{1+\ln x} + \int \frac{\ln x + 1}{1+\ln x}\,dx$$

$$= -\frac{x\ln x}{1+\ln x} + x + c$$

$$= \frac{x}{1+\ln x} + c$$

ii

$$\therefore \int \frac{\ln x}{(1+\ln x)^2}\,dx = \int \frac{1+\ln x - 1}{(1+\ln x)^2}\,dx$$

$$= \int \frac{(1+\ln x)(1) - (x)\left(\frac{1}{x}\right)}{(1+\ln x)^2}\,dx$$

$$= \int \frac{d}{dx}\left(\frac{x}{1+\ln x}\right)dx$$

$$= \frac{x}{1+\ln x} + c$$

Although this solution is simpler, it would be very difficult to see how to proceed under exam conditions.

EXERCISE 4.8

BASIC

Use Integration by parts to find/evaluate the following integrals, unless told otherwise.

1 $\int x^2 \ln x \, dx$

2 $\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$

3 $\int x^3 \sin x^2 \, dx$

4 $\int \ln x \, dx$

Hint: let $u = \ln x$ and $\frac{dv}{dx} = 1$.

You may wish to memorise this result.

5 $\int xe^{2x} \, dx$

6 $\int_0^{\pi} x \cos x \, dx$

7 $\int_0^2 te^{-t} \, dt$

8 $\int_0^1 \tan^{-1} x \, dx$

MEDIUM

9 $\int \ln|1 + x| \, dx$

10 $\int x2^x \, dx$

CHALLENGING

11 Find $\int x^2\sqrt{x-1} \, dx$

i Using IBP **ii** Using a u^2 substitution

12 Find $\int \frac{\ln x - 2}{(\ln x - 1)^2} dx$

i Using IBP

ii By using the quotient rule to find a function whose derivative equals the integrand

13 Using the result from Q4, repeat Q1 but ignore DETAIL and let $u = x^2$ and $\frac{dv}{dx} = \ln x$

14 $\int x \sin x \cos x \, dx$

15 $\int \frac{xe^x}{(1 + x)^2} dx$

1

$$\int x^2 \ln x \, dx$$

$$= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \times \frac{1}{x} dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2 \, dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$$

$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + c$$

$$u = \ln x \qquad \frac{dv}{dx} = x^2$$
$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^3}{3}$$

2

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \left(\left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx\right)$$

$$u = \cos x \qquad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = -\sin x \quad v = e^x$$

$$u = \sin x \qquad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = \cos x \qquad v = e^x$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \left[e^x \sin x\right]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \frac{1}{2}\left(\left(0 - 1\right) + \left(e^{\frac{\pi}{2}} - 0\right)\right)$$

$$= \frac{e^{\frac{\pi}{2}} - 1}{2}$$

3

$$\int x^3 \sin x^2 \, dx$$

$$= -\frac{x^2 \cos x^2}{2} + \int x \cos x^2 \, dx$$

$$= -\frac{x^2 \cos x^2}{2} + \frac{1}{2}\int 2x \cos x^2 \, dx$$

$$= -\frac{x^2 \cos x^2}{2} + \frac{\sin x^2}{2} + c$$

$$u = \frac{x^2}{2} \qquad \frac{dv}{dx} = 2x \sin x^2$$
$$\frac{du}{dx} = x \qquad v = -\cos x^2$$

4

$$\int \ln x \, dx$$

$$= x \ln x - \int x \times \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c$$

$$u = \ln x \qquad \frac{dv}{dx} = 1$$
$$\frac{du}{dx} = \frac{1}{x} \qquad v = x$$

5

$$\int xe^{2x}\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$$

$$\boxed{\begin{array}{ll} u = x & dv = e^{2x} \\ \dfrac{du}{dx} = 1 & v = \dfrac{1}{2}e^{2x} \end{array}}$$

6

$$\int_0^{\pi} x\cos x\,dx$$

$$= \Big[x\sin x\Big]_0^{\pi} - \int_0^{\pi}\sin x\,dx$$

$$= 0 - 0 + \Big[\cos x\Big]_0^{\pi}$$

$$= -1 - 1$$

$$= -2$$

$$\boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = 1 & v = \sin x \end{array}}$$

7

$$\int_0^{2} te^{-t}\,dt$$

$$= -\Big[te^{-t}\Big]_0^{2} + \int_0^{2} e^{-t}\,dt$$

$$= -\frac{2}{e^2} - \Big[e^{-t}\Big]_0^{2}$$

$$= -\frac{2}{e^2} - \frac{1}{e^2} + 1$$

$$= 1 - \frac{3}{e^2}$$

$$\boxed{\begin{array}{ll} u = t & \dfrac{dv}{dt} = e^{-t} \\ \dfrac{du}{dx} = 1 & v = -e^{-t} \end{array}}$$

8

$$\int_0^{1} \tan^{-1}x\,dx$$

$$= \Big[x\tan^{-1}x\Big]_0^{1} - \int_0^{1}\frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - 0 - \frac{1}{2}\Big[\ln|1+x^2|\Big]_0^{1}$$

$$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0)$$

$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$

$$\boxed{\begin{array}{ll} u = \tan^{-1}x & dv = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x^2} & v = x \end{array}}$$

9

$$\int \ln(1+x)\,dx$$

$$\boxed{\begin{array}{ll} u = \ln(1+x) & dv = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x} & v = x \end{array}}$$

$$= x\ln(1+x) - \int \frac{x}{1+x}\,dx$$
$$= x\ln(1+x) - \int \frac{1+x-1}{1+x}\,dx$$
$$= x\ln(1+x) - \int 1 - \frac{1}{1+x}\,dx$$
$$= x\ln(1+x) - x + \ln(1+x) + c$$
$$= (x+1)\ln(1+x) - x + c$$

10

$$\int x2^x\,dx$$

$$\boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = 2^x \\ \dfrac{du}{dx} = 1 & v = \dfrac{2^x}{\ln 2} \end{array}}$$

$$= \frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\,dx$$
$$= \frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\left(\frac{2^x}{\ln 2}\right) + c$$
$$= \frac{x2^x}{\ln 2} - \frac{2^x}{\ln^2 2} + c$$

11

i $\int x^2\sqrt{x-1}\,dx$

$$\boxed{\begin{array}{ll} u = x^2 & dv = (x-1)^{\frac{1}{2}} \\ \dfrac{du}{dx} = 2x & v = \dfrac{2}{3}(x-1)^{\frac{3}{2}} \end{array}}$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\int x(x-1)^{\frac{3}{2}}\,dx$$
$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\left[\frac{2x(x-1)^{\frac{5}{2}}}{5} - \frac{2}{5}\int (x-1)^{\frac{5}{2}}\,dx\right]$$
$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{8x(x-1)^{\frac{5}{2}}}{15} + \frac{16}{105}(x-1)^{\frac{7}{2}} + c$$
$$= \frac{2x^2\sqrt{(x-1)^3}}{3} - \frac{8x\sqrt{(x-1)^5}}{15} + \frac{16\sqrt{(x-1)^7}}{105} + c$$

ii $\int x^2\sqrt{x-1}\,dx$

$$\boxed{\begin{array}{c} u^2 = x-1 \\ 2u\,du = dx \end{array}}$$

$$= \int x^2 u \times 2u\,du$$
$$= 2\int (u^2+1)^2 u^2\,du$$
$$= 2\int (u^6 + 2u^4 + u^2)\,du$$
$$= \frac{2u^7}{7} + \frac{4u^5}{5} + \frac{2u^3}{3} + c$$
$$= \frac{2\sqrt{(x-1)^7}}{7} + \frac{4\sqrt{(x-1)^5}}{5} + \frac{2\sqrt{(x-1)^3}}{3} + c$$

12

i $\int \frac{\ln x - 2}{(\ln x - 1)^2}\,dx$

$$u = x(\ln x - 2) \qquad \frac{dv}{dx} = \frac{1}{x}(\ln x - 1)^{-2}$$

$$\frac{du}{dx} = x\left(\frac{1}{x}\right) + (\ln x - 2)(1) \qquad v = -(\ln x - 1)^{-1}$$

$$= \ln x - 1 \qquad = -\frac{1}{\ln x - 1}$$

$$= -\frac{x(\ln x - 2)}{\ln x - 1} + \int \frac{\ln x - 1}{\ln x - 1}\,dx$$

$$= -\frac{x(\ln x - 2)}{\ln x - 1} + \int dx$$

$$= \frac{2x - x\ln x}{\ln x - 1} + x + c$$

$$= \frac{2x - x\ln x}{\ln x - 1} + \frac{x\ln x - x}{\ln x - 1} + c$$

$$= \frac{x}{\ln x - 1} + c$$

ii

$$\therefore \int \frac{\ln x - 2}{(\ln x - 1)^2}\,dx = \int \frac{\ln x - 1 - 1}{(\ln x - 1)^2}\,dx$$

$$= \int \frac{(\ln x - 1)(1) - (x)\left(\frac{1}{x}\right)}{(\ln x - 1)^2}\,dx$$

$$= \int \frac{d}{dx}\left(\frac{x}{\ln x - 1}\right)dx$$

$$= \frac{x}{\ln x - 1} + c$$

13

$$u = x^2 \qquad \frac{dv}{dx} = \ln x$$

$$\frac{du}{dx} = 2x \qquad v = x\ln x - x$$

$$\int x^2 \ln x\,dx = x^2(x\ln x - x) - 2\int x(x\ln x - x)\,dx$$

$$= x^3\ln x - x^3 - 2\int (x^2\ln x - x^2)\,dx$$

$$= x^3\ln x - x^3 - 2\int x^2\ln x\,dx + 2\int x^2 dx$$

$$\therefore 3\int x^2\ln x\,dx = x^3\ln x - x^3 + \frac{2x^3}{3} + c$$

$$\int x^2\ln x\,dx = \frac{x^3\ln x}{3} - \frac{x^3}{9} + c$$

14

$$I = \int x \sin x \cos x \, dx$$

$$u = x \sin x \qquad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = \sin x + x \cos x \qquad v = \sin x$$

$$= x \sin^2 x - \int (\sin^2 x + x \sin x \cos x) \, dx$$

$$= x \sin^2 x - \int \sin^2 x \, dx - \int x \sin x \cos x \, dx$$

$$\therefore 2I = x \sin^2 x - \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$I = \frac{x \sin^2 x}{2} - \frac{1}{4}\left(x - \frac{1}{2} \sin 2x\right) + c$$

$$= \frac{x \sin^2 x}{2} - \frac{x}{4} + \frac{\sin 2x}{8} + c$$

15

$$\int \frac{xe^x}{(1+x)^2} dx$$

$$u = xe^x \qquad \frac{dv}{dx} = \frac{1}{(1+x)^2}$$

$$\frac{du}{dx} = xe^x + e^x \qquad v = -\frac{1}{1+x}$$

$$= e^x(x+1)$$

$$= -\frac{xe^x}{1+x} + \int e^x \, dx$$

$$= -\frac{xe^x}{1+x} + e^x + c$$

$$= \frac{-xe^x + e^x + xe^x}{1+x} + c$$

$$= \frac{e^x}{1+x} + c$$

Alternatively

$$\int \frac{xe^x}{(1+x)^2} dx$$

$$= \int \frac{(1+x)e^x - e^x}{(1+x)^2} dx$$

$$= \int \frac{d}{dx}\left(\frac{e^x}{1+x}\right) dx$$

$$= \frac{e^x}{1+x} + c$$

In Lesson 9 we look at Recurrence Relationships, commonly known as reduction formula:

We will cover:

- How to determine whether to use Integration by Parts or not
- Recurrence Relationships using Integration by parts
- Recurrence Relationships using other methods

RECURRENCE RELATIONSHIPS

Recurrence Relationships express an integral (where the integrand includes a function to the power of n) as a function of another integral whose integrand includes the same function to a different power of n. For example we can prove that

$$\int_0^1 x^n\sqrt{1-x^2}\,dx = \left(\frac{n-1}{n+2}\right)\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx$$

The first integral includes x^n, and is expressed in terms of the integral including x^{n-2}. The power in the second integral is usually lower like in this example, which gives us the common alternative name of reduction formula.

Typically we use abbreviations such as I_n for the integrals, and will find a relationship between I_n and I_{n-1}, or I_n and I_{n-2}, so we would write the reduction formula in the above example as

$$I_n = \left(\frac{n-1}{n+2}\right)I_{n-2}$$

We can use letters other than I for the integral, such as U or A, or leave the integral written in complete form.

We often use the recurrence relationships to evaluate integrals with higher powers by breaking them down into functions of a lower powered integral that we can integrate.

INTEGRATION BY PARTS OR NOT?

Recurrence Relationships are normally considered to be of medium to higher difficulty, as it is very easy to choose the wrong method and have to start again. Success has traditionally required much practise and a deep knowledge of differentiation and integration. Part of the problem is that while most Recurrence Relationships are found using integration by parts, others can only be solved using standard integration techniques.

There is however a very easy way to choose the correct method, which converts the topic to a simple mechanical process. This is great for quickly gaining marks in an exam, but our understanding suffers! We will also try to attain a deeper understanding of the work.

In Appendix 1 we will look at Tabular Integration by Parts which can also be used for those Recurrence Relationships where IBP applies, but is only an interesting aside - I recommend you use the traditional method shown in the examples.

CHOOSING THE CORRECT METHOD

As a rule of thumb, look at the coefficient of the lower powered integral on the RHS (I_{n-1} or I_{n-2}) and see if it involves n. Here are 3 examples:

$$(\mathbf{A})\ I_n = \frac{e}{2} \overset{\boxed{-n}}{\overset{\downarrow}{-\,n}} I_{n-1}$$

$$(\mathbf{B})\ I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \overset{\boxed{\frac{n-2}{n-1}}}{\overset{\downarrow}{\frac{n-2}{n-1}}} I_{n-2}$$

$$(\mathbf{C})\ I_n = \frac{1}{2n-1} \overset{\boxed{-1}}{\overset{\downarrow}{-}} I_{n-1}$$

Integration by Parts

If the coefficient involves n (examples A and B above) then this tells us to use integration by parts. The numerator of the coefficient also gives us an important shortcut - use it (or the multiple closest to the power in the original integral) as the power to use for u.

Other Methods

If the coefficient does not involve n (example C above) then we do not use integration by parts. It is normally easiest to move both integrals to the one side and simplify.

The rule of thumb requires that the integral with the higher power (usually n) is the subject of the reduction formula, so on the LHS.

Original Integral **Recurrence Relationship to be Proved**

$I_n = \int_1^{e^2} (\log_e x)^n \, dx$ $\qquad I_n = e^2 2^n - nI_{n-1}$

- The coefficient of I_{n-1} involves n so use IBP.
- The numerator is n, so let $u = (\log_e x)^n$

$I_n = \int_0^x \sec^n t \, dt$ $\qquad I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

- The coefficient of I_{n-2} involves n so use IBP.
- The numerator is $n-2$, so let $u = \sec^{n-2} t$.

$I_n = \int_0^1 x^{2n+1} e^{x^2} \, dx$ $\qquad I_n = \dfrac{e}{2} - nI_{n-1}$

- The coefficient of I_{n-1} involves n so use IBP.
- The numerator is n, and in this case we use a multiple of n, so $u = x^{2n}$.
- Where there are two functions in the integral, use the function that is already to the power of some function of n.

$I_n = \int_0^{\frac{\pi}{4}} \tan^{2n} \theta \, d\theta$ $\qquad I_n = \dfrac{1}{2n-1} - I_{n-1}$

- The coefficient of I_{n-1} is -1. As it does not involve n we don't use integration by parts.
- The easiest method is to rearrange the expression and first prove that

$$I_n + I_{n-1} = \frac{1}{2n-1}$$

$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$ $\qquad I_n = \left(\dfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

The numerator $n(n-1)$ is the product of two consecutive terms, which indicates that we will integrate by parts twice:

- the first time using $u = x^n$
- the second time using $u = x^{n-1}$

Note that we use the higher term for the first IBP.

Example 1

If $I_n = \int x^n e^x \, dx$ prove that $I_n = x^n e^x - nI_{n-1}$

This example is also done using Tabular IBP in Appendix 1

Solution

$$I_n = \int x^n e^x \, dx$$

$$\boxed{\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = nx^{n-1} & v = e^x \end{array}}$$

$$= x^n e^x - \int nx^{n-1} e^x \, dx$$

$$= x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - nI_{n-1}$$

The coefficient of I_{n-1} involves n, so use IBP.

The numerator is n which gives us the power we want, so we let $u = x^n$.

Example 2

If $I_n = \int \cos^n x \, dx$ prove that $I_n = \dfrac{\cos^{n-1} x \sin x}{n} + \dfrac{n-1}{n} I_{n-2}$

Solution

$$I_n = \int \cos^n x \, dx$$

$$\boxed{\begin{array}{ll} u = \cos^{n-1} x & dv = \cos x \ dx \\ \dfrac{du}{dx} = -(n-1)\cos^{n-2} x \sin x & v = \sin x \end{array}}$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin x \sin x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, (1 - \cos^2 x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$\therefore I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$nI_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

The coefficient of I_{n-2} involves n, so use IBP.

Looking at the final answer we see a numerator of $n-1$ which gives us the power we want, so we let $u = \cos^{n-1} x$.

Example 3

If $I_n = \int \tan^n x \, dx$ prove that $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$

Solution

The coefficient of I_{n-2} does not involve n, so move both integrals to the one side and simplify.

$I_n + I_{n-2}$

$= \int \tan^n x \, dx + \int \tan^{n-2} x \, dx$

$= \int \tan^{n-2} x \, (\tan^2 x + 1) \, dx$

$= \int \tan^{n-2} x \sec^2 x \; dx$

$= \int \frac{d}{dx}\left(\frac{\tan^{n-1} x}{n-1}\right) dx$

$= \frac{1}{n-1}\tan^{n-1} x$

$\therefore I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$

Example 4

In Example 1 we saw that if $I_n = \int x^n e^x \, dx$ then $I_n = x^n e^x - nI_{n-1}$

Use this recurrence relationship to evaluate $\int x^3 e^x \, dx$

Solution

$$I_0 = \int x^0 e^x \, dx$$

$$= \int e^x \, dx$$

$$= e^x + c$$

1. We usually start by evaluating I_0, as it will be the easiest integral to evaluate. Sometimes I_1 or rarely I_2 will be easy to integrate.
2. Determine which integral you are trying to find.
3. Use the reduction formula to express the integral you are trying to find in terms of I_0.
4. Evaluate.

$$\int x^3 e^x \, dx = I_3$$

$$\begin{aligned} I_3 &= x^3 e^x - 3I_2 \\ &= x^3 e^x - 3(x^2 e^x - 2I_1) \\ &= x^3 e^x - 3x^2 e^x + 6(x^1 e^x - 1I_0) \\ &= (x^3 - 3x^2 + 6x)e^x - 6I_0 \end{aligned}$$

$$\begin{aligned} \therefore \int x^3 e^x \, dx &= (x^3 - 3x^2 + 6x)e^x - 6(e^x + c) \\ &= (x^3 - 3x^2 + 6x - 6)e^x + c \end{aligned}$$

WHAT IF THE RECURRENCE RELATIONSHIP IS NOT GIVEN?

In all recent HSC questions the recurrence relationship has been given in the question, so we can use the tricks from this lesson to quickly choose whether to use IBP, and if so the correct value of u.

But how would we proceed if the recurrence relationship was not given? This might occur if the examiners want to create a much more difficult question for use late in the paper.

In our shortcut that we have used so far this lesson we have focused on u, but if the recurrence relationship is not known we shift our focus back to $\frac{dv}{dx}$, choosing it so that it is the most complicated function that we can easily integrate. This often uses IBP with the Reverse Chain Rule that we used last lesson, or is sometimes equal to 1. For trig functions we often need to split the powers so we can use the Pythagorean identities. It is important in all types that the n stays as a power of u, not becoming part of $\frac{dv}{dx}$.

Example 5

Find a recurrence relationship for $I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx, \qquad \text{for } n = 0, 1, 2..$

Solution

$$I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx, \qquad \text{for } n = 0, 1, 2\ldots$$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^{n-1} & \frac{dv}{dx} = 2x(x^2+1)^{-2} \\ \frac{du}{dx} = \frac{1}{2}(n-1)x^{n-2} & v = -(x^2+1)^{-1} \end{array}}$$

$$\begin{aligned}
&= -\frac{1}{2}\left[\frac{x^{n-1}}{x^2+1}\right]_0^1 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{x^2+1}\,dx \\
&= -\frac{1}{4} - 0 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}(x^2+1)}{(x^2+1)^2}\,dx \\
&= -\frac{1}{4} + \frac{n-1}{2}\int_0^1 \frac{x^n}{(x^2+1)^2}\,dx + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{(x^2+1)^2}\,dx \\
&= -\frac{1}{4} + \frac{n-1}{2}I_n + \frac{n-1}{2}I_{n-2}
\end{aligned}$$

$$\therefore \frac{3-n}{2}I_n = -\frac{1}{4} + \frac{n-1}{2}I_{n-2}$$

$$I_n = \frac{1}{2(n-3)} - \frac{n-1}{n-3}I_{n-2}$$

Using the RCR

$$\frac{dv}{dx} = 2x(x^2+1)^{-2}$$

EXERCISE 4.9

BASIC

1 If $I_n = \int x^n e^{2x}\,dx$ prove that $I_n = \frac{x^n e^{2x}}{2} - \frac{n}{2} I_{n-1}$

2 If $I_n = \int \sin^n x\,dx$ prove that $I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$

3 If $I_n = \int \cot^n x\,dx$ prove that $I_n = -\frac{1}{n-1}\cot^{n-1} x - I_{n-2}$

4 In Question 1 we saw that if $I_n = \int x^n e^{2x}\,dx$ then $I_n = \frac{x^n e^{2x}}{2} - \frac{n}{2} I_{n-1}$. Find $\int x^2 e^{2x}\,dx$

MEDIUM

5 If $I_n = \int_1^{e^2} (\log_e x)^n\,dx$ prove that $I_n = 2^n e^2 - nI_{n-1}$

6 **i** Let $I_n = \int_0^x \sec^n t\,dt$ where $0 \le x \le \frac{\pi}{2}$. Show that $I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

ii Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sec^4 t\,dt$

7 Prove $\int x \ln^n x\,dx = \frac{x^2 \ln^n x}{2} - \frac{n}{2}\int x \ln^{n-1} x\,dx$ and hence find $\int x \ln^2 x\,dx$

8 If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$ prove that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

9 If $I_n = \int x^n \sqrt{2x+1}\,dx$ prove that $I_n = \frac{x^n\sqrt{(2x+1)^3}}{2n+3} - \frac{n}{2n+3} I_{n-1}$

CHALLENGING

10 Find a recurrence relationship for $I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx$

11 Find a recurrence relationship for $I_m = \int_0^1 x^m (x^2-1)^5\,dx$

12 Find a recurrence relationship for $I_n = \int_{-3}^0 x^n \sqrt{x+3}\,dx$

13 Find a recurrence relationship for $I_n = \int \frac{dx}{\sin^n x}$

1

$$I_n = \int x^n e^{2x}\,dx$$

$u = x^n$	$\dfrac{dv}{dx} = e^{2x}$
$\dfrac{du}{dx} = nx^{n-1}$	$v = \dfrac{1}{2}e^{2x}$

$$= \frac{x^n e^{2x}}{2} - \frac{n}{2}\int x^{n-1}e^{2x}\,dx$$

$$= \frac{x^n e^{2x}}{2} - \frac{n}{2}I_{n-1}$$

2

$$I_n = \int \sin^n x\,dx$$

$u = \sin^{n-1} x$	$dv = \sin x\ dx$
$\dfrac{du}{dx} = (n-1)\sin^{n-2} x\cos x$	$v = -\cos x$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos x\cos x\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$\therefore I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

$$I_n = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}I_{n-2}$$

3

$$I_n + I_{n-2}$$

$$= \int (\cot^n x + \cot^{n-2} x)\,dx$$

$$= \int \cot^{n-2} x\,(\cot^2 x + 1)\,dx$$

$$= \int \operatorname{cosec}^2 x\cot^{n-2} x\ \ dx$$

$$= -\frac{1}{n-1}\cot^{n-1} x$$

$$\therefore I_n = -\frac{1}{n-1}\cot^{n-1} x - I_{n-2}$$

4

$$I_0 = \int x^0 e^{2x}\,dx$$

$$= \int e^{2x}\,dx$$

$$= \frac{1}{2}e^{2x} + c$$

$$\int x^2 e^{2x}\,dx = I_2$$

$$= \frac{x^2 e^{2x}}{2} - \frac{2}{2}I_{2-1}$$

$$= \frac{x^2 e^{2x}}{2} - \left(\frac{xe^{2x}}{2} - \frac{1}{2}I_0\right)$$

$$= \frac{x^2 e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{1}{4}e^{2x} + c$$

5

$$I_n = \int_1^{e^2} (\log_e x)^n\,dx$$

$$u = (\log_e x)^n \qquad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = n(\log_e x)^{n-1} \times \frac{1}{x} \qquad v = x$$

$$= \Big[x(\log_e x)^n\Big]_1^{e^2} - n\int_1^{e^2} (\log_e x)^{n-1}\,dx$$

$$= e^2 \times 2^n - 0 - nI_{n-1}$$

$$\therefore I_n = 2^n e^2 - nI_{n-1}$$

6

i $I_n = \displaystyle\int_0^x \sec^n t\,dt$

$$u = \sec^{n-2} t \qquad \frac{dv}{dt} = \sec^2 t$$

$$\frac{du}{dt} = (n-2)\sec^{n-3} t \sec t \tan t \qquad v = \tan t$$

$$= (n-2)\sec^{n-2} t \tan t$$

$$= \Big[\sec^{n-2} t \tan t\Big]_0^x - (n-2)\int \sec^{n-2} t \tan^2 t\,dt$$

$$= \sec^{n-2} x \tan x - 0 - (n-2)\int \sec^{n-2} t\,(\sec^2 t - 1)\,dt$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n t\,dt + (n-2)\int \sec^{n-2} t\,dt$$

$$\therefore I_n = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}I_{n-2}$$

ii $\displaystyle\int_0^{\frac{\pi}{3}} \sec^4 t\,dt$

$$= I_4$$

$$= \frac{\sec^2\frac{\pi}{3}\tan\frac{\pi}{3}}{3} + \frac{2}{3}I_2$$

$$= 2^2 \times \frac{\sqrt{3}}{3} + \frac{2}{3}\left(\frac{1 \times \tan\frac{\pi}{3}}{1} + 0\right)$$

$$= \frac{4\sqrt{3}}{3} + \frac{2}{3}\left(\sqrt{3}\right)$$

$$= 2\sqrt{3}$$

7

$$I_n = \int x \ln^n x \, dx = \frac{x^2 \ln^n x}{2} - \frac{n}{2}\int \frac{\ln x}{x} \times x^2 \, dx$$

$$= \frac{x^2 \ln^n x}{2} - \frac{n}{2}\int x \ln^{n-1} x \, dx$$

$$\boxed{\begin{array}{ll} u = \ln^n x & \dfrac{dv}{dx} = x \\ \dfrac{du}{dx} = \dfrac{n \ln^{n-1} x}{x} & v = \dfrac{x^2}{2} \end{array}}$$

$$\therefore I_n = \frac{x^2 \ln^n x}{2} - \frac{n}{2} I_{n-1}$$

$$I_0 = \int x \, dx = \frac{x^2}{2} + c$$

$$I_2 = \frac{x^2 \ln^2 x}{2} - \frac{2}{2} I_1$$

$$= \frac{x^2 \ln^2 x}{2} - \left(\frac{x^2 \ln x}{2} - \frac{1}{2} I_0\right)$$

$$= \frac{x^2 \ln^2 x}{2} - \frac{x^2 \ln x}{2} + \frac{1}{2}\left(\frac{x^2}{2} + c\right)$$

$$= \frac{x^2 \ln^2 x}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + c$$

8

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$$

$$\boxed{\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = n x^{n-1} & v = \sin x \end{array}}$$

$$\boxed{\begin{array}{ll} u = x^{n-1} & \dfrac{dv}{dx} = \sin x \\ \dfrac{du}{dx} = (n-1) x^{n-2} & v = -\cos x \end{array}}$$

$$= \Big[x^n \sin x\Big]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$= \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$= \left(\frac{\pi}{2}\right)^n + n\left\{\Big[x^{n-1} \cos x\Big]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx\right\}$$

$$= \left(\frac{\pi}{2}\right)^n + n(0-0) - n(n-1) I_{n-2}$$

$$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$$

9

$$u = x^n \qquad \frac{dv}{dx} = (2x+1)^{\frac{1}{2}}$$
$$\frac{du}{dx} = nx^{n-1} \qquad v = \frac{1}{3}(2x+1)^{\frac{3}{2}}$$

$$I_n = \int x^n\sqrt{2x+1}\,dx$$

$$= \frac{x^n(2x+1)^{\frac{3}{2}}}{3} - \frac{n}{3}\int x^{n-1}(2x+1)^{\frac{3}{2}}\,dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}\int (2x+1)x^{n-1}(2x+1)^{\frac{1}{2}}\,dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}\int 2x\times x^{n-1}(2x+1)^{\frac{1}{2}}\,dx - \frac{n}{3}\int x^{n-1}(2x+1)^{\frac{1}{2}}\,dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{2n}{3}\int x^n\sqrt{2x+1}\,dx - \frac{n}{3}\int x^{n-1}\sqrt{2x+1}\,dx$$

$$\therefore I_n = \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{2n}{3}I_n - \frac{n}{3}I_{n-1}$$

$$\frac{2n+3}{3}I_n = \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}I_{n-1}$$

$$\therefore I_n = \frac{x^n\sqrt{(2x+1)^3}}{2n+3} - \frac{n}{2n+3}I_{n-1}$$

10

$$u = \frac{1}{2}x^{n-1} \qquad \frac{dv}{dx} = 2x(x^2+1)^{-2}$$
$$\frac{du}{dx} = \frac{1}{2}(n-1)x^{n-2} \qquad v = -(x^2+1)^{-1}$$

$$I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx, \qquad \text{for } n = 0, 1, 2\ldots$$

$$= -\frac{1}{2}\left[\frac{x^{n-1}}{x^2+1}\right]_0^1 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{x^2+1}\,dx$$

$$= -\frac{1}{4} - 0 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}(x^2+1)}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}\int_0^1 \frac{x^n}{(x^2+1)^2}\,dx + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}I_n + \frac{n-1}{2}I_{n-2}$$

$$\therefore \frac{3-n}{2}I_n = -\frac{1}{4} + \frac{n-1}{2}I_{n-2}$$

$$I_n = \frac{1}{2(n-3)} - \frac{n-1}{n-3}I_{n-2}$$

11

$$I_m = \int_0^1 x^m (x^2-1)^5\,dx$$

$$u = \frac{1}{2}x^{m-1} \qquad \frac{dv}{dx} = 2x(x^2-1)^5$$
$$\frac{du}{dx} = \frac{m-1}{2}x^{m-2} \qquad v = \frac{(x^2-1)^6}{6}$$

$$= \frac{1}{12}\Big[x^{m-1}(x^2-1)^6\Big]_0^1 - \frac{(m-1)}{12}\int_0^1 x^{m-2}(x^2-1)^6\,dx$$

$$= 0 - \frac{m-1}{12}\int_0^1 x^{m-2}(x^2-1)(x^2-1)^5\,dx$$

$$= -\frac{m-1}{12}\int_0^1 x^m(x^2-1)^5\,dx + \frac{m-1}{12}\int_0^1 x^{m-2}(x^2-1)^5\,dx$$

$$\therefore \frac{m+11}{12}I_m = \frac{m-1}{12}I_{m-2}$$

$$\therefore I_m = \frac{m-1}{m+11}I_{m-2}$$

12

$$I_n = \int_{-3}^0 x^n\sqrt{x+3}\,dx$$

$$u = x^n \qquad \frac{dv}{dx} = (x+3)^{\frac{1}{2}}$$
$$\frac{du}{dx} = nx^{n-1} \qquad v = \frac{2}{3}(x+3)^{\frac{3}{2}}$$

$$= \frac{2}{3}\Big[x^n(x+3)^{\frac{3}{2}}\Big]_{-3}^0 - \frac{2n}{3}\int_{-3}^0 x^{n-1}(x+3)^{\frac{3}{2}}\,dx$$

$$= 0 - \frac{2n}{3}\int_{-3}^0 x^{n-1}(x+3)(x+3)^{\frac{1}{2}}\,dx$$

$$= -\frac{2n}{3}\int_{-3}^0 x^n\sqrt{x+3}\,dx - 2n\int_{-3}^0 x^{n-1}\sqrt{x+3}\,dx$$

$$\therefore I_n = -\frac{2n}{3}I_n - 2nI_{n-1}$$

$$\frac{2n+3}{3}I_n = -2nI_{n-1}$$

$$I_n = -\frac{6n}{2n+3}I_{n-1}$$

13

$$I_n = \int \frac{dx}{\sin^n x}$$

$$u = \operatorname{cosec}^{n-2} x \qquad \frac{dv}{dx} = \operatorname{cosec}^2 x$$

$$\frac{du}{dx} = (n-2)\operatorname{cosec}^{n-3} x\,(-\operatorname{cosec} x \cot x) \qquad v = -\cot x$$

$$= \int \operatorname{cosec}^n x\,dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\,dx$$

$$= -\frac{\cos x}{\sin x} \times \frac{1}{\sin^{n-2} x} - (n-2)\int \operatorname{cosec}^{n-2} x\,(\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\frac{\cos x}{\sin^{n-1} x} - (n-2)\int \operatorname{cosec}^n x\,dx + (n-2)\int \operatorname{cosec}^{n-2} x\,dx$$

$$\therefore I_n = -\frac{\cos x}{\sin^{n-1} x} - (n-2)I_n + (n-2)I_{n-2}$$

$$(n-1)I_n = -\frac{\cos x}{\sin^{n-1} x} + (n-2)I_{n-2}$$

$$\therefore I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}$$

In Lesson 10 we look at a variety of techniques relating to definite integrals.

We will cover:

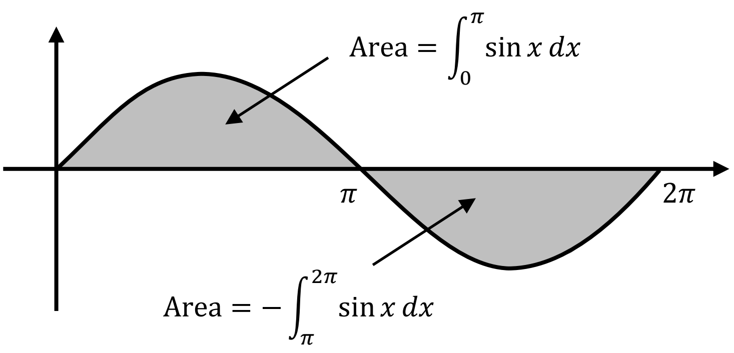
- Revision of Mathematics Advanced work on Integrals
- Working with Odd and/or Even Functions
- Reflections and translations

AREAS ABOVE OR BELOW THE X-AXIS

Part of this topic involves an understanding of how to manipulate integrals. This combines a basic understanding of what integrals are, with algebraic or geometric manipulation.

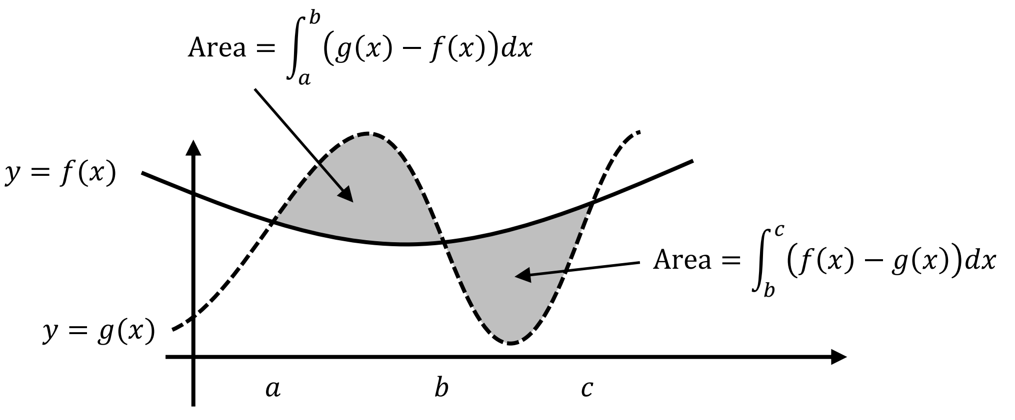
Where a function is above the x-axis the integral will be positive and be equal to the area between the curve and the x-axis. Where the function is below the x-axis it will be negative and be equal to the area between the curve and the x-axis multiplied by -1.

For example the function $y = \sin x$ is above the x-axis between 0 and π, and below the x-axis between π and 2π, so the respective areas would be $\int_0^{\pi} \sin x \, dx$ and $-\int_{\pi}^{2\pi} \sin x \, dx$



DIFFERENCE OF TWO FUNCTIONS

Where we have the difference of two functions, it is equal to the area from the first mentioned function down to the second one. If the second function is higher than the first function the integral will be negative.

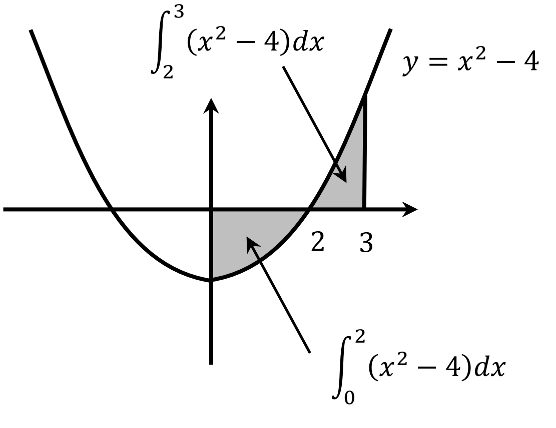


SPLITTING INTEGRALS

We can split an integral at any point, and the sum of the two integrals will equal the original. We will split an integral either where the curve cuts the x-axis, or at x=0 to compare the left and right hand sides of even or odd functions.

For example

$$\int_0^3 (x^2 - 4)dx$$
$$= \int_0^2 (x^2 - 4)dx + \int_2^3 (x^2 - 4)dx$$



SWITCHING LIMITS

If we switch the upper and lower limits of an integral the answer changes sign

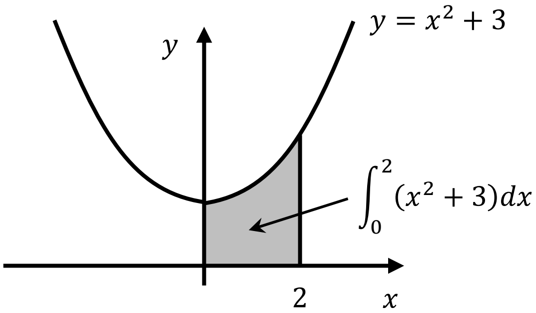
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

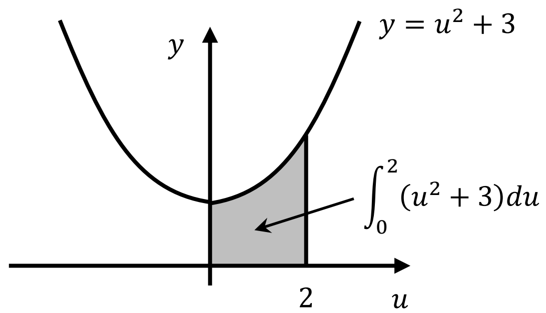
CHANGING VARIABLES

The value of an integral is independent of the variable used.

$$\int_a^b f(u)du = \int_a^b f(x)dx$$

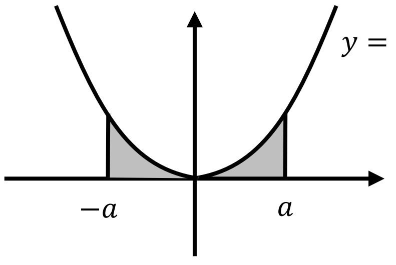
This is an important technique used in proofs.





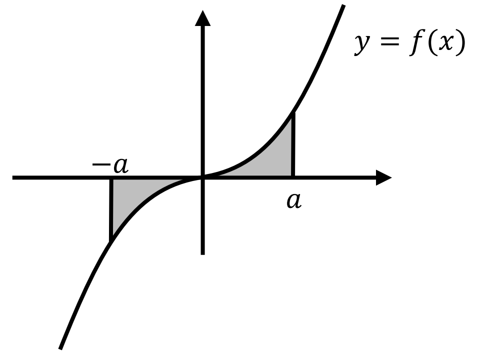
$$\int_0^2 (x^2+3)dx = \int_0^2 (u^2+3)du$$

Even Functions



$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)\,dx \text{ if } x \text{ is even}$$

Odd Functions



$$\int_{-a}^{a} f(x)dx = 0 \quad \text{ if } f(x) \text{ is odd}$$

When multiplying functions:

even × even = even

even × odd = odd

odd × even = odd

odd × odd = even

The same rules apply for division

When adding or subtracting functions:

even + even = even

even + odd = neither

odd + even = neither

odd + odd = odd

The same rules apply for subtraction

For many (but not all) compound functions:

If the inner and outer functions are both odd then the compound function is odd.

If either function is even the compound function is even.

TRANSFORMATIONS

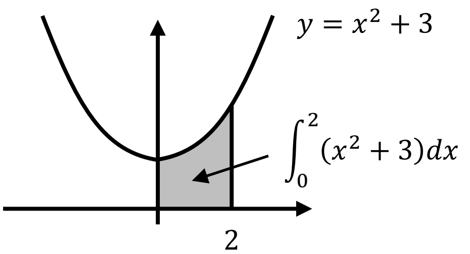
In Year 11 we looked at how to quickly sketch transformations of a curve. The transformations that moved a function left or right or reversed it horizontally, without changing the shape of the curve, can help us with areas under a curve, and thus with integrals in general.

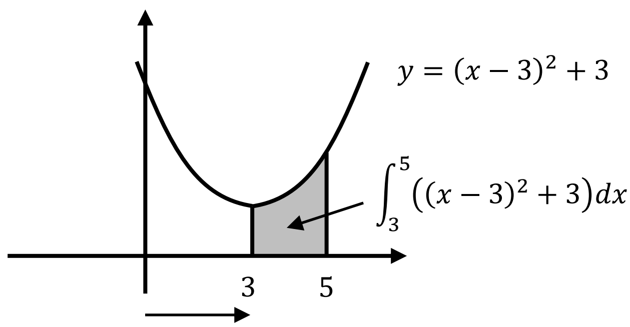
As a recap:

- $f(x-a)$ is $f(x)$ moved 'a' units to the right
- $f(-x)$ is $f(x)$ reflected about the y-axis
- $f(2a-x)$ is $f(x)$ reflected about $x=a$.

It is not only the functions that move, but also the areas under them, which we can use to find equivalent integrals. For example the area under the curve $y=x^2+3$ from $x=0$ to $x=2$, must be the same as the area under the curve $y=(x-3)^2+3$ from $x=3$ to $x=5$, as the curve is moved 3 units to the right, as are the upper and lower limits. We could thus say:

$$\int_0^2 (x^2+3)dx = \int_3^5 \big((x-3)^2+3\big)dx$$

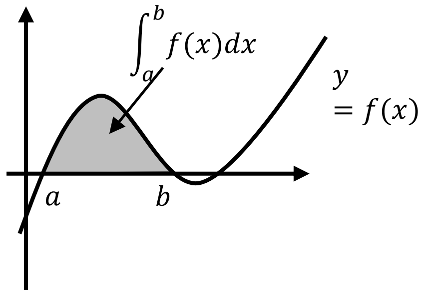


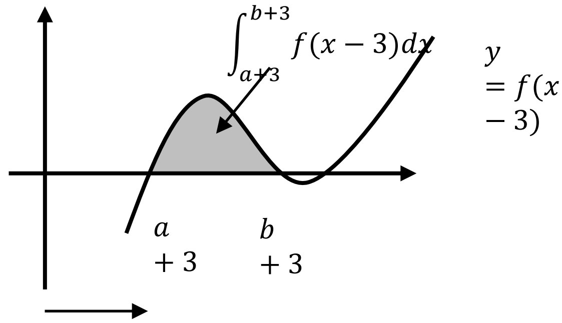


The curve and the limits have all been moved 3 units to the right.

We can generalize to find equivalent integrals like:

$$\int_a^b f(x)dx = \int_{a+3}^{b+3} f(x-3)dx$$





The curve and the limits have all been moved 3 units to the right.

Example 1

Prove $\int_0^a f(a-x)dx = \int_0^a f(x)dx$ and hence find $\int_0^\pi x\sin x\,dx$

Solution

$$\int_0^a f(a-x)dx$$

$$\boxed{\begin{aligned} u &= a-x \\ du &= -dx \\ dx &= -du \end{aligned}}$$

$$= \int_a^0 f(u)\times(-du)$$

$$= \int_0^a f(u)du$$

$$= \int_0^a f(x)dx$$

$$\therefore \int_0^\pi x\sin x\,dx = \int_0^\pi (\pi-x)\sin(\pi-x)\,dx$$

$$\int_0^\pi x\sin x\,dx = \int_0^\pi \pi\sin(\pi-x)\,dx - \int_0^\pi x\sin(\pi-x)\,dx$$

$$2\int_0^\pi x\sin x\,dx = \pi\int_0^\pi \sin x\,dx$$

$$\int_0^\pi x\sin x\,dx = \frac{\pi}{2}\int_0^\pi \sin x\,dx$$

$$= \frac{\pi}{2}\Big[-\cos x\Big]_0^\pi$$

$$= \frac{\pi}{2}\big(-(-1)-(-1)\big)$$

$$= \pi$$

Example 2

Evaluate $\int_0^1 (1-x)^{99}x\,dx$

Solution

$$\int_0^1 (1-x)^{99}x\,dx$$

$$= \int_0^1 x^{99}(1-x)\,dx$$

$$= \int_0^1 (x^{99}-x^{100})\,dx$$

$$= \left[\frac{x^{100}}{100}-\frac{x^{101}}{101}\right]_0^1$$

$$= \left(\frac{1}{100}-\frac{1}{101}\right)-\left(0-0\right)$$

$$= \frac{1}{10100}$$

Example 3

Evaluate $\int_{-a}^{a} \sin x \cos x \, dx$

Solution

Sine is an odd function and cosine is an even function, so the product is an odd function.

When $f(x)$ is an odd function $\int_{-a}^{a} f(x) dx = 0$

$\therefore \int_{-a}^{a} \sin x \cos x \, dx = 0$

Example 4

Evaluate $\int_{0}^{\frac{\pi}{2}} \tan\left(x - \frac{\pi}{4}\right) \sec\left(x - \frac{\pi}{4}\right) dx$

Solution

$$\int_{0}^{\frac{\pi}{2}} \tan\left(x - \frac{\pi}{4}\right) \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x \sec x \, dx$$

$= 0$ $\quad \tan x \sec x$ is odd since $\tan x$ is odd and $\sec x$ is even

BASIC

1 Prove $\int_0^a f(a-x)dx = \int_0^a f(x)dx$ and hence find $\int_0^{2\pi} x\cos x\,dx$

2 Evaluate $\int_0^2 x\sqrt{2-x}\,dx$

3 Evaluate $\int_{-2}^{2}(x+x^3+x^5)(1+x^2+x^4)dx$

4 Evaluate $\int_{-\pi}^{0}\sin\left(x+\frac{\pi}{2}\right)\cos\left(x+\frac{\pi}{2}\right)dx$

MEDIUM

5 A function $f(x)$ has the property that $f(x)+f(a-x)=f(a)$.

Given $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ prove that $\int_0^a f(x)\,dx = \frac{a}{2}f(a)$

6 Without evaluating the integrals, which one of the following integrals is greater than zero?

(A) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x^2}\cos x\,dx$ (B) $\int_{-\pi}^{\pi} x^3\cos x\,dx$ (C) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\sin^2 x-\cos^2 x)\,dx$ (D) $\int_{-1}^{1}\sin^{-1}(x^3)\,dx$

7 Which integral is necessarily equal to $\int_{-a}^{a} f(x)\,dx$

(A) $\int_0^a f(x-a)-f(-x)\,dx$ (B) $\int_0^a f(a+x)-f(x)\,dx$

(C) $\int_0^a f(x)+f(-x)\,dx$ (D) $\int_0^a f(x-a)-f(a-x)\,dx$

8 It is given that $f(x)$ is a non-zero even function and $g(x)$ is a non-zero odd function.

Which expression is equal to $\int_{-a}^{a} f(x)+g(x)\,dx$?

(A) $\int_0^a g(x)+g(-x)\,dx$ (B) $2\int_0^a g(x)+g(-x)\,dx$

(C) $\int_0^a f(x)+f(-x)\,dx$ (D) $2\int_0^a f(x)+f(-x)\,dx$

9 Which of these integrals has the smallest value?

A $\int_0^{\frac{\pi}{6}} \sin x \, dx$ **B** $\int_0^{\frac{\pi}{6}} \sin^2 x \, dx$ **C** $\int_0^{\frac{\pi}{6}} (1 - \sin x) \, dx$ **D** $\int_0^{\frac{\pi}{6}} (1 - \sin^2 x) \, dx$

10 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} \, dx$

11 Evaluate $\int_0^2 (1 + \sin(\pi(1 - x)^3)) \, dx$

CHALLENGING

12 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} \, dx$

1

$$\int_0^a f(a-x)dx$$

$$\boxed{\begin{aligned} u &= a-x \\ du &= -dx \\ dx &= -du \end{aligned}}$$

$$= \int_a^0 f(u) \times (-du)$$

$$= \int_0^a f(u)du$$

$$= \int_0^a f(x)dx$$

$$\int_0^{2\pi} x\cos x\,dx$$

$$= \int_0^{2\pi} (2\pi - x)\cos(2\pi - x)\,dx$$

$$= 2\pi\int_0^{2\pi} \cos(2\pi - x)\,dx - \int_0^{2\pi} x\cos(2\pi - x)\,dx$$

$$= 2\pi\int_0^{2\pi} \cos x\,dx - \int_0^{2\pi} x\cos x\,dx$$

$$\therefore 2\int_0^{2\pi} x\cos x\,dx = 2\pi\int_0^{2\pi} \cos x\,dx$$

$$\int_0^{2\pi} x\cos x\,dx = \pi\Big[\sin x\Big]_0^{2\pi}$$

$$= \pi(0-0)$$

$$= 0$$

2

$$\int_0^2 x\sqrt{2-x}\,dx$$

$$= \int_0^2 (2-x)\sqrt{x}\,dx$$

$$= \int_0^2 \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$$

$$= \frac{4}{3} \times 2\sqrt{2} - \frac{2}{5} \times 4\sqrt{2}$$

$$= \frac{16\sqrt{2}}{15}$$

3

$$\int_{-2}^{2} (x + x^3 + x^5)(1 + x^2 + x^4)dx = 0$$

[since an odd function × an even function is an odd function].

4

$$\int_{-\pi}^{0} \sin\left(x + \frac{\pi}{2}\right)\cos\left(x + \frac{\pi}{2}\right)dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\cos x\,dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx \quad \text{where } f(x) \text{ is odd}$$

$$= 0$$

5

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore 2\int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(a-x)\,dx$$

$$\begin{aligned}\int_0^a f(x)\,dx &= \frac{1}{2}\int_0^a \big(f(x)+f(a-x)\big)\,dx\\ &= \frac{1}{2}\int_0^a f(a)\,dx\\ &= \frac{f(a)}{2}\Big[x\Big]_0^a\\ &= \frac{f(a)}{2}(a-0)\\ &= \frac{a}{2}f(a)\end{aligned}$$

6

A: $e^{x^2} > 0$ and $\cos x \geq 0$ in the domain. True.

B: $x^3 \cos x$ is odd since x^3 is odd and $\cos x$ is even. False

C: $\sin^2 x - \cos^2 x = -\cos 2x$ which is negative in the domain. False

D; $\sin^{-1}(x^3)$ is the odd function of an odd function so is odd. False.

ANSWER (A)

7

$$\int_{-a}^{a} f(x)\,dx$$

$$= \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

ANSWER (C)

8

$$\int_{-a}^{a} f(x) + g(x)\,dx$$

$$= \int_{-a}^{a} f(x)\,dx \quad \text{since } g(x) \text{ is odd}$$

$$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \quad \text{from Q7}$$

ANSWER (C)

9

for $0 \leq x \leq \frac{\pi}{6}$ $\sin^2 x < \sin x < 1 - \sin x < 1 - \sin^2 x$

$$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x\,dx \text{ is the smallest integral}$$

ANSWER (B)

10

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx \qquad (1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx \qquad (2)$$

From (1) and (2):

$$2\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{1}{2}\Big[x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

11

$$\int_0^2 (1 + \sin(\pi(1 - x)^3))\,dx$$

$$= \int_0^2 (1 - \sin(\pi(x - 1)^3))\,dx$$

$$= \int_{-1}^1 (1 - \sin \pi x^3)\,dx$$

$$= \int_{-1}^1 1\,dx - \int_{-1}^1 \sin \pi x^3\,dx$$

$$= \Big[x\Big]_{-1}^1 - 0$$

$$= \big(1 - (-1)\big)$$

$$= 2$$

12

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x + \cos^2 x - 2\sin x \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\sin x - \cos x)^2}\,dx \qquad (1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right)}{1 + \left(\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)\right)^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + (\cos x - \sin x)^2}\,dx \qquad (2)$$

(1) + (2):

$$2\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1 + (\sin x - \cos x)^2}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\,dx = \frac{1}{2}\Big[\tan^{-1}(\sin x - \cos x)\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{4}$$

APPENDIX 1: TABULAR INTEGRATION BY PARTS

In a number of the lessons through this topic we have dealt with an integrand which was the product of two functions. We dealt with many of these using u or u^2 substitutions while others needed integration by parts.

A little known variation to integration by parts is tabular integration by parts, which we can use as a checking method whenever the answer we are trying to find is not given in the question. It is a process that does not provide any understanding, so really only recommended for checking an answer obtained using any of the usual methods.

We start by choosing the functions for $\frac{dv}{dx}$ and u using DETAIL - if you need the Reverse Chain Rule this method will quickly reach a dead end.

We then continually differentiate u and integrate $\frac{dv}{dx}$ until two terms on the same line create an integral we can deal with.

This happens when either we:

- differentiate u to reach 0 (as it is very easy to differentiate 0) - Example 1
- create a simple integral we can solve - Example 2
- create an integral that matches the original (so that we can swing it back with the original) - Example 3
- reach the integral with the lower power we need for the Reduction Formula - Example 4

Example 3

Find $\int x^2 \sin x \, dx$

Solution

Using DETAIL we let $\frac{dv}{dx} = \sin x$ and so $u = x^2$

Find $\frac{dv}{dx}$ and u as we would in IBP.

Differentiate u	Integrate $\frac{dv}{dx}$
$+ \quad x^2$	$\sin x$
$- \quad 2x$	$-\cos x$
$+ \quad 2$	$-\sin x$
0	$\cos x$

Differentiate u and integrate $\frac{dv}{dx}$ until the derivative of u becomes zero.

$\int x^2 \sin x \, dx$

$= +(x^2)(-\cos x) - (2x)(-\sin x) + (2)(\cos x) + c$

$= (2 - x^2)\cos x + 2x \sin x + c$

The integral equals the product of the diagonals as marked with an arrow. Every second arrow needs a negative product. Add c for indefinite integrals.

Example 2

Find $\int x \ln x \, dx$

Solution

Using DETAIL we let $\frac{dv}{dx} = x$ and so $u = \ln x$

Find $\frac{dv}{dx}$ and u as we would in IBP.

	Differentiate u	Integrate $\frac{dv}{dx}$
$+$	$\ln x$	x
$-$	$\frac{1}{x}$	$\frac{x^2}{2}$

Differentiate u and integrate $\frac{dv}{dx}$ until the product of the derivative of u and the integral of $\frac{dv}{dx}$ is easily integrated.

$$\int x \ln x \, dx$$

$$= +(\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right) dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

The integral equals the product of the diagonals as marked with an arrow. Every second arrow needs a negative product. Add c for indefinite integrals.

Example 3

Evaluate $\int_0^{\frac{\pi}{2}} e^x \sin x\, dx$

Solution

Using DETAIL we let $\frac{dv}{dx} = e^x$ and so $u = \sin x$

Find $\frac{dv}{dx}$ and u as we would in IBP.

	Differentiate u	Integrate $\frac{dv}{dx}$
$+$	$\sin x$	e^x
$-$	$\cos x$	e^x
$+$	$-\sin x$	e^x

Differentiate u and integrate $\frac{dv}{dx}$ until the product of the derivative of u and the integral of $\frac{dv}{dx}$ is a multiple of the original integral.

$$\int_0^{\frac{\pi}{2}} e^x \sin x\, dx = \Big[+(\sin x)(e^x) - (\cos x)(e^x)\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (-\sin x)(e^x)\ dx$$

$$= \Big[e^x(\sin x - \cos x)\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x\, dx$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \sin x\, dx = e^{\frac{\pi}{2}}(0+1) - 1(0-1)$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \sin x\, dx = \frac{e^{\frac{\pi}{2}}+1}{2}$$

The integral equals the product of the diagonals as marked with an arrow. Every second arrow needs a negative product. Add c for indefinite integrals.

Example 4

If $I_n = \int x^n e^x \, dx$ prove that $I_n = x^n e^x - nI_{n-1}$

Solution

Using DETAIL we let $\frac{dv}{dx} = e^x$ and so $u = x^n$

Find $\frac{dv}{dx}$ and u as we would in IBP.

	Differentiate u	Integrate $\frac{dv}{dx}$
$+$	x^n	e^x
$-$	nx^{n-1}	e^x

Differentiate u and integrate $\frac{dv}{dx}$ until the product of the derivative of u and the integral of $\frac{dv}{dx}$ is the lower powered integral required.

$$I_n = \int x^n e^x \, dx$$

$$= +(x^n)(e^x) - \int (nx^{n-1})(e^x) \, dx$$

$$\therefore I_n = x^n e^x - n\int x^{n-1} e^x \, dx$$

$$= x^n e^x - nI_n$$

The integral equals the product of the diagonals as marked with an arrow. Every second arrow needs a negative product. Add c for indefinite integrals.

Standard Integrals - Trigonometry

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\tan f(x)\,dx = \int \frac{f'(x)\sin f(x)}{\cos f(x)}dx = -\ln|\cos f(x)| + c$$

$$\int f'(x)\operatorname{cosec} f(x)\,dx = -\ln|\operatorname{cosec} f(x) + \cot f(x)| + c$$

$$\int f'(x)\sec f(x)\,dx = \ln|\sec f(x) + \tan f(x)| + c$$

$$\int f'(x)\cot f(x)\,dx = \int \frac{f'(x)\cos f(x)}{\sin f(x)}dx = \ln|\sin f(x)| + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)\operatorname{cosec}^2 f(x)\,dx = -\cot f(x) + c$$

$$\int f'(x)\sec f(x)\tan f(x)\,dx = \sec f(x) + c$$

$$\int f'(x)\operatorname{cosec} f(x)\cot f(x)\,dx = -\operatorname{cosec} f(x) + c$$

Standard Integrals - Other

$$\int f'(x)\,[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq 1$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c$$

$$\int f'(x)\,a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}dx = \ln\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + c$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}dx = \ln\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + c$$

Definite Integrals

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(u)du = \int_a^b f(x)dx$$

For even functions $[f(-x) = f(x)]$:

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

For odd functions $[f(-x) = -f(x)]$:

$$\int_{-a}^{a} f(x)dx = 0$$

Trig Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

$$\cot^2 x = \operatorname{cosec}^2 x - 1$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2\cos^2 x - 1$$

$$= 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Product to Sum Identities

$$\cos A\cos B = \tfrac{1}{2}\Big[\cos(A - B) + \cos(A + B)\Big]$$

$$\sin A\sin B = \tfrac{1}{2}\Big[\cos(A - B) - \cos(A + B)\Big]$$

$$\sin A\cos B = \tfrac{1}{2}\Big[\sin(A + B) + \sin(A - B)\Big]$$

$$\cos A\sin B = \tfrac{1}{2}\Big[\sin(A + B) - \sin(A - B)\Big]$$

t-results

$$t = \tan\frac{x}{2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1}{2}(1 + t^2)$$

$$dx = \frac{2dt}{1 + t^2}$$

$$\sin x = \frac{2t}{1 + t^2}$$

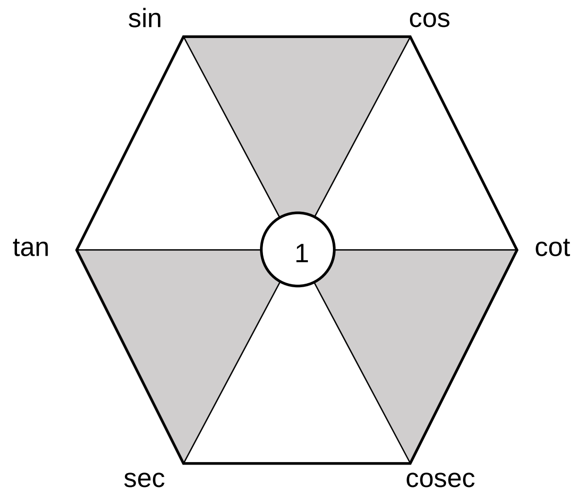
$$\cos x = \frac{1 - t^2}{1 + t^2}$$

$$\tan x = \frac{2t}{1 - t^2}$$

Hexagon Mnemonic

Co-function Relations: The trigonometric functions cosine, cotangent, and cosecant on the right of the hexagon are co-functions of sine, tangent, and secant on the left respectively.

$$\sin(90-\theta)=\cos\theta\,;\tan(90-\theta)=\cot\theta\,;\sec(90-\theta)=\operatorname{cosec}\theta$$



Reciprocal identities: The two trigonometric functions of any diagonal are reciprocals of each other.

$$\operatorname{cosec}\theta=\frac{1}{\sin\theta}\,;\sec\theta=\frac{1}{\cos\theta}\,;\cot\theta=\frac{1}{tan\,\theta}$$

Product identities: Along the outside edges of the hexagon any trigonometric function equals the product of the functions of the adjacent vertices.

$$\tan\theta\times\cos\theta=\sin\theta\,;\sec\theta\times\cot\theta=\operatorname{cosec}\theta\ \ \text{etc}$$

Quotient identities: along the outside edges of the hexagon, any trigonometric function equals the quotient of the next two trigonometric functions going either clockwise or counter clock wise.

$$\sin\theta=\tan\theta\div\sec\theta\,;\cot\theta\div\operatorname{cosec}\theta=\cos\theta\ \ etc$$

Pythagorean Identities: For each shaded triangle, the upper left function squared plus the upper right function squared, equals the bottom function squared.

$$\sin^2\theta+\cos^2\theta=1\,;\tan^2\theta+1=\sec^2\theta\ \,;1+\cot^2\theta=\operatorname{cosec}^2\theta$$

To rearrange any identity, drop any term from the top line to the right end of the bottom line with a minus in front.

$$\sin^2\theta=1-\cos^2\theta\ \,;\tan^2\theta=\sec^2\theta-1\,;\operatorname{cosec}^2\theta\ -\cot^2\theta=1$$

Product to Sum Mnemonic

The fours Product to Sum Identities are very similar, so to save us looking at the Reference Sheet each time, try:

Factors | Difference | Sum

Factors are the same so use cos on RHS

$$\cos A\cos B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

$$\sin A\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$

Factors are different so use sin on RHS

$$\sin A\cos B=\frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$$

$$\cos A\sin B=\frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$$

If cos is the second factor add the ratios on the RHS, if sin then subtract

Sin or Cos on the RHS?

- If the factors use the same ratio then use cos on the RHS, if the factors use different ratios then use sin

Sum or Difference of Angles first on RHS?

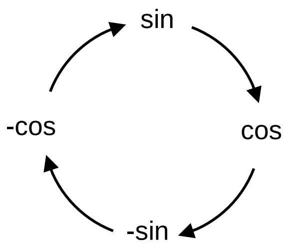
- If using cos then difference of angles is used first, if using sin then sum of angles is used first

Add or Subtract the Ratios on RHS?

- If cos is the second factor then add the ratios on RHS, if sin is the second factor then subtract

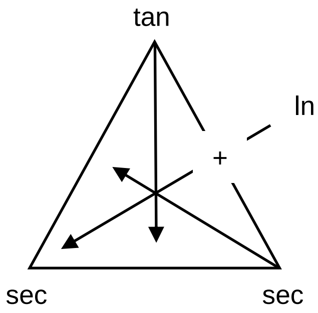
The arrows show differentiation, but the bottom two triangles show rules that are more often used in integration.

When differentiating:
- any of the complementary ratios ($\cos\theta$, $\cot\theta$ or $\operatorname{cosec}\theta$) the sign changes.
- any ratio ending in 'c' ($\sec\theta$, $\operatorname{cosec}\theta$) gives two terms.

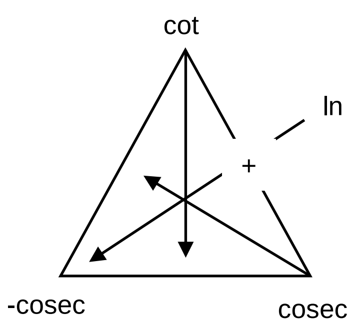


Function	**Derivative**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$-\sin x$	$-\cos x$
$-\cos x$	$\sin x$

In the diagrams below:
- an arrow pointing at a side means the derivative is the product of the vertices, so the derivative of $\sec x$ is $\sec x \tan x$
- the plus sign on the RH side means to take the ln of the sum of the vertices, so the derivative of $\ln|\tan x + \sec x|$ is $\sec x$



Function	**Derivative**
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\ln(\tan x + \sec x)$	$\sec x$



Function	**Derivative**
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln(\cot x + \operatorname{cosec} x)$	$-\operatorname{cosec} x$

Chapter 5
Vectors

MEX-V1 Further Work with Vectors

Vectors is a great topic for Extension 2 students as

- it provides another close link between geometry and algebra
- the topic has a wide variety of applications so questions do not need to be repetitive
- many of the questions can involve both theoretical and practical applications of mathematics, leading us towards the possibilities in university and beyond

LESSONS

Vectors is covered in 6 lessons.

5.1 Three Dimensional Vectors

5.2 Geometric Proofs

5.3 Vector Equation of a Line

5.4 Properties of Lines

5.5 Spheres and Basic Two Dimensional Curves

5.6 Harder Two Dimensional Curves and Three Dimensional Curves

REVISION QUESTIONS

In ‘1000 Extension 2 Revision Questions’, the revision book that goes with this textbook you will find the following questions matching this chapter:

- Revision Exercise 5

  100 graded questions on this topic only
- Revision Exercises 7 (Basic), 8 (Medium) and 9 (Challenging)

  Another 100 questions mixed through other topics for when you finish the course.

Don’t forget to do any questions from the exercises in this textbook you haven’t done.

5.1 THREE DIMENSIONAL VECTORS

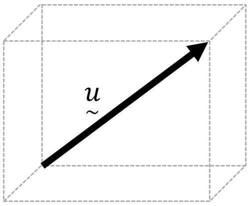
In Lesson 1 we extend some of our work from two-dimensional vectors in Extension 1 to three-dimensional vectors. This lesson is fairly basic once you have covered the same work for two-dimensional vectors, *with some sections being almost identical*. We cover:

- Three dimensional coordinate system
- Standard unit vectors in three dimensions
- Notation
- Addition, subtraction and scalar multiplication
- Magnitude
- Unit vectors
- Scalar product

VECTORS IN THREE DIMENSIONS

A vector in one, two or three dimensions is really the same - all we are really doing is moving or twisting it in relation to some arbitrary set of axes. As such many of our rules for three-dimensional vectors are identical to those we have seen for two-dimensional vectors.

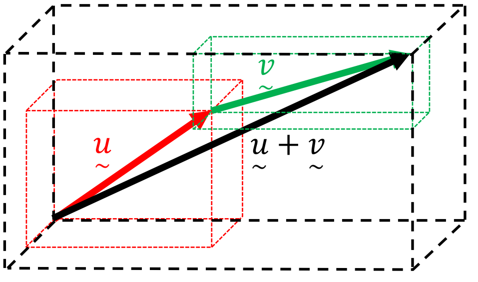
To indicate that a vector is three-dimensional we can draw it as the diagonal of a rectangular prism, or draw it on a three-dimensional coordinate system as we will see soon.



Just like for two dimensional vectors:

- Equal vectors have the same magnitude and direction
- Negative vectors have the same magnitude but opposite direction
- The scalar multiple of a vector is a vector parallel to the original.
- Collinear points involve one vector being a scalar multiple of another vector, with one point in common.
- Subtraction of Vectors is done either by adding the negative of a vector, or using tip minus tail.
- A vector can be split into component vectors, usually parallel to the three axes.

The Triangle, Parallelogram and Polygon Laws are used to add vectors. It can often help to imagine boxes being 'stacked' so that the corners just touch (very hard to draw clearly). The sum of the vectors being the diagonal of the box that would stretch from the start of the first box to the end of the last box.

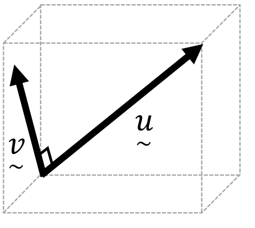


$\underset{\sim}{u} + \underset{\sim}{v}$ is the long diagonal of the box containing both smaller boxes.

We mainly use the laws algebraically rather than geometrically, if for no other reason that they are very hard to draw clearly as we can see!

The dot product of two perpendicular vectors is zero,

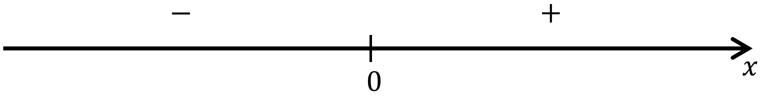
so here $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$.



THREE-DIMENSIONAL COORDINATE SYSTEM

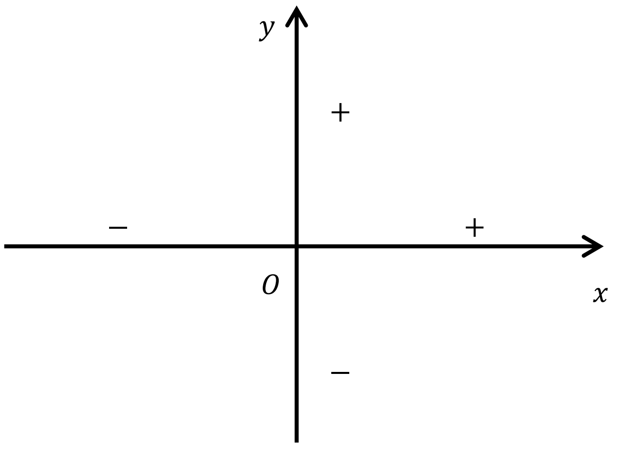
We are quite used to using coordinates in one and two dimensions.

In the one dimensional number line, the positive direction of the x-axis extends to the right from zero (think eastwards).

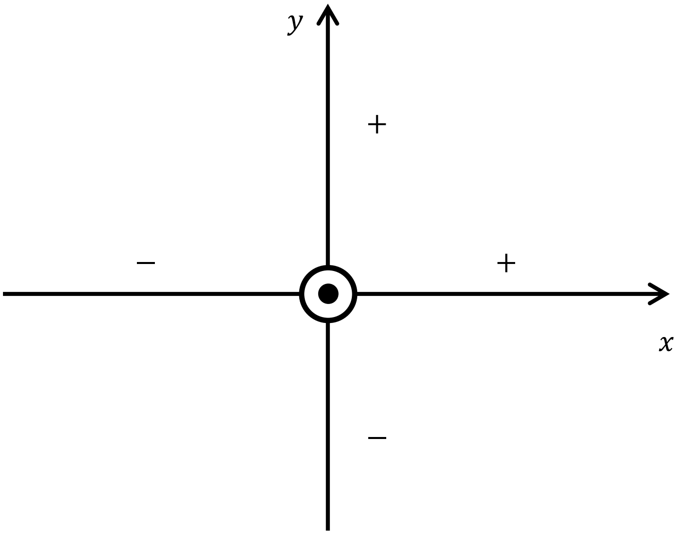


One dimensional Number Line

We extend the number line into the two dimensional Cartesian Plane (number plane) by extending the positive direction of the y-axis up the page from zero (think northwards).



We then extend the Cartesian Plane into three dimensions by having the positive direction of the z-axis come straight up out of the page - so if the page is flat on a desk the z-axis is truly vertical and comes up and pokes you in the eye!

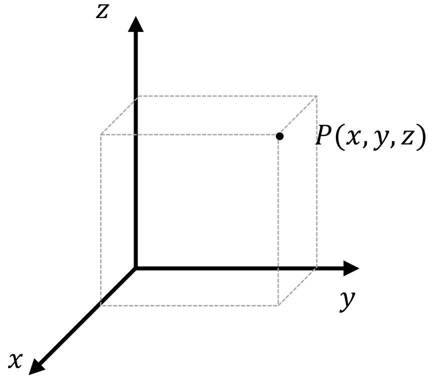


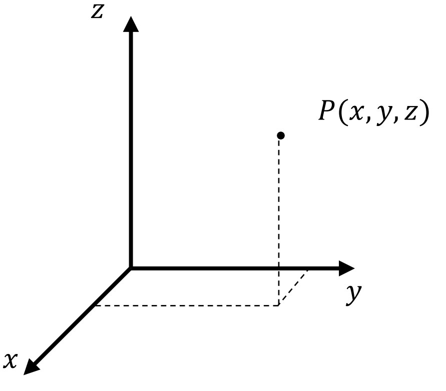
Now we can show that the z-axis is coming up out of the page by drawing a circle with a dot in the centre to look like the tip of an arrow coming towards us (think bow and arrow). We won't draw the axes like this, but it does help us realise why the three axes are in the order they are.

The three dimensional axis we have drawn is a right handed axes, for if we start with the back of our right fist flat on the table:

- opening out our thumb until it is flat, this gives the positive direction of the x-axis
- opening out our pointer finger until it is flat, this gives the positive direction of the y-axis
- opening out our middle finger until it is vertical, this gives us the positive direction of the z-axis

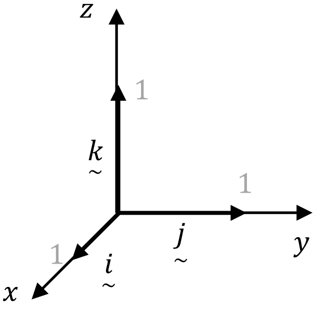
When drawing a point it helps to draw a rectangular prism stretching from the origin to the point so that we can more easily see the three dimensions. To make the diagram clearer we can just draw the base of the rectangular prism and the vertical line up to the point.





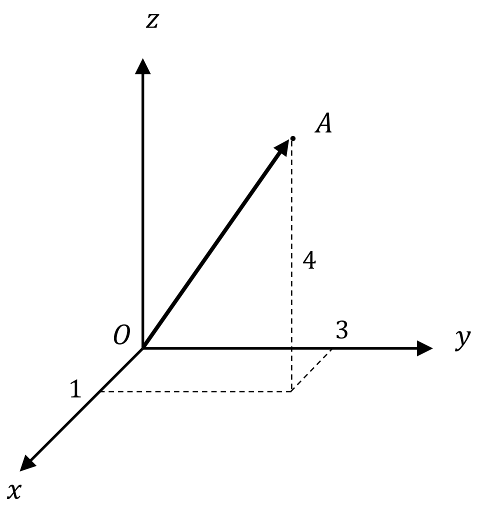
STANDARD UNIT VECTORS

We have already seen that the unit vectors along the x and y-axes are $\underset{\sim}{i}$ and $\underset{\sim}{j}$ respectively. To this we add the unit vector along the z-axis, which is $\underset{\sim}{k}$.



NOTATION

We can extend our notation for vectors to three dimensions.



We can say that $\overrightarrow{OA}$ is equal to:

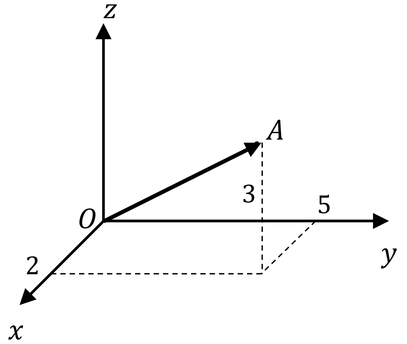
$\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$ in component form

$(1,3,4)$ as an ordered triple

$\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ in column vector notation

Example 1

Write $\overrightarrow{OA}$ in component form, as an ordered triple and in column vector notation



We can say that $\overrightarrow{OA}$ is equal to:

$2\underset{\sim}{i} + 5\underset{\sim}{j} + 3\underset{\sim}{k}$ in component form

$(2,5,3)$ as an ordered triple

$\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$ in column vector notation

ADDITION, SUBTRACTION AND SCALAR MULTIPLICATION

We can also extend our methods for addition, subtraction or scalar multiplication to three-dimensions using similar methods. The methods are exactly the same as for two dimensional vectors, so we won't go through any examples here.

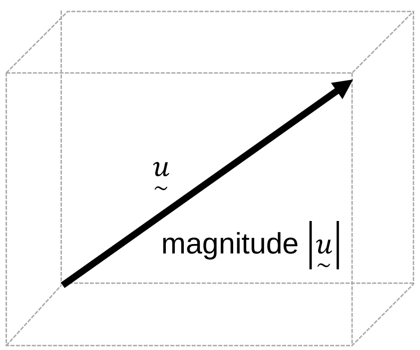
MAGNITUDE OF THREE DIMENSIONAL VECTORS

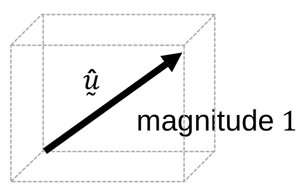
Extending our formula for the magnitude of a vector, we can say that if $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ then $\left|\underset{\sim}{u}\right| = \sqrt{x^2 + y^2 + z^2}$. Again it is very similar so we won't go through an example.

UNIT VECTORS

We have had a look at the standard unit vectors running parallel to the x, y and z-axes, $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$ respectively, but there are times when it is useful to define a unit vector running in the direction of the vector itself, rather than one of the axes.

$$\boxed{\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{\left|\underset{\sim}{u}\right|}}$$





Example 2

Find the unit vector for each vector.

a $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ **b** $\underset{\sim}{b} = (4, -3, -1)$ **c** $\underset{\sim}{c} = \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}$

a $\left|\underset{\sim}{a}\right| = \sqrt{17}, \quad \hat{\underset{\sim}{a}} = \frac{2}{\sqrt{17}}\underset{\sim}{i} + \frac{3}{\sqrt{17}}\underset{\sim}{j} + \frac{2}{\sqrt{17}}\underset{\sim}{k}$

b $\left|\underset{\sim}{b}\right| = \sqrt{26}, \quad \hat{\underset{\sim}{b}} = \left(\frac{4}{\sqrt{26}}, -\frac{3}{\sqrt{26}}, -\frac{1}{\sqrt{26}}\right)$

c $\left|\underset{\sim}{c}\right| = 7, \quad \hat{\underset{\sim}{c}} = \begin{pmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ \frac{2}{7} \end{pmatrix}$

Example 3

Find the vector $\underset{\sim}{v}$ parallel to $\underset{\sim}{u} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ that has a magnitude of 2.

Find the unit vector parallel to $3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and multiply it by ± 2

$$\left|\underset{\sim}{u}\right| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$$

$$\hat{\underset{\sim}{u}} = \frac{3}{\sqrt{14}}\underset{\sim}{i} + \frac{2}{\sqrt{14}}\underset{\sim}{j} - \frac{1}{\sqrt{14}}\underset{\sim}{k}$$

$$\underset{\sim}{v} = \hat{\underset{\sim}{u}} \times (\pm 2)$$

$$= \frac{6}{\sqrt{14}}\underset{\sim}{i} + \frac{4}{\sqrt{14}}\underset{\sim}{j} - \frac{2}{\sqrt{14}}\underset{\sim}{k}, \qquad -\frac{6}{\sqrt{14}}\underset{\sim}{i} - \frac{4}{\sqrt{14}}\underset{\sim}{j} + \frac{2}{\sqrt{14}}\underset{\sim}{k}$$

SCALAR (DOT) PRODUCT OF VECTORS

The product of two vectors can either be a scalar or a vector. We will only deal with the scalar product in the Extension 1 and 2 courses. A vector product (cross product) can only exist in $\mathbb{R}^3$ or $\mathbb{R}^7$, and is well beyond the Extension 2 course.

The scalar product, also known as the dot product, is important as it allows us to find the angle between two vectors and is one way to project one vector onto another. It has other uses which are beyond the syllabus, dealing with lengths, areas and volumes.

There are two versions of the rule. The first involves the vectors in component form, and extends the rule from two dimensions:

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2 + z_1z_2$$

The second involves the magnitudes of the two vectors and the angle between them, and is the same as for two-dimensional vectors.

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right| \left|\underset{\sim}{v}\right| \cos\theta$$

Example 4

Find the following scalar products

a $\left(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}\right) \cdot \left(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

b $(4, -3, 2) \cdot (2, 0, -1)$

c $\begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$

a $2 \times 1 + 3 \times (-2) + (-1) \times 3 = -7$

b $4 \times 2 + (-3) \times 0 + 2 \times (-1) = 6$

c $(-6) \times 3 + 3 \times 6 + 2 \times 0 = 0$

ANGLE BETWEEN TWO VECTORS

Extending the formula for the angle between two vectors from two dimensions to three we see:

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|}$$

We will prove this in the exercises using the formulae for the dot product.

> **Example 5**
>
> Find the angle between the vectors $\begin{pmatrix}1\\2\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ to the nearest degree.

$$\begin{aligned}\cos\theta &= \frac{x_1x_2 + y_1y_2 + z_1z_2}{\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|}\\ &= \frac{(1)(-1) + (2)(1) + (2)(0)}{\sqrt{1^2 + 2^2 + 2^2} \times \sqrt{(-1)^2 + 1^2 + 0^2}}\\ &= \frac{1}{3\sqrt{2}}\\ \theta &= 76^\circ \ \text{(nearest degree)}\end{aligned}$$

PARALLEL VECTORS

Two vectors are parallel if the angle between them is 0° or 180°, and $\cos\theta = 1$ and $\cos\theta = -1$ respectively.

PERPENDICULAR VECTORS

Two vectors are perpendicular if the angle between them is 90°, in which case $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$, which makes $\cos\theta = 0$. To prove two non-zero vectors are perpendicular we can prove either

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ or $\cos\theta = 0$.

ORTHOGONAL VECTORS

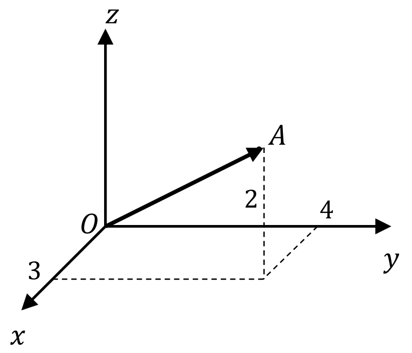
Two vectors are orthogonal if their scalar product is zero. This includes the case where two vectors are perpendicular, plus the case where one or both of the vectors are the zero vector.

So we can say that two vectors are orthogonal if $\underset{\sim}{u} \perp \underset{\sim}{v}$ or $\underset{\sim}{u} = 0$ or $\underset{\sim}{v} = 0$.

EXERCISE 5.1

BASIC

1 Write $\overrightarrow{OA}$ in component form, as an ordered triple and in column vector notation



2 Simplify the following in each of the three forms.

a $\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) + \left(2\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}\right)$ **b** $(1,3,-2) + (2,-4,1)$ **c** $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$

3 Simplify the following in each of the three forms.

a $\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) - \left(2\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}\right)$ **b** $(1,3,-2) - (2,-4,1)$ **c** $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$

4 Simplify.

a $3\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$ **b** $3(1,3,-2)$ **c** $3\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

5 Find the magnitude of these vectors:

a $\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$ **b** $(4,3,-2)$ **c** $\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$

6 Find the unit vector for each vector in Question 5.

MEDIUM

7 Prove that if $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ then $\left|\underset{\sim}{u}\right| = \sqrt{x^2 + y^2 + z^2}$

8 If $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 7 \\ 6 \\ 2 \end{pmatrix}$ find $\left|\overrightarrow{PQ}\right|$

9 Show that $\underset{\sim}{a} = \frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$ is a unit vector

10 Find the vector $\underset{\sim}{v}$ parallel to $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$ that has a magnitude of 3.

11 Find the following scalar products

a $\left(\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} - 4\underset{\sim}{j} + 6\underset{\sim}{k}\right)$ **b** $(3,2,-6) \cdot (2,-4,1)$ **c** $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

12 Use the two formula for the dot product to prove that the angle between

$\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$ is given by

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|}$$

13 Find the angle between

a $3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$

b $\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $-2\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

c $\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{i} + \underset{\sim}{j} + 0\underset{\sim}{k}$

d $0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k}$ and $\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

CHALLENGING

14 A triangular based pyramid has three of its vertices at $A(2,0,0), B(0,2,0)$ and $C(0,0,2)$. If its fourth vertex is at $D(a,a,a)$, where $a > 0$, find the value of a. You are given the three triangles forming the sides of the pyramid are equilateral.

15 A rectangular prism with sides of length 6, 8 and 10 units has both ends of one of its longest diagonals along the x-axis. Prove that all points on the surface of the prism satisfy $|z| \leq 5\sqrt{2}$.

1

$$\overrightarrow{OA} = 3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k} = (3,4,2) = \begin{pmatrix}3\\4\\2\end{pmatrix}$$

2

a $(1+2)\underset{\sim}{i} + (3-4)\underset{\sim}{j} + (-2+1)\underset{\sim}{k}$

$= 3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

b $(1+2, 3-4, -2+1)$

$= (3,-1,-1)$

c $\begin{pmatrix}1+2\\3-4\\-2+1\end{pmatrix} = \begin{pmatrix}3\\-1\\-1\end{pmatrix}$

3

a $(1-2)\underset{\sim}{i} + (3+4)\underset{\sim}{j} + (-2-1)\underset{\sim}{k}$

$= -\underset{\sim}{i} + 7\underset{\sim}{j} - 3\underset{\sim}{k}$

b $(1-2, 3+4, -2-1)$

$= (-1,7,-3)$

c $\begin{pmatrix}1-2\\3+4\\-2-1\end{pmatrix} = \begin{pmatrix}-1\\7\\-3\end{pmatrix}$

4

a $3\underset{\sim}{i} + 9\underset{\sim}{j} - 6\underset{\sim}{k}$

b $(3,9,-6)$

c $\begin{pmatrix}3\\9\\-6\end{pmatrix}$

5

a $\left|\underset{\sim}{a}\right| = \sqrt{1^2+3^2+(-2)^2} = \sqrt{14}$

b $\left|\underset{\sim}{b}\right| = \sqrt{4^2+3^2+(-2)^2} = \sqrt{29}$

c $\left|\underset{\sim}{c}\right| = \sqrt{1^2+5^2+(-2)^2} = \sqrt{30}$

6

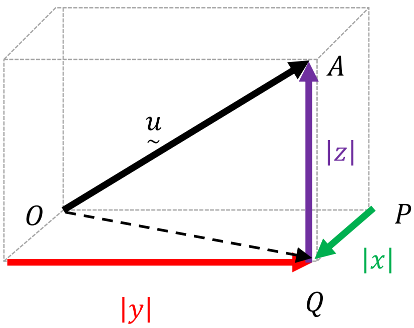
a $\left|\underset{\sim}{a}\right| = \sqrt{14}, \qquad \hat{\underset{\sim}{a}} = \frac{1}{\sqrt{14}}\underset{\sim}{i} + \frac{3}{\sqrt{14}}\underset{\sim}{j} - \frac{2}{\sqrt{14}}\underset{\sim}{k}$

b $\left|\underset{\sim}{b}\right| = \sqrt{29}, \qquad \hat{\underset{\sim}{b}} = \left(\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\right)$

c $\left|\underset{\sim}{c}\right| = \sqrt{30}, \qquad \hat{\underset{\sim}{c}} = \begin{pmatrix}\frac{1}{\sqrt{30}}\\ \frac{5}{\sqrt{30}}\\ -\frac{2}{\sqrt{30}}\end{pmatrix}$

7

Splitting $\underset{\sim}{u}$ into its component vectors $x\underset{\sim}{i}$, $y\underset{\sim}{j}$ and $z\underset{\sim}{k}$, we can see that ΔOPQ is right angled with hypotenuse $|\overrightarrow{OQ}|$ and short sides of $\left|x\underset{\sim}{i}\right| = |x|$ and $\left|y\underset{\sim}{j}\right| = |y|$



$\therefore |\overrightarrow{OQ}|^2 = |x|^2 + |y|^2 \quad \text{(Pythagoras)}$

Similarly we have ΔOAQ also right angled

$$\begin{aligned}\left|\underset{\sim}{u}\right|^2 &= |\overrightarrow{OQ}|^2 + |\overrightarrow{AQ}|^2\\ &= |x|^2 + |y|^2 + |z|^2\\ &= x^2 + y^2 + z^2\end{aligned}$$

$\therefore \left|\underset{\sim}{u}\right| = \sqrt{x^2+y^2+z^2} \qquad \square$

8

$$\overrightarrow{PQ} = \begin{pmatrix}7\\6\\2\end{pmatrix} - \begin{pmatrix}1\\5\\-2\end{pmatrix} = \begin{pmatrix}6\\1\\4\end{pmatrix}$$

$$|\overrightarrow{PQ}| = \sqrt{6^2+1^2+4^2} = \sqrt{53}$$

9

$$\begin{aligned}\left|\underset{\sim}{a}\right| &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\\ &= \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}}\\ &= 1\end{aligned}$$

$\therefore \underset{\sim}{a}$ is a unit vector

10 Find the unit vector parallel to $2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$ and multiply it by ± 3

$$\left|\underset{\sim}{u}\right| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$$

$$\hat{\underset{\sim}{u}} = \frac{2}{\sqrt{21}}\underset{\sim}{i} - \frac{1}{\sqrt{21}}\underset{\sim}{j} + \frac{4}{\sqrt{21}}\underset{\sim}{k}$$

$$\underset{\sim}{v} = \hat{\underset{\sim}{u}} \times (\pm 3)$$

$$= \pm\left(\frac{6}{\sqrt{21}}\underset{\sim}{i} - \frac{3}{\sqrt{21}}\underset{\sim}{j} + \frac{12}{\sqrt{21}}\underset{\sim}{k}\right)$$

11 **a** $1(2) + 3(-4) + 2(6) = 2$
b $3(2) + 2(-4) - 6(1) = -8$
c $1(2) + 3(0) - 2(1) = 0$

12 From the scalar product we have

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2 + z_1z_2 \quad (1)$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta \quad (2)$$

$$\therefore \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2 + z_1z_2$$

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|} \qquad \square$$

13

a $\cos\theta = \dfrac{3(2) + 1(-3) - 2(-1)}{\sqrt{3^2 + 1^2 + (-2)^2} \times \sqrt{2^2 + (-3)^2 + (-1)^2}}$

$\theta = 69°05'$

b $\cos\theta = \dfrac{1(-2) + 2(-4) - (2)}{\sqrt{1^2 + 2^2 + (-1)^2} \times \sqrt{(-2)^2 + (-4)^2 + 2^2}}$

$\theta = 180 \quad (\text{parallel vectors})$

c $\cos\theta = \dfrac{1(1) - (1) + (0)}{\sqrt{4^2 + (-1)^2 + 3^2} \times \sqrt{2^2 + 8^2} + 0^2}$

$\theta = 90° \quad (\text{perpendicular vectors})$

d The first vector is the zero vector. There is no angle, as an angle needs two arms and the zero vector does not form a line. The vectors are orthogonal.

14 D is the far corner of a cube whose other end is at the origin, and using the other vertices we can easily see $a = 2$.

15 The longest diagonal is $\sqrt{6^2 + 8^2 + 10^2} = \sqrt{200} = 10\sqrt{2}$. The centre of the prism is on the long diagonal, so lies on the x-axis, and every point on the prism must be at most half of the length of the long diagonal from the centre, and thus $|z| \leq 5\sqrt{2}$. We could also say $|y| \leq 5\sqrt{2}$.

In Lesson 2 we look at geometric proofs. We cover:

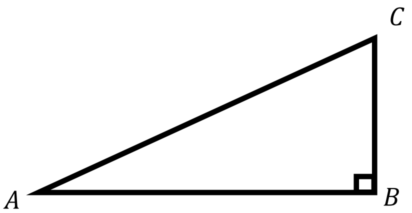
- Harder geometric proofs in the plane
- Proofs in three dimensions

GEOMETRIC PROOFS IN THE PLANE

There isn't a lot of difference in the syllabus dot points in Extension 1 and 2, so in Extension 2 we can expect mainly harder two dimensional proofs with a few three dimensional proofs thrown in.

Example 1

Prove that the square of the hypotenuse equals the sum of the squares of the other two sides in a right angled triangle.



Let ΔABC be right angled at B.

$$
\begin{aligned}
AC^2 &= \left|\overrightarrow{AC}\right|^2 \\
&= \left|\overrightarrow{AB} + \overrightarrow{BC}\right|^2 \\
&= \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \\
&= \overrightarrow{AB} \cdot \overrightarrow{AB} + 2\left(\overrightarrow{AB} \cdot \overrightarrow{BC}\right) + \overrightarrow{BC} \cdot \overrightarrow{BC} \\
&= \overrightarrow{AB} \cdot \overrightarrow{AB} + \overrightarrow{BC} \cdot \overrightarrow{BC} \qquad \text{since } \overrightarrow{AB} \perp \overrightarrow{BC} \quad \therefore \overrightarrow{AB} \cdot \overrightarrow{BC} = 0 \\
&= \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 \\
&= AB^2 + BC^2
\end{aligned}
$$

∴ The square on the hypotenuse of a right angled triangle equals the sum of the squares on the other two sides. □

** Thanks to Luke W, one of my students, who simplified this proof.

Example 2

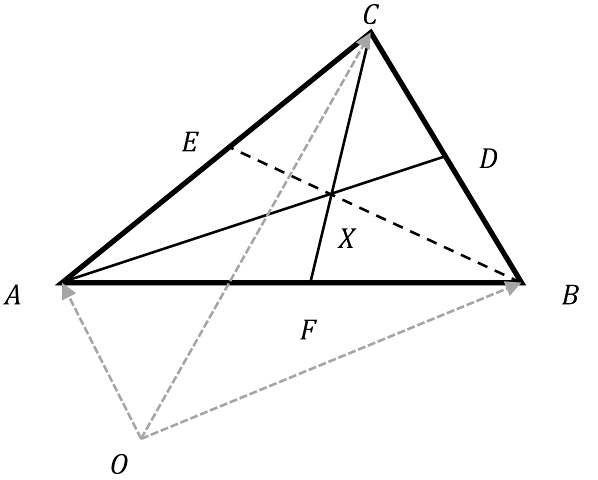
Prove that the medians of a triangle are concurrent.

We will let X be the intersection of AD and CF and prove that it lies on BE.

Let $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

In ΔABC let D, E, F be the midpoints of the sides BC, AC and AB respectively.

$\therefore \overrightarrow{OE} = \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2}, \overrightarrow{OD} = \frac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$ and $\overrightarrow{OF} = \frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$



$$\begin{aligned}
\overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX} \\
&= \underset{\sim}{a} + \lambda\overrightarrow{AD} \\
&= \underset{\sim}{a} + \lambda\left(\frac{\underset{\sim}{b}+\underset{\sim}{c}}{2} - \underset{\sim}{a}\right) \\
&= (1-\lambda)\underset{\sim}{a} + \frac{\lambda}{2}\underset{\sim}{b} + \frac{\lambda}{2}\underset{\sim}{c} \quad (1)
\end{aligned}$$

Also

$$\begin{aligned}
\overrightarrow{OX} &= \overrightarrow{OC} + \overrightarrow{CX} \\
&= \underset{\sim}{c} + \mu\overrightarrow{CF} \\
&= \underset{\sim}{c} + \mu\left(\frac{\underset{\sim}{a}+\underset{\sim}{b}}{2} - \underset{\sim}{c}\right) \\
&= \frac{\mu}{2}\underset{\sim}{a} + \frac{\mu}{2}\underset{\sim}{b} + (1-\mu)\underset{\sim}{c} \quad (2)
\end{aligned}$$

A median is a line joining a vertex to the midpoint of the opposite side

From (1) and (2):

$$1-\lambda = \frac{\mu}{2} \qquad \frac{\lambda}{2} = \frac{\mu}{2} \qquad \frac{\lambda}{2} = 1-\mu$$

$$\lambda = \mu$$

$$\begin{aligned}
1-\lambda &= \frac{\lambda}{2} \\
1 &= \frac{3\lambda}{2} \\
\lambda = \mu &= \frac{2}{3}
\end{aligned}$$

$$\therefore \overrightarrow{OX} = \frac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3}$$

$$\begin{aligned}
\overrightarrow{BX} &= \frac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3} - \underset{\sim}{b} \\
&= \frac{\underset{\sim}{a}+\underset{\sim}{c}-2\underset{\sim}{b}}{3} \\
\overrightarrow{XE} &= \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2} - \frac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3} \\
&= \frac{\underset{\sim}{a}+\underset{\sim}{c}-2\underset{\sim}{b}}{6} \\
&= \frac{1}{2}\overrightarrow{BX}
\end{aligned}$$

$\therefore B, E, X$ are collinear

$\therefore$ the medians of a triangle are concurrent.

GEOMETRIC PROOFS IN THREE DIMENSIONS

Many proofs in two dimensions can be repeated in three dimensions simply by twisting the axes with respect to the vectors - the solutions end up being exactly the same using vector notation. Let's focus on some proofs using three dimensional coordinates.

Example 3

Three vertices of a parallelogram are $\mathrm{O}(0,0,0), A(1,2,3)$ and $\mathrm{B}(3,2,1)$. Find the possible positions of the fourth vertex.

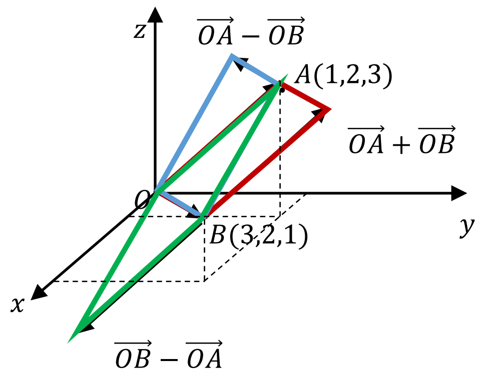
There are three possible vectors for $\overrightarrow{OD}$ that would create a parallelogram: $\overrightarrow{OA}+\overrightarrow{OB}, \overrightarrow{OA}-\overrightarrow{OB}$ and $\overrightarrow{OB}-\overrightarrow{OA}$.

$\therefore \overrightarrow{OD}\& = (1,2,3)+(3,2,1) = (4,4,4)$ or

$= (1,2,3)-(3,2,1) = (-2,0,2)$ or

$= (3,2,1)-(1,2,3) = (2,0,-2)$

The fourth vertex is at $(4,4,4), (-2,0,2)$ or $(2,0,-2)$.



Example 4

A mass exerts a downward force of $100\ \mathrm{N}$. It is being held in a steady position by four drones, exerting the forces in Newtons of $(25,25,25)$, $(30,-30,30), (-15,15,15)$ and (a,b,c). Find the value of a, b and c.

$$(25,25,25)+(30,-30,30)+(-15,15,15)+(a,b,c) = (0,0,100)$$

$$(25+30-15+a, 25-30+15+b, 25+30+15+c) = (0,0,100)$$

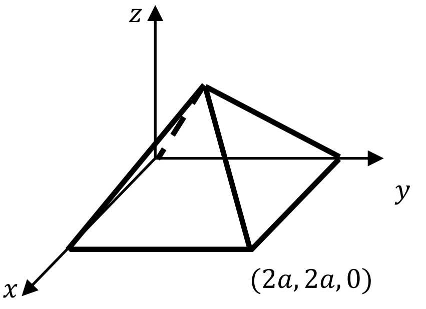
$$(40+a, 10+b, 70+c) = (0,0,100)$$

$\therefore a=-40, b=-10$ and $c=30$.

Example 5

A square based pyramid has its base on the $x-y$ plane, with opposite corners of the base at the origin and $(2a, 2a, 0)$. Its height is b, and the four triangles forming its sides are equilateral. Find b in terms of a.

The apex of the pyramid is (a, a, b).



The magnitude of the slant height must be 2a, as the triangles are equilateral.

$$\therefore \sqrt{a^2 + a^2 + b^2} = 2a$$
$$2a^2 + b^2 = 4a^2$$
$$b^2 = 2a^2$$
$$b = \sqrt{2}a$$

Example 6

The points $A(1,2,3)$, $B(4,5,z)$ and $C(7,8,9)$ form a right angled triangle, with $\angle ABC = 90°$. Prove that $z = 6 \pm 3\sqrt{3}$.

$$\begin{aligned}
AB^2 &= (4-1)^2 + (5-2)^2 + (z-3)^2 \\
&= 9 + 9 + z^2 - 6z + 9 \\
&= z^2 - 6z + 27 \\
BC^2 &= (7-4)^2 + (8-5)^2 + (9-z)^2 \\
&= 9 + 9 + 81 - 18z + z^2 \\
&= z^2 - 18z + 99 \\
AC^2 &= (7-1)^2 + (8-2)^2 + (9-3)^2 \\
&= 36 + 36 + 36 \\
&= 108
\end{aligned}$$

$$\therefore z^2 - 6z + 27 + z^2 - 18z + 99 = 108 \qquad \text{Pythagoras}$$

$$2z^2 - 24z + 18 = 0$$

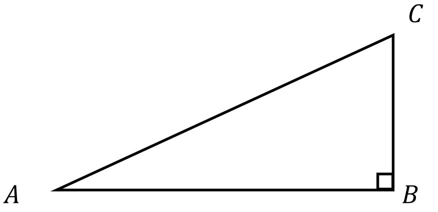
$$z^2 - 12z + 9 = 0$$

$$\begin{aligned}
z &= \frac{12 \pm \sqrt{(-12)^2 - 4(1)(9)}}{2(1)} \\
&= \frac{12 \pm \sqrt{108}}{2} \\
&= \frac{12 \pm 6\sqrt{3}}{2} \\
&= 6 \pm 3\sqrt{3}
\end{aligned}$$

EXERCISE 5.2

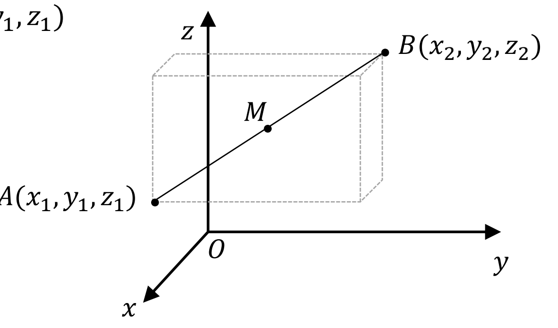
BASIC

1 Prove that the sum of the square of the hypotenuse equals the sum of the squares of the other two sides in a right angled triangle.



2 What type of triangle is formed by the points $\mathrm{A}(1,1,1), B(1,-1,1)$ and $O(0,0,0)$?

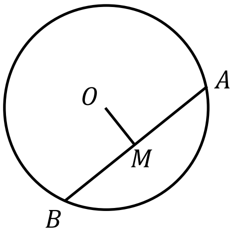
3 Prove that the midpoint of the interval from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$ is $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$



4 A mass exerts a downward force of $50\ \mathrm{N}$. It is being held in a steady position by four drones, exerting forces in Newtons of $(10,20,10), (20,-10,20), (-15,-15,20)$ and (a, b, c). Find the value of a, b and c.

MEDIUM

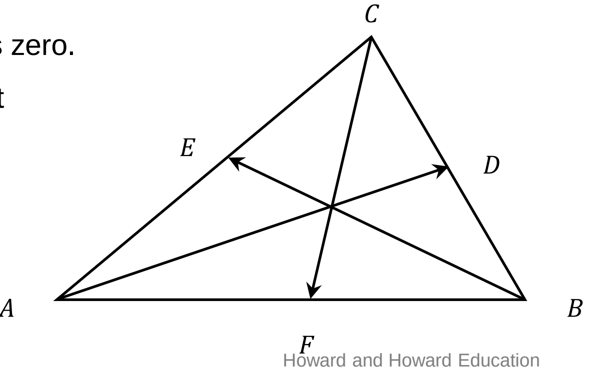
5 Prove that the line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.



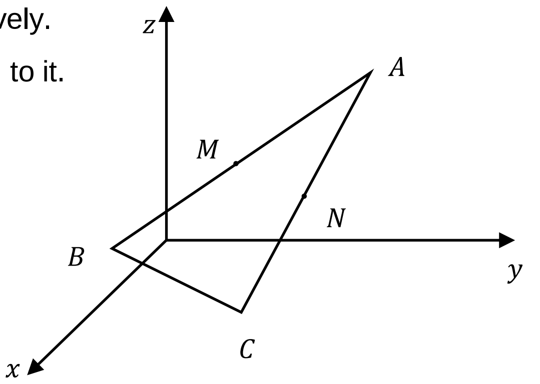
6 Find the point that divides $P(1,-2,4)$ and $Q(5,6,0)$ in the ratio $1:3$.

7 Three vertices of a parallleogram are $O(0,0,0), A(1,1,1)$ and $\mathrm{B}(1,-1,1)$. Find the possible positions of the fourth vertex.

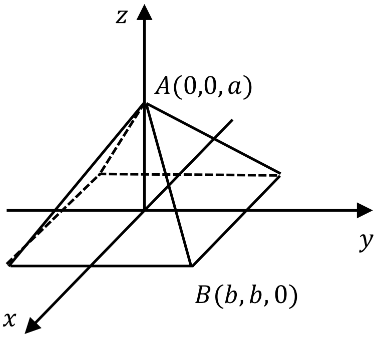
8 Prove that the sum of the medians of a triangle is zero. A median is a line joining a vertex to the midpoint of the opposite side as shown.



9 M and N are the midpoints of AB and AC respectively. Prove that MN is half the length of BC and parallel to it.



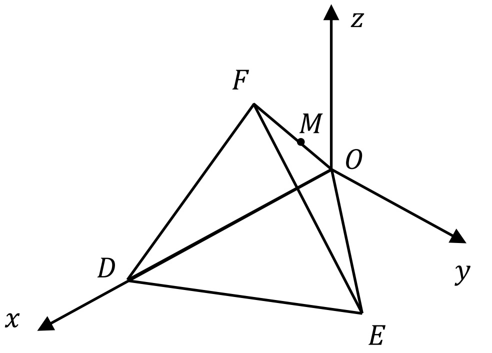
10 A square based pyramid has its base on the $x - y$ plane, with its apex at $A(0,0,a)$. The four triangles forming its sides are isosceles with sides in the ratio $2:2:1$, the short side being the bottom side. One of the four vertices of the square base is $\mathrm{B}(b,b,0)$, where $b > 0$. Find b in terms of a.



CHALLENGING

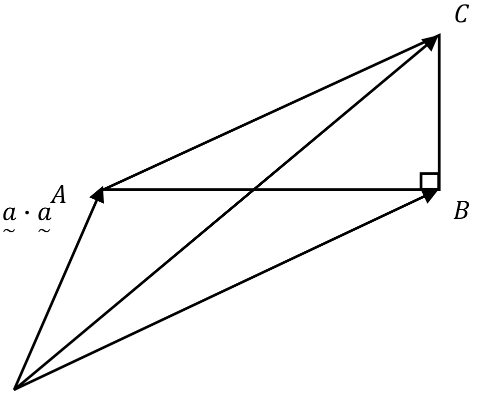
11 Prove that the medians of a triangle are concurrent (intersect at one point).

12 The faces of tetrahedron $ODEF$ are comprised of equilateral triangles of side length 1 unit. Its base lies flat on the $x - y$ plane with vertices at O, $\mathrm{D}(1,0,0)$ and $E\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$ as shown. Prove the coordinates of M, the midpoint of FO, is $\left(\frac{1}{4}, \frac{\sqrt{3}}{12}, \frac{\sqrt{6}}{6}\right)$.



1

$$\overrightarrow{BC}\cdot\overrightarrow{AB}=0\quad\text{(perpendicular)}$$
$$\left(\underset{\sim}{c}-\underset{\sim}{b}\right)\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)=0$$
$$\underset{\sim}{c}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{c}-\underset{\sim}{b}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{b}=0$$
$$\underset{\sim}{a}\cdot\underset{\sim}{c}=\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{c}-\underset{\sim}{b}\cdot\underset{\sim}{b}\quad(1)$$



$$\begin{aligned}\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{AB}\right|^2&=\left(\underset{\sim}{c}-\underset{\sim}{b}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{b}\right)+\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\\
&=\underset{\sim}{c}\cdot\underset{\sim}{c}-2\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{b}-2\,\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{a}\\
&=\underset{\sim}{c}\cdot\underset{\sim}{c}+2\left(\underset{\sim}{b}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{b}-\underset{\sim}{b}\cdot\underset{\sim}{c}\right)+\underset{\sim}{a}\cdot\underset{\sim}{a}\\
&=\underset{\sim}{c}\cdot\underset{\sim}{c}-2\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{a}\cdot\underset{\sim}{a}\\
&=\left(\underset{\sim}{c}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{a}\right)\\
&=\left|\overrightarrow{AC}\right|^2\end{aligned}$$

∴ The square on the hypotenuse of a right angled triangle equals the sum of the squares on the other two sides. □

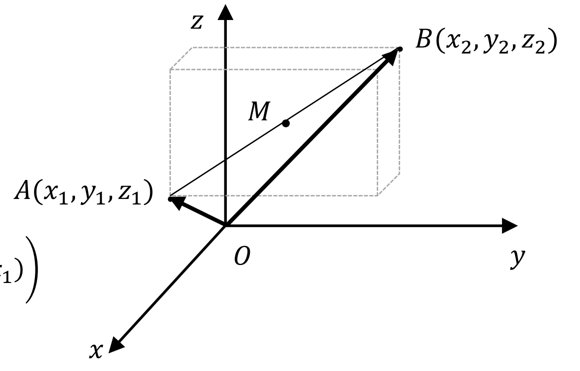
2

$$\left|\overrightarrow{OA}\right|=\sqrt{(1-0)^2+(1-0)^2+(1-0)^2}=\sqrt{3}$$
$$\left|\overrightarrow{OB}\right|=\sqrt{(1-0)^2+(-1-0)^2+(1-0)^2}=\sqrt{3}$$
$$\left|\overrightarrow{AB}\right|=\sqrt{(1-1)^2+(1+1)^2+(1-1)^2}=2$$

ΔABC is isosceles.

3

$$\begin{aligned}\overrightarrow{OM}&=\overrightarrow{OA}+\overrightarrow{AM}\\
&=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}\\
&=(x_1,y_1,z_1)+\frac{1}{2}(x_2-x_1,y_2-y_1,z_2-z_1)\\
&=\left(x_1+\frac{1}{2}(x_2-x_1),x_1+\frac{1}{2}(x_2-x_1),x_1+\frac{1}{2}(x_2-x_1)\right)\\
&=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)\quad\square\end{aligned}$$



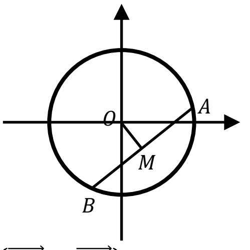
4

$$(10,20,10)+(20,-10,20)+(-15,-15,20)+(a,b,c)+(0,0,-50)=(0,0,0)$$
$$(10+20-15+a+0,20-10-15+b+0,10+20+20+c-50)=(0,0,0)$$
$$(15+a,b-5,c)=(0,0,0)$$

∴ $a=-15,b=5$ and $c=0$.

5

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$
$$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$
$$= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$
$$= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$$



$$\overrightarrow{OM} \cdot \overrightarrow{AB} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right) \cdot \left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$
$$= \frac{1}{2}\left(\left(\overrightarrow{OB}\right) \cdot \left(\overrightarrow{OB}\right) - \left(\overrightarrow{OA}\right) \cdot \left(\overrightarrow{OA}\right)\right)$$
$$= \frac{1}{2}\left(\left|\overrightarrow{OB}\right|^2 - \left|\overrightarrow{OA}\right|^2\right)$$
$$= \frac{1}{2}(r^2 - r^2)$$
$$= 0$$

$\therefore OM \perp AB$

6

Let $X(a, b, c)$ be the point that divides PQ in the ratio $1:3$.

$$\therefore \overrightarrow{PX} = \frac{1}{4}\overrightarrow{PQ}$$

$$\begin{pmatrix} a-1 \\ b+2 \\ c-4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5-1 \\ 6+2 \\ 0-4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$\therefore a - 1 = 1 \quad \rightarrow \quad a = 2$

$b + 2 = 2 \quad \rightarrow \quad b = 0$

$c - 4 = -1 \quad \rightarrow \quad c = 3$

$\therefore X(2,0,3)$

7

There are three possible vectors for $\overrightarrow{OD}$ that would create a parallelogram: $\overrightarrow{OA} + \overrightarrow{OB}, \overrightarrow{OA} - \overrightarrow{OB}$ and $\overrightarrow{OB} - \overrightarrow{OA}$.

$\therefore \overrightarrow{OD} = (1,1,1) + (1,-1,1) = (2,0,2)$ or

$= (1,1,1) - (1,-1,1) = (0,2,0)$ or

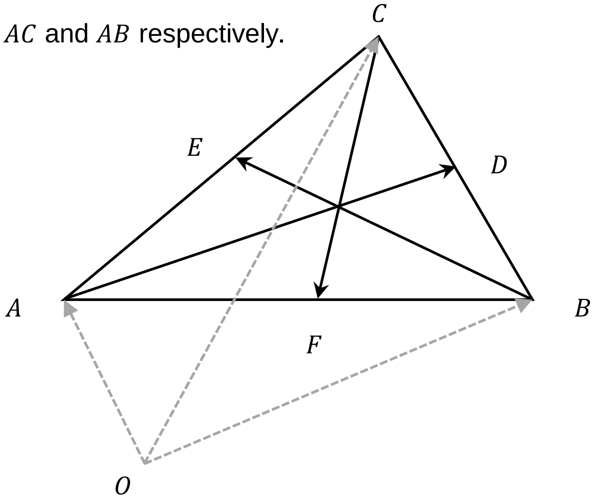
$= (1,-1,1) - (1,1,1) = (0,-2,0)$

The fourth vertex is at $(2,0,2), (0,2,0)$ or $(0,-2,0)$.

8

Let $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

In ΔABC let D, E, F be the midpoints of the sides BC, AC and AB respectively.



$\therefore \overrightarrow{OE} = \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2}, \overrightarrow{OD} = \frac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$ and $\overrightarrow{OF} = \frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$

$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$

$$= \left(\frac{\underset{\sim}{b}+\underset{\sim}{c}}{2} - \underset{\sim}{a}\right) + \left(\frac{\underset{\sim}{a}+\underset{\sim}{c}}{2} - \underset{\sim}{b}\right) + \left(\frac{\underset{\sim}{a}+\underset{\sim}{b}}{2} - \underset{\sim}{c}\right)$$

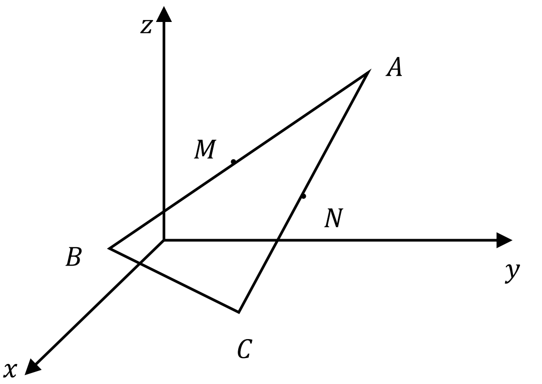
$$= \left(-1 + \frac{1}{2} + \frac{1}{2}\right)\underset{\sim}{a} + \left(\frac{1}{2} - 1 + \frac{1}{2}\right)\underset{\sim}{b} + \left(\frac{1}{2} + \frac{1}{2} - 1\right)\underset{\sim}{c}$$

$$= 0$$

9

$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{MA} + \overrightarrow{AN}\\ &= \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}\\ &= \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{AC}\right)\\ &= \frac{1}{2}\overrightarrow{BC}\end{aligned}$$

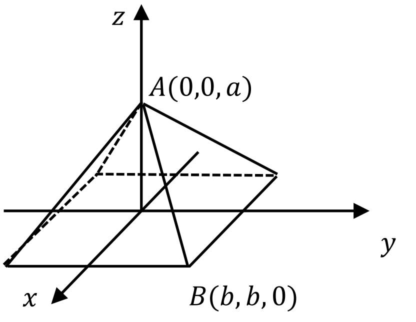
$\therefore$ MN is half the length of BC and parallel to it.



10

$\left|\overrightarrow{AB}\right| = \sqrt{(-b)^2 + (-b)^2 + a^2} = \sqrt{a^2 + 2b^2}$

The side length of the base is $2b$.

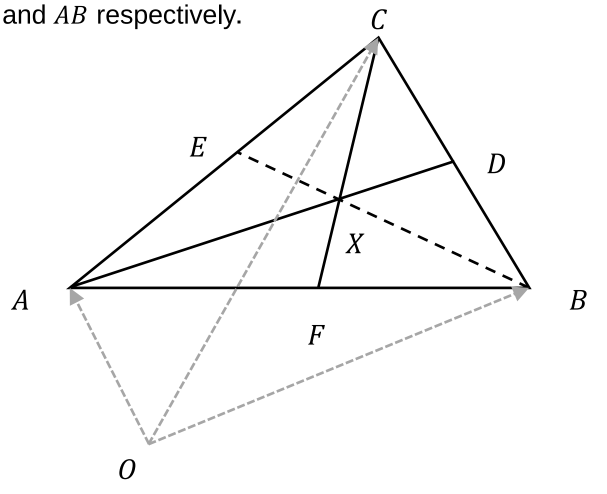


$$\begin{aligned}\therefore 2(2b) &= \sqrt{a^2 + 2b^2}\\ 4b &= \sqrt{a^2 + 2b^2}\\ 16b^2 &= a^2 + 2b^2\\ a^2 &= 14b^2\\ b^2 &= \frac{a^2}{14}\\ b &= \frac{a}{\sqrt{14}}\end{aligned}$$

11 We will let X be the intersection of AD and CF and prove that it lies on BE.

Let $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

In ΔABC let D, E, F be the midpoints of the sides BC, AC and AB respectively.



$\therefore \overrightarrow{OE} = \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2}, \overrightarrow{OD} = \frac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$ and $\overrightarrow{OF} = \frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX}\\ &= \underset{\sim}{a} + \lambda\overrightarrow{AD}\\ &= \underset{\sim}{a} + \lambda\left(\frac{\underset{\sim}{b}+\underset{\sim}{c}}{2} - \underset{\sim}{a}\right)\\ &= (1-\lambda)\underset{\sim}{a} + \frac{\lambda}{2}\underset{\sim}{b} + \frac{\lambda}{2}\underset{\sim}{c} \quad (1)\end{aligned}$$

Also

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OC} + \overrightarrow{CX}\\ &= \underset{\sim}{c} + \mu\overrightarrow{CF}\\ &= \underset{\sim}{c} + \mu\left(\frac{\underset{\sim}{a}+\underset{\sim}{b}}{2} - \underset{\sim}{c}\right)\\ &= \frac{\mu}{2}\underset{\sim}{a} + \frac{\mu}{2}\underset{\sim}{b} + (1-\mu)\underset{\sim}{c} \quad (2)\end{aligned}$$

From (1) and (2):

$$1-\lambda = \frac{\mu}{2} \qquad \frac{\lambda}{2} = \frac{\mu}{2} \qquad \frac{\lambda}{2} = 1-\mu$$

$$\lambda = \mu$$

$$\begin{aligned}1-\lambda &= \frac{\lambda}{2}\\ 1 &= \frac{3\lambda}{2}\\ \lambda &= \mu = \frac{2}{3}\end{aligned}$$

$$\therefore \overrightarrow{OX} = \frac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3}$$

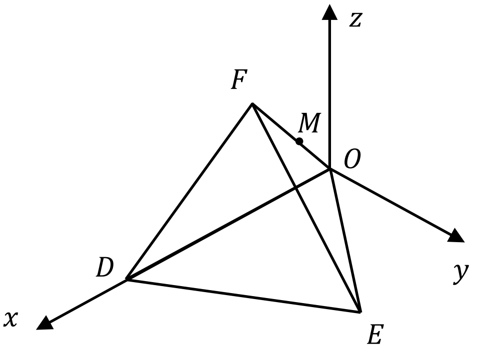
$$\begin{aligned}\overrightarrow{BX} &= \frac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3} - \underset{\sim}{b}\\ &= \frac{\underset{\sim}{a}+\underset{\sim}{c}-2\underset{\sim}{b}}{3}\end{aligned}$$

$$\begin{aligned}\overrightarrow{XE} &= \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2} - \frac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3}\\ &= \frac{\underset{\sim}{a}+\underset{\sim}{c}-2\underset{\sim}{b}}{6}\\ &= \frac{1}{2}\overrightarrow{BX}\end{aligned}$$

$\therefore B, E, X$ are collinear

$\therefore$ the medians of a triangle are concurrent

12 Let $M(a,b,c)$



In ΔMOD

$$\cos\frac{\pi}{3}=\frac{(a,b,c)\cdot(1,0,0)}{\frac{1}{2}\times 1}$$

$$\frac{1}{4}=a+0+0$$

$$a=\frac{1}{4}$$

In ΔMOE

$$\cos\frac{\pi}{3}=\frac{(a,b,c)\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right)}{\frac{1}{2}\times 1}$$

$$\frac{1}{4}=\frac{a}{2}+\frac{\sqrt{3}}{2}b+0$$

$$\frac{\sqrt{3}}{2}b=\frac{1}{4}-\frac{a}{2}$$

$$b=\frac{2}{\sqrt{3}}\left(\frac{1}{4}-\frac{1}{2}\times\frac{1}{4}\right)$$

$$=\frac{1}{4\sqrt{3}}$$

$$=\frac{\sqrt{3}}{12}$$

$$a^2+b^2+c^2=\left(\frac{1}{2}\right)^2$$

$$\left(\frac{1}{4}\right)^2+\left(\frac{\sqrt{3}}{12}\right)^2+c^2=\frac{1}{4}$$

$$\frac{1}{16}+\frac{3}{144}+c^2=\frac{1}{4}$$

$$c^2=\frac{1}{6}$$

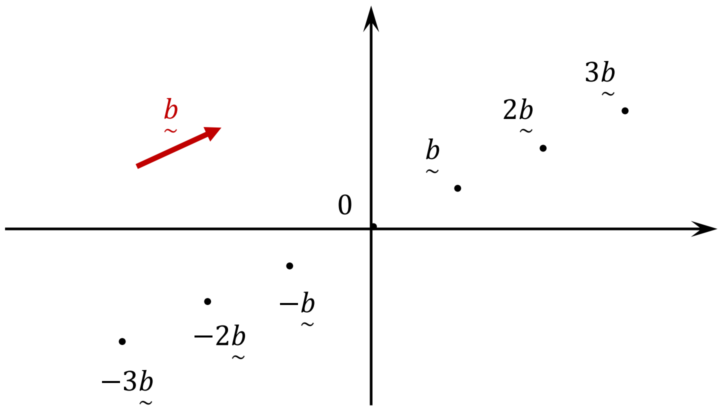
$$c=\frac{\sqrt{6}}{6}$$

$$\therefore M\left(\frac{1}{4},\frac{\sqrt{3}}{12},\frac{\sqrt{6}}{6}\right)$$

In Lesson 3 we start to look at the vector equation of lines in two and three dimensions, which continues next lesson.

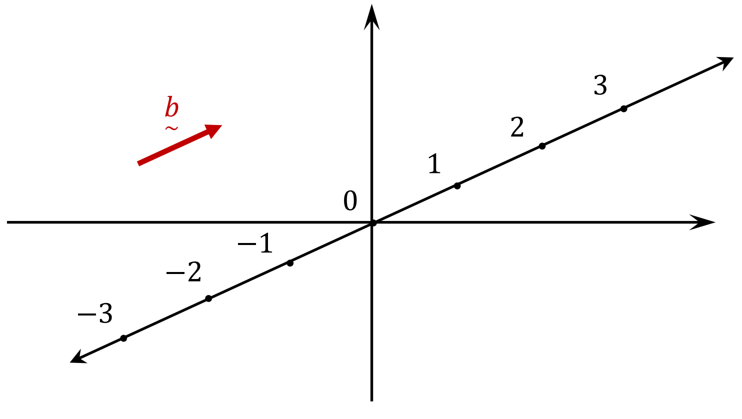
VECTOR EQUATION OF A LINE IN TWO DIMENSIONS

Plotting the tip of each of the scalar multiples of $\underset{\sim}{b}$ we see the following:



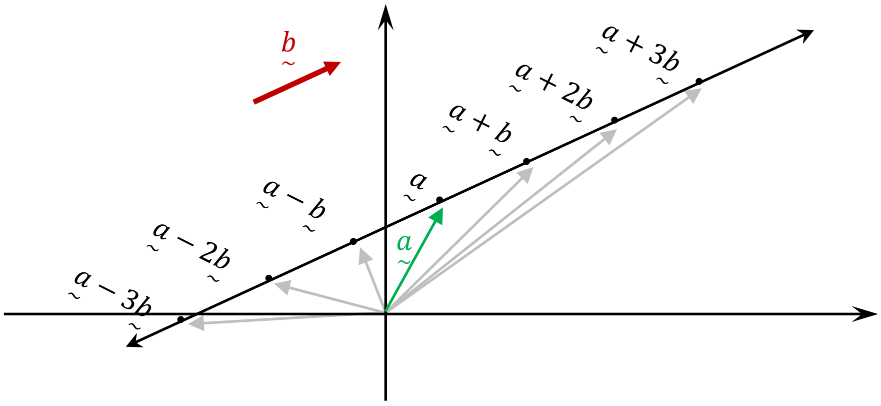
Now if we were to plot the tips of all scalar multiples of $\underset{\sim}{b}$ we would get a straight line through the origin, parallel to $\underset{\sim}{b}$. If we let λ represent any scalar then we could say that the new line is $\underset{\sim}{r} = \lambda\underset{\sim}{b}$, where $\underset{\sim}{r}$ is the position vector $\lambda\underset{\sim}{b}$.

If we plot the values of λ along the line we see the following:



So we could also think of the line $\underset{\sim}{r} = \lambda\underset{\sim}{b}$ as a number line parallel to $\underset{\sim}{b}$ with the distance between integers being $\left|\underset{\sim}{b}\right|$, with 0 at the origin. Note that the gradient of $\underset{\sim}{b}$ determines the gradient of the line.

But could we move the 0 mark away from the origin? Very easily - just choose the vector from the origin to where the 0 mark will be and add it on. By tradition we state it first and add the scalar multiple of the second vector to it, so the position vector to the starting point is $\underset{\sim}{a}$.

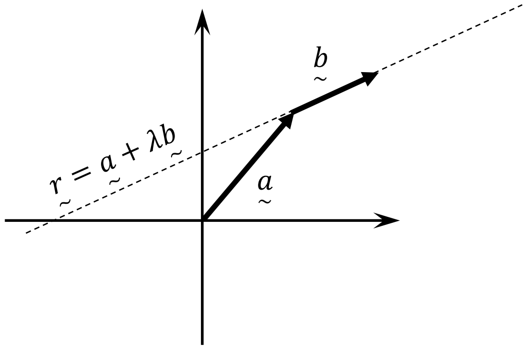


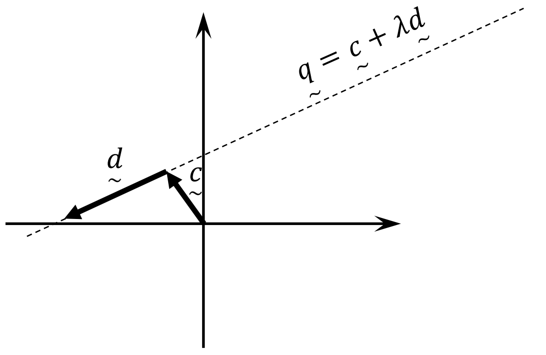
Note that the line is formed by the tip of the position vectors shown in grey and green, although we will normally only draw the line itself.

So the line above is $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$. We can see that $\underset{\sim}{a}$ tells us where the 0 mark is while $\underset{\sim}{b}$ sets the gradient and how far apart the integers on the 'number line' are. So a vector equation is similar to the point-gradient form of a line.

It is important to realise that λ is a parameter, with each value of λ representing a single point on the line.

There is no unique vector form of a line, just like we have more than one parametric form of a Cartesian equation. So we could move the starting point represented by $\underset{\sim}{a}$ to any point on the line we want, and use any vector parallel to the line (of any length and pointing in either direction along the line). We can see two different vector equations for the same line below.





We can replace $\underset{\sim}{b}$ with any parallel vector, so any vector having its components in the same ratio. So for example the following vector equations are all equal - focus on the vector after λ:

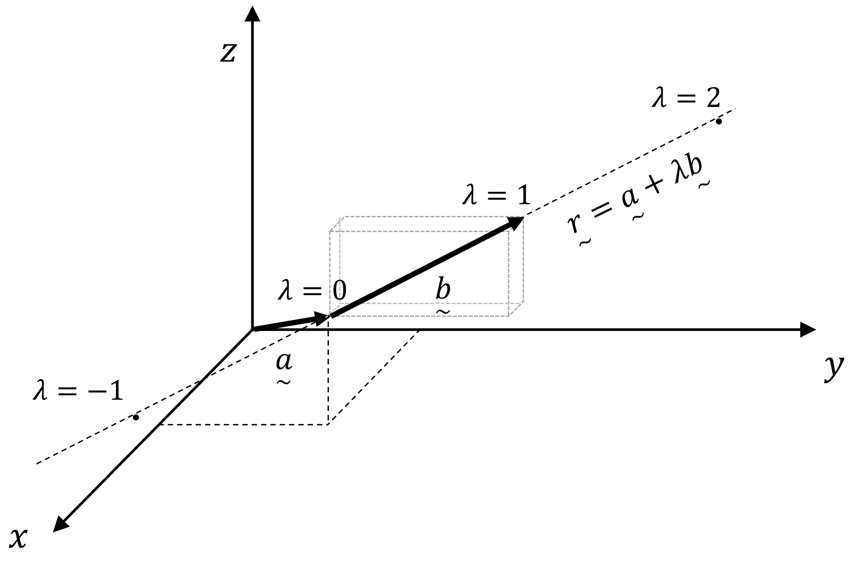
$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \end{pmatrix} \qquad \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 8 \end{pmatrix} \qquad \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

If we think of them in as number lines then:

1. The numbers on the second number line would be twice as far apart as for the first one
2. The third number line would have its positive direction opposite that of the other two lines.

LINES IN THREE DIMENSIONS

The vector equation for a line in three dimensions is exactly the same as it is for two dimensions, and the line is again like a number line with the 0 at the tip of $\underset{\sim}{a}$, 1 at the tip of $\underset{\sim}{a} + \underset{\sim}{b}$ etc. The line is made of the tip of the position vectors, although they are not shown in the diagram below.



Due to the difficulty of drawing lines in three dimensions we often work with them algebraically only.

SKETCHING A VECTOR EQUATION

We have two methods to sketch a line. Consider $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$, which could be a two or three dimensional vector.

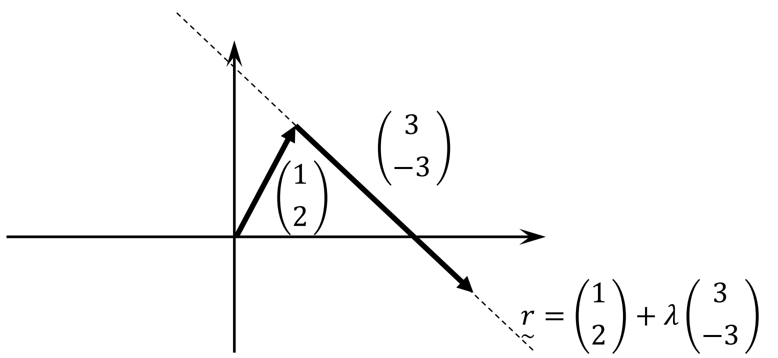
Method 1 treats the vector equation as a point ($\underset{\sim}{a}$) and a gradient ($\underset{\sim}{b}$). We first draw a position vector representing $\underset{\sim}{a}$, then from the tip of this vector draw the vector $\underset{\sim}{b}$. Now draw a line that passes through the tips of both vectors and we have our line.

Method 2 uses the vector equation to find two points on the line, then plots a line through these two points. It easiest to use $\lambda = 0$ and $\lambda = 1$, but we could use any values.

Example 1

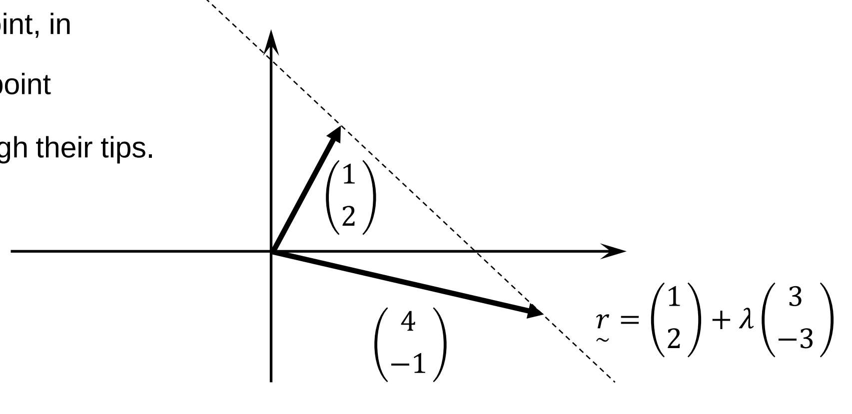
Sketch $\underset{\sim}{r} = \binom{1}{2} + \lambda\binom{3}{-3}$

Method 1



Method 2

Let $\lambda = 1$ to find a second point, in this case $\binom{4}{-1}$, and plot this point and $\binom{1}{2}$, drawing a line through their tips.



We can use the same two methods to sketch a three dimensional line.

SKETCHING AN INTERVAL

Sketching an interval (line segment) is very similar to Method 2, but we use the two values of λ given in the question and only go from tip to tip. Let's sketch a three dimensional interval.

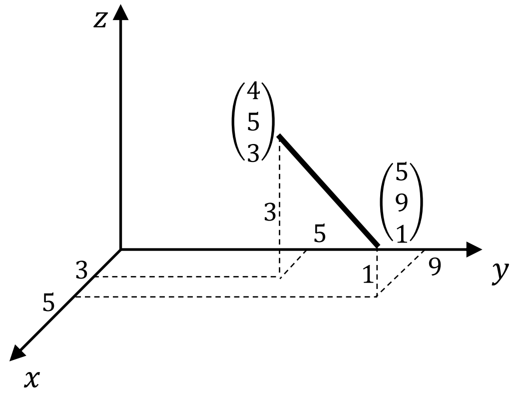
Example 2

Sketch the interval $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ for $1 \leq \lambda \leq 3$

Substitute $\lambda = 1$ and $\lambda = 3$ to find the end points of the interval.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix}$$



FINDING A VECTOR EQUATION OF A LINE THROUGH TWO POINTS

To find the equation of a line through two points, say A and B, first find the vector $\overrightarrow{AB}$, then rewrite the equation as $\underset{\sim}{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$.

Note that there are four easy possible answers here that are all correct, plus an infinite number of other correct answers. Including the above, four possible correct answers are:

$$\underset{\sim}{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} \qquad \underset{\sim}{r} = \overrightarrow{OA} + \lambda\overrightarrow{BA} \qquad \underset{\sim}{r} = \overrightarrow{OB} + \lambda\overrightarrow{AB} \qquad \underset{\sim}{r} = \overrightarrow{OB} + \lambda\overrightarrow{BA}$$

So we can start with either vector, $\overrightarrow{OA}$ or $\overrightarrow{OB}$ and add either vector between them, $\overrightarrow{AB}$ or $\overrightarrow{BA}$.

Sketching an interval is quite similar, but we need to calculate the parameter values at each end of the interval, and write them at the end like a domain.

Example 3

a Find a vector equation of the line through $\binom{2}{3}$ and $\binom{5}{7}$.

b Find a vector equation for the interval from $\binom{2}{3}$ to $\binom{5}{7}$.

a

$$\begin{pmatrix}5\\7\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$$

$\therefore \underset{\sim}{r} = \binom{2}{3} + \lambda\binom{3}{4}$ is one correct answer.

Some other possible answers are

$$\underset{\sim}{r} = \begin{pmatrix}2\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\-4\end{pmatrix}, \underset{\sim}{r} = \begin{pmatrix}5\\7\end{pmatrix} + \lambda\begin{pmatrix}3\\4\end{pmatrix} \text{ and } \underset{\sim}{r} = \begin{pmatrix}5\\7\end{pmatrix} + \lambda\begin{pmatrix}-3\\-4\end{pmatrix}$$

b Letting $\lambda = 0$ in $\underset{\sim}{r} = \binom{2}{3} + \lambda\binom{3}{4}$ gives us $\binom{2}{3}$, and $\lambda = 1$ gives $\binom{5}{7}$, so one vector equation for the interval is

$$\underset{\sim}{r} = \begin{pmatrix}2\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\4\end{pmatrix}, \qquad 0 \le \lambda \le 1$$

PARALLEL AND PERPENDICULAR LINES

Two lines $\underset{\sim}{a}_1 + \lambda \underset{\sim}{b}_1$ and $\underset{\sim}{a}_2 + \lambda \underset{\sim}{b}_2$ are:

- parallel if $\underset{\sim}{b}_1 = k\underset{\sim}{b}_2$.
- perpendicular if $\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_2 = 0$, or if we find the gradients first and check as normal. Finding the dot product is much easier.

Example 4
Prove the following lines are parallel: $\underset{\sim}{r} = \binom{3}{-2} + \lambda\binom{2}{-4}$ and $\underset{\sim}{q} = \binom{4}{1} + \lambda\binom{-1}{2}$.

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are parallel.

The syllabus states that we need to check whether intersecting lines are perpendicular, so we don't need to worry about skew lines.

Example 5

Prove the lines $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ are perpendicular.

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 2(-1) + 3(1) - 1(1) = 0$$

$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are perpendicular

EXERCISE 5.3

BASIC

1 Sketch $\underset{\sim}{r} = \binom{1}{1} + \lambda\binom{2}{-1}$

2 Sketch $\underset{\sim}{r} = \begin{pmatrix}1\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$

3 Sketch the interval $\underset{\sim}{r} = \binom{1}{1} + \lambda\binom{2}{-1}$ for $-1 \le \lambda \le 1$

4 Sketch the interval $\underset{\sim}{r} = \begin{pmatrix}1\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$ for $-1 \le \lambda \le 2$

5 **a** Find a vector equation of the line through $\binom{1}{1}$ and $\binom{2}{3}$.
b Find a vector equation for the interval from $\binom{1}{1}$ to $\binom{2}{3}$.

6 Consider the points $A\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $B\begin{pmatrix}1\\2\\1\end{pmatrix}$.
a Find a vector equation of the line through A and B.
b Find a vector equation for the interval from A to B.

7 Prove the following lines are parallel: $\underset{\sim}{r} = \binom{1}{-1} + \lambda\binom{1}{2}$ and $\underset{\sim}{q} = \binom{3}{1} + \lambda\binom{-2}{-4}$.

8 Prove the following lines are parallel: $\underset{\sim}{r} = \begin{pmatrix}2\\1\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} + \lambda\begin{pmatrix}-3\\3\\-9\end{pmatrix}$

9 Prove the lines $\underset{\sim}{r} = \binom{2}{1} + \lambda\binom{3}{2}$ and $\underset{\sim}{q} = \binom{4}{-2} + \lambda\binom{-2}{3}$ are perpendicular.

10 Prove the lines $\underset{\sim}{r} = \begin{pmatrix}-1\\2\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix}-2\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$ are perpendicular.

MEDIUM

11 Find the vector equation of the line through $\mathrm{A}\binom{-1}{2}$ parallel to $\overrightarrow{BC}$ with $\mathrm{B}\binom{2}{1}$ and $\mathrm{C}\binom{1}{2}$

12 Find the vector equation of the line through $\mathrm{A}\begin{pmatrix}1\\2\\1\end{pmatrix}$ parallel to $\overrightarrow{BC}$ with $\mathrm{B}\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathrm{C}\begin{pmatrix}3\\3\\1\end{pmatrix}$.

13 Find a vector equation for the line through $\binom{0}{1}$ with gradient $m = -2$

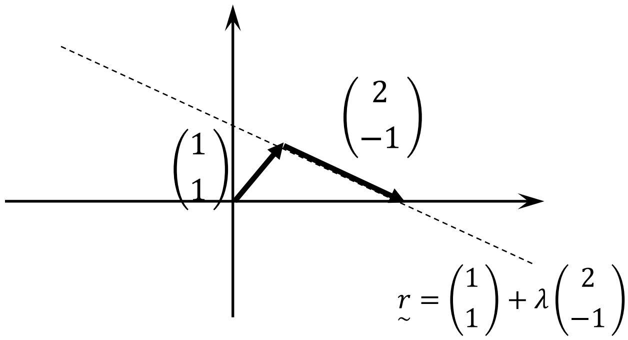
14 The lines $\underset{\sim}{r} = \lambda\binom{2}{3}$ and $\underset{\sim}{q} = \binom{4}{1} + \lambda\binom{p}{2}$ are perpendicular. Find p.

15 The lines $\underset{\sim}{r} = \lambda\begin{pmatrix}-2\\1\\-p\end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$ are perpendicular. Find p.

CHALLENGING

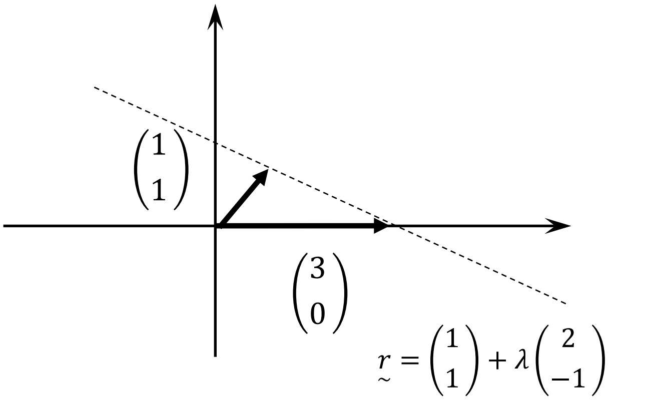
16 A cube has opposite vertices at the origin and $(2,2,2)$. State the equations of the four diagonals. Are the diagonals perpendicular?

1 Method 1

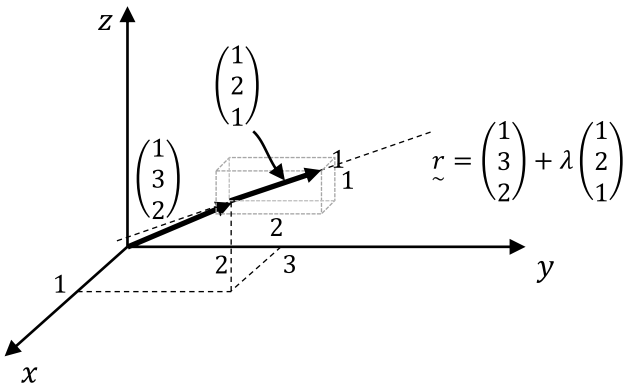


Method 2

Let $\lambda = 1$ to find a second point, in this case $\binom{3}{0}$, and plot this point and $\binom{1}{2}$, drawing a line through their tips.

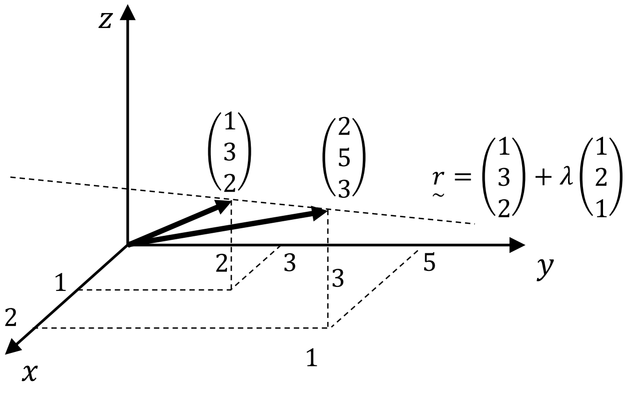


2 Method 1



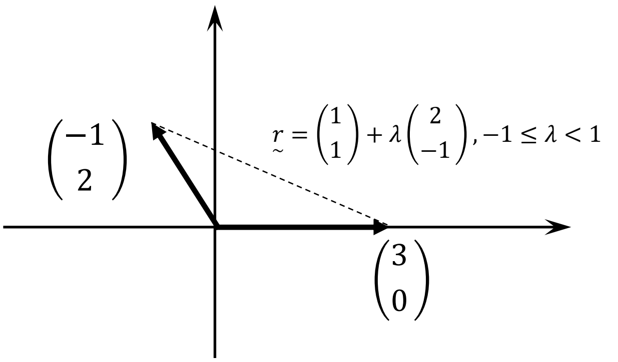
Method 2

Let $\lambda = 1$ to find a second point, in this case $\begin{pmatrix}2\\5\\3\end{pmatrix}$, and plot this point and $\begin{pmatrix}1\\3\\2\end{pmatrix}$, drawing a line through their tips.



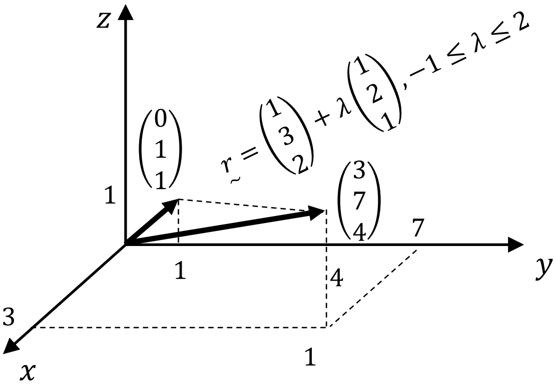
3 Substitute $\lambda = -1$ and $\lambda = 1$ to find the end points of the interval.

$$\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$$
$$\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$



4 Substitute $\lambda = -1$ and $\lambda = 2$ to find the end points of the interval.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix}$$



5 **a**
$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\therefore \underset{\sim}{r} = \binom{1}{1} + \lambda\binom{1}{2}$ is one correct answer.

b Letting $\lambda = 0$ in $\underset{\sim}{r} = \binom{1}{1} + \lambda\binom{1}{2}$ gives us $\binom{1}{1}$, and $\lambda = 1$ gives $\binom{2}{3}$, so one vector equation for the interval is

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad 0 \le \lambda \le 1$$

6 **a**
$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is one correct answer.

b Letting $\lambda = 0$ in $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ gives us $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\lambda = 1$ gives $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, so one vector equation for the interval is

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad 0 \le \lambda \le 1$$

7
$$\begin{pmatrix} -2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are parallel.

8
$$\begin{pmatrix} -3 \\ 3 \\ -9 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$
$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are parallel

9
$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} = -6 + 6 = 0$$
$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are perpendicular

10
$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 1(-1) + 2(1) - 1(1) = 0$$
$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are perpendicular

11
$$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\therefore \underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

12
$$\overrightarrow{BC} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$
$$\therefore \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

13 The gradient of -2 can be represented by the vector $\begin{pmatrix}1\\-2\end{pmatrix}$, or any vector where the y-value is minus two times the x-value.

$$\therefore \underset{\sim}{r} = \begin{pmatrix}0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\end{pmatrix}$$

14

$$\begin{aligned}\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}p\\2\end{pmatrix} &= 0\\ 2p+6 &= 0\\ 2p &= -6\\ p &= -3\end{aligned}$$

15

$$\begin{aligned}\begin{pmatrix}-2\\1\\-p\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix} &= 0\\ -2(2)+1(1)-p(-2) &= 0\\ -3+2p &= 0\\ \therefore p &= \frac{3}{2}\end{aligned}$$

16 All diagonals pass through $(1,1,1)$, plus one of the four base vertices, $A(0,0,0), B(2,0,0), C(2,2,0)$ and $D(0,2,0)$

$$\underset{\sim}{r_1} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1-0\\1-0\\1-0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\underset{\sim}{r_2} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1-2\\1-0\\1-0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\underset{\sim}{r_3} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1-2\\1-2\\1-0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$$\underset{\sim}{r_4} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1-0\\1-2\\1-0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

None of the dot products of the direction vectors give zero, so the diagonals are not perpendicular.

For example $\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix} = -1+1+1 = 1 \neq 0$

In Lesson 4 we take a further look at the properties of two or three dimensional lines. We cover:

- Does a point lie on a line?
- The Point of Intersection of Two Lines
- The connection between the vector equation and the gradient-intercept form of a line in two dimensions

DOES A POINT LIE ON A LINE?

To determine whether a point lies on a line represented by a vector equation, we check to see if the x and y values, plus the z value if needed, can be created using the same value of the parameter λ - if they can the point is on the line, while if different values of λ are needed it is not on the line. This works for two and three dimensional lines.

Example 1

Prove the point $\begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix}$ lies on the line $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 + 3\lambda \\ 3 + 4\lambda \\ -2 - 2\lambda \end{pmatrix}$$

$$\begin{aligned} -4 &= -1 + 3\lambda \rightarrow \quad \lambda = -1 \\ -1 &= 3 + 4\lambda \rightarrow \quad \lambda = -1 \\ 0 &= -2 - 2\lambda \rightarrow \quad \lambda = -1 \end{aligned}$$

In each case $\lambda = -1$

$\therefore \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix}$ lies on the line $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$

POINT OF INTERSECTION

To find the point of intersection of two lines, we must first use different parameters for the two lines. If we used the same parameter for both lines then all we would be checking is whether the intersection at the same parameter value, not whether they intersect at all. If the lines are written with the same parameter then change one of them to a different parameter.

We then solve the equations simultaneously, needing to find only one result, then substitute it into either original equation to find the point of intersection.

Example 2

Find the point of intersection of the lines $\underset{\sim}{r} = \binom{2}{3} + \lambda\binom{-1}{2}$ and $\underset{\sim}{q} = \binom{6}{9} + \lambda\binom{3}{1}$

Change the second parameter to μ

$$\begin{pmatrix} 2-\lambda \\ 3+2\lambda \end{pmatrix} = \begin{pmatrix} 6+3\mu \\ 9+\mu \end{pmatrix}$$

$2-\lambda = 6+3\mu \;\; (1) \quad 3+2\lambda = 9+\mu \;\; (2)$

$(2) + 2 \times (1)$:

$7 = 21 + 7\mu \;\; \rightarrow \;\; 7\mu = -14 \;\; \rightarrow \;\; \mu = -2$

The point of intersection is $\binom{6}{9} - 2\binom{3}{1} = \binom{0}{7}$.

Note that we do not need to find the value of both scalars.

CONVERTING BETWEEN CARTESIAN AND VECTOR EQUATIONS

For two dimensional lines we can convert from a vector equation to its Cartesian equation by finding the parametric equations for x and y, then solving simultaneously. Generally rearranging the equation for x to make the parameter the subject then substituting into the equation for y is easier.

Example 3

Rewrite the following vector equations in Cartesian form by first finding expressions for x and y in terms of λ.

a $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

b $\underset{\sim}{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

a

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + 3\lambda \\ 2 - 3\lambda \end{pmatrix}$$

$$\begin{aligned} x &= 1 + 3\lambda \ \rightarrow \ \lambda = \frac{x-1}{3} \\ y &= 2 - 3\lambda \\ &= 2 - 3\left(\frac{x-1}{3}\right) \\ &= 2 - x + 1 \\ \therefore y &= -x + 3 \end{aligned}$$

b

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 + 4\lambda \\ 3 + 2\lambda \end{pmatrix}$$

$$\begin{aligned} x &= -2 + 4\lambda \ \rightarrow \ \lambda = \frac{x+2}{4} \\ y &= 3 + 2\lambda \\ &= 3 + 2\left(\frac{x+2}{4}\right) \\ &= 3 + \frac{x}{2} + 1 \\ \therefore y &= \frac{1}{2}x + 4 \end{aligned}$$

EXERCISE 5.4

BASIC

1 Prove the point $\binom{1}{-4}$ lies on the line $\underset{\sim}{r} = \binom{3}{-1} + \lambda\binom{2}{3}$

2 Prove the point $\begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix}$ lies on the line $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

3 Prove the point $\binom{1}{2}$ does not lie on $\underset{\sim}{r} = \binom{3}{-1} + \lambda\binom{2}{3}$

4 Prove the point $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ does not lie on $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

MEDIUM

5 Find the point of intersection of the lines $\underset{\sim}{r} = \binom{1}{1} + \lambda\binom{2}{3}$ and $\underset{\sim}{q} = \binom{4}{2} + \lambda\binom{-1}{2}$

6 Find the point of intersection of the lines
$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

7 Rewrite the following vector equations in Cartesian form by first finding expressions for x and y in terms of λ.
a $\underset{\sim}{r} = \binom{0}{1} + \lambda\binom{2}{-1}$
b $\underset{\sim}{r} = \binom{2}{-3} + \lambda\binom{-1}{-3}$

8 Prove that the vector equations $\underset{\sim}{r} = \binom{2}{4} + \lambda\binom{1}{3}$ and $\underset{\sim}{q} = \binom{0}{-2} + \lambda\binom{-2}{-6}$ have the same Cartesian equation.

9 Rewrite $y = 4x + 5$ as a vector equation.

CHALLENGING

10 Prove the lines $y = 2x + 1$ and $y = -\frac{1}{2}x$ are perpendicular
a Using the product of their gradients
b By first converting them to vector form

11 A triangle has vertices $A(0,0,0), B(0,2,4)$ and $C(4,2,0)$. Find the equations of the three medians and show that they are concurrent.

1
$$\begin{pmatrix}1\\-4\end{pmatrix}=\begin{pmatrix}3+2\lambda\\-1+3\lambda\end{pmatrix}$$
$$1=3+2\lambda\ \rightarrow \lambda=-1$$
$$-4=-1+3\lambda\ \rightarrow \lambda=-1$$
$\therefore \binom{1}{-4}$ lies on the line $\underset{\sim}{r}=\binom{3}{-1}+\lambda\binom{2}{3}$

2
$$\begin{pmatrix}4\\3\\-3\end{pmatrix}=\begin{pmatrix}2\lambda\\1+\lambda\\-1-\lambda\end{pmatrix}$$
$$4=2\lambda\rightarrow\lambda=2$$
$$3=1+\lambda\rightarrow\lambda=2$$
$$-3=-1-\lambda\ \rightarrow\ \lambda=2$$
$\therefore \begin{pmatrix}4\\3\\-3\end{pmatrix}$ lies on the line $\underset{\sim}{r}=\begin{pmatrix}0\\1\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$

3
$$\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3+2\lambda\\-1+3\lambda\end{pmatrix}$$
$$1=3+2\lambda\ \rightarrow\lambda=-1$$
$$2=-1+3\lambda\ \rightarrow\lambda=1$$
$\therefore\binom{1}{2}$ does not lie on $\underset{\sim}{r}=\binom{3}{-1}+\lambda\binom{2}{3}$

4
$$\begin{pmatrix}3\\4\\2\end{pmatrix}=\begin{pmatrix}3\lambda\\1+2\lambda\\2+\lambda\end{pmatrix}$$
$$3=3\lambda\ \rightarrow\lambda=1$$
$$4=1+2\lambda\ \rightarrow\lambda=\frac{3}{2}$$
$\therefore\begin{pmatrix}3\\4\\2\end{pmatrix}$ does not lie on $\underset{\sim}{r}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}$

5
Change the second parameter to μ
$$\begin{pmatrix}1+2\lambda\\1+3\lambda\end{pmatrix}=\begin{pmatrix}4-\mu\\2+2\mu\end{pmatrix}$$
$1+2\lambda=4-\mu\ \ (1)\qquad 1+3\lambda=2+2\mu\ \ (2)$
$(2)+2\times(1)$:
$3+7\lambda=10\ \rightarrow\ 7\lambda=7\ \rightarrow\ \lambda=1$
The point of intersection is $\binom{1}{1}+\binom{2}{3}=\binom{3}{4}$

6
$$\begin{pmatrix}-\lambda\\3+\lambda\\-5+2\lambda\end{pmatrix}=\begin{pmatrix}-1-\mu\\2+2\mu\\3-\mu\end{pmatrix}$$
$-\lambda=-1-\mu\qquad(1)$
$3+\lambda=2+2\mu\qquad(2)$
$-5+2\lambda=3-\mu\qquad(3)$
$(1)+(2)$:
$3=1+\mu$
$\mu=2$

The point of intersection is
$$\begin{pmatrix}-1\\2\\3\end{pmatrix}+2\begin{pmatrix}-1\\2\\-1\end{pmatrix}=\begin{pmatrix}-3\\6\\1\end{pmatrix}$$

7 **a**
$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\lambda\\1-\lambda\end{pmatrix}$$
$$x=2\lambda\ \rightarrow\ \lambda=\frac{x}{2}$$
$$y=1-\lambda=1-\frac{x}{2}$$
$$\therefore y=-\frac{1}{2}x+1$$

b
$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2-\lambda\\-3-3\lambda\end{pmatrix}$$
$$x=2-\lambda\ \rightarrow\ \lambda=2-x$$
$$y=-3-3\lambda=-3-3(2-x)$$
$$\therefore y=3x-9$$

8
For $\underset{\sim}{r}=\binom{2}{4}+\lambda\binom{1}{3}$, we have $m=\frac{3}{1}=3$ and $b=4-2(3)=-2$, so its Cartesian equation is $y=3x-2$.
For $\underset{\sim}{q}=\binom{0}{-2}+\lambda\binom{-2}{-6}$, we have $m=\frac{-6}{-2}=3$ and $b=-2-0(3)=-2$, so its Cartesian equation is $y=3x-2$..
Both vector equations have the same Cartesian equation.

9
The y-intercept is $(0,5)$ so let $\underset{\sim}{a}=\binom{0}{5}$.
The gradient is $\frac{4}{1}$ so let $\underset{\sim}{b}=\binom{1}{4}$.

$y=4x+5$ is equivalent to $\underset{\sim}{r}=\binom{0}{5}+\lambda\binom{1}{4}$

10 **a**

$m_1 = 2, m_2 = -\frac{1}{2}$

$m_1 \times m_2 = 2 \times \left(-\frac{1}{2}\right) = -1$

∴ The two lines are perpendicular

b

In vector form $y = 2x + 1 \quad \rightarrow \quad \underset{\sim}{r_1} = \binom{0}{1} + \lambda \binom{1}{2}$ and $y = -\frac{1}{2}x \quad \rightarrow \quad \underset{\sim}{r_2} = \lambda \binom{-2}{1}$

$\binom{1}{2} \cdot \binom{-2}{1} = -2 + 2 = 0$

∴ The two lines are perpendicular

11 The midpoints are $M_{AB} = (0,1,2), M_{AC} = (2,1,0), M_{BC} = (2,2,2)$. The equations of the medians through each vertex are:

$\underset{\sim}{r_A} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda \begin{pmatrix}2-0\\2-0\\2-0\end{pmatrix} = \begin{pmatrix}2\lambda\\2\lambda\\2\lambda\end{pmatrix} \qquad (1) \qquad \underset{\sim}{r_B} = \begin{pmatrix}0\\2\\4\end{pmatrix} + \lambda \begin{pmatrix}2-0\\1-2\\0-4\end{pmatrix} = \begin{pmatrix}2\lambda\\2-\lambda\\4-4\lambda\end{pmatrix} \qquad (2)$

$\underset{\sim}{r_C} = \begin{pmatrix}4\\2\\0\end{pmatrix} + \lambda \begin{pmatrix}0-4\\1-2\\2-0\end{pmatrix} = \begin{pmatrix}4-4\lambda\\2-\lambda\\2\lambda\end{pmatrix} \qquad (3)$

For (1) and (2):

$2\lambda = 2\mu \quad 2\lambda = 2 - \mu \quad 2\lambda = 4 - 4\mu$

$\lambda = \mu \quad 3\lambda = 2 \ \rightarrow \lambda = \frac{2}{3}$

∴ (1) and (2) intersect at $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

For (1) and (3):

$2\lambda = 4 - 4\mu \quad 2\lambda = 2 - \mu \quad 2\lambda = 2\mu$

$3\lambda = 2 \ \rightarrow \lambda = \frac{2}{3} \quad \lambda = \mu$

∴ (1) and (3) intersect at$\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

Since (2) and (3) both intersect (1) at the same point, they must intersect there as well.

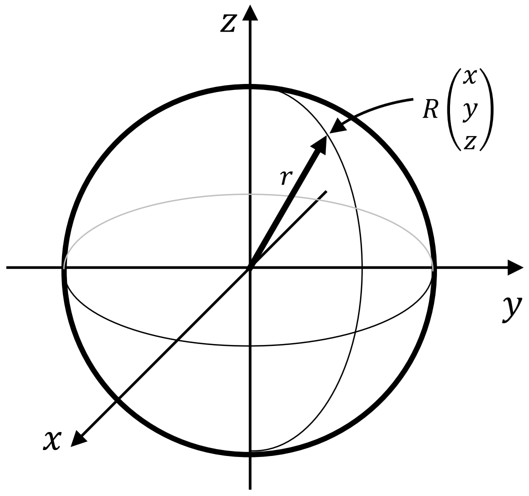
∴ all three medians intersect at $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

In Lesson 5 we look at spheres and basic two dimensional curves. We cover:

- Spheres
- Curves in Two Dimensions
  - Common Curves
  - Transformations
  - Restrictions

SPHERES

Every point on a sphere is exactly r units from the centre, where r is the radius of the sphere.



Letting any point on the surface of the sphere be $R\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, this means that the magnitude of $\overrightarrow{OR}$ is r, so using our formula for magnitude we see

$$\sqrt{x^2+y^2+z^2}=r$$
$$\therefore x^2+y^2+z^2=r^2$$

This is the Cartesian equation of a sphere. We can adjust the formula easily enough using transformations if the centre is not at the origin.

When drawing a sphere, it can be easiest to draw a circle, with an ellipse used to indicate the equator, and one line of longitude, as above.

Example 1
Find the equations of the spheres with:
a radius 4 and centre at the origin
b radius $\sqrt{2}$ and centre at $(1,-2,1)$

a $x^2+y^2+z^2=16$

b $(x-1)^2+(y+2)^2+(z-1)^2=2$

Example 2
Complete the square to determine the radius and centre of the sphere.

$$x^2+2x+y^2-6y+z^2+8z+10=0$$

$$\begin{aligned} x^2+2x+y^2-6y+z^2+8z+10&=0 \\ x^2+2x+1+y^2-6y+9+z^2+8z+16&=16 \\ (x+1)^2+(y-3)^2+(z+4)^2&=16 \end{aligned}$$

A sphere with radius 4 and centre $(-1,3,-4)$.

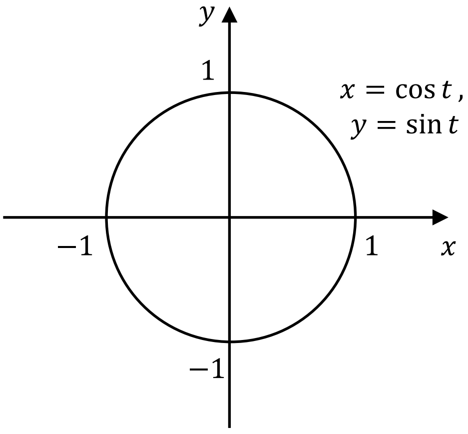
CURVES IN TWO DIMENSIONS

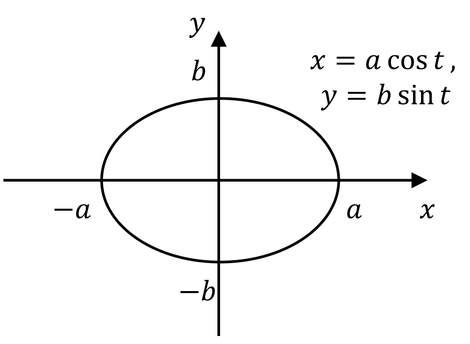
We will start by looking at the common curves you should know by sight. We will look at how to convert them to a Cartesian equation and the important features of their sketches. We will then look at how to transform them (without converting to Cartesian form) and how to restrict them. We will then look at how to approach the sketching of unusual curves.

COMMON CURVES

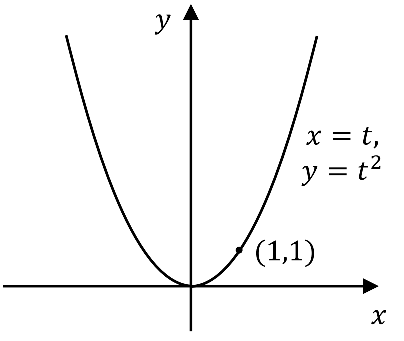
You should be able to recognise and sketch the following curves from their parametric equations, without needing to convert them to Cartesian form first, plus be able to convert each of these into Cartesian form if needed.

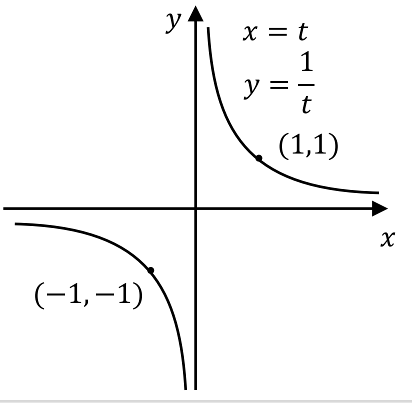
Once you are familiar with them you will be able to use them as the basis of harder curves.

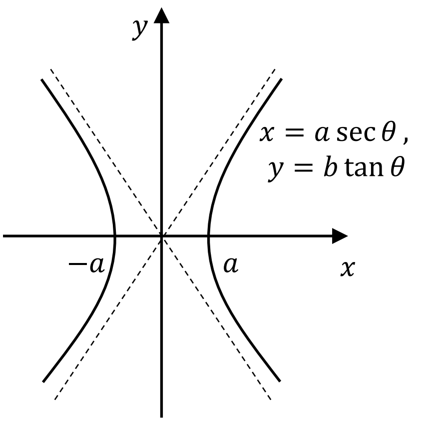
<table>
<tr><th>Shape</th><th>Parametric Equations</th><th>Cartesian Equation</th><th>Sketch</th></tr>
<tr><td>Unit Circle</td><td>$x = \cos t$
$y = \sin t$</td><td>$x^2 + y^2 = 1$</td><td rowspan="2">
</td></tr>
<tr><td colspan="3">A circle centred at the origin with radius 1.</td></tr>
</table>

<table>
<tr><th>Shape</th><th>Parametric Equations</th><th>Cartesian Equation</th><th>Sketch</th></tr>
<tr><td>Ellipse</td><td>$x = a\cos t$
$y = b\sin t$</td><td>$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$</td><td rowspan="2">
</td></tr>
<tr><td colspan="3">An ellipse centred at the origin with semi-major axis a and semi-minor axis b.

An ellipse could instead be viewed as a circle stretched by different factors vertically and horizontally.</td></tr>
</table>

Shape	Parametric Equations	Cartesian Equation	Sketch
Parabola	$x = t$ $y = t^2$	$y = x^2$	
A concave up parabola with vertex at the origin.			

Shape	Parametric Equations	Cartesian Equation	Sketch
Rectangular Hyperbola	$x = t$ $y = \frac{1}{t}$	$y = \frac{1}{x}$ or $xy = 1$	
A rectangular hyperbola with branches in the 1st and 3rd quadrants, passing through $(1,1)$ and $(-1,-1)$.			

Shape	Parametric Equations	Cartesian Equation	Sketch
Hyperbola	$x = a\sec t$ $y = b\tan t$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	
A hyperbola with its axis along the x-axis, with x-intercepts $x = \pm a$ and asymptotes $y = \pm\frac{b}{a}x$. Probably beyond the syllabus but worth covering just in case.			

TRANSFORMATIONS

The easiest way to sketch most curves is to recognise the parametric equations as a transformation of one of the common curves above, and sketch it directly.

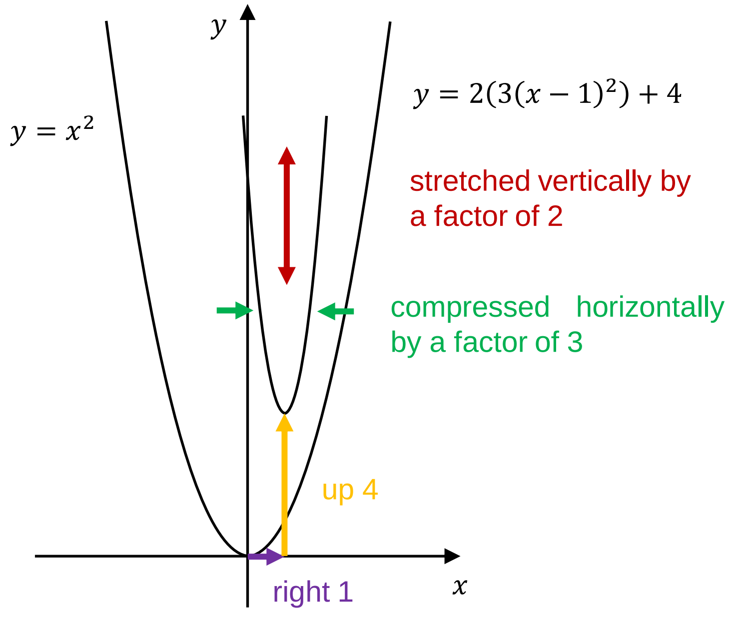
We need to be careful, as it is easy to get the transformation wrong due to some leftover baggage from our work with transforming Cartesian equations.

Consider the Cartesian equation $y = 2(3(x-1))^2 + 4$. Note that y is the subject while the x is over with all the numbers. For this reason the vertical changes (relating to the y value) are as expected while the horizontal changes (relating to the x value) are counterintuitive.

We can sketch this function by transforming its basic curve $y = x^2$. To do this we note that we perform two operations before squaring (subtracting 1 and multiplying by 3) and two operations after squaring (multiplying by 2 and adding 4).

The operations made before squaring cause horizontal changes that are counterintuitive - so subtracting 1 moves the curve 1 unit to the right (not the left) and multiplying by 3 compresses it horizontally (not stretching it).

The operations made after squaring cause vertical changes in the expected directions - so multiplying by 2 stretches it vertically and adding 4 moves it up.



So how does this affect our sketches of parametric equations?

The parametric equations have x and y as the subject (so all the other numbers on the opposite side), so any operations will cause changes as expected - they will not be counterintuitive!

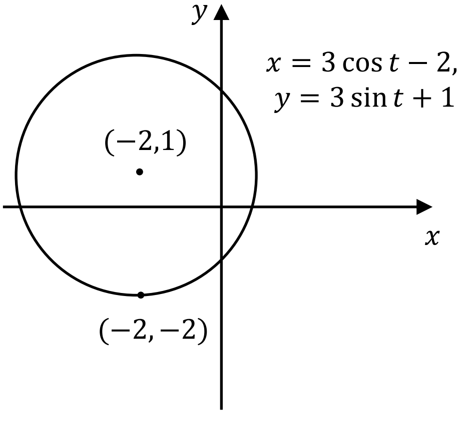
So if we consider $x = t - 1, y = 2t^2$ this will take the basic parabola $y = x^2$ and move it 1 unit to the left and stretch it vertically by a factor of 2.

Now these rules for transformations only apply to simple equations - as soon as an operation occurs to t before the main function occurs then it is safest to find the Cartesian equation. So for example $x = t - 1, y = 2(t + 3)^2$ actually moves $y = x^2$ four units to the left - not nearly what we might expect.

Example 3

Sketch $x = 3\cos t - 2, y = 3\sin t + 1$

This is the circle with radius 3 moved 2 units to the left and 1 unit up.



RESTRICTIONS

With the exception of the hyperbola, where $t \neq 0$, the curves we have seen so far have had no restrictions given in the question or implied by one of the parametric equations.

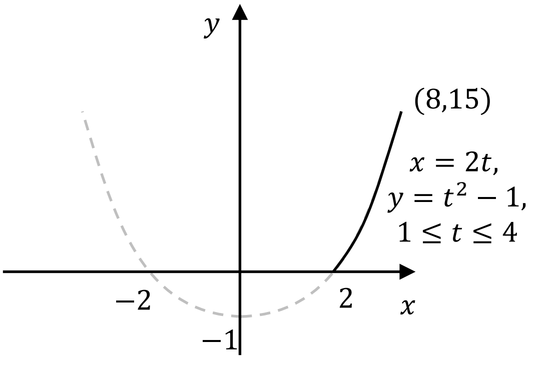
Let's have a look at some examples where we do not sketch all the curve.

In the first example t will be restricted in the question.

Example 4
Sketch $x = 2t, y = t^2 - 1$ for $1 \leq t \leq 4$

This is the basic parabola stretched horizontally by a factor of 2 and moved down 1.

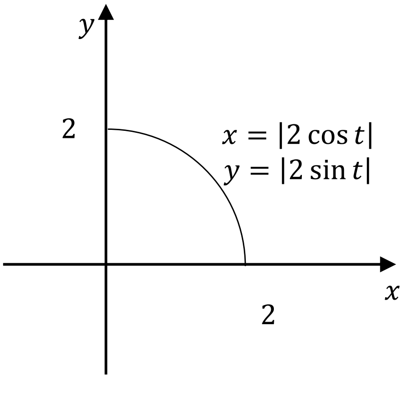
We only take the section from $t = 1$ to $t = 4$, so from $(2,0)$ to $(8,15)$



Example 5

Sketch $x = |2\cos t|, y = |2\sin t|$

$0 \leq x \leq 2$ and $0 \leq y \leq 2$, so the top right quadrant of a circle of radius 2 centred at the origin.



SPIRALS

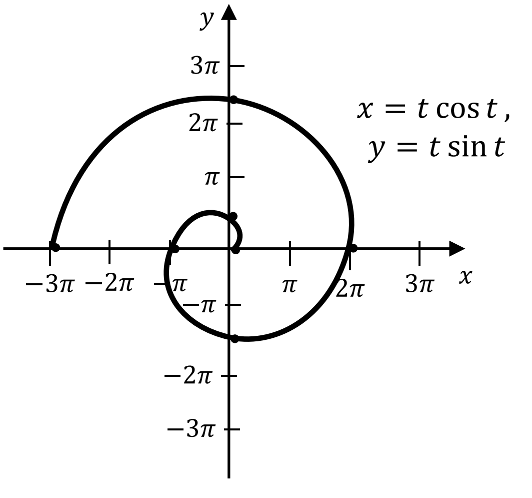
A spiral can be thought of as a point moving around a circle, but where the radius increases. There are many types of spiral - we will cover the simplest, the Archimedean Spiral, in the next example.

When sketching spirals it is best to sketch a few points to determine the pattern.

Example 6 The Archimedean Spiral

Sketch $x = t\cos t\,, y = t\sin t$ for $t \geq 0$

t	x	y
0	0	0
$\frac{\pi}{2}$	0	$\frac{\pi}{2}$
π	$-\pi$	0
$\frac{3\pi}{2}$	0	$-\frac{3\pi}{2}$
2π	2π	0
3π	-3π	0



Note in this case that the magnitude of each successive intercept increases by $\frac{\pi}{2}$.

EXERCISE 5.5

BASIC

1 Find the equations of the spheres with:
a radius 2 and centre at the origin
b radius 3 and centre at $(-1,1,3)$

2 Sketch $x = 3\cos t\,, y = 2\sin t$

3 Sketch $x = t^2, y = t$

4 Sketch $x = 5\cos t - 2, y = 4\sin t + 1$

5 Sketch $x = t^2 - 1, y = t + 1$

6 Sketch $x = \frac{1}{t}, y = t + 2$

7 Sketch $x = \sec\theta\,, y = \tan\theta$

8 Sketch $x = t, y = t^2$ for $0 \le t \le 2$

MEDIUM

9 Complete the square to determine the radius and centre of the sphere

$$x^2 - 2x + y^2 + z^2 + 4z + 4 = 0$$

10 The parameterised equation of a sphere is $x = r\sin\alpha\sin\beta\,, y = r\cos\alpha\,, z = r\sin\alpha\cos\beta$. Prove that it satisfies $x^2 + y^2 + z^2 = r^2$.

11 Sketch $x = -|2\cos t|, y = |2\sin t|$

12 Sketch $x = t^2, y = \frac{1}{t^2}$

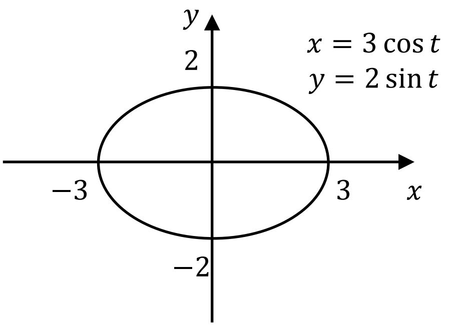
13 Sketch $x = t\sin t\,, y = t\cos t$

CHALLENGING

14 The parametric equations $x = \cos t\,, y = \sin t$ gives a unit circle, and as t increases from zero the point moves anticlockwise from $(1,0)$. Find the parametric equations of a circle where as t increases from zero the point moves clockwise from $\left(\sqrt{3}, 1\right)$, centred about the origin.

1 **a** $x^2 + y^2 + z^2 = 4$ **b** $(x+1)^2 + (y-1)^2 + (z-3)^2 = 9$

2 From the common curves this is an ellipse centred at the origin with a horizontal semi-major axis of 3 and a semi-minor axis of 3.



Alternatively

$x = 3\cos t \ \rightarrow \cos t = \dfrac{x}{3}$

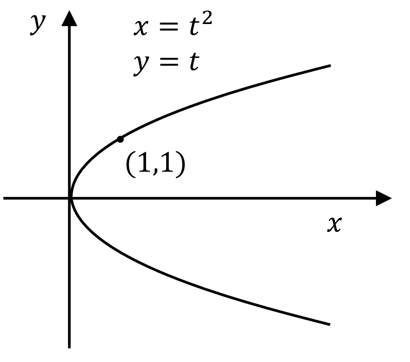
$y = 2\sin t \ \rightarrow \sin t = \dfrac{y}{2}$

$\sin^2\theta + \cos^2\theta = 1$ Pythagorean Identity

$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Which is the ellipse centred at the origin with horizontal semi major axis 3 and semi minor axis 2

3 From the common curves this is a parabola with x and y values swapped, so concave right with vertex at the origin



Alternatively

$x = t^2 \ \rightarrow \ t = \pm\sqrt{x} \quad (1)$

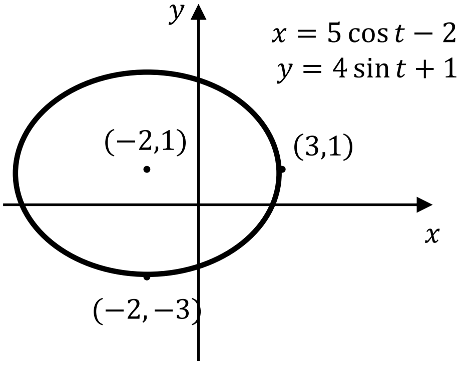
$y = t \quad (2)$

sub (1) in (2)

$y = \pm\sqrt{x}$

This is the parabola which is concave right with vertex at the origin.

4 This is the ellipse with horizontal semi major axis 5 and semi minor axis 4 moved 2 units to the left and 1 unit up.

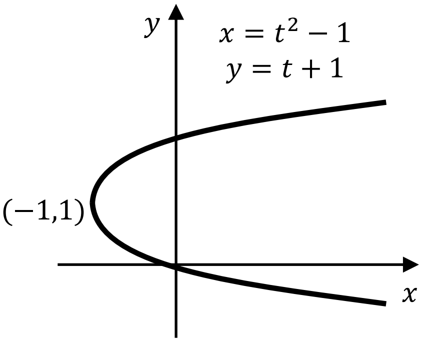


5 Safest to find the Cartesian equation:

$x = t^2 - 1 \ \rightarrow t = \pm\sqrt{x+1}$

$y = t + 1 = \pm\sqrt{x+1} + 1$

This is the concave right parabola moved 1 unit to the left and up 1.

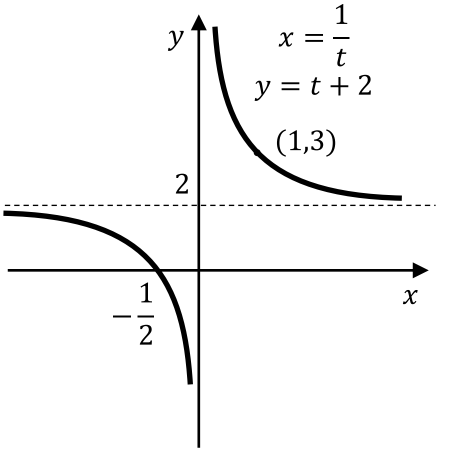


6 This is the basic rectangular hyperbola moved 2 units up.

Alternatively we could find the Cartesian equation:

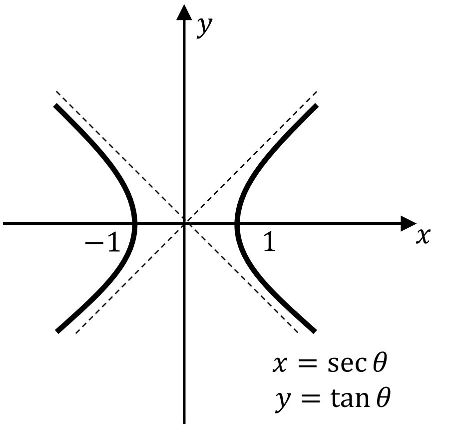
$x = \frac{1}{t} \rightarrow t = \frac{1}{x}$

$y = t + 2 = \frac{1}{x} + 2$



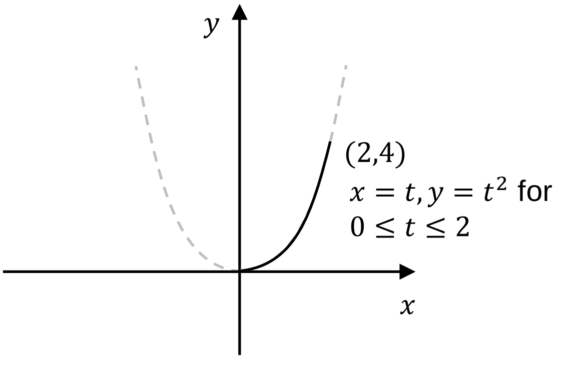
7 This is the hyperbola $x^2 - y^2 = 1$.

The asymptotes are $y = \pm x$



8 This is the basic parabola.

We only take the section from $t = 0$ to $t = 2$, so from $(0,0)$ to $(2,4)$



9

$$x^2 - 2x + y^2 + z^2 + 4z + 4 = 0$$
$$x^2 - 2x + 1 + y^2 + z^2 + 4z + 4 = 1$$
$$(x-1)^2 + y^2 + (z+2)^2 = 1$$

A sphere with radius 1 and centre $(1,0,-2)$.

10 $x^2 + y^2 + z^2$

$= (r\sin\alpha\sin\beta)^2 + (r\cos\alpha)^2 + (r\sin\alpha\cos\beta)^2$

$= r^2\sin^2\alpha\sin^2\beta + r^2\cos^2\alpha + r^2\sin^2\alpha\cos^2\beta$

$= r^2(\sin^2\alpha\,(\sin^2\beta + \cos^2\beta) + \cos^2\alpha)$

$= r^2(\sin^2\alpha \times (1) + \cos^2\alpha)$

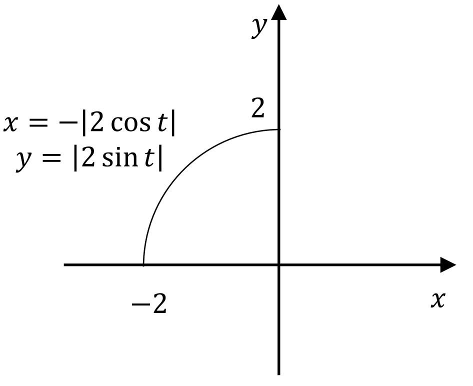
$= r^2(\sin^2\alpha + \cos^2\alpha)$

$= r^2$

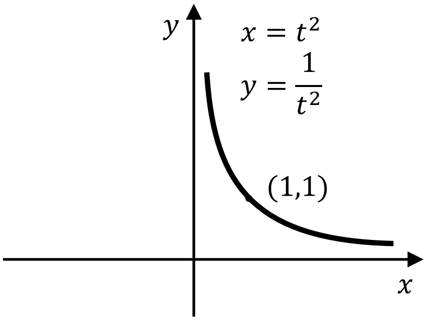
$\therefore x = r\sin\alpha\sin\beta,\, y = r\cos\alpha,\, z = r\sin\alpha\cos\beta$ satisfies

$x^2 + y^2 + z^2 = r^2$

11 $-2 \le x \le 0$ and $0 \le y \le 2$, so the top left quadrant of a circle of radius 2 centred at the origin.

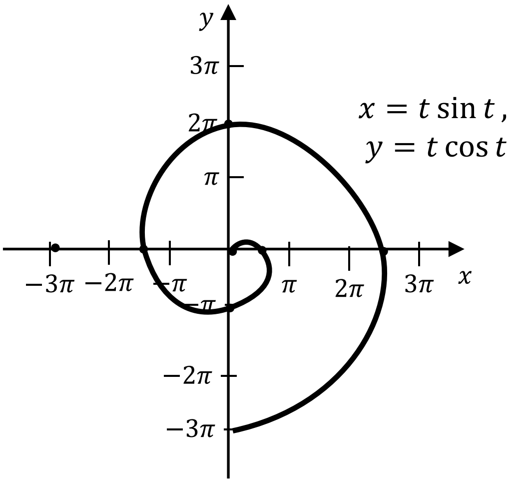


12 Since $t^2 \ge 0$, so $x > 0$ and $y > 0$. This is the top right branch of the hyperbola.



13 A clockwise Archimedean spiral.

t	x	y
0	0	0
$\frac{\pi}{2}$	$\frac{\pi}{2}$	0
π	0	$-\pi$
$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	0
2π	0	2π
3π	0	-3π



14

The radius of the new circle is $r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$.

For the point to move clockwise we swap sine and cosine, so $x = 2\sin\left(f(t)\right), y = 2\cos\left(f(t)\right)$.

To move the starting position to $\left(\sqrt{3}, 1\right)$, or $\left(2\sin\frac{\pi}{3}, 2\cos\frac{\pi}{3}\right)$ we use $t + \frac{\pi}{3}$.

$\therefore x = 2\sin\left(t + \frac{\pi}{3}\right), y = 2\cos\left(t + \frac{\pi}{3}\right)$

5.6 HARDER TWO DIMENSIONAL CURVES + 3D CURVES

In Lesson 6 we look at harder two dimensional curves and three dimensional curves. We cover:

- More Curves in Two Dimensions
  - Harder Curves
- Curves in Three Dimensions
  - Helixes
  - Harder Curves

HARDER TWO DIMENSIONAL CURVES

We saw a variety of common curves last lesson, but it is quite likely that you will come across unusual curves and have to sketch them, particularly late in the exam.

For unknown curves it can help to first work out the domain and range and lightly draw a box on the number plane as a guide for the sketch. The second step is to create a table of values to sketch part of the curve, while symmetry or a bit of thought can help finish off the rest of the sketch.

In the exercises we will also sketch some of the Lissajous Curves, of which the ABC logo is one. They are possibly beyond what would be tested in an exam as they take some time to sketch accurately.

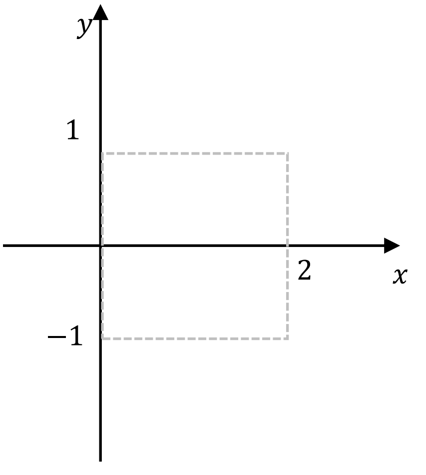
Example 1
Sketch $x = 2\sin^2 t\,, y = \cos t$

With this unknown curve we will start by considering the domain and range then use a table to plot some points.

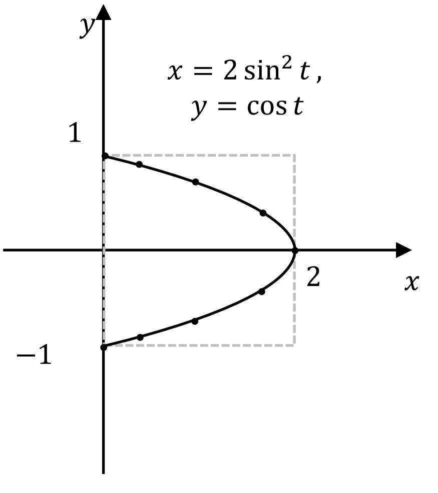
domain: $[0,2]$ since $0 \le \sin^2 t \le 1$
range: $[-1,1]$ since $-1 \le \cos t \le 1$

The sketch must fit within the square of side length 2 centred at $(1,0)$ as shown.

We then plot some positive values of t, making them multiples of π.



t	x	y
0	0	1
$\frac{\pi}{8}$	0.3	0.9
$\frac{\pi}{4}$	1	0.7
$\frac{3\pi}{8}$	1.7	0.4
$\frac{\pi}{2}$	2	0



Plot these points, then also reflect them over the x-axis as $\cos t$ can be positive or negative, then draw a smooth curve through the points, filling the box.

CURVES IN THREE DIMENSIONS

The parameterised curves in two dimensions that we have just looked at involve a point moving along a curve on the xy plane. We can visualise this as the point moving on a piece of paper on a desk. The point will often follow a circle, ellipse, parabola or hyperbola.

For easier examples of parameterised curves in three dimensions, we can think of the point moving around the xy plane, but also rising or falling above or below the desk. This elevation is the z value. Often the point will still be following a circle in the xy plane, but we could use an ellipse, parabola or hyperbola. The elevation will either be a multiple of the parameter, so it will rise steadily, or a function of sine or cosine so it rises and falls.

If we want to make the question harder then we can make the point follow a regular shape in the xz or yz plane, with the linear function the equation of x or y.

If you get stuck then it can help to focus on only two of the parameters, using them to draw a top view (xy plane), front view (xz plane) and side view (yz plane) like you would have done with solids in Year 7.

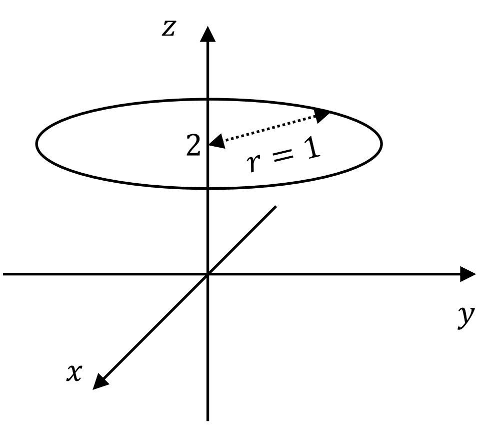
CIRCLES

Let's start with a circle in three dimensional space. We could use the same techniques for any of the shapes we have dealt with earlier.

Example 2

Sketch $x = \cos t\,, y = \sin t\,, z = 2$

We see that in the xy plane this is the unit circle, and its elevation is constant at $z = 2$, so it is the unit circle but raised 2 units.



HELIXES

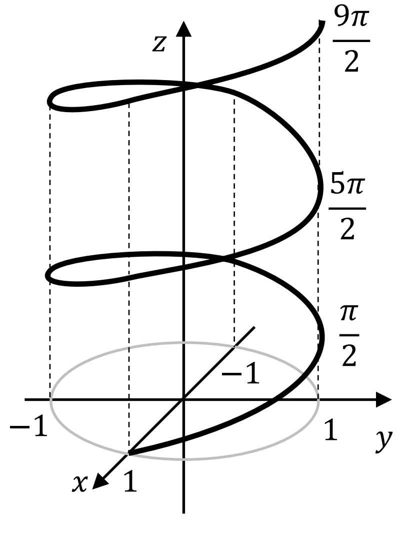
Now if the elevation is increasing as the point moves around the circle we will get a helix (think a coil or spring).

We will start by lightly drawing the unit circle on the xy plane, then test some values around the circle to find the elevation. This method will help us with some of the harder sketches to come.

Example 3

Sketch $x = \cos t\,, y = \sin t\,, z = t$ for $t \geq 0$.

As we move around the unit circle the elevation increases from 0 at a steady rate. Note that the parameter is used for the angle and elevation, which helps us plot it. Our initial point is at 1 on the x-axis, then as we pass above the y and x axes as we rotate anticlockwise we are at elevations of $\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ etc.



The example above has its axis along the z-axis, and if we look at the parameterised equations we can see that $z = \mathrm{t}$. We can make the helix have its axis along either of the other axes by letting $x = t$ or $y = t$. We can also make the helix turn the other way by swapping sine and cosine.

UNUSUAL CURVES

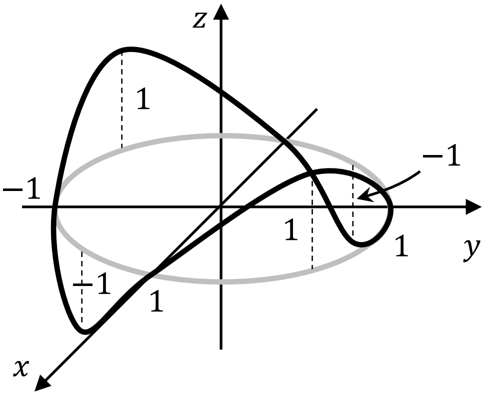
Now let's have a look at what happens if the z value varies up and down.

Example 4

Sketch $x = \cos t\,, y = \sin t\,, z = \sin 2t$

Similar to the last example, but the elevation completes 2 cycles of the sine curve t goes from 0 to 2π. The two peaks will occur at $t = \frac{\pi}{4}, \frac{5\pi}{4}$ and the two troughs at $t = \frac{3\pi}{4}, \frac{7\pi}{4}$.

The curve is shaped like a saddle.

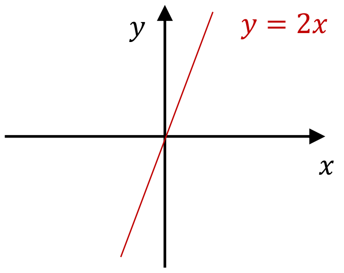


When we get an unusual curve and are not sure what it looks like, one method is to draw at least two of the two dimensional views, then combine them to create the three dimensional sketch.

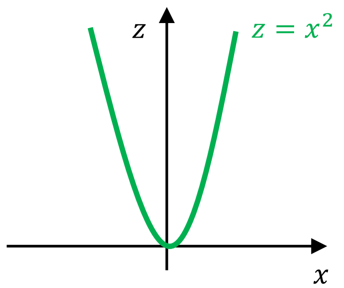
Example 5

Sketch $x = t, y = 2t, z = t^2$

Using $x = t, y = 2t$ we can rearrange to get $y = 2x$. This means that in the xy plane (so viewed from above) the curve is a straight line through the origin.

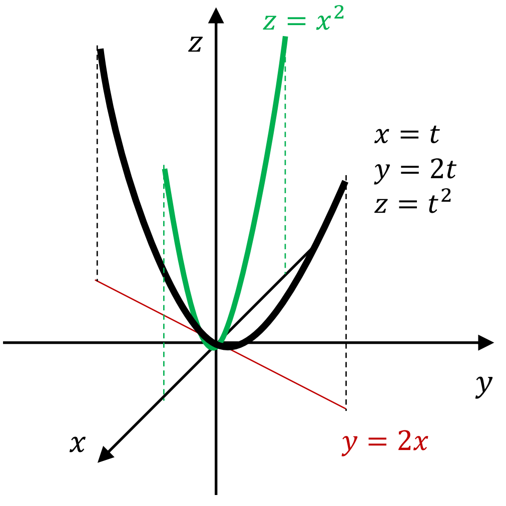


Using $x = t, z = t^2$ we can rearrange to get $z = x^2$. This means that in the xz plane (so viewed from the right) the curve is a concave up parabola with vertex at the origin.



So now we convert our two lines into three dimensional space. Remember that the first quadrant is now bottom right, so the line $y = 2x$ has a different look to what we just drew.

So the only curve that would like the line $y = 2x$ from above and a parabola from the right would be a parabola (shown in black) that is above the line $y = 2x$ as shown.



EXERCISE 5.6

BASIC

Sketch the following curves

1 $x = 0, y = \cos t, z = \sin t$

2 $x = \cos t, y = -1, z = \sin t$

3 $x = \sin t, y = \cos t, z = 1$

MEDIUM

4 $x = 2\cos^2 t, y = \sin t$

5 $x = 2\cos t, y = 1 + \sin^2 t$

6 $x = \sin t, y = \cos t, z = t$ for $t \geq 0$.

7 $x = \cos t, y = t, z = \sin t$ for $t \leq 0$.

8 $x = \sin t, y = \cos t, z = \sin t$

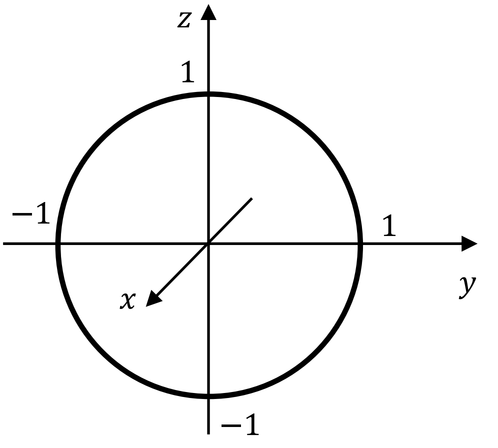
9 $x = \cos t, y = \sin t, z = \sin 3t$

CHALLENGING

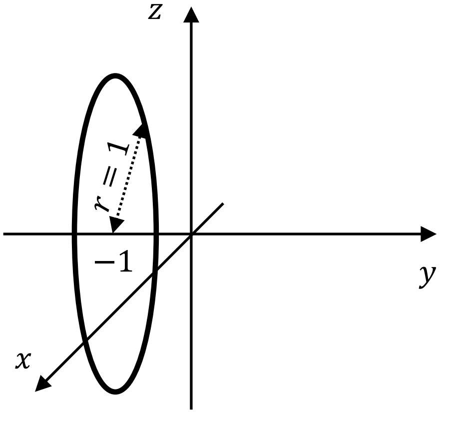
10 $x = \cos t, y = \sin 2t$

11 $x = t, y = t^2, z = t^2$

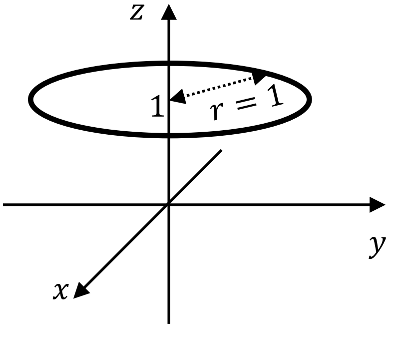
1 We see that in the yz plane this is the unit circle, with an x-value of zero, so it is a vertical unit circle.



2 We see that in the xz plane this is the unit circle, but with $y = -1$, so it is a vertical unit circle but moved 1 unit to the left.



3 Swapping $\cos t$ and $\sin t$ makes no difference to the final curve, so here we have a unit circle in the xy plane, but moved 1 unit up.

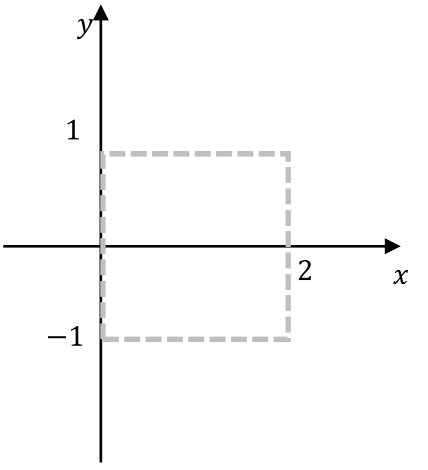


4 With this unknown curve we will start by considering the domain and range then use a table to plot some points. There no restrictions for t, though we note in the domain that $t^2 \geq 0$.

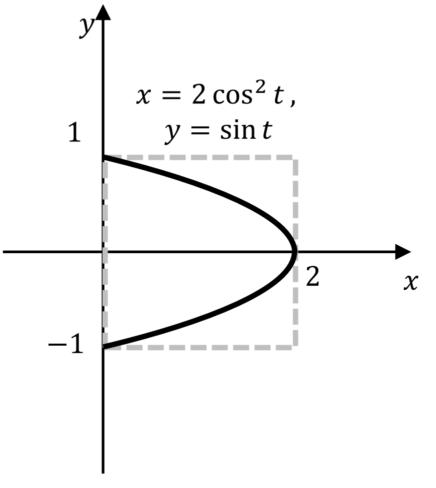
domain: $[0,2]$ since $0 \leq \cos^2 t \leq 1$
range: $[-1,1]$ since $-1 \leq \sin t \leq 1$

The sketch must fit within the square of side length 2 centred at $(1,0)$ as shown.

We then plot some positive values of t, making them multiples of π.



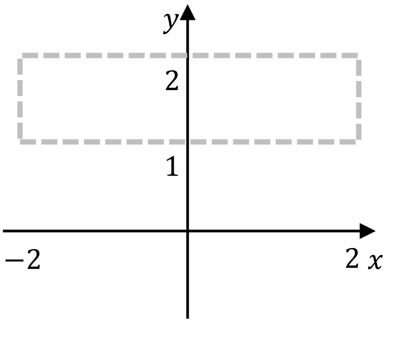
t	x	y
0	2	0
$\frac{\pi}{8}$	1.7	0.4
$\frac{\pi}{4}$	1	0.7
$\frac{3\pi}{8}$	0.3	0.9
$\frac{\pi}{2}$	0	1



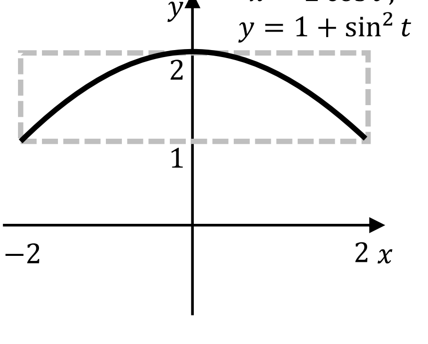
Plot these points, then also reflect them over the x-axis as $\sin t$ can be positive or negative, then draw a smooth curve through the points, filling the box.

5 domain: $[-2,2]$ since $-1 \leq \cos t \leq 1$
range: $[1,2]$ since $0 \leq \sin^2 t \leq 1$

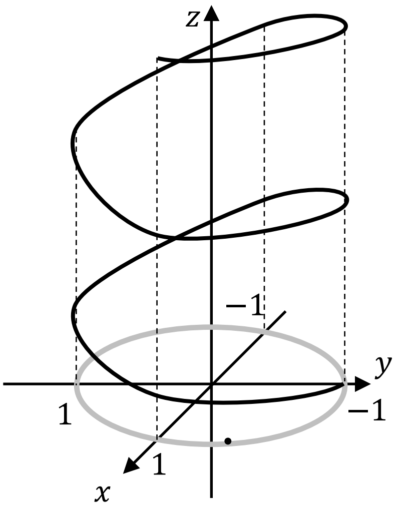
The sketch must fit within the rectangle as shown.



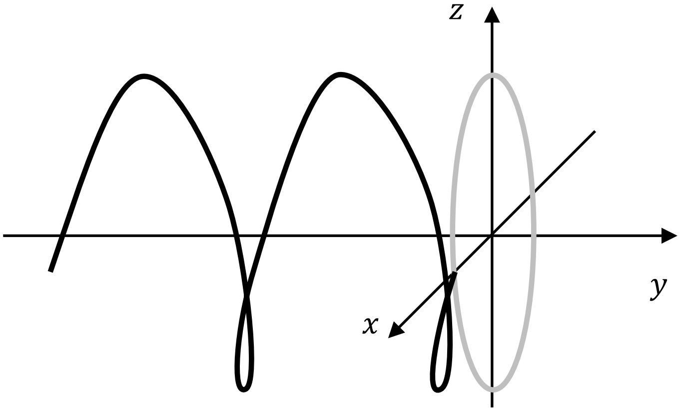
t	x	y
0	2	1
$\frac{\pi}{8}$	1.8	1.1
$\frac{\pi}{4}$	1.4	1.5
$\frac{3\pi}{8}$	0.8	1.9
$\frac{\pi}{2}$	0	2



6 Swapping sine and cosine causes the helix to start at $(0,1,0)$ an spiral in the opposite direction.

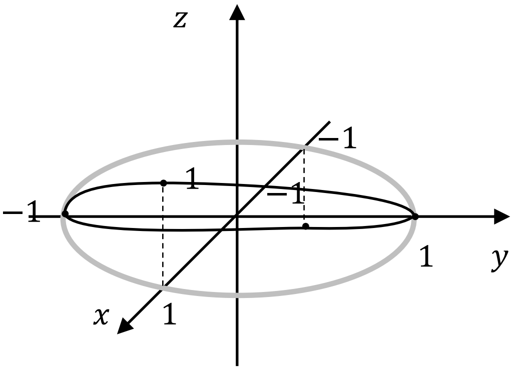


7 The helix spirals to the left around the y-axis.



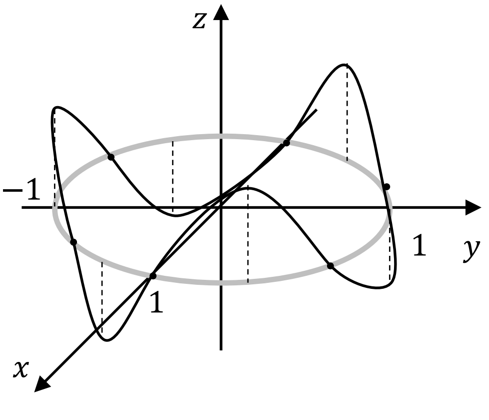
8 As we move around the unit circle the elevation is 0 where it crosses the y-axis, and 1 or -1 where it crosses the x-axis.

The shape is an ellipse tilted $45°$ about the y-axis. By Pythagoras we could see that $a = \sqrt{2}$ and $b = 1$. Very difficult to see from this perspective!



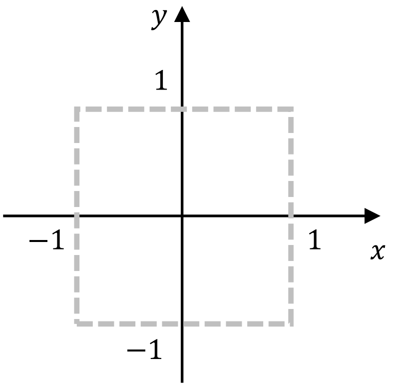
9 The elevation completes 3 cycles of the sine curve, with peaks at a height of 1 at $\frac{\pi}{6}, \frac{5\pi}{6}$ and $\frac{3\pi}{2}$, and troughs at $\frac{\pi}{2}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The two peaks will occur at $t = \frac{\pi}{4}, \frac{5\pi}{4}$ and the two troughs at $t = \frac{3\pi}{4}, \frac{7\pi}{4}$, and the elevation will be zero at $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

Viewed from the side we see the ABC logo.



10 domain: $[-1,1]$ since $-1 \leq \cos t \leq 1$

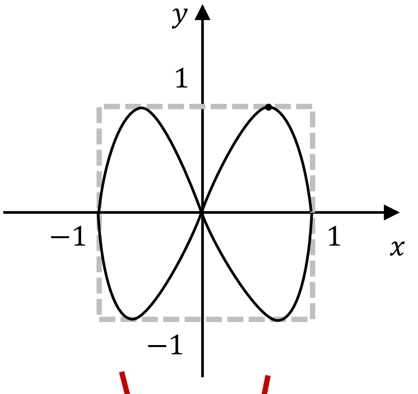
range: $[-1,1]$ since $0 \leq \sin 2t \leq 1$



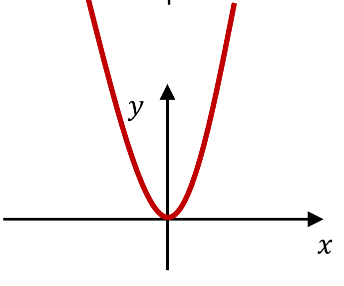
The sketch must fit within the rectangle as shown.

The 2 in $2t$ indicates that there will be two humps in the Lissajous curve. Let $y = 1 \rightarrow$ $\sin 2t = 1 \rightarrow \ t = \frac{\pi}{4} \ \rightarrow x = \cos\frac{\pi}{4} = 0.7$.

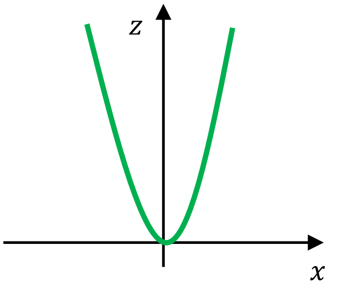
Arrange other humps symmetrically.



11 Using $x = \mathrm{t}, y = t^2$ we can rearrange to get $y = x^2$. This means that in the xy plane (so viewed from above) the curve is a parabola. Think of this like its shadow.

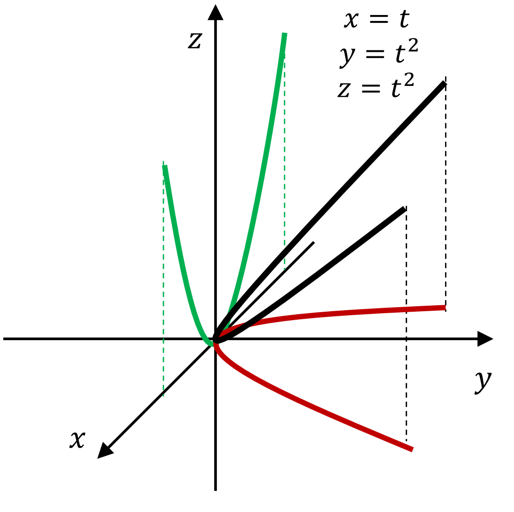


Using $x = t, z = t^2$ we can rearrange to get $\mathrm{z} = x^2$. This means that in the xz plane (so viewed from the right) the curve is a concave up parabola with vertex at the origin.



So now we convert our two lines into three dimensional space. Remember that the first quadrant is now bottom right, so the parabola looks concave right.

So the only curve that would like a parabola from above and from the right is a parabola with its axis along the line $y = z$, shown in black.



Chapter 6
Mechanics

MEX-M1 Applications of Calculus to Mechanics

In Mechanics we bring many of our calculus and algebra skills together at a high level, focusing on their practical applications.

LESSONS

Mechanics is covered in 8 lessons.

6.1 Motion in a Straight Line

6.2 Motion Without Resistance

6.3 Simple Harmonic Motion

6.4 Harder Simple Harmonic Motion

6.5 Horizontal Resisted Motion

6.6 Vertical Resisted Motion

6.7 Further Projectile Motion - Cartesian Equations

6.8 Projectile Motion with Resistance

REVISION QUESTIONS

In ‘1000 Extension 2 Revision Questions’, the revision book that goes with this textbook you will find the following questions matching this chapter:

- Revision Exercise 6

    100 graded questions on this topic only
- Revision Exercises 7 (Basic), 8 (Medium) and 9 (Challenging)

    Another 100 questions mixed through other topics for when you finish the course.

Don’t forget to do any questions from the exercises in this textbook you haven’t done.

6.1 MOTION IN A STRAIGHT LINE

CALCULUS IN MECHANICS

Mechanics involves applying calculus to solve real world problems, rather than being about physics.

Integration, and especially solving differential equations, is the most important skill you will need to achieve success in the topic, although your ability to manipulate algebra is also very important.

The differentiation involved is generally handled easily by most students, although there may be algebraic manipulation required there as well.

In Advanced and Extension 1 the equations of motion have always been functions of time, and we used simple differentiation or integration to convert between displacement, velocity and acceleration.

In Extension 2 questions, the equation of motion can be:

- Harder examples of displacement or velocity as functions of time
- Velocity as a function of displacement
- Acceleration as a function of displacement
- Acceleration as a function of velocity.

We will have to learn new integration techniques for equations of motion that are not functions of time.

Through the chapter you will see some common approaches to questions from lesson to lesson, even as the context changes.

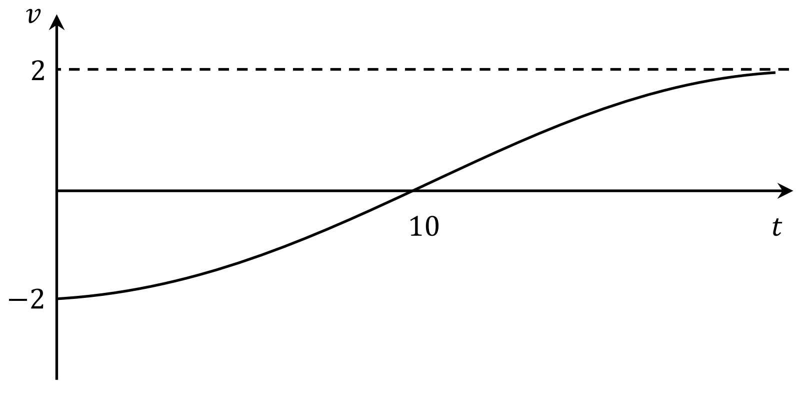
The syllabus also places more emphasis on using graphs to understand motion. Let’s start by looking at an example.

GRAPHING DISPLACEMENT, VELOCITY AND TIME

We start by using displacement-time and velocity-time graphs, to describe the motion and any forces involved.

Example 1

The graph below shows the velocity of a particle moving horizontally along the x-axis over time.



a What can we say about the initial velocity and acceleration?
b Can we tell the initial displacement without further information?
c When is the particle furthest to the left?
d When is the force at a minimum/maximum?
e When is the force directed to the left/right?
f What would the graph of the displacement look like as $t \to \infty$?

Solution

a The particle is initially at a velocity of 2 metres per second to the left (since the height is -2) with a slightly positive acceleration (since the curve then has a slightly positive gradient)
b No, as there could be an infinite number of displacement-time graphs where the gradient matches the height of this velocity-time graph.
c After 10 seconds when the particle is at rest. The particle moved left for the first ten seconds, then moves to the right.
d The force (and acceleration) is at a minimum at the start and end of the motion when the gradient is flattest, and at a maximum at about $t = 10$ when it is steepest (point of inflexion).
e The force is always to the right since the gradient is always positive, although it approaches zero as $t \to \infty$.
f Since velocity is approaching a constant value of 2, the gradient of the displacement would be approaching a straight line with gradient 2. In other words there would be a slant asymptote with gradient 2.

SOLVING DIFFERENTIAL EQUATIONS

Many of the equations of motion we work with in Mechanics are differential equations, and they often have features that are not obvious from the equation itself.

Looking at their slope fields can help explain why velocity or displacement approaches a limit (as an asymptote). We will most often see this as a terminal velocity.

The variable (velocity or displacement) cannot reach that limit, and so cannot equal any value on the other side.

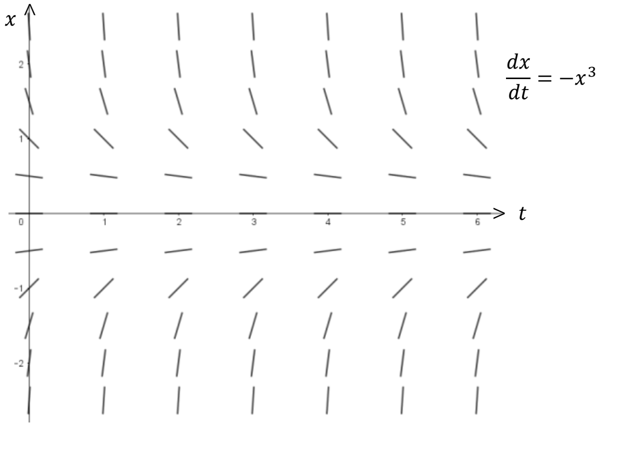
Most questions involve an initial condition which allows us to find which side of this limiting value we are on.

If we are given an initial condition, then in solving differential equations:

- We take the positive square root or the negative square root, not both.
- When integrating an integrand of the form $\frac{f'(x)}{f(x)}$ we get $\ln\big(f(x)\big)$ rather than $\ln\big|f(x)\big|$. Theoretically we could get $\ln\big(-f(x)\big)$ if $f(x) < 0$, but that situation rarely occurs.

In the next example we are given the equation of motion $\dot{x} = -x^3$ and told that the particle is initially at $x = 2$. During the solution we take the positive square root while ignoring the negative square root – why do we do this?

Looking at the slope field for $\frac{dx}{dt} = -x^3$ on the next slide, we can see a line of horizontal slopes along the horizontal axis, so at $x = 0$. This means that the curve will approach $x = 0$ as an asymptote. As a result the particle never reaches the origin or crosses from a positive displacement to a negative displacement, or vice versa.

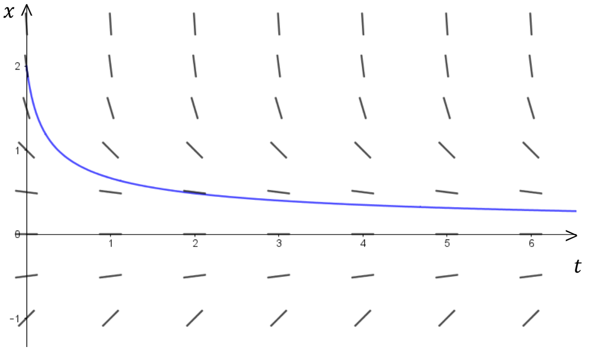


Given the slope field, an object starting with a positive displacement will approach the origin from the positive side, but never reach it. The horizontal axis is an asymptote.

Similarly an object starting with a negative displacement will approach the origin from the negative side, but never reach it.

If an object was to start at the origin it would never move from there – its graph would be a horizontal line along the axis.

In the next example the particle starts at $x = 2$. If we graph the slope field on geogebra and find the solution which passes through $(0,2)$ we get the curve at right.



As we will see in the example, this is the function $x = \frac{2}{\sqrt{8t+1}}$, not the function $x = -\frac{2}{\sqrt{8t+1}}$ (which would apply if the particle started at $x = -2$), or $x = \pm\frac{2}{\sqrt{8t+1}}$ which is two functions.

VELOCITY, DISPLACEMENT AND TIME

At times we want to know the velocity at a displacement rather than at a point in time. Let's look at how velocity, displacement and time interact. We need to use examples with constant or variable acceleration. In most Mechanics questions it helps to start by drawing a diagram, although for simple questions it is not needed.

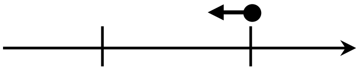
In the examples below see that using a definite integral is often more efficient than using an indefinite integral and solving for c. There will be a deeper explanation after the second example. We will use definite integrals as our default method in this topic, but HSC solutions usually use indefinite integrals.

Example 2

A particle has velocity given by $\dot{x} = -x^3$. If it is initially at $x = 2$, find the displacement of the particle after 1 second.

Solution

$\dot{x} = -x^3$



$$\frac{dx}{dt} = -x^3$$

$$\frac{dt}{dx} = -x^{-3}$$

$$t = -\int_2^x x^{-3}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{x^2}\right]_2^x$$

$$= \frac{1}{2x^2} - \frac{1}{8}$$

From time 0 to t the particle's displacement changes from 2 to x

Alternatively:

$$t = -\int x^{-3}\,dx$$

$$= \frac{1}{2x^2} + c$$

Let $t = 0, x = 2$

$$0 = \frac{1}{2 \times 2^2} + c$$

$$c = -\frac{1}{8}$$

$$\therefore t = \frac{1}{2x^2} - \frac{1}{8}$$

$$\frac{8t+1}{8} = \frac{1}{2x^2}$$

$$x^2 = \frac{4}{8t+1}$$

$$x = \frac{2}{\sqrt{8t+1}} \quad \text{positive root since } \frac{dt}{dx} = -\frac{1}{x^3} \to x \neq 0, \therefore\ x > 0 \text{ for all } t$$

Let $t = 1$

$$x = \frac{2}{\sqrt{8(1)+1}}$$

$$= \frac{2}{3}$$

The particle has a displacement of $\frac{2}{3}$ after 1 second

A NOTE ON INTEGRATION

In the last example we saw the following step:

$$\frac{dt}{dx} = -x^{-3}$$

$$t = -\int_2^x x^{-3}\,dx$$

This is how the official HSC solutions do similar steps and is perfectly fine, but what is happening behind the scenes? How can we integrate the LHS with respect to x from $x = 2$ to $x = x$ and seemingly ignore the limits? What really happened and how does it work? Let's have a look.

$$\frac{dt}{dx} = -x^{-3}$$

1. Separate the variables.

$$dt = -x^{-3}dx$$

2. Integrate each side, using the start and end values as the limits. Time changes from 0 to t while displacement changes from 2 to x.

$$\int_0^t dt = -\int_2^x x^{-3}\,dx$$

3. Take the primitive of the LHS.
This will always give us the variable, in this case t.

$$\Big[t\Big]_0^t = -\int_2^x x^{-3}\,dx$$

4. Substituting the limits we get the change in the time, $t - 0$, which equals the variable t. We are actually getting the change in time rather than t itself though in this case they are the same since we started from $t = 0$.

$$t - 0 = -\int_2^x x^{-3}\,dx$$

$$t = -\int_2^x x^{-3}\,dx$$

If the initial value of the variable on the LHS was not 0, then when we integrated $\frac{dt}{dx}$ we would have $t - t_0$ rather than t on the LHS.

ACCELERATION AND VELOCITY

When velocity and acceleration are functions of time then we can simply differentiate or integrate to convert from one to the other.

So we have already seen that $\boxed{\ddot{x} = \dfrac{dv}{dt}}$

If we have velocity as a function of displacement we have two ways to find acceleration, each useful in different situations.

$$\boxed{\ddot{x} = v\frac{dv}{dx} \quad \text{or} \quad \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)}$$

When we need to find acceleration from velocity, use:

$\frac{dv}{dt}$ if velocity is given as a function of t

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ if we are given v^2 as a function of x,

or v as the square root of a function of x

$v\frac{dv}{dx}$ if velocity is any other function of x

Proof 3

Prove $\ddot{x} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

Solution

$$\begin{aligned}
\ddot{x} &= \frac{d^2x}{dt^2} \\
&= \frac{d}{dt}\left(\frac{dx}{dt}\right) \\
&= \frac{d}{dt}(v) \\
&= \frac{dv}{dt} && (1) \\
&= \frac{dv}{dx} \times \frac{dx}{dt} \\
&= \frac{dv}{dx} \times v && (2) \\
&= \frac{dv}{dx} \times \frac{d}{dv}\left(\frac{1}{2}v^2\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}v^2\right) && (3)
\end{aligned}$$

$$\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) \quad \text{from } (1), (2), (3)$$

Example 4

The velocity of a particle is given by $\dot{x} = x^2 + 1$ metres per second. What is the acceleration when the particle is at $x = 2$?

Solution

$$\begin{aligned}\ddot{x} &= v\frac{dv}{dx} \\ &= (x^2+1)\times(2x) \\ &= 2x^3 + 2x\end{aligned}$$

Alternatively:

$$\begin{aligned}\ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\ &= \frac{1}{2}\times\frac{d}{dx}(x^4+2x^2+1) \\ &= \frac{1}{2}(4x^3+4x) \\ &= 2x^3+2x\end{aligned}$$

Let $x = 2$

$$\begin{aligned}\ddot{x} &= 2(2)\left((2)^2+1\right) \\ &= 20\ \text{ms}^{-2}\end{aligned}$$

Example 5

The acceleration of a particle is given by $\ddot{x} = -e^{-2x}$ metres per second squared.

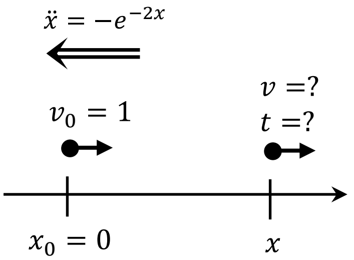
It is initially at the origin with a velocity of 1 metre per second.

i Show that $\dot{x} = e^{-x}$

ii Hence show that $x = \ln(t+1)$

Solution

i



$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -e^{-2x}$$

$$\frac{1}{2}v^2 - \frac{1}{2}(1)^2 = -\int_0^x e^{-2x}\,dx$$

$$\frac{1}{2}v^2 - \frac{1}{2} = \frac{1}{2}\left[e^{-2x}\right]_0^x$$

$$v^2 - 1 = e^{-2x} - 1$$

$$v^2 = e^{-2x}$$

$$\dot{x} = e^{-x} \qquad \text{positive root since } \dot{x} = 1 \text{ when } x = 0$$

Alternatively???

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}(e^{-x})^2\right)$$

$$= \frac{1}{2} \times \frac{d}{dx}(e^{-2x})$$

$$= -e^{-2x}$$

$$\therefore \dot{x} = e^{-x}$$

Note that this alternative has only proved that $\dot{x} = e^{-x}$ is a possible solution, not that it is the only possible solution. Has it answered the question??

ii

$$\frac{dx}{dt} = e^{-x}$$

$$\frac{dt}{dx} = e^{x}$$

$$t = \left[e^{x}\right]_0^x$$

$$t = e^x - 1$$

$$e^x = t + 1$$

$$x = \ln(t+1)$$

No absolute value sign is needed when the term is positive

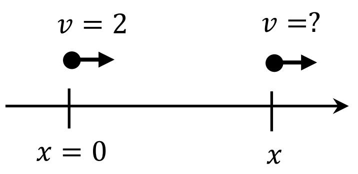
Example 6

The acceleration of a particle is given by $\ddot{x} = 4v^2 + 2v$. If the particle has a velocity of $2\ \mathrm{ms}^{-1}$ at the origin, find an expression for the velocity in terms of displacement.

Solution

$$\ddot{x} = 4v^2 + 2v$$

$$\Longrightarrow$$



$$v\frac{dv}{dx} = 4v^2 + 2v$$

We have acceleration as a function of velocity, which we most commonly see with resisted motion.

In these cases we replace acceleration with $v\frac{dv}{dx}$.

$$\frac{dv}{dx} = 4v + 2$$

$$\frac{dx}{dv} = \frac{1}{4v + 2}$$

$$x = \frac{1}{2}\int_2^v \frac{1}{2v + 1}\,dv$$

$$= \frac{1}{4}\Big[\ln(2v + 1)\Big]_2^v$$

The particle is initially moving to the right, and so $a \geq 0$ so the velocity is positive and will continue to increase.

$$= \frac{1}{4}\Big(\ln(2v + 1) - \ln 5\Big)$$

$$4x = \ln(2v + 1) - \ln 5$$

$$\ln(2v + 1) = \ln 5 + 4x$$

$$2v + 1 = e^{\ln 5 + 4x}$$

$$= 5e^{4x}$$

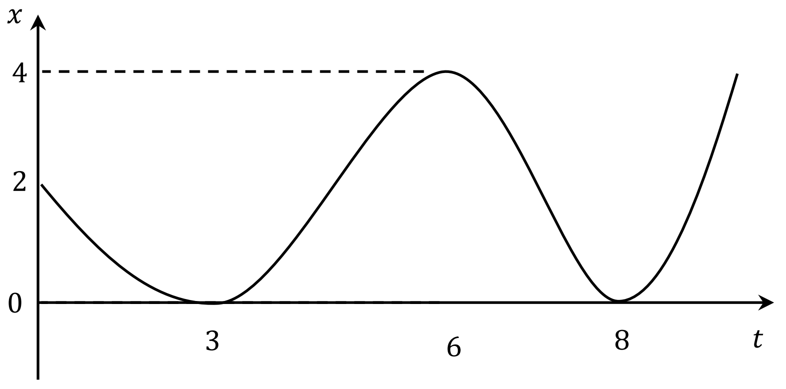
$$2v = 5e^{4x} - 1$$

$$v = \frac{5e^{4x} - 1}{2}$$

EXERCISE 6.1

BASIC

1 The graph below shows the displacement of a particle moving horizontally along the x-axis over time.

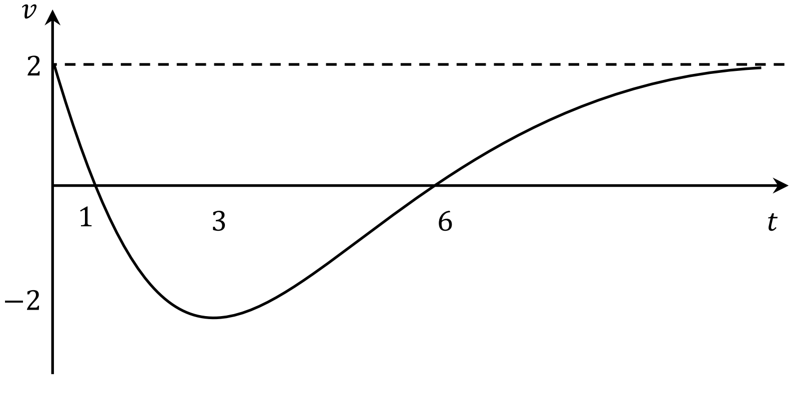


a What can we determine about the initial displacement, velocity and acceleration of the particle?
b When does the resultant force on the particle equal zero?
c When is the force directed to the right?

For the rest of the question assume that the displacement function is a polynomial of degree 4.

d What are the degrees of the functions of velocity and acceleration?
e How many times is the particle at the origin?
f How many times is the particle at rest?

2 The graph below shows the velocity of a particle moving horizontally along the x-axis over time.



a What is the initial velocity and acceleration of the particle?
b Can we tell the initial displacement without further information?
c When is the particle furthest to the left?
d When is the force at a minimum/maximum?
e When is the force directed to the left/right?
f What would the graph of the displacement of the particle look like as $t \to \infty$?

3 Prove $\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

4 The velocity of a particle is given by $\dot{x} = 2x^2 + 3$ metres per second. What is the acceleration when the particle is at $x = 1$?

5 The velocity of a particle, in metres per second, is given by $v = x^2 + 2$, where x is its displacement in metres from the origin. What is the acceleration of the particle at $x = 1$?

6 At time t the displacement, x, of a particle satisfies $t = 4 - e^{-2x}$.
Find the acceleration of the particle as function of x.

MEDIUM

7 A particle moves along a straight line with displacement x metre and velocity v metres per second. The acceleration of the particle is given by $\ddot{x} = 2 - e^{-\frac{x}{2}}$. Given that $v = 4$ when $x = 0$, express v^2 in terms of x.

8 A particle moves on the x-axis with velocity v. The particle is initially at rest at $x = 1$. Its acceleration is given by $\ddot{x} = x + 4$. Find the speed of the particle at $x = 2$.

9 A particle has velocity given by $\dot{x} = -x^2$. If it is initially at $x = 2$, find the displacement of the particle after 1 second.

10 The acceleration of a particle is given by $\ddot{x} = v^2 + v$. If the particle has a velocity of $2\ \mathrm{ms}^{-1}$ at the origin, find an expression for the velocity in terms of displacement.

11 A particle moves in a straight line. At time t seconds the particle has a displacement of x metres, a velocity of v metres per second and acceleration of a metres per second squared. Initially the particle has displacement $0\ \mathrm{m}$ and velocity of $2\ \mathrm{ms}^{-1}$. The acceleration is given by $a = -2e^{-x}$. The velocity of the particle is always positive.

i Show that $v = 2e^{-\frac{x}{2}}$

ii Find an expression for x as a function of t.

12 A particle is moving horizontally. Initially the particle is at the origin O moving with velocity $1\ \mathrm{ms}^{-1}$. The acceleration of the particle is given by $\ddot{x} = x - 1$, where x is its displacement at time t.

i Show that the velocity of the particle is given by $\dot{x} = 1 - x$.

ii Find an expression for x as a function of t.

iii Find the limiting position of the particle.

13 A particle is moving so that $\ddot{x} = 18x^3 + 27x^2 + 9x$. Initially $x = -2$ and the velocity, v, is -6.

i Show that $v^2 = 9x^2(1+x)^2$

ii Hence, or otherwise, show that $\int \frac{1}{x(1+x)}\,dx = -3t$

iii It can be shown that for some constant c, $\log_e\left(1+\frac{1}{x}\right) = 3t + c$ (Do NOT prove this)

Using this equation and the initial conditions, find x as a function of t.

14 The acceleration of a particle moving along a straight path is given by $\ddot{x} = -\frac{e^x + 1}{e^{2x}}$ where x is in metres. Initially the particle is at the origin with a velocity of $2\ \mathrm{ms}^{-1}$, and its velocity remains positive.

i Show that $v = e^{-x} + 1$

ii Find the equation of the displacement, x, in terms of t.

15 **i** Prove $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

ii Prove $\frac{d}{dx}(x\ln x) = 1 + \ln x$

iii The acceleration of a particle moving in a straight line and starting from rest at $1\ \mathrm{cm}$ on the positive side of the origin is given by $\frac{d^2x}{dt^2} = 1 + \ln x$

(α) Derive the equation relating v and x

(β) Hence, evaluate v when $x = e^2$.

1 a The particle is initially 2 metres to the right of the origin, moving to the left (since the gradient is negative), with positive acceleration (since it is concave up).

b After approximately 4.5 and 7 seconds at the points of inflexion, since the curve is neither concave up or down so the net force is zero.

c From $t = 0$ to approx. $t = 4.5$ seconds, and from $t = 7$ onwards since the curve is concave up.

d The velocity is of degree 3 and acceleration of degree 2, since they are the first and second derivative of displacement.

e Only the two times shown, as the particle will continue to move to the right.

f Only the three turning points shown, as the particle will continue moving to the right.

2 a The particle is initially at a velocity of 2 metres per second to the right (since the height is 2) with negative acceleration (since the curve then has a negative gradient)

b No, as there could be an infinite number of displacement-time graphs where the gradient matches the height of this velocity-time graph.

c After 6 seconds when the particle is at rest, after having negative velocity.

d The force (and acceleration) is at a minimum at the turning point (approx. $t = 3$), and at a maximum at about $t = 0$ when it is steepest.

e The force is always to the right from $t = 0$ to $t = 3$ where the gradient of v is negative and to the right for $t > 3$ as the velocity increases.

f Since velocity is approaching a constant value of 2, the gradient of the displacement would be approaching a straight line with gradient 2.

3

$$\begin{aligned}
\ddot{x} &= \frac{d^2x}{dt^2} \\
&= \frac{d}{dt}\left(\frac{dx}{dt}\right) \\
&= \frac{d}{dt}(v) \\
&= \frac{dv}{dt} && (1) \\
&= \frac{dv}{dx} \times \frac{dx}{dt} \\
&= \frac{dv}{dx} \times v \;\rightarrow= v\frac{dv}{dx} && (2) \\
&= \frac{dv}{dx} \times \frac{d}{dv}\left(\frac{1}{2}v^2\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}v^2\right) && (3)
\end{aligned}$$

$$\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) \quad \text{from } (1), (2), (3)$$

4

$$\begin{aligned}
\ddot{x} &= v\frac{dv}{dx} \\
&= (2x^2 + 3) \times (4x) \\
&= 8x^3 + 12x
\end{aligned}$$

Let $x = 1$

$$\begin{aligned}
\ddot{x} &= 8(1)^3 + 12(1) \\
&= 20 \text{ ms}^{-2}
\end{aligned}$$

5

$$\begin{aligned}
\ddot{x} &= v\frac{dv}{dx} \\
&= (x^2 + 2)(2x) \\
&= (1^2 + 2)\big(2(1)\big) \\
&= 6 \text{ ms}^{-2}
\end{aligned}$$

6

$$\begin{aligned}
t &= 4 - e^{-2x} \\
\frac{dt}{dx} &= 2e^{-2x} \\
v &= \frac{dx}{dt} \\
&= \frac{e^{2x}}{2} \\
a &= v\frac{dv}{dx} \\
&= \frac{e^{2x}}{2} \times e^{2x} \\
&= \frac{e^{4x}}{2}
\end{aligned}$$

7

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= 2 - e^{-\frac{x}{2}} \\
\frac{1}{2}v^2 - \frac{1}{2}(4)^2 &= \int_0^x \left(2 - e^{-\frac{x}{2}}\right) dx \\
\frac{1}{2}v^2 - 8 &= \left[2x + 2e^{-\frac{x}{2}}\right]_0^x \\
\frac{1}{2}v^2 &= \left(2x + 2e^{-\frac{x}{2}}\right) - (0 + 2) + 8 \\
&= 2x^{-\frac{x}{2}} + 2x + 6 \\
v^2 &= 4e^{-\frac{x}{2}} + 4x + 12
\end{aligned}$$

8

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= x + 4 \\
\frac{1}{2}v^2 &= \int_1^x (x + 4)\, dx \\
&= \left[\frac{x^2}{2} + 4x\right]_1^x \\
v^2 &= 2\left(\left(\frac{x^2}{2} + 4x\right) - \left(\frac{1}{2} + 4\right)\right) \\
&= x^2 + 8x - 9
\end{aligned}$$

when $x = 2$

$$\begin{aligned}
v^2 &= 2^2 + 8(2) - 9 \\
&= 11
\end{aligned}$$

$\therefore$ speed $= \sqrt{11}$

9

$$\begin{aligned}
\frac{dx}{dt} &= -x^2 \\
\frac{dt}{dx} &= -x^{-2} \\
t &= -\int_2^x x^{-2}\, dx \\
&= \left[\frac{1}{x}\right]_2^x \\
&= \frac{1}{x} - \frac{1}{2} \\
\frac{2t + 1}{2} &= \frac{1}{x} \\
x &= \frac{2}{2t + 1}
\end{aligned}$$

Let $t = 1$

$$x = \frac{2}{2(1) + 1} = \frac{2}{3}$$

10

$$\begin{aligned}
v\frac{dv}{dx} &= v^2 + v \\
\frac{dv}{dx} &= v + 1 \\
\frac{dx}{dv} &= \frac{1}{v + 1} \\
x &= \int_2^v \frac{1}{v + 1}\, dv \\
&= \Big[\ln(v + 1)\Big]_2^v \\
&= \ln(v + 1) - \ln 3 \\
\ln(v + 1) &= x - \ln 3 \\
v + 1 &= e^{x - \ln 3} \\
&= 3e^x \\
v &= 3e^x - 1
\end{aligned}$$

11 **i**

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= -2e^{-x} \\
\frac{1}{2}v^2 - \frac{1}{2}(2)^2 &= -2\int_0^x e^{-x}\, dx \\
\frac{1}{2}v^2 - 2 &= 2\left[e^{-x}\right]_0^x \\
&= 2(e^{-x} - 1) \\
\therefore \frac{1}{2}\, v^2 &= 2e^{-x} \\
v^2 &= 4e^{-x} \\
\therefore v &= 2e^{-\frac{x}{2}}
\end{aligned}$$

positive root since $v = 2$ when $x = 0$

ii

$$\begin{aligned}
\therefore \frac{dx}{dt} &= 2e^{-\frac{x}{2}} \\
\frac{dt}{dx} &= \frac{1}{2}e^{\frac{x}{2}} \\
t &= \frac{1}{2}\int_0^x e^{\frac{x}{2}}\, dx \\
&= \left[e^{\frac{x}{2}}\right]_0^x \\
&= e^{\frac{x}{2}} - 1 \\
e^{\frac{x}{2}} &= t + 1 \\
\frac{x}{2} &= \ln(t + 1) \\
x &= 2\ln(t + 1)
\end{aligned}$$

12 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x - 1$$
$$\frac{1}{2}v^2 - \frac{1}{2}(1)^2 = \int_0^x (x-1)\,dx$$
$$\frac{1}{2}v^2 - \frac{1}{2} = \left[\frac{x^2}{2} - x\right]_0^x$$
$$\frac{1}{2}v^2 - \frac{1}{2} = \frac{x^2}{2} - x$$
$$\frac{1}{2}v^2 = \frac{x^2}{2} - x - \frac{1}{2}$$
$$v^2 = x^2 - 2x + 1$$
$$= (x-1)^2$$
$$v = -(x-1) = 1 - x$$

negative root since $\dot{x} = 1$ when $x = 0$

ii

$$\frac{dx}{dt} = 1 - x$$
$$\frac{dt}{dx} = \frac{1}{1-x}$$
$$t = \int_0^x \frac{dx}{1-x}$$
$$= -\Big[\ln(1-x)\Big]_0^x$$
$$= -\ln(1-x)$$
$$e^{-t} = 1 - x$$
$$x = 1 - e^{-t}$$

iii

as $t \to \infty$ $e^{-t} \to 0$ $\therefore x \to 1$

13 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 18x^3 + 27x^2 + 9x$$
$$\frac{1}{2}v^2 - \frac{1}{2}(-6)^2 = \int_{-2}^x (18x^3 + 27x^2 + 9x)\,dx$$
$$\frac{1}{2}v^2 - 18 = \left[\frac{9}{2}x^4 + 9x^3 + \frac{9}{2}x^2\right]_{-2}^x$$
$$v^2 - 36 = (9x^4 + 18x^3 + 9x^2) - (144 - 144 + 36)$$
$$v^2 = 9x^4 + 18x^3 + 9x^2$$
$$= 9x^2(x^2 + 2x + 1)$$
$$= 9x^2(x+1)^2$$

ii

$\therefore v = -3x(1+x)$

(negative root since when $x = -2$ $v = -6$)

$$\frac{dx}{dt} = -3x(1+x)$$
$$\frac{dt}{dx} = -\frac{1}{3} \times \frac{1}{x(1+x)}$$
$$t = -\frac{1}{3}\int \frac{1}{x(1+x)}\,dx$$
$$\therefore \int \frac{1}{x(1+x)}\,dx = -3t$$

iii

$$\log_e\left(1 + \frac{1}{x}\right) = 3t + c$$

at $t = 0$ $x = -2$:

$$\log_e\left(1 - \frac{1}{2}\right) = 0 + c$$
$$c = -\log_e 2$$
$$\log_e\left(1 + \frac{1}{x}\right) = 3t - \log_e 2$$
$$1 + \frac{1}{x} = e^{3t - \log_e 2}$$
$$1 + \frac{1}{x} = \frac{e^{3t}}{2}$$
$$\frac{1}{x} = \frac{e^{3t} - 2}{2}$$
$$x = \frac{2}{e^{3t} - 2}$$

14 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{e^x+1}{e^{2x}}$$
$$\frac{1}{2}v^2 - \frac{1}{2}(2)^2 = -\int_0^x \frac{e^x+1}{e^{2x}}\,dx$$
$$v^2 - 4 = -2\int_0^x \left(e^{-x}+e^{-2x}\right)dx$$
$$= \left[2e^{-x}+e^{-2x}\right]_0^x$$
$$= \left(2e^{-x}+e^{-2x}\right) - \left(2+1\right)$$
$$v^2 = e^{-2x}+2e^{x}+1$$
$$\therefore v = e^{-x}+1$$

positive root since $v = 2$ when $x = 0$.

ii

$$\frac{dx}{dt} = e^{-x}+1$$
$$= \frac{1+e^x}{e^x}$$
$$\frac{dt}{dx} = \frac{e^x}{1+e^x}$$
$$t = \int_0^x \frac{e^x}{1+e^x}\,dx$$
$$= \left[\ln(e^x+1)\right]_0^x$$
$$= \ln(e^x+1) - \ln 2$$
$$\ln(e^x+1) = t + \ln 2$$
$$e^x + 1 = e^{t+\ln 2}$$
$$e^x + 1 = 2e^t$$
$$e^x = 2e^t - 1$$
$$x = \ln(2e^t - 1)$$

15 **i**

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$
$$= \frac{d}{dt}(v)$$
$$= \frac{dv}{dx}\times\frac{dx}{dt}$$
$$= \frac{dv}{dx}\times v$$
$$= \frac{dv}{dx}\times\frac{d}{dv}\left(\frac{1}{2}v^2\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

ii

$$\frac{d}{dx}(x\ln x) = \ln x \times 1 + x \times \frac{1}{x}$$
$$= \ln x + 1$$

iii

α

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1 + \ln x$$
$$\frac{1}{2}v^2 = \int_1^x (1+\ln x)\,dx$$
$$v^2 = 2\left[x\ln x\right]_1^x$$
$$= 2x\ln x$$

β

Let $x = e^2$

$$v^2 = 2(e^2)\ln(e^2)$$
$$= 4e^2$$
$$v = 2e$$

Positive root given initial conditions

FURTHER MOTION IN A STRAIGHT LINE

In the last lesson we looked at questions where particles were moving in a straight line and the equations of motion were given in the question. In this lesson we look at questions where we are given information about force or acceleration and we have to find the equation of motion. None of the questions in this lesson involve resistance to motion, though it is important in most of the remaining lessons in this topic.

HINTS FOR SUCCESS

Many of the Mechanics questions in the rest of this topic seem more difficult than they end up being, mainly due to the large number of different constants and variables in the equations, and the length of the questions.

- There is commonly a mixture of lower and upper case pronumerals with Greek letters, which is particularly confusing until you become familiar with it.
- There are often square roots, natural logarithms or inverse tangents to make it look more complex.

To successfully solve Mechanics questions:

- Look carefully to see which are the variables and which are the constants, as this can help break down the question into more manageable chunks.
- Remember in exams that if you don't get part (i) you can still use the result you were given to prove in part (ii) etc - this is particularly important in Mechanics.
- Possibly the greatest difficulty with solving mechanics questions comes from choosing whether acceleration needs to be replaced with $\frac{dv}{dt}$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ or $v\frac{dv}{dx}$.

The flowchart on the next page shows how we choose the correct version when integrating from acceleration, and are different to the choices we use when differentiating from velocity to find acceleration. With practice it becomes easy to select the correct version.

Equation of Motion

- **Acceleration as a function of time** $a = f(t)$
  - Sometimes used in Motion without Resistance
  - sub $a = \dfrac{dv}{dt}$
- **Acceleration as a function of displacement** $a = f(x)$
  - Often used for Motion without Resistance
  - sub $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$
- **Acceleration as a function of velocity** $a = f(v)$
  - All Resisted Motion is a function of velocity. Also some questions on Motion without Resistance are functions of velocity.
    - if proof involves displacement and velocity → sub $a = v\dfrac{dv}{dx}$
    - if proof involves time and velocity → sub $a = \dfrac{dv}{dt}$
    - if proof involves time and displacement → sub $a = \dfrac{dv}{dt}$ → (find velocity as a function of time first then integrate for displacement as a function of time)

UNIT OF FORCE

We measure force in Newtons (N). One Newton is equal to a mass of 1 kilogram being accelerated at 1 ms^{-2}. To give you a rough idea, a mandarin has a mass of about 100 grams, so its weight on Earth is about 1 Newton. So to imagine a force of 50 Newtons we can either:

- imagine the weight of 50 mandarins
- divide 50 by 10 to get a mass of 5 kg and imagine something of the same mass.

NEWTON'S LAWS

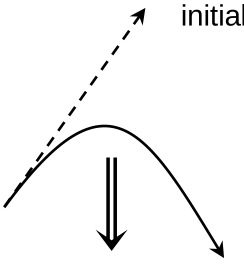
Newton's Laws underpin much of our work in the rest of this chapter, and much of the work we have already done in projectile motion. A knowledge of the laws allows us to do unusual questions. As stated in the syllabus Newton's Law are:

1. Unless acted upon by a resultant force, a body remains at rest or in uniform motion in a straight line.
2. The acceleration of a body is proportional to the resultant force that acts on the body and inversely proportional to the mass of the body.
3. For every action, there is an equal and opposite reaction.

NEWTON'S FIRST LAW

The first law is also known as inertia - a particle will remain stationary if it is at rest or keep moving in a straight line at a constant velocity if it is already moving, unless there is a resultant force.

So with projectile motion the particle would continue in a straight line at the initial velocity, but the resultant force (gravity) bends the trajectory downwards.



So if we want the velocity to stay constant (like with terminal velocity later in the chapter) we have to make the resultant force zero.

NEWTON'S SECOND LAW

The second law gives us the formula $F = ma$, or in our case $m\ddot{x} = F$ where F is the resultant force. This is a common starting point for solving questions, which we rearrange to make $\ddot{x}$ the subject.

In this course we deal with constant mass, but theoretically we could be given questions with variable mass – say a rocket losing mass as its burns fuel. For a given force the rocket will have greater acceleration as the mass decreases.

NEWTON'S THIRD LAW

The third law leads us into the concepts of Normal forces, Tension forces and friction, which are often needed in more involved questions. These three types of forces are adjustable - they create an equal and opposite force which can increase from zero up to some natural limit before something breaks.

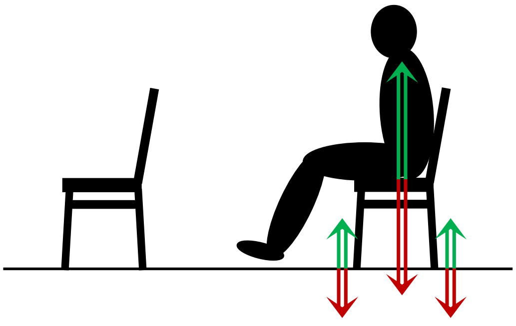
Let's look at them in more detail.

NORMAL FORCE - AN ADJUSTABLE PUSHING FORCE

Looking round your classroom you might see an empty chair. Let's pretend for the moment that it has no mass, so exerts no force on the floor beneath.

When someone sits on the chair their weight presses down on the chair, matched by an equal normal force from the chair - otherwise they would fall to the floor. Their weight transfers through the four legs of the chair, with each leg receiving a matching normal force from the floor so the chair does not fall through the floor.

Normal forces are variable, up to the breaking strength of the object. The chair or the floor can exert any required force as lighter or heavier people sit on it, but will eventually break if there is enough force.



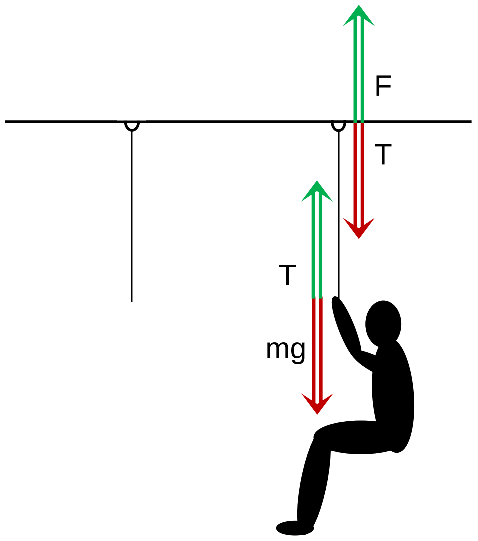
The top of the seat exerts a normal force equal to the gravitational force (weight) of the person.

The floor is also exerting a normal force on each of the legs of the chair.

The force is at right angles to the surface, which is why it is called a Normal force - just like normals to a curve, which are lines perpendicular to the tangent.

TENSION - AN ADJUSTABLE PULLING FORCE

When two objects are joined together by a rope or cable and a force is trying to move them apart, then the force transmitted through the rope to keep them stationary is a tension force. The rope exerts an equal tension force on both objects - the tension pulling in each direction is equal. A single rope must have equal tension throughout.



The rope exerts a tension force on the person and on the hook in the ceiling, equal to the gravitational force (weight) of the person.

The ceiling is also applying an equal force on the hook.

Imagine a rope (of no mass) attached to a hook in the ceiling. Given our theoretical rope has no mass then there will be no tension - it will be loose or floppy.

Now get a person to hang off it - the tension in the rope will increase until it matches the new gravitational force (weight) of the person. It will also pull down on the hook with that same tension, so there is some type of force F of equal magnitude holding the rope up. Depending on how the hook is attached this could be compression or tension.

If it was the same person sitting on our chair from the previous page, then the Tension force in the rope would be the same magnitude as the Normal force from the chair.

Tension forces are variable, up to the breaking strength of the object. A rope can exert any required tension force as lighter or heavier people hang off it, but will eventually break if there is enough force.

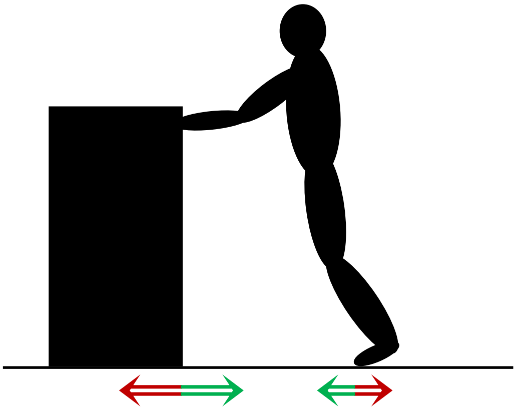
We could have tension forces in a chair (imagine two students pulling on each end of a chair), but we cannot have normal forces with a rope as it would collapse in a pile (imagine pushing the two ends of a rope together).

FRICTION - AN ADJUSTABLE LATERAL PUSHING FORCE

When a force is trying to move an object along the surface of another object, a force preventing that movement is friction. It operates in a similar manner to a Normal force, in that it can adjust to different levels, but it operates parallel to the surface rather than perpendicular. The other big difference is that it can change direction, as it always operates against the direction of any possible movement. It is called a resistive force, as it resists movement.

Imagine a heavy box being pushed along a floor. While the box is stationary, the floor is exerting a friction force equal to the force with which the box is being pushed, in the opposite direction.

The floor is also creating a friction force against the motion of the person's feet. The feet are pushing away from the box, so the two friction forces act in opposite directions.



The friction forces act against the direction of push.

Friction forces are variable, up to the limit of the gripping strength of the surfaces. At a certain amount of force either the box or the feet will slide!

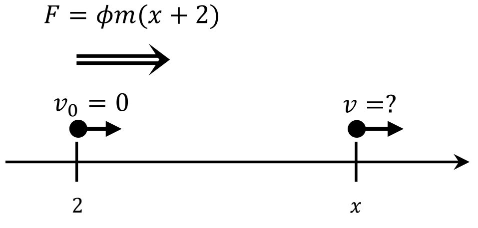
Let's start with an example where we need to prove a result for velocity as a function of displacement, which is generally the easiest type of result to prove.

Example 1

A particle of mass m is moving in a straight line under the action of a force, $F = \phi m(x+2)$, where ϕ is a positive constant.

Given the particle starts from rest at $x = 2$, prove $v = \sqrt{\phi(x^2 + 4x - 12)}$.

Solution



$$m\ddot{x} = \phi m(x+2)$$
$$\ddot{x} = \phi(x+2)$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \phi(x+2)$$
$$\frac{1}{2}v^2 = \phi\int_2^x (x+2)\,dx$$
$$= \phi\left[\frac{x^2}{2} + 2x\right]_2^x$$
$$= \phi\left(\left(\frac{x^2}{2} + 2x\right) - \left(\frac{2^2}{2} + 2(2)\right)\right)$$
$$= \phi\left(\frac{x^2}{2} + 2x - 6\right)$$
$$v^2 = \phi(x^2 + 4x - 12)$$
$$\therefore v = \sqrt{\phi(x^2 + 4x - 12)}$$

Always start by drawing a forces diagram, clearly showing displacement, velocity and force/ acceleration separately. Here displacement is shown against a number line, velocity with an single arrow and force with a double arrow.

Motion is given as a function of displacement, so use $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Alternatively we could use an indefinite integral as below.

$$\frac{1}{2}v^2 = \phi\left(\frac{x^2}{2} + 2x\right) + C$$

when $x = 2, v = 0$

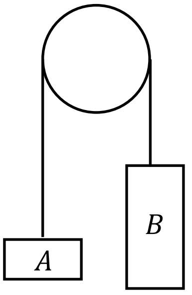
$$0 = \phi\left(\frac{2^2}{2} + 2(2)\right) + C \Rightarrow C = -6\phi$$
$$\frac{1}{2}v^2 = \phi\left(\frac{x^2}{2} + 2x - 6\right)$$
$$v^2 = \phi(x^2 + 4x - 12)$$

Note we have taken the positive square root only - why is this? Since the particle starts to the right, force will always be positive, so the particle will keep going faster and faster to the right, and velocity can never be negative.

Tension also allows us to redirect forces, as we can see in the example below. Another interesting aspect of the next example is that since we get an expression for acceleration which is a constant we can use $\ddot{x} = v\frac{dv}{dx}$ or $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

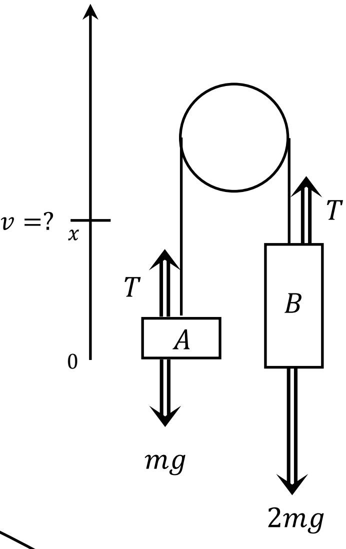
Example 2

Particle A of mass m kg and Particle B of mass $2m$ kg are connected by a light inextensible string passing over a frictionless pulley. Initially the particles are at rest. After Particle A has travelled x metres in an upwards direction it is travelling at v metres per second.



Prove $v = \sqrt{\dfrac{2gx}{3}}$

Solution



$$(m+2m)\ddot{x} = (2mg - T) - (mg - T)$$
$$3m\ddot{x} = mg$$
$$\ddot{x} = \frac{g}{3}$$

$$\left.\begin{aligned}
\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{g}{3} \\
\frac{1}{2}v^2 &= \frac{g}{3}\int_0^x dx \\
v^2 &= \frac{2g}{3}x
\end{aligned}\right\}$$

$$v = \sqrt{\frac{2gx}{3}} \quad \text{positive root since Particle A will only move up}$$

Alternatively

$$\begin{aligned}
v\frac{dv}{dx} &= \frac{g}{3} \\
\frac{dv}{dx} &= \frac{g}{3v} \\
\frac{dx}{dv} &= \frac{3v}{g} \\
x &= \frac{3}{g}\int_0^v v\,dv \\
&= \frac{3}{2g}\Big[v^2\Big]_0^v \\
&= \frac{3v^2}{2g} \\
v^2 &= \frac{2gx}{3}
\end{aligned}$$

Example 3

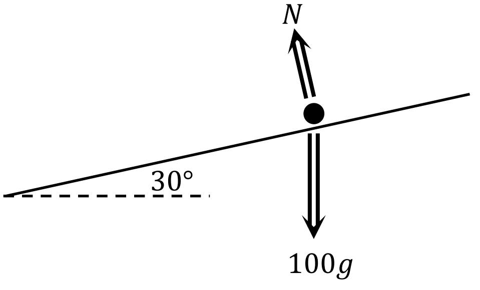
A 100 kilogram box sits on a slippery ramp which is inclined at an angle of $30°$ to the horizontal. If the box starts from rest, find:

a its velocity after t seconds.

b its displacement after t seconds

Solution

a



The weight of $100g$ can be split into components parallel to the slope and perpendicular to the slope. We can ignore the component perpendicular to the slope as it will be balanced by the normal reactive force.

$$100\ddot{x} = 100g \times \sin 30°$$

$$\frac{dv}{dt} = \frac{g}{2}$$

$$v = \frac{g}{2}\int_0^t dt$$

$$= \frac{gt}{2}$$

b

$$\therefore \frac{dx}{dt} = \frac{gt}{2}$$

$$x = \frac{g}{2}\int_0^t t\,dt$$

$$= \frac{g}{4}\left[t^2\right]_0^t$$

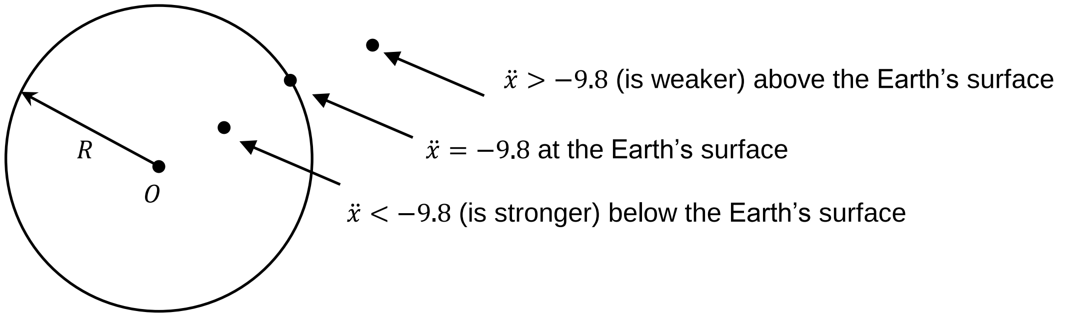
$$= \frac{gt^2}{4}$$

GRAVITY

In the old syllabus HSC questions on motion without resistance occurred every few years, and often dealt with gravity. It is important to note that gravity is only constant close to the Earth's surface. Most questions that we deal with in Mechanics are close to the Earth, which will normally be specified in the question by stating gravitational acceleration is $g, -g, 9.8\ \mathrm{ms}^{-2}$ or $10\ \mathrm{ms}^{-2}$.

Gravity is inversely proportional to the square of the distance from the centre of the Earth - gravity gets weaker as you go in to space until eventually it is negligible. If you could head deeper into the Earth it would get stronger until you were crushed by the forces.

We often use R to indicate the radius of the Earth.



A common mistake is to measure distance from the surface of the Earth rather than the centre.

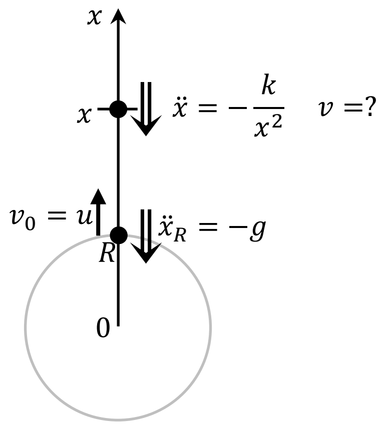
Since for gravity acceleration is a function of displacement we are unable to use the formulae $\frac{dv}{dt}$, so we will usually only prove results for velocity in terms of displacement

Example 4

A body is projected vertically upwards from the surface of the Earth with initial speed u. The acceleration due to gravity at any point on its path is inversely proportional to the square of its distance from the centre of the Earth. Let the radius of the Earth be R, and the acceleration due to gravity at the surface be $-g$.

Prove that the speed at any position x is given by $v^2 = u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

Solution



Let x be the distance from the centre of the Earth.

$$\ddot{x} = -\frac{k}{x^2}$$

at the surface of the Earth $x = R$ and $\ddot{x} = -g$

$$\therefore -g = -\frac{k}{R^2}$$

$$k = gR^2$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^2}{x^2}$$

$$= -gR^2x^{-2}$$

$$\therefore \frac{1}{2}v^2 - \frac{1}{2}u^2 = -\int_R^x gR^2x^{-2}\,dx$$

$$v^2 - u^2 = -2\left[gR^2 \times \frac{x^{-1}}{-1}\right]_R^x$$

$$= 2gR^2\left[\frac{1}{x}\right]_R^x$$

$$= 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

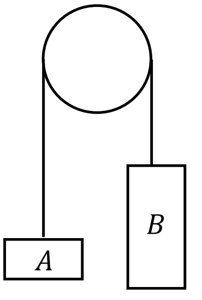
$$v^2 = u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

EXERCISE 6.2

BASIC

1 A particle of mass m is moving in a straight line under the action of a force, $F = \phi m(x-2)$, where ϕ is a positive constant. Given the particle starts from rest at the origin, prove $v = -\sqrt{\phi(x^2-4x)}$.

2 Particle A of mass m kg and Particle B of mass $5m$ kg are connected by a light inextensible string passing over a frictionless pulley. Initially the particles are at rest. After Particle A has travelled x metres in an upwards direction it is travelling at v metres per second.



Prove $v = \sqrt{\dfrac{4gx}{3}}$

3 A body is projected vertically downwards from a height of $2R$ (from the centre of the Earth) with initial speed u. The acceleration due to gravity at any point on its path is inversely proportional to the square of its distance from the centre of the Earth. Let the radius of the Earth be R, and the acceleration due to gravity at the surface be $-g$.

Prove that the speed at any position x is given by $v^2 = u^2 + 2gR^2\left(\dfrac{1}{x} - \dfrac{1}{2R}\right)$

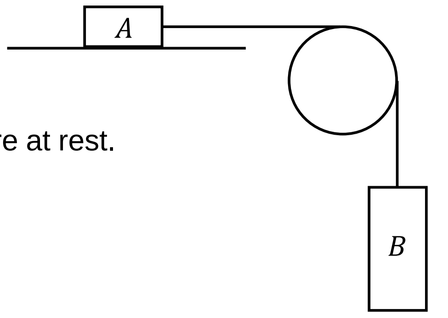
4 A particle of mass m is moving in a straight line under the action of a force, $F = \frac{m}{x^3}(6+10x)$. Find an expression for velocity as a function of its displacement x, if the particle starts from rest at $x = 1$.

5 A $20\sqrt{2}$ kilogram box sits on a slippery ramp which is inclined at an angle of $45°$ to the horizontal. If the box starts from rest, find:

a its velocity after t seconds.

b its displacement after t seconds

6 Particle A of mass m kg and Particle B of mass $2m$ kg are connected by a light inextensible string passing over a frictionless pulley. Particle A is on a horizontal surface, with constant friction F*. Initially the particles are at rest. After Particle A has travelled x metres to the right it is travelling at v metres per second.



Given $v = \sqrt{\frac{gx}{2}}$, find an expression for F as a function of m and g.

* Friction isn't constant, so this is not a real world example, but good integration practice!

7 A particle of mass m moves in a straight line under the action of a resultant force F where $F = F(x)$. Given that the velocity v is v_0 when the position x is x_0, and that v is v_1 when x is x_1, prove $|v_1| = \sqrt{\frac{2}{m}\int_{x_0}^{x_1} F(x)\, dx + v_0^2}$

CHALLENGING

8 A particle is initially at rest at the point B which is b metres to the right of O. The particle then moves in a straight line towards O. For $x \neq 0$, the acceleration of the particle is given by $-\frac{\mu^2}{x^2}$, where x is the distance from O and μ is a positive constant.

i Prove that $\frac{dx}{dt} = -\mu\sqrt{2}\sqrt{\frac{b-x}{bx}}$

ii Using the substitution $x = b\cos^2\theta$, show that the time taken to reach a distance d metres to the right of O is given by $t = \frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cos^2\theta\, d\theta$

iii It can be shown that $t = \frac{1}{\mu}\sqrt{\frac{b}{2}}\left(\sqrt{bd - d^2} + b\cos^{-1}\sqrt{\frac{d}{b}}\right)$ (Do NOT prove this.)

What is the limiting time taken for the particle to reach O?

9 In an alien universe, the gravitational attraction between two bodies is proportional to x^{-3}, where x is the distance between their centres. A particle is projected upward from the surface of the planet with velocity u at time $t = 0$. Its distance x from the centre of the planet satisfies the equation $\ddot{x} = -\frac{k}{x^3}$.

i Show that $k = gR^3$, where g is the magnitude of the acceleration due to gravity at the surface of the planet and R is the radius of the planet.

ii Show that v, the velocity of the particle, is given by $v^2 = \dfrac{gR^3}{x^2} - (gR - u^2)$

iii It can be shown that $x = \sqrt{R^2 + 2uRt - (gR - u^2)t^2}$. (Do NOT prove this.)

Show that if $u \geq \sqrt{gR}$ the particle will not return to the planet.

iv If $u < \sqrt{gR}$ the particle reaches a point whose distance from the centre of the planet is D, and then falls back.

(α) Use the formula in part (ii) to find D in terms of u, R and g.

(β) Use the formula in part (iii) to find the time taken for the particle to return to the surface of the planet in terms of u, R and g.

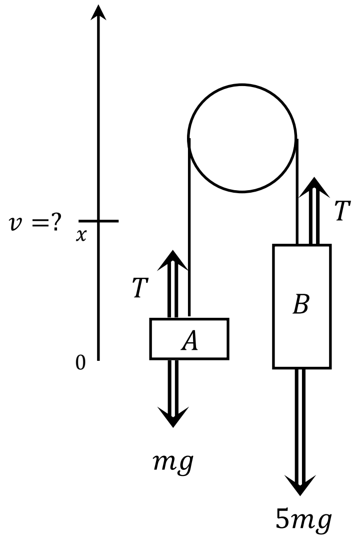
1

$$m\ddot{x} = \phi m(x-2)$$
$$\ddot{x} = \phi(x-2)$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \phi(x-2)$$
$$\frac{1}{2}v^2 = \phi\int_0^x (x-2)\,dx$$
$$v^2 = 2\phi\left[\frac{x^2}{2} - 2x\right]_0^x$$
$$\therefore v^2 = \phi x^2 - 4\phi x$$
$$v = -\sqrt{\phi(x^2-4x)}$$

negative root since initially $\ddot{x} < 0$ and $\dot{x} = 0$ at the origin so the particle moves left.

2

$$(m+5m)\ddot{x} = (5mg - T) - (mg - T)$$
$$6m\ddot{x} = 4mg$$
$$\ddot{x} = \frac{2g}{3}$$
$$v\frac{dv}{dx} = \frac{2g}{3}$$
$$\frac{dv}{dx} = \frac{2g}{3v}$$
$$\frac{dx}{dv} = \frac{3v}{2g}$$
$$x = \frac{3}{2g}\int_0^v v\,dv$$
$$= \frac{3}{4g}\left[v^2\right]_0^v$$
$$= \frac{3v^2}{4g}$$
$$v^2 = \frac{4gx}{3}$$
$$v = \sqrt{\frac{4gx}{3}} \quad \text{since Particle A only moves up}$$



3

$$\ddot{x} = -\frac{k}{x^2}$$
when $x = R$ $\ddot{x} = -g$
$$-g = -\frac{k}{R^2}$$
$$k = gR^2$$
$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^2}{x^2}$$
$$= -gR^2x^{-2}$$
$$\therefore \frac{1}{2}v^2 - \frac{1}{2}u^2 = -gR^2\int_{2R}^x x^{-2}\,dx$$
$$v^2 - u^2 = -2gR^2\left[\frac{x^{-1}}{-1}\right]_{2R}^x$$
$$v^2 = 2gR^2\left(\frac{1}{2R} - \frac{1}{x}\right) + u^2$$
$$= u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{2R}\right)$$

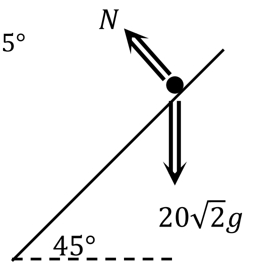
4

$$a = \frac{F}{m}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^{-3} + 10x^{-2}$$
$$\frac{1}{2}v^2 = \int_1^x (6x^{-3} + 10x^{-2})\,dx$$
$$v^2 = 2\left[-3x^{-2} - 10x^{-1}\right]_1^x$$
$$= 2\big((-3x^{-2} - 10x^{-1}) - (-3 - 10)\big)$$
$$= \frac{2(-3 - 10x + 13x^2)}{x^2}$$
$$v = \frac{1}{x}\sqrt{26x^2 - 20x - 6}$$

positive root since initially $F > 0$ and $\dot{x} = 0$ so the particle moves to the right.

5

a

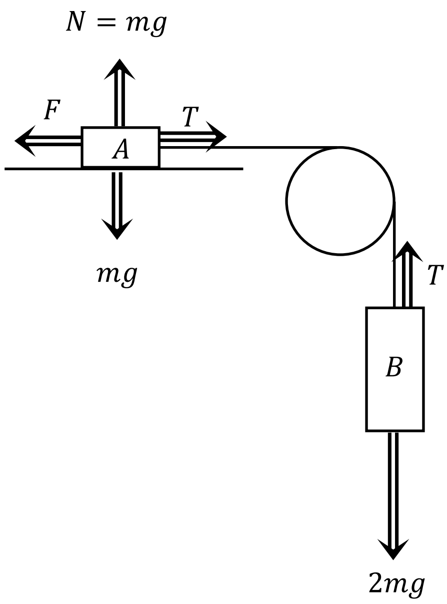


$$20\sqrt{2}\ddot{x} = 20\sqrt{2}g \times \sin 45°$$
$$\frac{dv}{dt} = \frac{g}{\sqrt{2}}$$
$$v = \frac{g}{\sqrt{2}}\int_0^t dt$$
$$= \frac{gt}{\sqrt{2}}$$

b

$$\therefore \frac{dx}{dt} = \frac{gt}{\sqrt{2}}$$
$$x = \frac{g}{\sqrt{2}}\int_0^t t\,dt$$
$$= \frac{g}{2\sqrt{2}}\left[t^2\right]_0^t$$
$$= \frac{gt^2}{2\sqrt{2}}$$

6



$$(m + 2m)\ddot{x} = (2mg - T) - (F - T)$$
$$3m\ddot{x} = 2mg - F$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{2mg - F}{3m}$$
$$\frac{d}{dx}\left(\frac{gx}{4}\right) = \frac{2mg - F}{3m}$$
$$\frac{g}{4} = \frac{2mg - F}{3m}$$
$$3mg = 8mg - 4F$$
$$4F = 5mg$$
$$F = \frac{5mg}{4}$$

7

$$m\ddot{x} = F(x)$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{m} \times F(x)$$

integrating from $x = x_0, v^2 = v_0^2$

to $x = x_1, v^2 = v_1^2$

$$\frac{1}{2}\int_{v_0^2}^{v_1^2} v^2\ \,dv^2 = \frac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$$
$$\frac{1}{2}\left[v^2\right]_{v_0^2}^{v_1^2} = \frac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$$
$$\frac{1}{2}v_1^2 - \frac{1}{2}v_0^2 = \frac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$$
$$v_1^2 - v_0^2 = \frac{2}{m}\int_{x_0}^{x_1} F(x)\,dx$$
$$v_1^2 = \frac{2}{m}\int_{x_0}^{x_1} F(x)\,dx + v_0^2$$
$$|v_1| = \sqrt{\frac{2}{m}\int_{x_0}^{x_1} F(x)\,dx + v_0^2}$$

8 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{\mu^2}{x^2}$$
$$\frac{1}{2}v^2 = -\mu^2\int_b^x \frac{1}{x^2}\,dx$$
$$\frac{1}{2}v^2 = \mu^2\left[\frac{1}{x}\right]_b^x$$
$$= \mu^2\left(\frac{1}{x}-\frac{1}{b}\right)$$
$$v^2 = 2\mu^2\left(\frac{b-x}{bx}\right)$$
$$\frac{dx}{dt} = -\mu\sqrt{2}\sqrt{\frac{b-x}{bx}}$$

Negative root since particle initially moves to the left.

ii

$$\frac{dt}{dx} = -\frac{1}{\mu\sqrt{2}}\sqrt{\frac{bx}{b-x}}$$
$$t = -\frac{1}{\mu\sqrt{2}}\int_b^d \sqrt{\frac{bx}{b-x}}\,dx \qquad \boxed{\begin{aligned} x &= b\cos^{\mathbf{2}}\theta \\ dx &= -2b\cos\theta\sin\theta\,d\theta \end{aligned}}$$
$$= -\frac{1}{\mu\sqrt{2}}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \sqrt{\frac{b^2\cos^2\theta}{b-b\cos^2\theta}}(-2b\cos\theta\sin\theta\,d\theta)$$
$$= \frac{2b}{\mu\sqrt{2}}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \sqrt{\frac{b\cos^2\theta}{\sin^2\theta}}\cos\theta\sin\theta\,d\theta$$
$$= \frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cos^2\theta\,d\theta$$

iii

Let $d = 0$ in the given equation

$$t = \frac{1}{\mu}\sqrt{\frac{b}{2}}\left(0 + b\left(\frac{\pi}{2}\right)\right)$$
$$= \frac{b\sqrt{b}\pi}{2\sqrt{2}\mu}\ sec$$

9 i When $x = R,\ \ddot{x} = -g$

$$\therefore -g = -\frac{k}{R^3}$$
$$k = gR^3$$

ii $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^3}{x^3}$$
$$\frac{1}{2}v^2 - \frac{1}{2}u^2 = -gR^3\int_R^x \frac{1}{x^3}\,dx$$
$$v^2 - u^2 = 2gR^3\left[\frac{1}{x^2}\right]_R^x$$
$$v^2 = 2gR^3\left(\frac{1}{x^2}-\frac{1}{R^2}\right) + u^2$$
$$= \frac{gR^3}{x^2} - (gR - u^2)$$

iii Let $x = R$

$$R = \sqrt{R^2 + 2uRt - (gR-u^2)t^2}$$
$$R^2 = R^2 + 2uRt - (gR-u^2)t^2$$
$$\therefore t(2uR - (gR-u^2)t) = 0$$
$$\therefore t_1 = 0 \text{ or } 2uR - (gR-u^2)t = 0$$
$$t_2 = \frac{2uR}{gR-u^2}$$

now $0 < \sqrt{gR} \leq u$ so the numerator is positive and the denominator negative so $t_2 < 0$ and is not a valid solution. The particle does not return to the planet.

$\boldsymbol{iv\ \alpha}$ *let* $v = 0$

$$0 = \frac{gR^3}{D^2} - (gR-u^2)$$
$$\frac{gR^3}{D^2} = gR - u^2$$
$$D^2 = \frac{gR^3}{gR-u^2}$$
$$D = \sqrt{\frac{gR^3}{gR-u^2}}$$

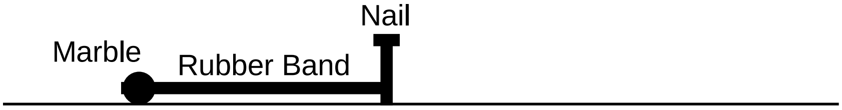
$$\boldsymbol{\beta} \therefore \sqrt{\frac{gR^3}{gR-u^2}} = \sqrt{R^2 + 2uRt - (gR-u^2)t^2}$$
$$gR^3 = gR^3 - u^2R^2 + 2uR(gR-u^2)t - (gR-u^2)^2t^2$$
$$\therefore (gR-u^2)^2t^2 - 2uR(gR-u^2)t + u^2R^2 = 0$$
$$\left((gR-u^2)t - uR\right)^2 = 0$$
$$t = \frac{uR}{gr-u^2}$$

UNDERSTANDING SIMPLE HARMONIC MOTION

Simple Harmonic Motion (SHM) describes particles that are oscillating in one dimension, usually either vertically or horizontally. It is a type of motion without resistance, which is why I prefer to teach it after the first two lessons rather than at the start as in the syllabus.

There are obvious examples of SHM such as the tides rising and falling, and less obvious examples such as the vibrations in a string. As a particle moves around a circle its x-value is in SHM and its y-value is in SHM, while the particle itself is in circular motion.

Before we look in more detail at SHM, let's consider a nail, a rubber band and a marble as shown below, so that we get a better feel for the behaviour of a particle in SHM.

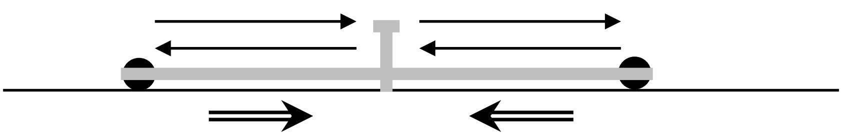


Now we have to use a bit of imagination here. First we will pretend that:

- When we release the marble it will somehow pass through the nail
- The marble will somehow stay within the rubber band forever
- No energy will be lost as heat, sound or friction, so the marble moves forever

So the marble will move from left to right until the end of time as two physical phenomena continually fight each other;

- The marble always wants to keep moving in a straight line at its current velocity (inertia)
- The rubber band always tries to bring the marble back towards the nail, so sometimes it is speeding the marble towards the nail, sometimes slowing it down once it has gone past.



Force When the rubber band is fully extended (at either end of the motion) it places the most force on the marble. As the marble moves towards the nail the rubber band slightly slackens and the force reduces, while as the marble passes the nail the rubber band is slack so there is temporarily no force. Once the marble passes the nail the rubber band stretches and the force increases, slowing the marble until it comes to a stop.

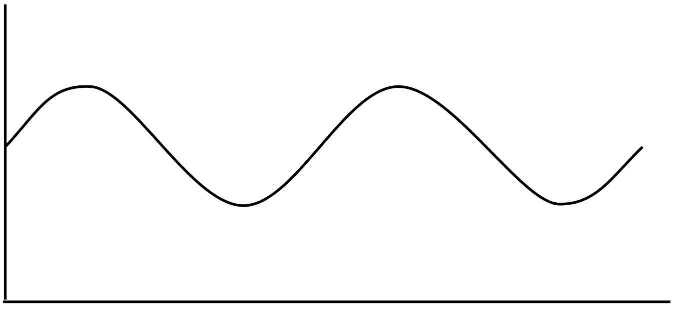
The acceleration of a particle in SHM is directly proportional to the distance from the centre of motion, and directed towards the centre.

Speed* At each end of the motion the marble temporarily stops, then speeds up towards the centre, then slows down once it passes the centre.

The speed of the particle is zero at each end of the motion and reaches a maximum at the centre.

***** When trying to understand SHM we use the term speed, but we are interested in velocity.

Now we have a better feel for the force and speed involved in SHM, but how could we model the motion? Let's imagine graphing the height of water during the day as the tide rises and falls with the tides. If we graph the depth of water against time we would see something like the diagram below.



Some characteristics of the depth of water would be:

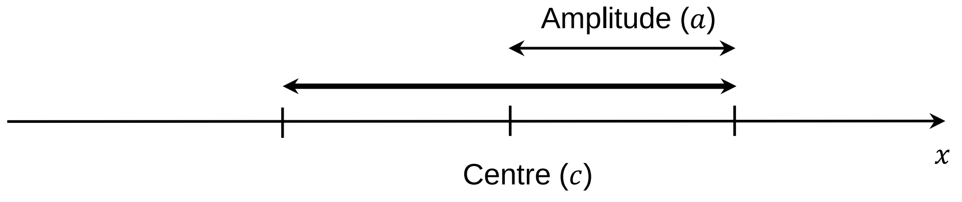
- The water depth rises and falls between the same two heights.
- The change of depth is slower near high tide and low tide
- The change of depth is greatest when the water is close to the central depth

Looking at the graph above we can see that the motion might be modelled by a sine curve, and shortly we will prove that this is true.

TERMINOLOGY AND DEFINITIONS

We start by assuming that the SHM is horizontal along a number line, with the positive direction to the right. We can easily adapt this to vertical motion.

The distance from the centre of the motion (c) to either end of the motion is called the amplitude (a). The centre of motion is also known as the equilibrium position.



The base definition of SHM is that the particle has acceleration which is directly proportional to the displacement from the centre, and in the opposite direction. In general we can state this as $\ddot{x} = -k(x - c)$, where k is a positive constant.

In a moment we will define the displacement as a function involving n, and we can define acceleration as:

$$\boxed{\ddot{x} = -n^2(x - c)}$$

If we are asked to prove a particle is in SHM then we need to prove that this formula holds for a given equation of motion.

In the old syllabus the only way to prove SHM was using acceleration as above. Given the wording of the new syllabus I think that if students can show that the equation of motion matches either of the forms for displacement we will see immediately below then that would also be accepted.

Students need to be able to prove that a particle is in SHM given an equation of motion for displacement, velocity or acceleration.

EQUATIONS OF MOTION

There are several different equations of motion that satisfy $\ddot{x} = -n^2(x - c)$. We will look at some this lesson and some more next lesson. The two most common equations we will use are $x = a\sin(nt + \alpha) + c$ and $x = a\cos(nt + \alpha) + c$, where x is the displacement from a fixed point (the origin), $\frac{2\pi}{n}$ is the period, $\frac{\alpha}{n}$ is the phase shift and c is the central point of motion. Let's prove that $x = a\sin(nt + \alpha) + c$ satisfies $\ddot{x} = -n^2(x - c)$, then look at the purpose of n and α.

Proof 1

Prove that a particle where $x = a\sin(nt + \alpha) + c$ is in Simple Harmonic Motion

Solution

$x = a\sin(nt + \alpha) + c$

$\dot{x} = an\cos(nt + \alpha)$

$\ddot{x} = -an^2\sin(nt + \alpha)$

$\quad = -n^2(a\sin(nt + \alpha) + c - c)$

$\quad = -n^2(x - c)$

$\therefore$ a particle where $= a\sin(nt + \alpha) + c$ is in Simple Harmonic Motion

Example 2

Prove that a particle where $\ddot{x} = -16x - 4$ is in Simple Harmonic Motion

Solution

$\ddot{x} = -16x - 4$

$$= -4^2\left(x + \frac{1}{4}\right)$$

$\therefore$ The particle is in SHM since it satisfies $\ddot{x} = -n^2(x - c)$

THE PURPOSE OF n AND α ON THE EQUATION OF MOTION

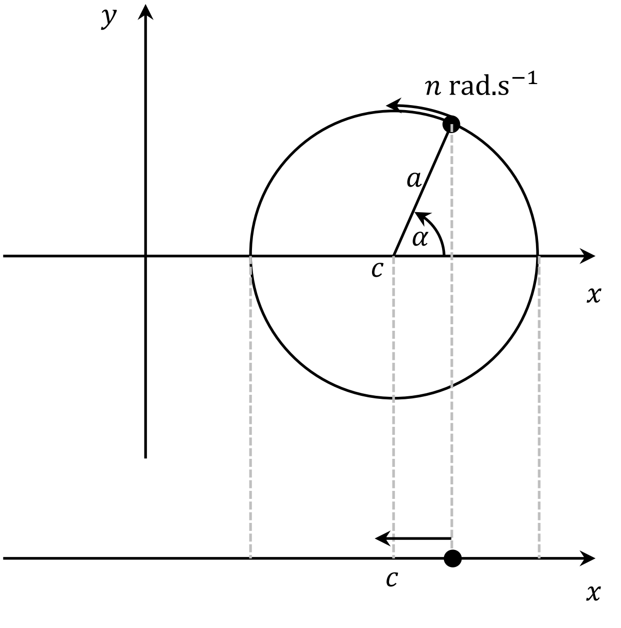
So we have a particle in SHM with amplitude a about a centre of motion c, but where did the particle start, what direction did it first move in and how fast is it travelling? To adjust our equation of motion to allow for these differences the variables n and α have been included. Be careful not to mix the amplitude a and the angle α as they can easily be mistaken as they look similar in many fonts.

Many questions do not give (or need) initial values, in which case let $\alpha = 0$ for ease of calculations.

Consider for a moment a particle moving around a circle of radius a with centre $(c, 0)$ - the centre could be at any height, but let's say 0 to make things easier. Let's say that it starts at an angle of α to the positive direction of the x-axis and rotates at a speed of n radians per second in an anti-clockwise direction, as shown in Figure 1. Figure 2 shows a view taken from directly above where we can see the horizontal motion but not the vertical - the particle is now in SHM. Dotted lines join the important points on the two figures.

Figure 1
Circular Motion

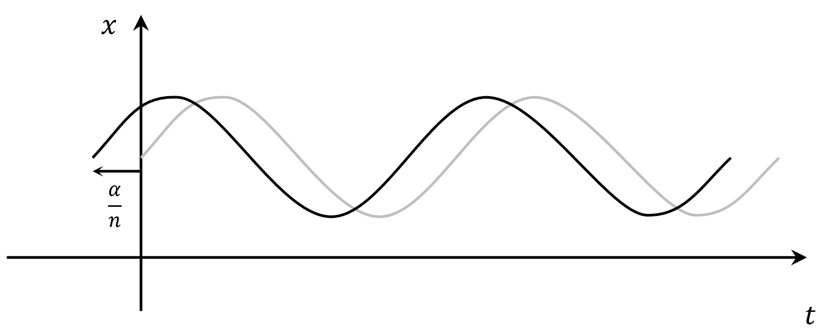
Figure 2
Simple Harmonic Motion



In Figure 1 the particle will move anticlockwise, which in Figure 2 means that the particle will initially move to the left. We can see how the radius of the circle matches the amplitude of the SHM. In the diagram as shown we could work out that the x-value would be given by
$x = a\cos(nt + \alpha) + c$, using simple trigonometry.

In Figure 1, to complete a full revolution of 2π radians at n radians per second would take $\frac{2\pi}{n}$ seconds, so the Period is $\frac{2\pi}{n}$. This would correspond to the time taken for the particle in SHM to complete one full cycle, so the period of a particle in SHM is also $\frac{2\pi}{n}$.

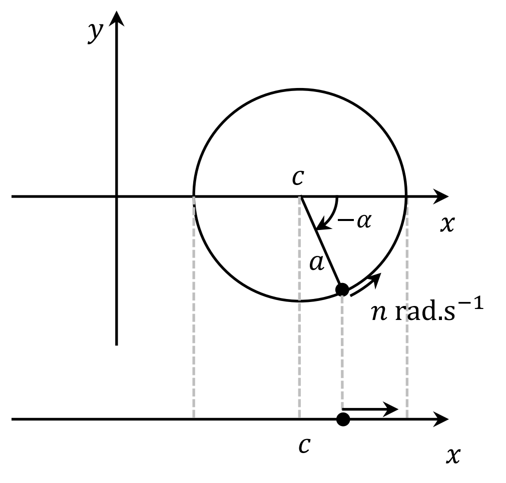
If we were to graph displacement against time for $x = a\sin(nt)$ in grey and $x = a\sin(nt + \alpha)$ in black as shown below, we would see that the curve $x = a\sin(nt)$ has been shifted left by $\frac{\alpha}{n}$ to get the curve $x = a\sin(nt + \alpha)$, so we say that the phase shift is $\frac{\alpha}{n}$.



Now the observant student might have a couple of questions by now, such as:

- what if we want the particle to start by moving in the other direction?
- what happens if we use the equation of motion with cosine instead of sine?

Firstly, by changing the value of α we can move to an equivalent starting position but where the particle would move in the opposite direction. So in Figure 1 if we start at $-\alpha$, then in Figure 2 we get the same starting position but initially move to the right. Theoretically we could also make n negative to move the other way, but we always take n as positive so that the period $\frac{2\pi}{n}$ remains positive.



Secondly, the cosine curve is the sine curve shifted to the left (or right), so we can use either version of the formula and get the same results - we would use a different value for α in each version so that the particle starts in the correct position with the correct initial direction.

Example 3

A particle moves in SHM about the centre of motion $x = 2$, with amplitude 3 and period $\frac{\pi}{2}$. Find a possible equation of motion.

Solution

$c = 2, a = 3$

$$T = \frac{2\pi}{n} = \frac{\pi}{2} \quad \rightarrow \quad n = 4$$

One possible equation of motion is $x = 3\sin(4t) + 2$.

Alternative solutions would swap the sine for cosine, and replace $4t$ with $4t + \alpha$ for any angle α.

Example 4

A particle moves in SHM about the origin, with a period of π seconds and amplitude 5 metres. Find

i The maximum and minimum displacement

ii The maximum and minimum velocity

iii The maximum and minimum acceleration

Solution

$$\frac{2\pi}{n} = \pi \quad \rightarrow \quad n = 2$$

i $x = 5\sin(2\mathrm{t})$

$\therefore -5 \le x \le 5$ since $-1 \le \sin\theta \le 1$

ii $\dot{x} = 10\cos(2t)$

$\therefore \ -10 \le \dot{x} \le 10$ since $-1 \le \cos\theta \le 1$

iii $\ddot{x} = -20\sin(2t)$

$\therefore -20 \le \ddot{x} \le 20$ since $-1 \le \sin\theta \le 1$

We used sine, as we don't have to deal with negatives until $\ddot{x}$, so less chance of mistakes. There were no initial conditions so we let $\alpha = 0$.

Example 5

A particle moves in SHM about the origin, with displacement given by $x = 3\sin\left(t + \frac{\pi}{4}\right)$.

i What is its initial displacement?

ii What is its initial velocity?

iii Once its velocity is zero, how long does it take to next reach the origin?

Solution

i Let $t = 0$

$$x_0 = 3\sin\left(0 + \frac{\pi}{4}\right)$$

$$= 3 \times \frac{1}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{2}$$

ii $\dot{x} = 3\cos\left(t + \frac{\pi}{4}\right)$

Let $t = 0$

$$\dot{x}_0 = 3\cos\left(0 + \frac{\pi}{4}\right)$$

$$= 3 \times \frac{1}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{2}$$

iii To get from one end of the motion to the centre takes one quarter of the period

$$t = \frac{1}{4} \times \frac{2\pi}{1}$$

$$= \frac{\pi}{2}\ \text{sec}$$

Example 6

A particle moves in SHM centred about the origin. When $x = 4$ the particle is at rest. When $x = 2$ the velocity of the particle is 4. Given the equation of motion is $x = a\sin(nt)$ find the values of a and n.

Solution

$a = 4$

$x = 4\sin(nt)$

$\dot{x} = 4n\cos(nt)$

Let $x = 2, \dot{x} = 4$

$$\therefore 2 = 4\sin(nt) \;\rightarrow\; \sin(nt) = \frac{1}{2} \qquad (1)$$

$$4 = 4n\cos(nt) \;\rightarrow \cos(nt) = \frac{1}{n} \qquad (2)$$

$\sin^2(nt) + \cos^2(nt) = 1 \qquad \text{Pythagorean Identity}$

$$\therefore \left(\frac{1}{2}\right)^2 + \left(\frac{1}{n}\right)^2 = 1$$

$$\frac{1}{4} + \frac{1}{n^2} = 1$$

$$n^2 + 4 = 4n^2$$

$$3n^2 = 4$$

$$n^2 = \frac{4}{3}$$

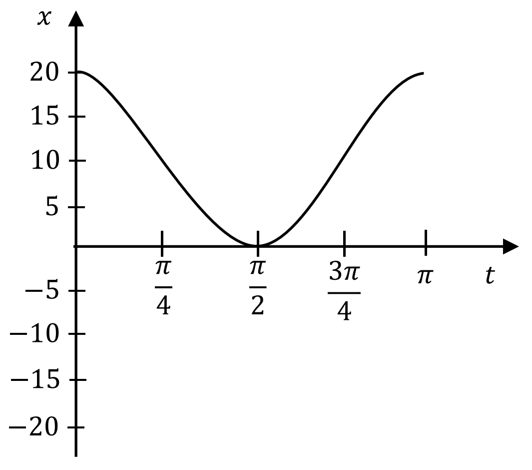
$$n = \frac{2}{\sqrt{3}}$$

GRAPHS OF SHM

The new syllabus places a far greater emphasis on sketching SHM and using sketches of SHM. The sketches involved can be displacement, velocity or acceleration, and involve finding equations of motion and proving that motion is simple harmonic. In the next lesson we will also look at sketches of velocity as a function of displacement.

Example 7

The graph below shows the displacement of a particle in SHM. Its equation of motion is given by $x = a\cos(nt + \alpha) + c$. Find the values of a, n, α and c.



Solution

The difference between the peak and the trough is 20, so $a = 10$.

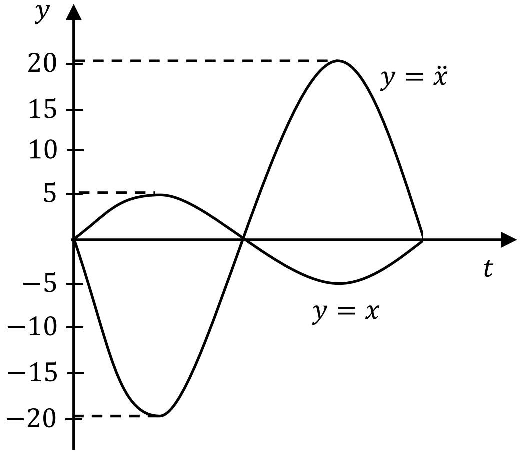
The motion has period π, so $\frac{2\pi}{n} = \pi \quad \rightarrow \quad n = 2$

The curve is the cosine curve shifted vertically, not horizontally, so $\alpha = 0$

The centre of motion is the average of the peak and the trough, so $c = \frac{20+0}{2} = 10$.

Example 8

The graph below shows part of the graph of the displacement and acceleration of a particle in SHM, with the horizontal scale missing.



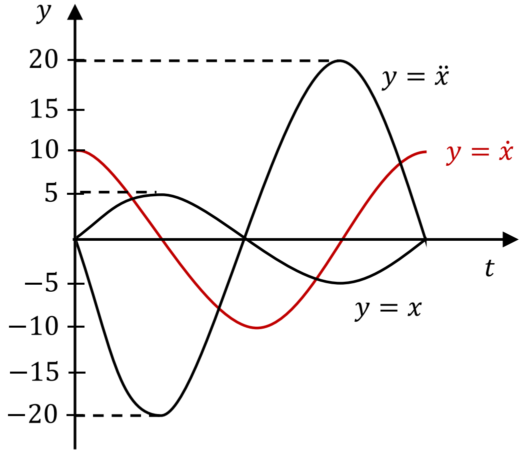
i Find the value of n

ii Sketch the velocity of the particle onto the graph.

Solution

i The amplitudes are 5 and 20 respectively with opposite sign, so $\ddot{x} = -4x$. $\therefore n^2 = 4 \;\rightarrow\; n = 2$

ii The graphs of displacement and acceleration are sine curves, so the graph of velocity will be a cosine curve. It will be zero when displacement is at a maximum or minimum, positive when displacement is increasing and negative when displacement is decreasing. Its amplitude will be na, so $2 \times 5 = 10$



EXERCISE 6.3

BASIC

1 Prove that a particle where $x = a\cos(nt + \alpha) + c$ is in Simple Harmonic Motion

2 A particle moves in SHM about the centre of motion $x = -1$, with amplitude 2 and period $\frac{3\pi}{2}$. Find a possible equation of motion.

3 A particle moves in SHM about the origin, with a period of $\frac{\pi}{3}$ seconds and amplitude 2 metres. Find

i The maximum and minimum displacement

ii The maximum and minimum velocity

iii The maximum and minimum acceleration

4 A particle moves in SHM about the origin, with displacement given by $x = 2\sin\left(t - \frac{\pi}{4}\right)$.

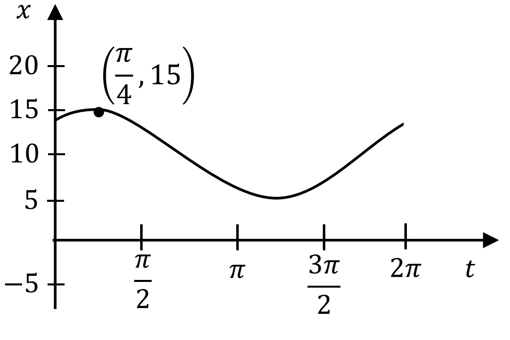
i What is its initial displacement?

ii What is its initial velocity?

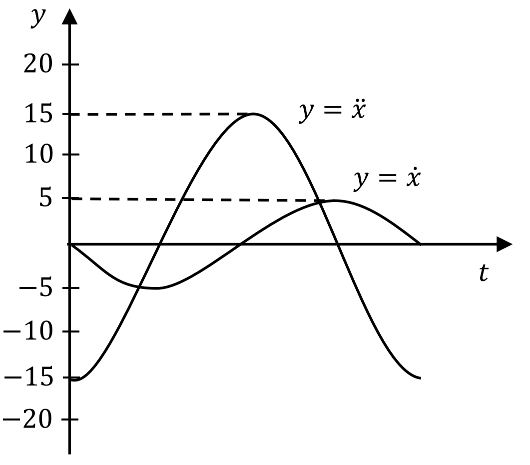
iii Once its velocity is first zero, how long does it take to reach the origin for the second time?

5 A particle moves in SHM centred about the origin. When $x = 2$ the particle is at rest. When $x = 1$ the velocity of the particle is 3. Given the equation of motion is $x = a\sin(nt)$ find the values of a and n.

6 The graph below shows the displacement of a particle in SHM. Its equation of motion is given by $x = a\cos(nt + \alpha) + c$. Find the values of a, n, α and c.



7 The graph below shows part of the graph of the velocity and acceleration of a particle in SHM, with the horizontal scale missing.



i Find the value of n

ii Sketch the displacement of the particle onto the graph.

8 A particle moves in SHM with $x = 2\cos t - 2$. Sketch displacement and acceleration on the same axes.

MEDIUM

9 A particle is undergoing simple harmonic motion on the x-axis about the origin. It is initially at its extreme positive position. The amplitude of motion is 18 and the particle returns to its initial position every 5 seconds.

i Write down an equation for the position of the particle at time t seconds.

ii How long does it take the particle to move from a rest position to the point halfway between the rest position and the equilibrium position?

10 Two particles oscillate horizontally. The displacement of the first is given by $x = 3\sin 4t$ and the displacement of the second is given by $x = a\sin nt$. In one oscillation, the second particle covers twice the distance of the first particle, but in half the time. What are the values of a and n?

11 The displacement, in metres, of a particle from a fixed point in time t, in seconds, $t \geq 0$, is given by $x = 2\cos 3t$. How many oscillations does the particle make per second?

12 The tide can be modelled using simple harmonic motion. At a particular location, the high tide is 9 metres and the low tide is 1 metre. At this location the tide completes 2 full periods every 25 hours. Let t be the time in hours after the first high tide today.

i Explain why the tide can be modelled by the function $x = 5 + 4\cos\left(\dfrac{4\pi}{25}t\right)$

ii The first high tide tomorrow is at 2 am. What is the earliest time tomorrow at which the tide is increasing at the fastest rate?

13 A particle is oscillating between A and B, 7 m apart, in Simple Harmonic Motion. The time for a particle to travel from B to A and back is 3 seconds. Find the velocity and acceleration at M, the midpoint of OB where O is the centre of AB.

14 A particle moving in simple harmonic motion oscillates about a fixed point O in a straight line with a period of 10 seconds. The maximum displacement of P from O is 5 m. Which of the following statements are true?

If initially the particle is at O moving to the right then 27 second later P will be:

(I) moving with a decreasing displacement

(II) moving with a decreasing speed

(III) moving with a decreasing acceleration

CHALLENGING

15 At the start of the observation yesterday, the upper deck of a ship, anchored at Sydney Wharf was 1.2 metres above the wharf at $6{:}13$ am, when the tide was at its lowest level. At $12{:}03$ pm at the following high tide the last observation record shows that the upper deck was 2.6 metres above the wharf. Considering that the tide moves in simple harmonic motion, find:

i At what time during the observation period, was the upper deck exactly 2 metres above the wharf?

ii What was the maximum rate at which the tide increased during this period of observation?

16 A particle moves in SHM with period T about a centre O. Its displacement at any time t is given by $x = A\sin nt$, where A is the amplitude.

i Draw a neat sketch of one period of this displacement-time equation, showing all intercepts.

ii Show that $\dot{x} = \dfrac{2\pi A}{T}\cos\left(\dfrac{2\pi t}{T}\right)$

iii The point P lies D units on the positive side of O. Let V be the velocity of the particle when it first passes through P. Show that the first time the particle is at P after passing through O is $t = \dfrac{T}{2\pi}\tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$

iv Show that the time between the first two occasions when the particle passes through P is $\dfrac{T}{\pi}\tan^{-1}\left(\dfrac{VT}{2\pi D}\right)$. You may assume that $\tan^{-1}x + \tan^{-1}\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$ for $x > 0$

1

$x = a\cos(nt + \alpha) + c$
$\dot{x} = -an\sin(nt + \alpha)$
$\ddot{x} = -an^2\cos(nt + \alpha)$
$= -n^2(a\cos(nt + \alpha) + c - c)$
$= -n^2(x - c)$
$\therefore$ a particle where $= a\cos(nt + \alpha) + c$ is in SHM

2

$c = -1, a = 2$
$T = \frac{2\pi}{n} = \frac{3\pi}{2} \rightarrow n = \frac{4}{3}$
One possible equation of motion is
$x = 2\sin\left(\frac{4t}{3}\right) - 1$.
Alternative solutions would swap the sine for cosine, and replace $\frac{4t}{3}$ with $\frac{4t}{3} + \alpha$ for any angle α

3

$\frac{2\pi}{n} = \frac{\pi}{3} \rightarrow n = 6$
i $x = 2\sin(6t)$
$\therefore -2 \le x \le 2$ since $-1 \le \sin\theta \le 1$
ii $\dot{x} = 12\cos(6t)$
$\therefore -12 \le \dot{x} \le 12$ since $-1 \le \cos\theta \le 1$
iii $\ddot{x} = -72\sin(2t)$
$\therefore -72 \le \ddot{x} \le 72$ since $-1 \le \sin\theta \le 1$

4

i Let $t = 0$
$x_0 = 2\sin\left(0 - \frac{\pi}{4}\right)$
$= 2 \times \left(-\frac{1}{\sqrt{2}}\right)$
$= -\sqrt{2}$

ii $\dot{x} = 2\cos\left(t - \frac{\pi}{4}\right)$
Let $t = 0$
$\dot{x}_0 = 2\cos\left(0 - \frac{\pi}{4}\right)$
$= 2 \times \frac{1}{\sqrt{2}}$
$= \sqrt{2}$

iii To get from one end of the motion to the other end of the motion then back to the centre takes three quarters of the period
$t = \frac{3}{4} \times \frac{2\pi}{1}$
$= \frac{3\pi}{2}$ sec

5

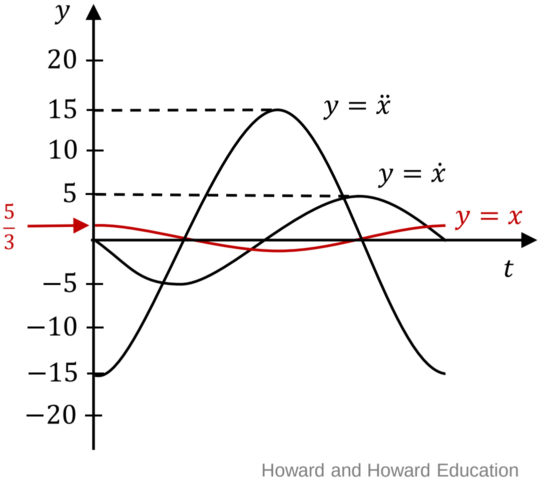
$a = 2$
$x = 2\sin(nt)$
$\dot{x} = 2n\cos(nt)$
Let $x = 1, \dot{x} = 3$
$\therefore 1 = 2\sin(nt) \rightarrow \sin(nt) = \frac{1}{2}$ (1)
$3 = 2n\cos(nt) \rightarrow \cos(nt) = \frac{3}{2n}$ (2)
$\sin^2(nt) + \cos^2(nt) = 1$
$\therefore \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2n}\right)^2 = 1$
$\frac{1}{4} + \frac{9}{4n^2} = 1$
$n^2 + 9 = 4n^2$
$3n^2 = 9$
$n^2 = 3$
$n = \sqrt{3}$

6

The difference between the peak and the trough is $15 - 5 = 10$, so $a = 5$.
The motion has period 2π, so $\frac{2\pi}{n} = 2\pi \rightarrow n = 1$
The curve is the cosine curve shifted vertically by 10 and right by $\frac{\pi}{4}$,
so $\frac{\alpha}{1} = -\frac{\pi}{4} \rightarrow \alpha = -\frac{\pi}{4}$
The centre of motion is the average of the peak and the trough, so $c = \frac{15+5}{2} = 10$.

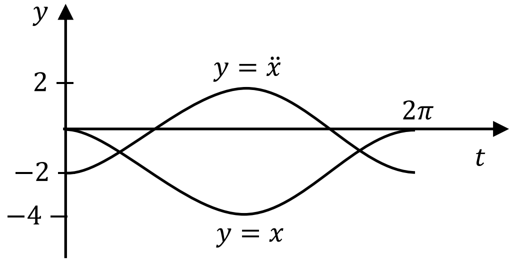
7

i The amplitudes are 5 and 15 respectively, and the amplitude of $\ddot{x}$ is the amplitude of $\dot{x}$ multiplied by n, so $n = 3$
ii The graph of displacement is the reflection of the graph of acceleration over the t-axis, but with amplitude $\frac{1}{n}$ that of velocity, so $\frac{5}{3}$.



8 Displacement is a cosine curve with an amplitude of 2, centre of motion -2 and $T = 2\pi$ since $n = 1$. So it is a cosine curve stretched vertically by a factor of 2 then moved down 2 units.

Acceleration is given by $\ddot{x} = -(x+2) = -x - 2$ since $n = 1$ and $c = -2$, so it is the displacement curve reflected over the x-axis then moved down 2 units. Notice that it is centred about 0, not 3, as acceleration is proportional to the distance from the centre of motion.



9 **i**

$$n = \frac{2\pi}{T} = \frac{2\pi}{5}$$

$at\ t = 0$ we want $\cos(nt + \alpha) = 1$

$(\text{ or } \sin(nt+\alpha) = 1)$

$$x = 18\cos\frac{2\pi}{5}t$$

$$\left(\text{or } 18\sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)\right)$$

ii

The particle is at rest at the extreme position, so at time $t = 0$.

$$9 = 18\cos\frac{2\pi}{5}t$$

$$\cos\frac{2\pi}{5}t = \frac{1}{2}$$

$$\frac{2\pi}{5}t = \frac{\pi}{3}$$

$$t = \frac{5}{6}\text{ sec}$$

10 Twice the distance means that the amplitude is doubled. Half the time means that the angle velocity (n) is doubled.

$\therefore a_2 = 6, n_2 = 8$

11

$$f = \frac{1}{T} = \frac{n}{2\pi} = \frac{3}{2\pi}$$

12 **i**

$$c = \frac{9+1}{2} = 5$$

$$a = \frac{1}{2}(9-1) = 4$$

The period of motion is

$$\frac{2\pi}{n} = \frac{25}{2} \rightarrow n = \frac{4\pi}{25}$$

$\therefore\ x = 5 + 4\cos\left(\frac{4\pi}{25}t\right)$ if $t = 0$ at high tide

ii

The tide increases fastest as it rises past its centre value, so $\frac{3}{4}$ of a period after 2 am:

$$2\text{ am} + \frac{3}{4}\times 12\frac{1}{2}\text{ hours} = 11{:}22{:}30\text{ am}$$

13 $a = \frac{7}{2}, \frac{2\pi}{n} = 3 \rightarrow n = \frac{2\pi}{3}$, at M $x = \frac{7}{4}$.

Let $x = \frac{7}{2}\sin\left(\frac{2\pi t}{3}\right)$

$$\dot{x} = \frac{7\pi}{3}\cos\left(\frac{2\pi t}{3}\right)$$

At M:

$$\frac{7}{2}\sin\left(\frac{2\pi t}{3}\right) = \frac{7}{4}$$

$$\sin\left(\frac{2\pi t}{3}\right) = \frac{1}{2}$$

$$\frac{2\pi t}{3} = \frac{\pi}{6}$$

$$t = \frac{1}{4}$$

$$\dot{x}_M = \frac{7\pi}{3}\cos\left(\frac{2\pi\left(\frac{1}{4}\right)}{3}\right) = \frac{7\pi}{3}\cos\frac{\pi}{6} = \frac{7\pi}{3}\times\frac{\sqrt{3}}{2}$$

$$= \frac{7\sqrt{3}\pi}{6}\text{ ms}^{-1}$$

$$\ddot{x}_M = -n^2x = -\left(\frac{2\pi}{3}\right)^2\times\frac{7}{4} = -\frac{7\pi^2}{9}$$

14 The period is 10 seconds, so after 27 seconds P will have completed to full cycles and be almost halfway through the third. Since it started at O moving right it will be to the right of O moving towards O when $t = 27$.

P is moving left so displacement is decreasing, ∴ (I) is true

P is moving left at an increasing speed, so velocity is becoming move negative, ∴ (II) is true

P is getting closer to the origin, so the magnitude of the acceleration is decreasing. Since acceleration is negative it is increasing, ∴ (III) is false.

15 **i** $a = \dfrac{2.6 - 1.2}{2} = 0.7$

$T = 2 \times (12{:}03 - 6{:}13) = 700$ min

$n = \dfrac{2\pi}{T} = \dfrac{\pi}{350}$; $c = \dfrac{2.6 + 1.2}{2} = 1.9$

$\therefore x = 1.9 - 0.7\cos\left(\dfrac{\pi}{350}t\right)$

where t is measured in minutes from $6{:}13$am.

$$\therefore 2 = 1.9 - 0.7\cos\left(\frac{\pi}{350}t\right)$$

$$\cos\left(\frac{\pi}{350}t\right) = -\frac{1}{7}$$

$$\frac{\pi}{350}t = \cos^{-1}\left(-\frac{1}{7}\right)$$

$$t = \frac{350}{\pi}\left(\cos^{-1}\left(-\frac{1}{7}\right)\right)$$

$= 190.97$

$= 3$ hours 11 minutes

The water is 2.0 m high at $9{:}24$ am

ii Maximum rate of increase when it passes the equilibrium point, which is when $t = 175$

$x = 1.9 - 0.7\cos\left(\dfrac{\pi}{350}t\right)$

$\dfrac{dx}{dt} = \dfrac{0.7\pi}{350}\sin\left(\dfrac{\pi}{350}t\right)$

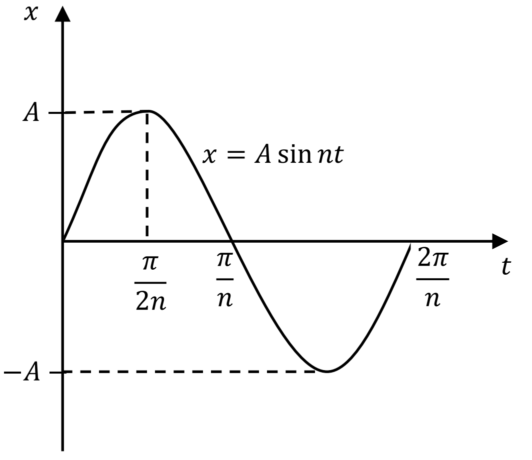
when $t = 175$

$\dfrac{dx}{dt} = \dfrac{0.7\pi}{350}\sin\left(\dfrac{\pi}{350} \times 175\right)$

$= 0.00628$ m/min

$= 37.7$ cm/hour

16 **i**



ii

$T = \dfrac{2\pi}{n} \ \rightarrow \ n = \dfrac{2\pi}{T}$

$x = A\sin nt$

$\quad = A\sin\left(\dfrac{2\pi}{T}t\right)$

$\dot{x} = \dfrac{2\pi A}{T}\cos\left(\dfrac{2\pi t}{T}\right)$

iii

Let $x = D$

$D = A\sin\left(\dfrac{2\pi t}{T}\right)$

$\dfrac{D}{A} = \sin\left(\dfrac{2\pi t}{T}\right) \qquad (1)$

Let $\dot{x} = V$

$V = \dfrac{2\pi A}{T}\cos\left(\dfrac{2\pi t}{T}\right)$

$\dfrac{VT}{2\pi A} = \cos\left(\dfrac{2\pi t}{T}\right) \qquad (2)$

$(1) \div (2)$:

$\dfrac{2\pi D}{VT} = \tan\left(\dfrac{2\pi t}{T}\right)$

$\dfrac{2\pi t}{T} = \tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$

$t = \dfrac{T}{2\pi}\tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$

This is the first time that the particle passes through P since all variables are positive.

iv

The particle will pass through P again when t_2 and $\frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$ add to half the period.

$\therefore t_2 = \dfrac{T}{2} - \dfrac{T}{2\pi}\tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$

The difference between the times is:

$\left(\dfrac{T}{2} - \dfrac{T}{2\pi}\tan^{-1}\left(\dfrac{2\pi D}{VT}\right)\right) - \dfrac{T}{2\pi}\tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$

$= \dfrac{T}{2} - \dfrac{T}{\pi}\tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$

$= \dfrac{T}{2} - \dfrac{T}{\pi}\left(\dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{VT}{2\pi D}\right)\right)$

$= \dfrac{T}{2} - \dfrac{T}{2} + \dfrac{T}{\pi}\tan^{-1}\left(\dfrac{VT}{2\pi D}\right)$

$= \dfrac{T}{\pi}\tan^{-1}\left(\dfrac{VT}{2\pi D}\right)$

ALTERNATIVE EQUATIONS OF MOTION

In the last lesson we used two possible equations of motion for SHM, $x = a\sin(nt+\alpha)+c$ and $x = a\cos(nt+\alpha)+c$, but there are other equations that also describe SHM. Remember that for a particle to be in SHM it must satisfy $\ddot{x} = -n^2(x-c)$.

In Extension 1 we have looked at harmonic addition, also known as the auxiliary angle method, where we add a sine function and a cosine function to create a new sine or cosine function. This means that functions of the form $x = A\sin(nt+\alpha)+B\cos(nt+\alpha)+c$ are in SHM.

Example 1

A particle moves with equation of motion $x = \sin 2t + \cos 2t + 3$ metres. Prove that the particle is in SHM, and find the centre and amplitude of its motion.

Solution

Let $\sin 2t + \cos 2t = R\sin(2t+\alpha)$

$\therefore R = \sqrt{1^2+1^2} = \sqrt{2}$

$\alpha = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\therefore x = \sqrt{2}\sin\left(2t+\frac{\pi}{4}\right)+3$

This is in the form $x = a\sin(nt+\alpha)+c$, so the particle is in SHM. The centre of motion is 3 and the amplitude is $\sqrt{2}$.

Alternatively

$x = \sin 2t + \cos 2t + 3$

$\dot{x} = 2\cos 2t - 2\sin 2t$

$\ddot{x} = -4\sin 2t - 4\cos 2t$

$= -4(\sin 2t + \cos 2t + 3 - 3)$

$= -2^2(x-3)$

$\therefore$ the particle is in SHM with centre 3.

Let $\sin 2t + \cos 2t = R\sin(2t+\alpha)$

$\therefore R = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore$ the amplitude of the motion is $\sqrt{2}$.

Other sources of equations of motion for SHM are based on the Pythagorean Identities. For instance $x = 4\cos^2 t + 1$ is in SHM since we can show that

$4\cos^2 t + 1 = 4\times\frac{1}{2}(1+\cos 2t)+1 = 2\cos 2t + 3$, so it has $a = 2, n = 2$ and $c = 3$

VELOCITY AS A FUNCTION OF DISPLACEMENT

There is a common formula linking velocity and displacement, $v^2 = n^2(a^2 - (x-c)^2)$, which is now no longer in the syllabus as such. Instead it is mentioned as something that you must derive given an expression for acceleration as a function of x, plus initial conditions. Let's prove the formula first in a couple of ways, then use it as intended in some examples.

Proof 2

Prove for a particle in SHM about a point c with amplitude a that $v^2 = n^2(a^2 - (x-c)^2)$.

Solution

$$\begin{aligned}
&\text{Let } x = a\sin(nt+\alpha) + c \\
&\therefore \dot{x} = an\cos(nt+\alpha) \\
&\therefore v^2 = a^2n^2\cos^2(nt+\alpha) \\
&\quad = a^2n^2\left(1 - \sin^2(nt+\alpha)\right) \\
&\quad = n^2(a^2 - a^2\sin^2(nt+\alpha)) \\
&\quad = n^2(a^2 - (a\sin(nt+\alpha) + c - c)^2) \\
&\quad = n^2(a^2 - (x-c)^2)
\end{aligned}$$

Alternatively

$$\text{Let } \ddot{x} = -n^2(x-c)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x-c)$$

Integrating from $x = c-a, v^2 = 0$ to $x = x, v^2 = v^2$

$$\frac{1}{2}v^2 = -n^2\int_{c-a}^{x}(x-c)\,dx$$

$$v^2 = 2n^2\left[\frac{x^2}{2} - cx\right]_x^{c-a}$$

$$\begin{aligned}
&= 2n^2\left(\left(\frac{(c-a)^2}{2} - c(c-a)\right) - \left(\frac{x^2}{2} - cx\right)\right) \\
&= n^2(c^2 - 2ac + a^2 - 2c^2 + 2ac - x^2 + 2cx) \\
&= n^2\left(a^2 - (x^2 - 2cx + c^2)\right) \\
&= n^2(a^2 - (x-c)^2)
\end{aligned}$$

Example 3

A particle is moving in SHM about the point $x = 2$ with period $\frac{\pi}{2}$, and initially the particle is at rest at the origin.

i Derive an equation for v^2 as a function of displacement, x.

ii Find all values of x for which the particle is at rest.

iii Find the maximum velocity of the particle

Solution

i $\frac{2\pi}{n} = \frac{\pi}{2} \rightarrow \quad n = 4$

$\therefore \ddot{x} = -16(x - 2)$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -16(x - 2)$$

$$\frac{1}{2}v^2 = -16\int_0^x (x - 2)\,dx$$

$$v^2 = 32\left[\frac{x^2}{2} - 2x\right]_x^0$$

$$= 32\left(0 - \left(\frac{x^2}{2} - 2x\right)\right)$$

$$= 64x - 16x^2$$

$$= 16x(4 - x)$$

ii Let $v = 0$

$\therefore 0^2 = 16x(4 - x)$

$x = 0,4$

iii The maximum velocity occurs at the centre of motion, $x = 2$, when the particle is moving to the right.

$v_{\text{max}}^2 = 16(2)\big(4 - (2)\big)$

$v_{\text{max}} = \sqrt{64}$

$= 8$

Example 4

A particle is moving in SHM with $v^2 = 24 - 8x - 2x^2$.

i Find an expression for the acceleration of the particle in terms of x.

ii Find the centre of motion and period

Solution

i

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(12 - 4x - x^2\right)$$

$$= -4 - 2x$$

$$= -2(x + 2)$$

$$= -\left(\sqrt{2}\right)^2\left(x - (-2)\right)$$

ii

$c = -2, n = \sqrt{2}$

$$T = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$$

Example 5

A particle is moving with equation of motion $v^2 + 9x^2 = k$, where k is a positive constant.

Show that the particle is in SHM with period $\frac{2\pi}{3}$.

Solution

$v^2 = k - 9x^2$

$$\begin{aligned}\ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\\ &= \frac{1}{2} \times \frac{d}{dx}\left(k - 9x^2\right)\\ &= \frac{1}{2}(-18x)\\ &= -3^2 x\end{aligned}$$

∴ the particle is in SHM with $n = 3$, so the period is $\frac{2\pi}{3}$.

Example 6

A particle is moving in SHM with $v^2 = 16(5 + 4x - x^2)$. Find a possible equation for displacement as a function of time.

Solution

$$\begin{aligned}v^2 &= 16(5 + 4x - x^2)\\ &= 4^2\left(-(x^2 - 4x - 5)\right)\\ &= 4^2\left(-(x^2 - 4x + 4 - 9)\right)\\ &= 4^2\left(3^2 - (x - 2)^2\right)\end{aligned}$$

Given $v^2 = n^2(a^2 - (x - c)^2)$ $\therefore n = 4, a = 3$ and $c = 2$

A possible equation of motion is $x = 3\sin(4t) + 2$.

EXERCISE 6.4

BASIC

1 A particle is moving in simple harmonic motion with displacement x. Its velocity v is given by $v^2 = 16(9 - x^2)$. What is the amplitude and period of the motion?

2 A particle is moving in simple harmonic motion. The displacement of the particle is x and its velocity, v, is given by the equation $v^2 = n^2(2kx - x^2)$, where n and k are constants. The particle is initially at $x = k$. Find a possible equation for the displacement of the particle as a function of time.

3 A particle moves with equation of motion $x = \sqrt{3}\cos 3t - \sin 3t - 2$ metres. Prove that the particle is in SHM, and find the centre and amplitude of its motion.

4 A particle is moving in SHM about the point $x = 1$ with period $\frac{\pi}{4}$, and initially the particle is at rest at the origin.

i Derive an equation for v^2 as a function of displacement, x.

ii Find all values of x for which the particle is at rest.

iii Find the maximum velocity of the particle

5 A particle is moving in SHM with $v^2 = 8 - 4x - 4x^2$.

i Find an expression for the acceleration of the particle in terms of x.

ii Find the centre of motion and period

6 A particle is moving with equation of motion $v^2 + x^2 = 4$.
Show that the particle is in SHM with period 2π.

7 A particle is moving in SHM with $v^2 = 25(3 - 2x - x^2)$. Find a possible equation for displacement as a function of time.

8 A particle is moving in SHM with $v^2 = -4(x - 5)(x + 1)$. For what value of x is acceleration a maximum.

9 The displacement x of a particle at time t is given by $x = 5\sin 4t + 12\cos 4t$. What is the maximum velocity of the particle?

MEDIUM

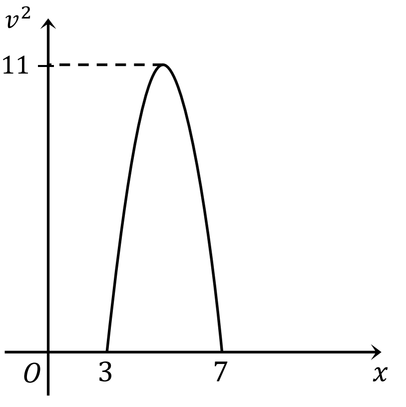
10 Prove for a particle in SHM about a point c with amplitude a that $v^2 = n^2(a^2 - (x - c)^2)$.

11 A particle is moving in a straight line according to the equation $x = 5 + 6\cos 2t + 8\sin 2t$, where x is the displacement in metres and t is the time in seconds.

i Prove that the particle is moving in simple harmonic motion by showing that x satisfies an equation of the form $\ddot{x} = -n^2(x - c)$.

ii When is the displacement of the particle zero for the first time?

12 A particle is moving along the x-axis in simple harmonic motion. The displacement of the particle is x metres and its velocity is $v\ \mathrm{ms}^{-1}$. The parabola at right shows v^2 as a function of x.



i For what value(s) of x is the particle at rest?

ii What is the maximum speed of the particle?

iii The velocity of the particle is given by the equation $v^2 = n^2(a^2 - (x-c)^2)$, where a, c and n are positive constants.

What are the values of a, c and n?

13 **i** Verify that a particle with displacement given by $x = A\cos nt + B\sin nt$, where A and B are constants, is in simple harmonic motion.

ii The particle is initially at the origin and moving with velocity $2n$.

Find the values of A and B.

iii When is the particle first at its greatest distance from the origin?

iv What is the total distance the particle travels between $t = 0$ and $t = \frac{2\pi}{n}$?

14 The equation of motion for a particle moving in simple harmonic motion is given by $\frac{d^2x}{dt^2} = -n^2x$, where n is a positive constant, x is the displacement of the particle and t is time.

i Show that the square of the velocity of the particle is given by $v^2 = n^2(a^2 - x^2)$, where $v = \frac{dv}{dt}$ and a is the amplitude of the motion.

ii Find the maximum speed of the particle.

iii Find the maximum acceleration of the particle.

iv The particle is initially at the origin. Write a formula for x as a function of t, and hence find the first time that the particle's speed is half its maximum speed.

15 The velocity, $v\ \mathrm{ms}^{-1}$, of a particle moving in simple harmonic motion along the x-axis is given by $v^2 = 8 - 2x - x^2$, where x is in metres.

i Find the centre of the motion, and the two extreme points of motion

ii Find the maximum speed

iii Find an expression for the acceleration of the particle in terms of x.

16 A particle P is moving in simple harmonic motion. At time t seconds, its acceleration is given by $\ddot{x} = -9(x-2)$, where x metres is the displacement from the origin O. Initially the particle is at O and its velocity is $8\ \mathrm{ms}^{-1}$.

i Find the centre and period of motion

ii Show that $v^2 = 64 + 36x - 9x^2$.

iii Find the maximum speed of the particle.

17 A particle is moving in a straight line and performing simple harmonic motion. At time t seconds it has displacement x metres from a fixed point O on the line, given by

$x = 2\cos\left(2t - \frac{\pi}{4}\right)$, velocity $v \text{ ms}^{-1}$ and acceleration $\ddot{x} \text{ ms}^{-2}$.

i Show that $v^2 - x\ddot{x} = 16$

ii Sketch the graph of x as a function of t for $0 \leq t \leq \pi$ clearly showing the coordinates of the endpoints.

iii Show that the particle first returns to its starting point after one quarter of its period.

iv Find the time taken by the particle to travel the first 100 metres of its motion.

18 A particle moves in such a way that its displacement, x cm, from the origin at any time is given by the function $x = 2 + \cos^2 t$, where t is in seconds.

i Show that acceleration is given by $\ddot{x} = 10 - 4x$

ii Prove $v^2 = -4x^2 + 20x - 24$

1

$v^2 = 4^2(3^2 - x^2)$

$\therefore n = 4, a = 3, T = \dfrac{2\pi}{n} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

2

$$\begin{aligned} v^2 &= n^2\big(-(-2kx + x^2)\big) \\ &= n^2\big(k^2 - (k^2 - 2kx + x^2)\big) \\ &= n^2\big(k^2 - (k - x)^2\big) \end{aligned}$$

$\therefore n^2(k^2 - (x - k)^2) \equiv n^2(a^2 - (x - b)^2)$

The amplitude is k and the centre of motion is k. Possible equations of motion include $x = k\sin nt + k$ and $x = k\cos\left(nt + \frac{\pi}{2}\right) + k$

3

Let $\sqrt{3}\cos 3t - \sin 3t = R\cos(3t + \alpha)$

$\therefore R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\alpha = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

$\therefore x = 2\cos\left(3t + \dfrac{\pi}{6}\right) - 2$

This is in the form $x = a\cos(nt + \alpha) + c$, so the particle is in SHM. The centre of motion is -2 and the amplitude is 2.

Alternatively

$$\begin{aligned} x &= \sqrt{3}\cos 3t - \sin 3t - 2 \\ \dot{x} &= -3\sqrt{3}\sin 3t - 3\cos 3t \\ \ddot{x} &= -9\sqrt{3}\cos 3t + 9\sin 3t \\ &= -9\big(\sqrt{3}\cos 3t - \sin 3t - 2 + 2\big) \\ &= -3^2(x + 2) \end{aligned}$$

$\therefore$ the particle is in SHM with centre -2.

Let $\sqrt{3}\cos 3t - \sin 3t = R\cos(3t + \alpha)$

$\therefore R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\therefore$ the amplitude of the motion is 2.

4

i $\frac{2\pi}{n} = \frac{\pi}{4} \rightarrow \quad n = 8$

$$\begin{aligned} \therefore \ddot{x} &= -64(x - 1) \\ \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= -64(x - 1) \\ \frac{1}{2}v^2 &= -64\int_0^x (x - 1)\,dx \\ v^2 &= -128\left[\frac{x^2}{2} - x\right]_0^x \\ &= -64x^2 + 128x \\ &= 64x(2 - x) \end{aligned}$$

ii Let $v = 0$

$\therefore 0^2 = 64x(2 - x)$

$x = 0,2$

iii The maximum velocity occurs at the centre of motion, $x = 1$, when the particle is moving to the right.

$v_{\text{max}}^2 = 64(1)\big(2 - (1)\big)$

$v_{\text{max}} = \sqrt{64}$

$= 8$

5

i

$$\begin{aligned} \ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\ &= \frac{d}{dx}\left(4 - 2x - 2x^2\right) \\ &= -2 - 4x \\ &= -4\left(x + \frac{1}{2}\right) \\ &= -2^2\left(x - \left(-\frac{1}{2}\right)\right) \end{aligned}$$

ii

$c = -\dfrac{1}{2}, n = 2$

$T = \dfrac{2\pi}{2} = \pi$

6

$$\begin{aligned} v^2 &= 4 - x^2 \\ \ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\ &= \frac{1}{2}\times\frac{d}{dx}\left(4 - x^2\right) \\ &= \frac{1}{2}(-2x) \\ &= -x \end{aligned}$$

$\therefore$ the particle is in SHM with $n = 1$, so the period is 2π.

7 $v^2 = 25(3 - 2x - x^2)$
$= 5^2\left(-(x^2 + 2x - 3)\right)$
$= 5^2\left(-(x^2 + 2x + 1 - 4)\right)$
$= 5^2\left(2^2 - (x + 1)^2\right)$

$\therefore n = 5, a = 2$ and $c = -1$
A possible equation of motion is
$x = 2\sin(5t) - 1$.

8 The extremes of the motion are at $x = 5, x = -1$, with maximum (positive) acceleration to the left, so $x = -1$

9
$x = 5\sin 4t + 12\cos 4t = r\sin(4t + \alpha)$
$r = \sqrt{5^2 + 12^2} = 13$
$x = 13\sin(4t + \alpha)$
$\dot{x} = 52\cos(4t + \alpha)$
since the maximum value of $\cos\theta$ is 1, the maximum value of $\dot{x}$ is 52

10

$$\begin{aligned}
&\text{Let } \ddot{x} = -n^2(x - c)\\
\therefore\ &\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x - c)\\
&\frac{1}{2}v^2 = -n^2\int_{c-a}^{x}(x - c)\,dx\\
&v^2 = -2n^2\left[\frac{x^2}{2} - cx\right]_{c-a}^{x}\\
&= -2n^2\left(\left(\frac{x^2}{2} - cx\right) - \left(\frac{1}{2}(c - a)^2 - c(c - a)\right)\right)\\
&= -2n^2\left(\frac{x^2}{2} - cx - \frac{1}{2}c^2 + ac - \frac{1}{2}a^2 + c^2 - ac\right)\\
&= n^2(-x^2 + 2cx - c^2 + a^2)\\
&= n^2\left(a^2 - (x^2 - 2cx + c^2)\right)\\
&= n^2(a^2 - (x - c)^2)
\end{aligned}$$

Alternatively

$$\begin{aligned}
&\text{Let } x = a\sin(nt + \alpha) + c\\
\therefore\ &\dot{x} = an\cos(nt + \alpha)\\
\therefore\ &v^2 = a^2n^2\cos^2(nt + \alpha)\\
&= a^2n^2\left(1 - \sin^2(nt + \alpha)\right)\\
&= n^2(a^2 - a^2\sin^2(nt + \alpha))\\
&= n^2(a^2 - (a\sin(nt + \alpha) + c\\
&= n^2(a^2 - (x - c)^2)
\end{aligned}$$

11 **i**
$x = 5 + 6\cos 2t + 8\sin 2t$
$\dot{x} = -12\sin 2t + 16\cos 2t$
$\ddot{x} = -24\cos 2t - 32\sin 2t$
$= -4(6\cos 2t + 8\sin 2t)$
$= -2^2(x - 5)$

ii
$5 + 6\cos 2t + 8\sin 2t = 0$
$6\cos 2t + 8\sin 2t = -5$
$r = \sqrt{6^2 + 8^2} = 10$
$\alpha = \tan^{-1}\left(\frac{8}{6}\right) = 0.9272\ldots$
$\therefore 10\cos(2t - 0.9272) = -5$
$\cos(2t - 0.9272) = -\frac{1}{2}$
$2t - 0.9272 = \frac{2\pi}{3}$
$t = \frac{1}{2}\left(\frac{2\pi}{3} + 0.9272\right)$
$= 1.510\ldots$
$t = 1.5\ s$ (1 dp)

12 **i**
$x = 3$ or 7

ii
$v = \sqrt{11}$

iii
$c = 5$ (centre of oscillation)
$a = 2$ (the amplitude of the motion)

$v^2 = n^2(a^2 - (x - c)^2)$
maximum velocity when $x = 5$

$11 = n^2(2^2 - (5 - 5)^2)$
$11 = 4n^2$
$n = \frac{\sqrt{11}}{2}$ $(n > 0)$

13 **i**
$x = A\cos nt + B\sin nt$
$\dot{x} = -An\sin nt + Bn\cos nt$
$\ddot{x} = -An^2\cos nt - Bn^2\sin nt$
$= -n^2(A\cos nt + B\sin nt)$
$= -n^2x$

ii
$0 = A\cos 0 + B\sin 0$
$0 = A + 0$
$A = 0$
$2n = -An\sin 0 + Bn\cos 0$
$2n = 0 + Bn$
$B = 2$

iii
Let $\dot{x} = 0$
$0\cos nt + 2n\cos nt = 0$
$\cos nt = 0$
$nt = \frac{\pi}{2}$
$t = \frac{\pi}{2n}$

iv
$t = 0\ to\ t = \frac{2\pi}{n}$ is one full cycle
$= 4 \times \text{amplitude} = 4 \times 2 = 8$

14 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x$$
$$\frac{1}{2}v^2 = -n^2\int_{-a}^{x} x\,dx$$
$$v^2 = -2n^2\left[\frac{x^2}{2}\right]_{-a}^{x}$$
$$= -2n^2\left(\frac{x^2}{2} - \frac{a^2}{2}\right)$$
$$= n^2(a^2 - x^2)$$

ii

Let $x = 0$

$v = \sqrt{n^2(a^2 - 0)} = an\ \ \text{ms}^{-1}$

iii

Let $x = a$

$$\frac{d^2x}{dt^2} = -n^2a$$

∴ maximum acceleration is n^2a

iv

$x = a\sin(nt)$

$\dot{x} = an\cos(nt)$

$let\ \dot{x} = \dfrac{an}{2}$

$\dfrac{an}{2} = an\cos(nt)\ \rightarrow\ \cos(nt) = \dfrac{1}{2}$

$nt = \dfrac{\pi}{3}\ \rightarrow\ t = \dfrac{\pi}{3n}$

15 **i**

$$v^2 = 8 - 2x - x^2$$
$$= -(x^2 + 2x - 8)$$
$$= -(x + 4)(x - 2)$$
$$= (x + 4)(2 - x)$$

The extremes of motion are $x = -4$ and $x = 2$, with the centre of motion halfway between at $c = \frac{-4+2}{2} = -1$.

ii

Maximum speed at the centre

$v^2_{\text{max}} = (-1 + 4)\big(2 - (-1)\big) = 9$

∴ $\text{speed}_{\text{max}} = 3\ \text{ms}^{-1}$

iii

$\ddot{x} = -n^2(x - c) = -(x + 1)$

16 **i**

$$\ddot{x} = -9(x - 2)$$
$$= -3^2(x - 2)$$

∴ $n = 3, c = 2$

$$T = \frac{2\pi}{3}$$

ii

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9(x - 2)$$
$$\frac{1}{2}v^2 - \frac{1}{2}(8)^2 = -9\int_{0}^{x}(x - 2)\,dx$$
$$v^2 - 64 = 18\left[\frac{x^2}{2} - 2x\right]_{x}^{0}$$
$$v^2 = \big(0 - (9x^2 - 36x)\big) + 64$$
$$= 64 + 36x - 9x^2$$

iii

Let $x = 2$

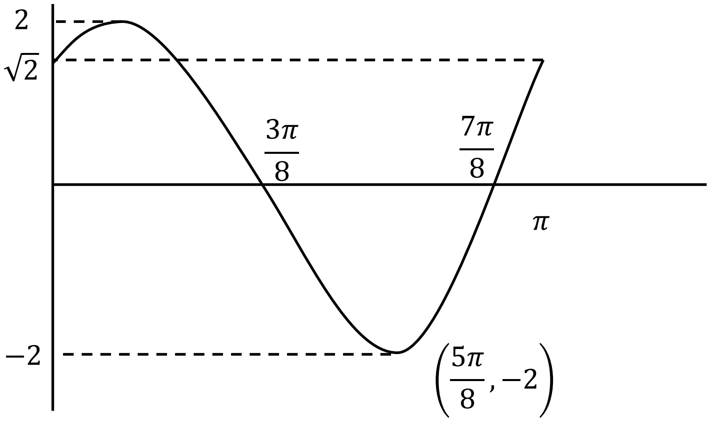
$$v^2_{\text{max}} = 64 + 36(2) - 9(2)^2$$
$$= 100$$

$\text{speed}_{\text{max}} = 10\ \text{ms}^{-1}$

17 i

$$x = 2\cos\left(2t - \frac{\pi}{4}\right)$$
$$v = -4\sin\left(2t - \frac{\pi}{4}\right)$$
$$\ddot{x} = -8\cos\left(2t - \frac{\pi}{4}\right)$$
$$\begin{aligned} v^2 - x\ddot{x} &= \left(-4\sin\left(2t - \frac{\pi}{4}\right)\right)^2 - 2\cos\left(2t - \frac{\pi}{4}\right)\left(-8\cos\left(2t - \frac{\pi}{4}\right)\right) \\ &= 16\sin^2\left(2t - \frac{\pi}{4}\right) + 16\cos^2\left(2t - \frac{\pi}{4}\right) \\ &= 16\left(\sin^2\left(2t - \frac{\pi}{4}\right) + \cos^2\left(2t - \frac{\pi}{4}\right)\right) \\ &= 16 \end{aligned}$$

ii



iii

$$\sqrt{2} = 2\cos\left(2t - \frac{\pi}{4}\right)$$
$$\cos\left(2t - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$2t - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \ldots$$
$$2t = 0, \frac{\pi}{2}, 2\pi, \ldots$$
$$t = 0, \frac{\pi}{4}, \pi, \ldots$$

The particle first returns to its starting point after $\frac{\pi}{4}$ seconds, which is one quarter of its period (from the graph).

iv

The amplitude is 2, so each cycle the particle travels $2 \times 4 = 8$ metres.
$\frac{100}{8} = 12.5$, so it will take 12.5 cycles to travel 100 metres

$$t = 12.5 \times T = \frac{25\pi}{2}$$

18 i

$$\begin{aligned} x &= 2 + \cos^2 t \\ \dot{x} &= -2\cos t \sin t \\ \ddot{x} &= -2\big(\cos t \times \cos t + \sin t \times (-\sin t)\big) \\ &= -2(\cos^2 t - \sin^2 t) \\ &= -2(2\cos^2 t - 1) \\ &= -2(2(2 + \cos^2 t - 2) - 1) \\ &= -4(x - 2) + 2 \\ &= -4x + 10 \\ &= 10 - 4x \end{aligned}$$

ii

$$\begin{aligned} x &= 2 + \cos^2 t \\ &= 2 + \frac{1}{2}(1 + \cos 2t) \\ &= \frac{1}{2}\cos 2t + \frac{5}{2} \end{aligned}$$
$$a = \frac{1}{2}, n = 2, c = \frac{5}{2}$$
$$\begin{aligned} v^2 &= (2)^2\left(\left(\frac{1}{2}\right)^2 - \left(x - \frac{5}{2}\right)^2\right) \\ &= 1 - (2x - 5)^2 \\ &= 1 - 4x^2 + 20x - 25 \\ &= -4x^2 + 20x - 24 \end{aligned}$$

6.5 RESISTED MOTION - HORIZONTAL

RESISTED MOTION - HORIZONTAL

In real life situations we also have to consider resisted motion – this is where the fluid (gas or liquid) the particle travels through, or the surface it travels over, provides a resistance to the motion. Resistance is a force, not an acceleration. Since the resistance increases as the velocity increases, acceleration is a function of velocity.

Resisted motion is covered in three lessons in this chapter:

- In this lesson we will consider motion along a horizontal line (where gravity is irrelevant)
- Next lesson we will consider motion up or down (where resistance acts with or against gravity)
- In the last lesson we will look at projectile motion in a resisted medium:
    - where motion is up then down
    - where motion starts at an angle to the horizontal

Resisted Motion questions were asked almost every year in the HSC in the old syllabus and we can expect this to continue, possible becoming more common as we also consider projectile motion with resistance.

We use either $v\frac{dv}{dx}$ or $\frac{dv}{dt}$ for motion with resistance, since acceleration is almost always a function of velocity. If the result you are trying to prove involves time then use $\frac{dv}{dt}$, otherwise use $v\frac{dv}{dx}$.

if result involves displacement and velocity* → sub $a = v\frac{dv}{dx}$

if result involves time and velocity → sub $a = \frac{dv}{dt}$

if result involves time and displacement → sub $a = \frac{dv}{dt}$ → find velocity as a function of time first then integrate for displacement as a function of time

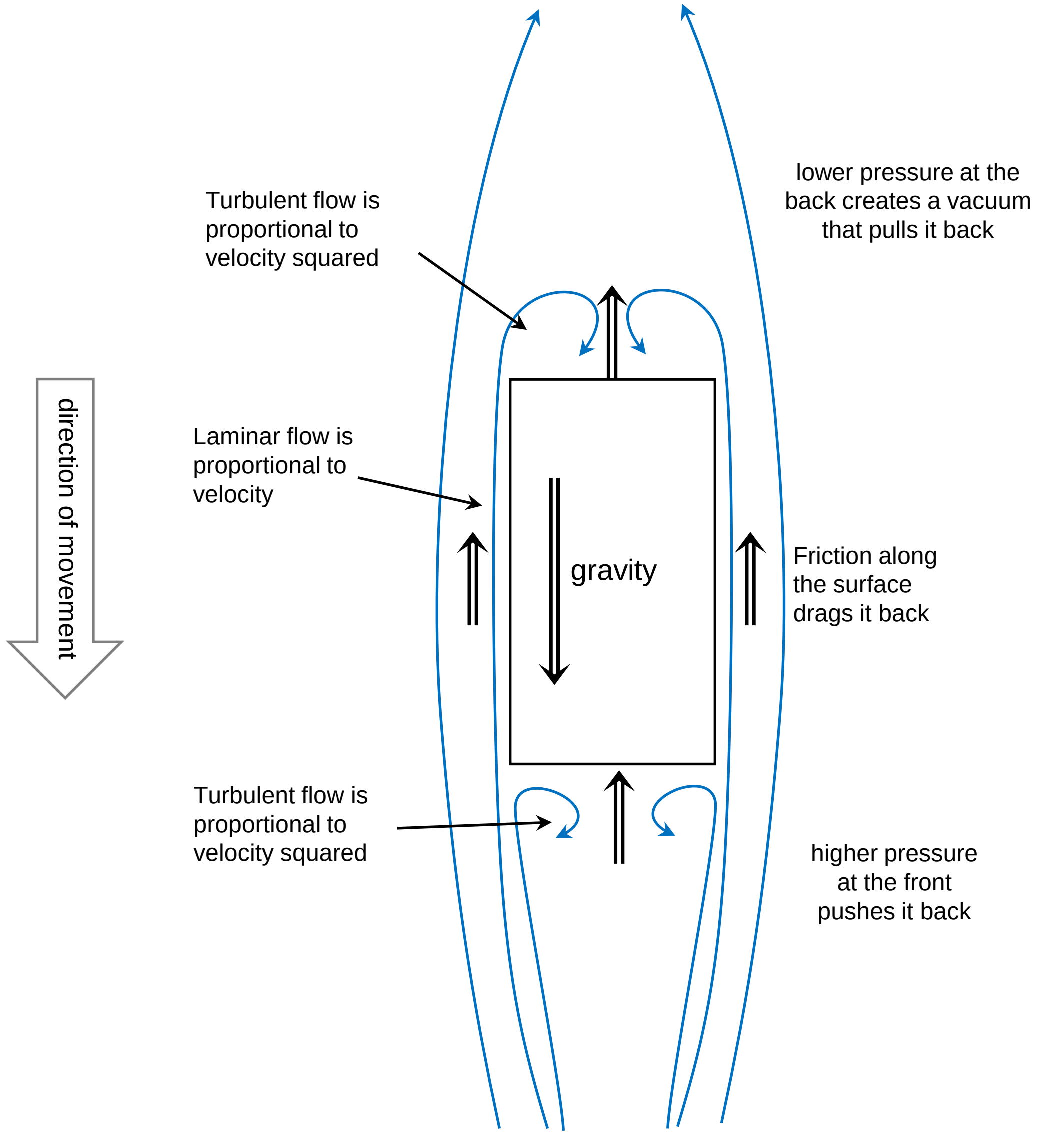
* We will see an example shortly where acceleration is constant, so we can use $v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

RESISTANCE AND HOW IT IS MODELLED

In real life we see that motion through a fluid (gas or a liquid), is resisted by a force that is:

- partly proportional to velocity (linear drag)
  - caused by friction from the laminar flow along the surface
- partly proportional to the square of velocity (quadratic drag)
  - caused by turbulence at the front and back

At lower speeds, or for very streamlined objects, air resistance is mainly proportional to velocity as there is little turbulence, so friction is most important. At higher speeds, or for less streamlined objects, turbulence occurs and is more important than friction, so resistance is mainly proportional to the square of velocity. The diagram below shows a brick falling through the air, although similar affects occur for horizontal motion.



You will come across questions where the resistance is proportional to some other power of velocity, or a combination of linear and quadratic resistance.

EQUATIONS OF MOTION

Every question will give you a simplified equation of motion to use, or enough information to create one. The resultant force is the force propelling the object (in the positive direction) plus the resistive force (which acts in the opposite direction so is negative).

LINEAR DRAG

$\underset{F}{\Longrightarrow} \qquad \underset{-kv}{\Leftarrow}$

The equation of motion is $m\ddot{x} = F - kv$

where F is the force that is propelling the object.

QUADRATIC DRAG

$\underset{F}{\Longrightarrow} \qquad \underset{-kv^2}{\Leftarrow}$

The equation of motion is $m\ddot{x} = F - kv^2$

where F is the force that is propelling the object.

RESISTANCE IS NOT A FUNCTION OF MASS

Resistance is related to surface area and shape, not mass, and so resistance is a constant times some power of velocity. Some sources of questions will include resistance in the form mkv or mkv^2 - this makes for easier calculations but is not realistic. It would imply that changing the mass would change the resistance which is usually false. Similarly we could be given resistance as an acceleration (aka retardation) of $-kv^2$ etc.

RESULTS INVOLVING DISPLACEMENT AND VELOCITY

The results that need to proved in Resisted Motion involving one of three pairs:

- displacement and velocity, for example prove $x = \frac{25}{k}\ln\left(\frac{200}{200 - kv^2}\right)$
- velocity and time, for example prove $v = 150\left(1 - e^{-\frac{t}{50}}\right)$
- displacement and time, for example prove $x = (u+2)(1 - e^{-t}) - 2t$

We will start with some examples involving displacement and time, which are generally the easiest types of results to prove (as we have seen in previous lessons). We generally replace acceleration with $v\frac{dv}{dx}$.

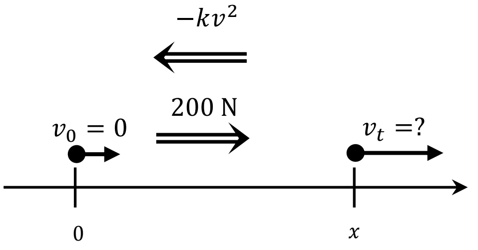
Example 1

A cart with a mass of 50 kg is pushed along a horizontal path with a force of 200 N. Friction causes a force acting against the cart's motion which is proportional to the square of the cart's velocity.

i Show that $\ddot{x} = 4 - \dfrac{kv^2}{50}$ where k is a positive constant

ii Show that the velocity of the cart, given it started from rest, is given by $x = \dfrac{25}{k}\ln\left(\dfrac{200}{200-kv^2}\right)$ where x is its displacement from its starting position.

Solution



i

$$m\ddot{x} = 200 - kv^2$$

$$\ddot{x} = \frac{200-kv^2}{50}$$

$$= 4 - \frac{kv^2}{50}$$

ii

$$\therefore v\frac{dv}{dx} = \frac{200-kv^2}{50}$$

$$\frac{dv}{dx} = \frac{200-kv^2}{50v}$$

$$\frac{dx}{dv} = \frac{50v}{200-kv^2}$$

$$x = \int_0^v \frac{50v}{200-kv^2}\,dv$$

$$= -\frac{25}{k}\int_0^v \frac{-2kv}{200-kv^2}\,dv$$

We don't need to use absolute values when integrating as $200 - kv^2 > 0$. We will look at this more closely when we look at terminal velocity.

$$= \frac{25}{k}\Big[\ln(200-kv^2)\Big]_v^0$$

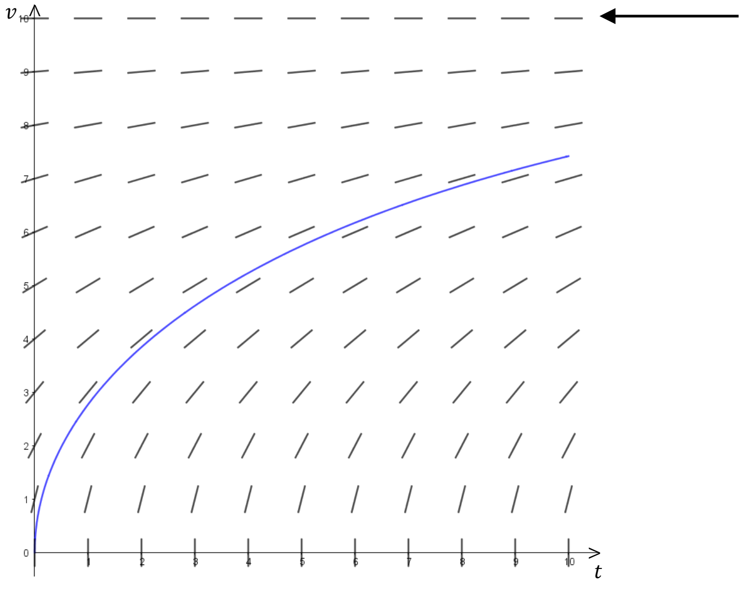
$$= \frac{25}{k}\left(\ln 200 - \ln(200-kv^2)\right)$$

$$= \frac{25}{k}\ln\left(\frac{200}{200-kv^2}\right)$$

EXPLAINING EXAMPLE 1

Letting $k = 2$, we can see that velocity is will reach a plateau of 10 m/s – this is the terminal velocity we will see next lesson.

This means that $200 - kv^2$ is always positive, which is why we did not need to take its absolute value in the example.



The line of horizontal slopes indicates an asymptote – the velocity will approach but never reach it.

RESULTS INVOLVING TIME AND VELOCITY

Examples involving velocity and time are generally a bit more involved. We generally replace acceleration with $\frac{dv}{dt}$. It is also the first step we use when finding results involving displacement and time.

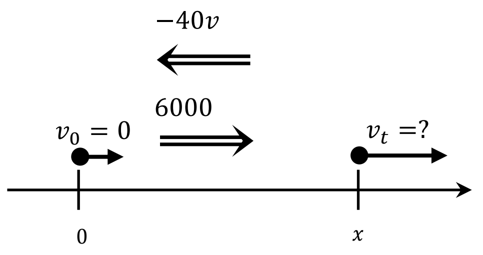
Example 2

A car of mass 2000 kg starts from rest and travels along a horizontal road. The engine produces a constant force of 6000 N, while a force of $40v$ N opposes the motion of the car.

i Show that $\dfrac{dv}{dt} = \dfrac{150 - v}{50}$

ii Prove that $v = 150\left(1 - e^{-\frac{t}{50}}\right)$

Solution



i

$$m\frac{dv}{dt} = F - R$$

$$2000\frac{dv}{dt} = 6000 - 40v$$

$$\frac{dv}{dt} = \frac{150 - v}{50}$$

ii

$$\frac{dv}{dt} = \frac{150 - v}{50}$$

$$\frac{dt}{dv} = \frac{50}{150 - v}$$

$$t = \int_0^v \frac{50}{150 - v}\,dv$$

$$= -50\Big[\ln(150 - v)\Big]_0^v$$

$$= -50\Big(\ln(150 - v) - \ln 150\Big)$$

$$= 50\ln\left(\frac{150}{150 - v}\right)$$

$$\frac{t}{50} = \ln\left(\frac{150}{150 - v}\right)$$

$$e^{\frac{t}{50}} = \frac{150}{150 - v}$$

$$150e^{\frac{t}{50}} - ve^{\frac{t}{50}} = 150$$

$$ve^{\frac{t}{50}} = 150\left(e^{\frac{t}{50}} - 1\right)$$

$$v = 150\left(1 - e^{-\frac{t}{50}}\right)$$

Example 3

A supercar has a mass of 500 kilograms and its engine generates a force of $1000\ \mathrm{N}$. Its motion is opposed by a resistive force of $\frac{v^2}{10}\ \mathrm{N}$.

a What is the maximum possible speed (terminal velocity) of the car on flat ground?

b If the car starts from rest, prove that the time taken to reach a speed of v, where $v < 100$, is given by $t = 25\ln\left(\dfrac{100+v}{100-v}\right)$

c How does this formula help support the idea that the car can never reach the terminal velocity?

Solution

a

$$500\ddot{x} = 1000 - \frac{v^2}{10}$$

$$\ddot{x} = \frac{10000 - v^2}{5000}$$

Let $\ddot{x} = 0, \therefore 10000 - v_T^2 = 0 \quad \rightarrow \quad v_T = \sqrt{10000} = 100\ \mathrm{ms}^{-1}$

b

$$\frac{dv}{dt} = \frac{10000 - v^2}{5000}$$

$$\frac{dt}{dv} = \frac{5000}{10000 - v^2}$$

$$t = 25\int_0^v \left(\frac{1}{100 - v} + \frac{1}{100 + v}\right) dv$$

$$= 25\Big[-\ln(100 - v) + \ln(100 + v)\Big]_0^v$$

$$= 25(-\ln(100 - v) + \ln(100 + v))$$

$$= 25\ln\left(\frac{100 + v}{100 - v}\right)$$

c

Let $t = 100$ in $t = 25\ln\left(\dfrac{100+v}{100-v}\right)$

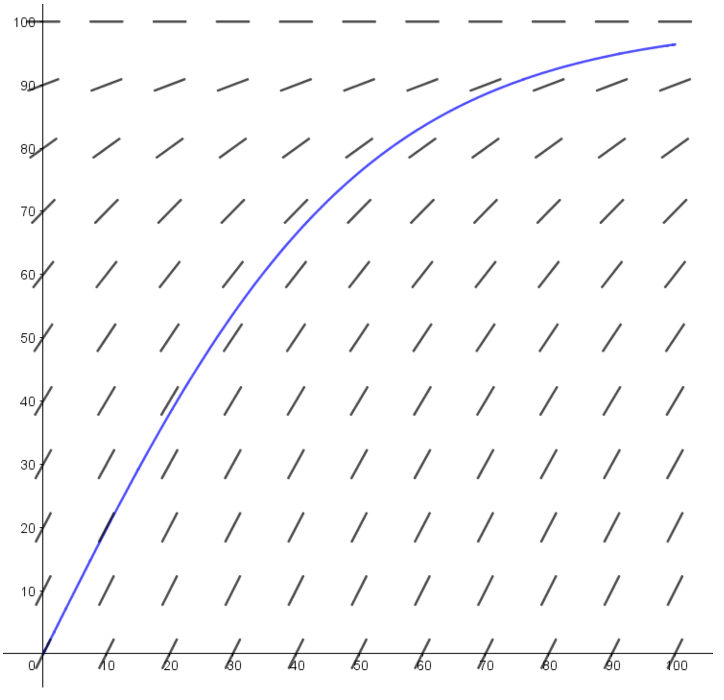
$$\therefore t = 25\ln\left(\frac{100 + v}{100 - v}\right)$$

$$= 25\ln\left(\frac{200}{0}\right) \quad \text{which is undefined}$$

As $v \rightarrow 100, \frac{100+v}{100-v} \rightarrow \infty$ so the time taken to reach terminal velocity is infinite – the car will never reach it.

EXPLAINING EXAMPLE 3

Graphing the slope field for $\frac{dv}{dt} = \frac{10000 - v^2}{5000}$, we can see that velocity is flattening out as it approaches 100 ms^{-1}.



Results Involving Time and Displacement

Examples involving displacement and time are generally the most involved. We generally replace acceleration with $\frac{dv}{dt}$ to first find an expression involving velocity as a function of time (as we have done in the last two examples) then integrate to find an expression for displacement in terms of time.

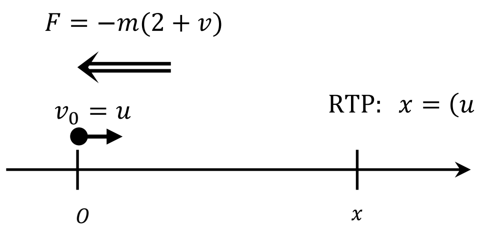
** In the next example we use a resultant force that is a function of mass. When students come across examples like this it is important to explain to them that this not reflective of real life, as resistance is unrelated to mass **

Example 4

A particle of mass m kg is set in horizontal motion with speed u ms^{-1} and experiences a resistive force and comes to rest. At time t seconds the particle has displacement x metres from its starting point O, velocity v ms^{-1} and acceleration a ms^{-2}. The resultant force acting on the particle directly opposes its motion and has magnitude $m(2+v)$ N.

Prove that $x = (u+2)(1-e^{-t}) - 2t$

Solution



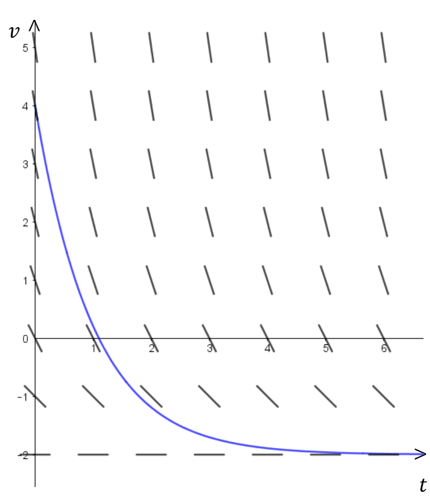
RTP: $x = (u+2)(1-e^{-t}) - 2t$

$$m\ddot{x} = -m(2+v)$$
$$\ddot{x} = -(2+v)$$
$$\frac{dv}{dt} = -(2+v)$$
$$\frac{dt}{dv} = -\frac{1}{2+v}$$
$$t = -\int_u^v \frac{1}{2+v}\,dv$$
$$= -\Big[\ln(2+v)\Big]_u^v$$
$$= \ln\frac{2+u}{2+v}$$
$$\therefore e^t = \frac{2+u}{2+v}$$
$$2e^t + ve^t = 2+u$$
$$ve^t = 2+u-2e^t$$
$$v = (2+u)e^{-t} - 2$$

$$x = \int_0^t \big((2+u)e^{-t} - 2\big)\,dt$$
$$= \Big[-(2+u)e^{-t} - 2t\Big]_0^t$$
$$= (-(2+u)e^{-t} - 2t) - (-(2+u) - 0)$$
$$= -(2+u)(e^{-t} - 1) - 2t$$
$$= (u+2)(1-e^{-t}) - 2t$$

EXPLAINING EXAMPLE 4

Letting $u = 4$ we get the slope field at right, which is only relevant until just after $t = 1$ when $v = 0$ (the object is at rest). The equation of motion then becomes irrelevant.



EXERCISE 6.5

BASIC

1** A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $mk(v+v^3)$ Newtons, its speed is v ms^{-1} and k is a positive constant. At time t seconds the particle has displacement x metres from a fixed point O on the line.

Prove $x = -\frac{1}{k}\int \frac{1}{1+v^2}dv$

2 A body is moving in a horizontal straight line. At time t seconds, its displacement is x metres from a fixed point O on the line, and its acceleration is $-\frac{1}{10}\sqrt{v}(1+\sqrt{v})$ where $v \geq 0$ is its velocity. The body is initially at O with velocity $V > 0$.

Show that $t = 20\log_e\left(\frac{1+\sqrt{V}}{1+\sqrt{v}}\right)$

MEDIUM

3 A high speed train of mass m starts from rest and moves along a straight track. At time t hours, the distance travelled by the train from its starting point is x km, and its velocity is v km/h. The train is driven by a constant force F in the forward direction. The resistive force in the opposite direction is Kv^2, where K is a positive constant. The terminal velocity of the train is 300 km/h.

i Show that the equation of motion for the train is $m\ddot{x} = F\left(1-\left(\frac{v}{300}\right)^2\right)$

ii Find, in terms of F and m, the time it takes the train to reach a velocity of 200 km/h.

4 A 20 kg trolley is pushed with a force of 100 N. Friction causes a resistive force which is proportional to the square of the trolley's velocity.

i Show that $\ddot{x} = 5 - \frac{kv^2}{20}$ where k is a positive constant.

ii If the trolley is initially stationary at the origin, show that the distance travelled when its speed V is given by

$$x = \frac{10}{k}\ln\left(\frac{100}{100-kV^2}\right)$$

5** A landing aeroplane of mass m kg is brought to rest by the action of two retarding forces: a force of $4m$ Newtons due to the reverse thrust of the engines; and a force due to the brakes of $\frac{mv^2}{40\,000}$ Newtons.

i Show that the aeroplane's equation of motion for its speed v at time t seconds after landing is

$$\dot{v} = -\frac{v^2+400^2}{40\,000}$$

ii Assuming the aeroplane lands at a speed of U m/s, find an expression for the time it takes to come to rest.

iii Show that, given a sufficiently long runway, no matter how fast its landing speed, it will always come to rest within approximately 2.6 minutes of landing.

** Resultant force is given as a function of mass which makes our calculations easier but is not reflective of real life, as resistance is unrelated to mass.

6 A car, starting from rest, moves along a straight horizontal road. The car's engine produces a constant horizontal force of magnitude 4000 Newtons. At time t seconds, the speed of the car is v ms^{-1} and a resistance force of magnitude $40v$ Newtons acts upon the car. The mass of the car is 1600 kg.

i Show that $\frac{dv}{dt} = \frac{100 - v}{40}$

ii Find the velocity of the car at time t.

7 A supercar has a mass of 1000 kilograms and its engine generates a force of 1125 N. Its motion is opposed by a resistive force of $\frac{v^2}{20}$ N.

a What is the maximum possible speed (terminal velocity) of the car on flat ground?

b If the car starts from rest, prove that the time taken to reach a speed of v, where $v < 150$, is given by $t = \frac{200}{3}\ln\left(\frac{150+v}{150-v}\right)$

c How does this formula help support the idea that the car can never reach the terminal velocity?

CHALLENGING

8 A fishing boat drifts with a current in a straight line across a fishing ground. The boat's velocity v, at time t after the start of this drift is given by $v = \mathrm{b} - (\mathrm{b} - v_0)e^{-\alpha t}$, where v_0, α and b are positive constants, and $v_0 < b$.

i Show that $\frac{dv}{dt} = \alpha(b - v)$

ii The physical significance of v_0 is that it represents the initial velocity of the boat. What is the physical significance of b?

iii Let x be the distance travelled by the boat from the start of the drift. Find x as a function of t. Hence show that

$$x = \frac{b}{\alpha}\log_e\left(\frac{b - v_0}{b - v}\right) + \frac{v^0 - v}{\alpha}$$

iv The initial velocity of the boat is $\frac{b}{10}$. How far has the boat drifted when $v = \frac{b}{2}$?

9 A particle of unit mass moves in a straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is traveling with velocity Q, where $Q > 0$.

i Explain why $\ddot{x} = -(v + v^3)$

ii Show that v is related to the displacement x by the formula $x = \tan^{-1}\left[\frac{Q - v}{1 + Qv}\right]$

iii Show that the time t which has elapsed when the particle is traveling with velocity V is given by $t = \frac{1}{2}\log_e\left[\frac{Q^2(1+V^2)}{V^2(1+Q^2)}\right]$

iv Find V^2 as a function of t.

10 A particle of mass one kg is moving in a straight line. It is initially at the origin and is travelling with velocity $\sqrt{3}$ ms^{-1}. The particle is moving against a resisting force $v + v^3$, where v is the velocity.

i Briefly explain why the acceleration of the particle is given by $\dfrac{dv}{dt} = -(v + v^3)$

ii Show that the displacement x of the particle from the origin is given by $x = \tan^{-1}\left(\dfrac{\sqrt{3} - v}{1 + v\sqrt{3}}\right)$

iii Show that the time t which has elapsed when the particle is travelling with velocity V is given by $t = \dfrac{1}{2}\log_e\left[\dfrac{3(1 + V^2)}{4V^2}\right]$

iv Find V^2 as a function of t.

v Hence find the limiting position of the particle as $t \to \infty$.

1

$$mv\frac{dv}{dx} = -mk(v + v^3)$$
$$\frac{dv}{dx} = -k(1 + v^2)$$
$$\frac{dx}{dv} = -\frac{1}{k} \times \frac{1}{1 + v^2}$$
$$x = -\frac{1}{k}\int \frac{1}{1 + v^2}\,dv$$

2

$$\frac{dv}{dt} = -\frac{1}{10}\sqrt{v}(1 + \sqrt{v})$$
$$\frac{dt}{dv} = -\frac{10}{\sqrt{v}(1 + \sqrt{v})}$$
$$t = -\int_V^v \frac{10}{\sqrt{v}(1 + \sqrt{v})}\,dv$$
$$= 20\int_v^V \frac{\frac{1}{2}v^{-\frac{1}{2}}}{1 + \sqrt{v}}\,dv$$
$$= 20\left[\ln(1 + \sqrt{v})\right]_v^V$$
$$= 20\left(\ln\left(1 + \sqrt{V}\right) - \ln(1 + \sqrt{v})\right)$$
$$= 20\ln\left(\frac{1 + \sqrt{V}}{1 + \sqrt{v}}\right)$$

3

i

The equation of motion is given by

$m\ddot{x} = F - kv^2$

At terminal velocity of 300 km/h $\ddot{x} = 0$

$\therefore 0 = F - k \times 300^2$

$k = \dfrac{F}{300^2}$

The equation of motion is:

$$m\ddot{x} = F - \frac{F}{300^2}v^2$$
$$= F\left[1 - \left(\frac{v}{300}\right)^2\right]$$

ii

$$m\ddot{x} = F\left[1 - \left(\frac{v}{300}\right)^2\right]$$
$$\frac{dv}{dt} = \frac{F}{m}\left[\frac{300^2 - v^2}{300^2}\right]$$
$$\frac{dt}{dv} = \frac{m}{F}\left(\frac{300^2}{300^2 - v^2}\right)$$
$$t = \frac{m}{F}\int_0^{200} \frac{300}{300^2 - v^2}\,dv$$
$$= \frac{300^2 m}{600F}\int_0^{200}\left(\frac{1}{300 + v} + \frac{1}{300 - v}\right)dv$$
$$= \frac{150m}{F}\left[\ln(300 + v) - \ln(300 - v)\right]_0^{200}$$
$$= \frac{150m}{F}\left[\ln\frac{300 + v}{300 - v}\right]_0^{200}$$
$$= \frac{150m}{F}\left(\ln\frac{500}{100} - \ln\frac{300}{300}\right)$$
$$= \frac{150m}{F}\ln 5 \text{ hours}$$

4

i

$$20\ddot{x} = 100 - kv^2$$
$$\ddot{x} = 5 - \frac{kv^2}{20}$$

ii

$$v\frac{dv}{dx} = 5 - \frac{kv^2}{20}$$
$$\frac{dv}{dx} = \frac{100 - kv^2}{20v}$$
$$\frac{dx}{dv} = \frac{20v}{100 - kv^2}$$
$$x = \int_0^V \frac{20v}{100 - kv^2}\,dv$$
$$= -\frac{10}{k}\left[\ln(100 - kv^2)\right]_0^V$$
$$= -\frac{10}{k}\left(\ln(100 - kV^2) - \ln 100\right)$$
$$= \frac{10}{k}\ln\left(\frac{100}{100 - kV^2}\right)$$

5 **i**

$$m\dot{v} = -\frac{mv^2}{40\ 000} - 4m$$
$$\dot{v} = -\frac{v^2 + 160\ 000}{40\ 000}$$
$$= -\frac{v^2 + 400^2}{40\ 000}$$

ii

$$\frac{dv}{dt} = -\frac{v^2 + 400^2}{40\ 000}$$
$$\frac{dt}{dv} = -\frac{40\ 000}{v^2 + 400^2}$$
$$t = -\int_U^0 \frac{40\ 000}{v^2 + 400^2}\, dv$$
$$= 40\ 000\ \left[\frac{1}{400}\tan^{-1}\frac{v}{400}\right]_0^U$$
$$= 100\left(\tan^{-1}\frac{U}{400} - 0\right)$$
$$= 100\tan^{-1}\frac{U}{400}\ \text{s}$$

iii

As $U \to \infty \quad \tan^{-1}\frac{U}{400} \to \frac{\pi}{2}$

$\therefore t \to 100 \times \frac{\pi}{2} \approx 157$ s ≈ 2.618 minutes

$\therefore$ The plane lands within approximately 2.6 minutes of landing regardless of speed.

6 **i**

$$1600\frac{dv}{dt} = 4000 - 40v$$
$$\frac{dv}{dt} = \frac{100 - v}{40}\ \text{ms}^{-2}$$

ii

$$\frac{dt}{dv} = \frac{40}{100 - v}$$
$$t = \int_0^v \frac{40}{100 - v}\, dv$$
$$= -40\left[\ln(100 - v)\right]_0^v$$
$$-\frac{t}{40} = \ln(100 - v) - \ln 100$$
$$\ln 100 - \frac{t}{40} = \ln(100 - v)$$
$$100e^{-\frac{t}{40}} = 100 - v$$
$$v = 100\left(1 - e^{-\frac{t}{40}}\right)\ \text{ms}^{-1}$$

7 **a**

$$1000\ddot{x} = 1125 - \frac{v^2}{20}$$
$$\ddot{x} = \frac{22500 - v^2}{20000}$$

Let $\ddot{x} = 0, \therefore 22500 - v_T^2 = 0 \quad \rightarrow \quad v_T = \sqrt{22500} = 150\ \text{ms}^{-1}$

b

$$\frac{dv}{dt} = \frac{22500 - v^2}{20000}$$
$$\frac{dt}{dv} = \frac{20000}{22500 - v^2}$$
$$t = \frac{200}{3}\int_0^v \left(\frac{1}{150 - v} + \frac{1}{150 + v}\right) dv$$
$$= \frac{200}{3}\left[-\ln(150 - v) + \ln(150 + v)\right]_0^v$$
$$= \frac{200}{3}(-\ln(150 - v) + \ln(150 + v))$$
$$= \frac{200}{3}\ln\left(\frac{150 + v}{150 - v}\right)$$

c

The time taken to reach $v = 150$ would be $t = \frac{200}{3}\ln\left(\frac{150+150}{150-150}\right) = \frac{200}{3}\ln\left(\frac{300}{0}\right)$ which is undefined, which is an indicator that velocity can never reach terminal velocity (slope fields give a better explanation).

8 **i**

$$\frac{dv}{dt} = \alpha(b - v_0)e^{-\alpha t}$$
$$= \alpha\left(b - (b - (b - v_0)e^{-\alpha t})\right)$$
$$= \alpha(b - v)$$

ii

'b' is the speed of the current – the boat slowly approaches the speed of the current, as a limiting value.

iii

$$v\frac{dv}{dx} = \alpha(b - v)$$
$$\frac{dv}{dx} = \frac{\alpha(b - v)}{v}$$
$$\frac{dx}{dv} = \frac{1}{\alpha} \times \frac{v}{b - v}$$
$$x = \frac{1}{\alpha}\int_{v_0}^{v} \frac{v}{b - v}\,dv$$
$$= \frac{1}{\alpha}\int_{v_0}^{v} \frac{-(b - v) + b}{b - v}\,dv$$
$$= \frac{1}{\alpha}\int_{v_0}^{v} \left(-1 + \frac{b}{b - v}\right)dv$$
$$= \frac{1}{\alpha}\Big[-v - b\ln(b - v)\Big]_{v_0}^{v}$$
$$= \frac{1}{\alpha}\left((-v - b\ln(b - v)) - (-v_0 - b\ln(b - v_0))\right)$$
$$= \frac{1}{\alpha}\left(b\ln\left(\frac{b - v_0}{b - v}\right) + v_0 - v\right)$$
$$= \frac{b}{\alpha}\ln\left(\frac{b - v_0}{b - v}\right) + \frac{v_0 - v}{\alpha}$$

iv

$$x = \frac{b}{\alpha}\ln\left(\frac{b - \frac{b}{10}}{b - \frac{b}{2}}\right) + \frac{\frac{b}{10} - \frac{b}{2}}{\alpha}$$
$$= \frac{b}{\alpha}\ln\left(\frac{\frac{9}{10}}{\frac{1}{2}}\right) + \frac{b}{\alpha}\left(-\frac{2}{5}\right)$$
$$= \frac{b}{\alpha}\left(\ln\frac{9}{5} - \frac{2}{5}\right)$$

9 **i**

$$m\ddot{x} = -m(v + v^3)$$
$$\ddot{x} = -(v + v^3)$$

ii

$$v\frac{dv}{dx} = -(v + v^3)$$
$$\frac{dv}{dx} = -(1 + v^2)$$
$$\frac{dx}{dv} = -\frac{1}{1 + v^2}$$
$$x = -\int_Q^v \frac{1}{1 + v^2}\,dv$$
$$= \Big[\tan^{-1} v\Big]_v^Q$$
$$= \tan^{-1} Q - \tan^{-1} v$$
$$= \tan^{-1}\left(\frac{\tan(\tan^{-1} Q) - \tan(\tan^{-1} v)}{1 + (\tan(\tan^{-1} Q))(\tan(\tan^{-1} v))}\right)$$
$$= \tan^{-1}\left(\frac{Q - v}{1 + Qv}\right)$$

iii

$$\frac{dv}{dt} = -(v + v^3)$$
$$\frac{dt}{dv} = -\frac{1}{v + v^3}$$
$$t = -\int_Q^V \frac{1}{v + v^3}\,dv$$
$$= \int_V^Q \left(\frac{1}{v} - \frac{v}{1 + v^2}\right) dv$$
$$= \left[\ln v - \frac{1}{2}\ln(1 + v^2)\right]_V^Q$$
$$= \left(\ln Q - \frac{1}{2}\ln(1 + Q^2)\right)$$
$$-\left(\ln V - \frac{1}{2}\ln(1 + V^2)\right)$$
$$= \left(\frac{1}{2}\ln Q^2 - \frac{1}{2}\ln(1 + Q^2)\right)$$
$$-\left(\frac{1}{2}\ln V^2 - \frac{1}{2}\ln(1 + V^2)\right)$$
$$= \frac{1}{2}\ln\frac{Q^2}{1 + Q^2} - \frac{1}{2}\ln\frac{V^2}{1 + V^2}$$
$$= \frac{1}{2}\ln\left(\frac{Q^2(1 + V^2)}{V^2(1 + Q^2)}\right)$$

iv

$$e^{2t} = \frac{Q^2(1 + V^2)}{V^2(1 + Q^2)}$$
$$V^2 e^{2t}(1 + Q^2) = Q^2 + Q^2V^2$$
$$V^2(e^{2t}(1 + Q^2) - Q^2) = Q^2$$
$$V^2 = \frac{Q^2}{e^{2t} + e^{2t}Q^2 - Q^2}$$

10 **i**

$$m\ddot{x} = -(v + v^3)$$
$$\ddot{x} = -(v + v^3)$$

ii

$$v\frac{dv}{dx} = -(v + v^3)$$
$$\frac{dv}{dx} = -(1 + v^2)$$
$$\frac{dx}{dv} = -\frac{1}{1 + v^2}$$
$$x = -\int_{\sqrt{3}}^v \frac{1}{1 + v^2}\,dv$$
$$= \Big[\tan^{-1} v\Big]_v^{\sqrt{3}}$$
$$= \tan^{-1}\sqrt{3} - \tan^{-1} v$$
$$= \tan^{-1}\left(\tan\left(\tan^{-1}\sqrt{3} - \tan^{-1} v\right)\right)$$
$$= \tan^{-1}\left(\frac{\tan\left(\tan^{-1}\sqrt{3}\right) - \tan(\tan^{-1} v)}{1 + \tan\left(\tan^{-1}\sqrt{3}\right) \times \tan(\tan^{-1} v)}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{3} - v}{1 + \sqrt{3}v}\right)$$

iii

$$\frac{dv}{dt} = -(v + v^3)$$
$$\frac{dt}{dv} = -\frac{1}{v + v^3}$$
$$t = -\int_{\sqrt{3}}^V \frac{1}{v + v^3}\,dv$$
$$= \int_V^{\sqrt{3}} \left(\frac{1}{v} - \frac{v}{1 + v^2}\right) dv$$
$$= \left[\ln v - \frac{1}{2}\ln(1 + v^2)\right]_V^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \frac{1}{2}\ln 4 - \ln V + \frac{1}{2}\ln(1 + V^2)$$
$$= \frac{1}{2}\ln 3 + \frac{1}{2}\ln(1 + V^2) - \frac{1}{2}\ln 4 - \frac{1}{2}\ln V^2$$
$$= \frac{1}{2}\ln\left(\frac{3(1 + V^2)}{4V^2}\right)$$

iv

$$e^{2t} = \frac{3(1 + V^2)}{4V^2}$$
$$4V^2e^{2t} = 3 + 3V^2$$
$$V^2(4e^{2t} - 3) = 3$$
$$V^2 = \frac{3}{4e^{2t} - 3}$$

v

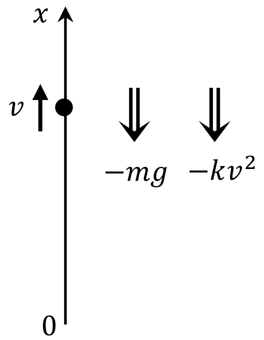
as $t \to \infty \quad e^{2t} \to \infty \quad \therefore V^2 \to 0 \quad \therefore V \to 0$

$\therefore x \to \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

VERTICAL RESISTED MOTION

When we consider objects moving vertically in a resisting medium we also need to take gravity into account. The resistive force always opposes motion, but gravity can either resist upward motion or assist downwards motion.

RISING OBJECTS

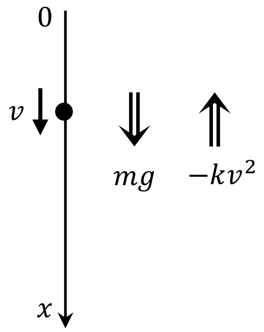


If an object is rising we take its starting point as the origin and measure upwards motion as positive.

The two forces on the object are a gravitational force, $-mg$, and a resistance, $-kv$ or $-kv^2$, given in the question. Both forces are negative as they go against the direction of motion.

The equation of motion is $m\ddot{x} = -(mg + kv^2)$ or $m\ddot{x} = -(mg + kv)$.

FALLING OBJECTS



If an object is falling we take its starting point as the origin and measure downwards motion as positive.

The two forces on the object are a gravitational force, mg, and a resistance, $-kv$ or $-kv^2$, given in the question. Gravitational force is positive since it is in the direction of motion, while resistance is negative since it is against the direction of motion.

The equation of motion is $m\ddot{x} = mg - kv^2$ or $m\ddot{x} = mg - kv$.

We will cover objects that rise then fall in Lesson 8.

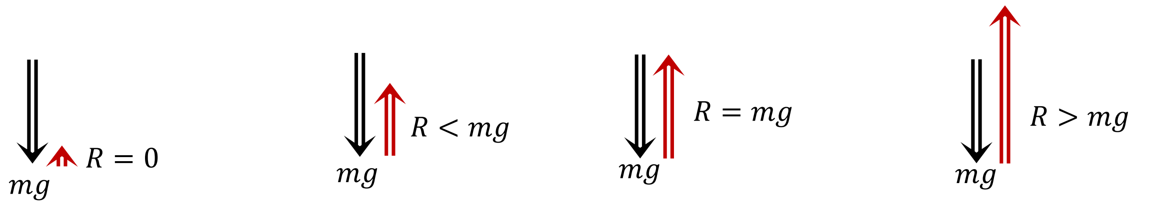
CONSTANT GRAVITATIONAL ACCELERATION

With resisted motion we assume that the object stays close enough to the surface of the Earth so that gravitational acceleration is constant, and we measure displacement from the surface of the Earth.

This is different to some of our work on Motion without Resistance where we looked at gravity being inversely proportional to the distance from the centre of the Earth. In that case the origin needs to be the centre of the Earth, with displacement measured away from the centre, as we then assume that we will travel far enough from the Earth's surface for gravitational acceleration to reduce.

TERMINAL VELOCITY

When an object is falling towards Earth, it goes faster and faster, but at some point the downward force of gravity is almost balanced by the upward force of air resistance, so the resultant force approaches zero and the velocity plateaus. This velocity is called the terminal velocity, and will be fixed for a given air resistance and mass.



When the object is first dropped it is at rest so there is no resistance. The object starts to accelerate at 9.8 ms^{-2}.

As the velocity increases the resistance increases. The acceleration reduces, but the object is still accelerating and going faster. Acceleration is less than 9.8 ms^{-2}.

The resistance continues to increase as velocity increases. Acceleration approaches zero but never quite reaches it.

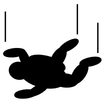
If the resistance was to suddenly increase (like opening a parachute) then resistance is greater than gravity, so the particle starts slowing towards the new terminal velocity.

Terminal Velocity is actually a limit for the velocity and is never reached - it is approached as an asymptote. In general a particle will quickly reach 90%, 99% or even 99.99% of the terminal velocity.

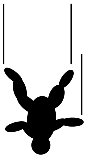
At terminal velocity acceleration (and force) is zero – this is an important technique in solving questions. Particles travelling horizontally with a constant force can also reach a terminal velocity.

UNDERSTANDING TERMINAL VELOCITY

Consider this situation:



A parachutist jumps out of a plane and starts falling in a horizontal position - they accelerate at first, with their velocity plateauing out as they approach their initial terminal velocity.



They could increase their terminal velocity by becoming more vertical, (which decreases their air resistance), causing them to accelerate towards a new higher terminal velocity.

Once they release their parachute the air resistance increases massively, so their final terminal velocity is much lower, and allows them to land at a safe speed. Note that when the parachute opens the parachutist is travelling faster than the new terminal velocity, so they have to slow towards the new terminal velocity - this is why you need to open a parachute far enough above the ground to slow to a safe speed for landing.

If they were able to change their mass on the way down (by jettisoning mass) this would also change their terminal velocity for any given position, as air resistance would become proportionally more important and slow them down.

RESULTS INVOLVING DISPLACEMENT AND VELOCITY

Again we will find that examples involving results involving displacement and velocity are generally the easiest.

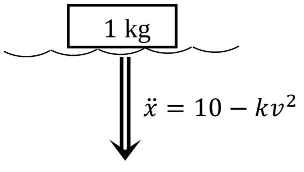
Example 1

A brick weighing 1 kg is placed on the surface of a lake then released, and sinks with an equation of motion of $\ddot{x} = 10 - kv^2$.

i If the terminal velocity of the brick is $2\ \mathrm{ms}^{-1}$, find the value of k.

ii How deep is the brick before it reaches 90% of its terminal velocity?

Solution



i

Let $\ddot{x} = 0$

$\therefore 10 - k(2^2) = 0$

$k = 2.5$

ii

$\ddot{x} = 10 - 2.5\,v^2$

$\therefore v\dfrac{dv}{dx} = 10 - 2.5v^2$

Note: the question involves displacement and velocity so we use $\ddot{x} = v\frac{dv}{dx}$.

$\dfrac{dv}{dx} = \dfrac{20 - 5v^2}{2v}$

$x = \displaystyle\int_0^{1.8} \frac{2v}{20 - 5v^2}\,dv$

90% of the terminal velocity is $1.8\ \mathrm{ms}^{-1}$

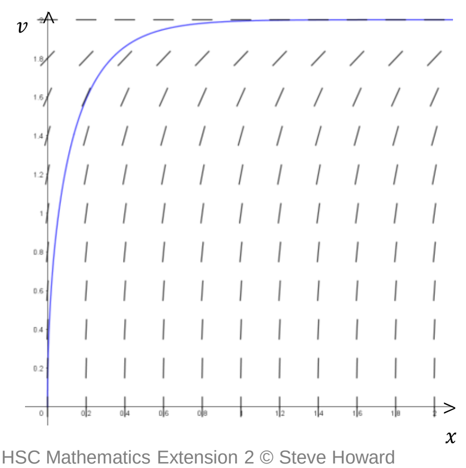
$= -\dfrac{1}{5}\Big[\ln(20 - 5v^2)\Big]_0^{1.8}$

$= -\dfrac{1}{5}\left(\ln(20 - 5(1.8)^2) - \ln 20\right)$

≈ 0.33 m

The brick reaches 90% of its terminal velocity when it is 33cm under the surface.

EXPLAINING EXAMPLE 1



The velocity approaches $2\ \mathrm{m/s}$, with the brick reaching 90% of that after moving 33 cm.

RESULTS INVOLVING TIME AND VELOCITY

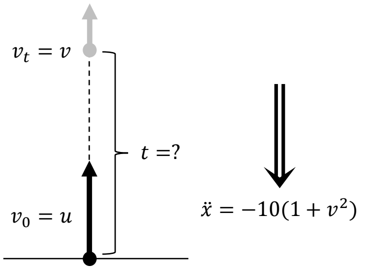
Again we will find that examples involving results involving time and velocity are generally a bit more involved.

Example 2

A particle of unit mass is projected vertically upwards with an initial velocity of u ms^{-1} in a medium in which resistance to the motion is proportional to the square of the velocity v ms^{-1} of the particle, or kv^2. Let x be the displacement in metres of the particle above the point of projection. Assume $k = 10$ and the acceleration due to gravity is 10 ms^{-2}, so that the equation of motion is $\ddot{x} = -10(1 + v^2)$.

Prove that the time of flight at any point on the upward flight can be calculated as $\dfrac{\tan^{-1} u - \tan^{-1} v}{10}$

Solution



$$\frac{dv}{dt} = -10(1 + v^2)$$

$$\frac{dt}{dv} = -\frac{1}{10} \times \frac{1}{1 + v^2}$$

$$t = -\frac{1}{10}\int_u^v \frac{1}{1 + v^2}\,dv$$

$$= -\frac{1}{10}\left[\tan^{-1} v\right]_u^v$$

$$= -\frac{1}{10}(\tan^{-1} v - \tan^{-1} u)$$

$$= \frac{\tan^{-1} u - \tan^{-1} v}{10}$$

RESULTS INVOLVING TIME AND DISPLACEMENT

Again we will find that examples involving results involving time and displacement are generally a bit more involved. We start by finding a function for velocity in terms of time then integrate.

Example 3

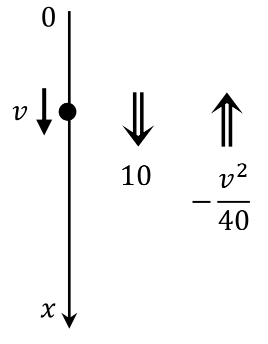
A particle of unit mass is dropped through oil which has a resistance to motion of $\frac{v^2}{40}$.
Assume $g = 10$.

i Prove that, $v = 20\left(1 - \dfrac{2}{e^t + 1}\right)$ where v is the velocity and t is the time in seconds.

ii Hence prove $x = 20\left(t + 2\ln\left(\dfrac{1 + e^{-t}}{2}\right)\right)$ where x is the distance travelled.

Solution

i



$$\begin{aligned}
\ddot{x} &= 10 - \frac{v^2}{40} \\
\frac{dv}{dt} &= \frac{400 - v^2}{40} \\
\frac{dt}{dv} &= \frac{40}{400 - v^2} \\
t &= \int_0^v \frac{40}{400 - v^2}\,dv \\
&= \int_0^v \left(\frac{1}{20 - v} + \frac{1}{20 + v}\right) dv \\
&= \Big[\ln(20 + v) - \ln(20 - v)\Big]_0^v \\
&= \ln\left(\frac{20 + v}{20 - v}\right) \\
e^t &= \frac{20 + v}{20 - v} \\
20e^t - ve^t &= 20 + v \\
v(1 + e^t) &= 20(e^t - 1) \\
v &= \frac{20(e^t - 1)}{e^t + 1} \\
&= \frac{20(e^t + 1 - 2)}{e^t + 1} \\
&= 20\left(1 - \frac{2}{e^t + 1}\right)
\end{aligned}$$

ii

$$\begin{aligned}
\frac{dx}{dt} &= 20\left(1 - \frac{2}{e^t + 1}\right) \\
x &= 20\int_0^t \left(1 - \frac{2}{e^t + 1}\right) dt \\
&= 20\int_0^t \left(1 - \frac{2e^{-t}}{1 + e^{-t}}\right) dt \\
&= 20\Big[t + 2\ln(1 + e^{-t})\Big]_0^t \\
&= 20\big((t + 2\ln(1 + e^{-t})) - (0 + 2\ln 2)\big) \\
&= 20\left(t + 2\ln\left(\frac{1 + e^{-t}}{2}\right)\right)
\end{aligned}$$

EXERCISE 6.6

BASIC

1 A mass of 1 kg is released from rest at the surface in which the retardation on the mass is proportional to the distance fallen (x). The net force for this motion is $g - kx$ Newtons, with the downward direction as positive. How far will the mass fall before acceleration is zero?

2 A particle of unit mass falls from rest from the top of a cliff in a medium where the resistive force is kv^2 . How far has it fallen when it reaches a speed half its terminal velocity?

3 A ball of mass m is projected vertically upwards with speed u. The acceleration acting against the ball is gravity plus air resistance proportional to its speed, kv. Find the time (t) taken to reach the greatest height.

MEDIUM

4 A particle of mass m is projected vertically upwards with an initial velocity of u ms^{-1} in a medium in which the resistance to the motion is proportional to the square of the velocity v ms^{-1} of the particle or mkv^2. Let x be the displacement in metres of the particle above the point of projection, O, so that the equation of motion is $\ddot{x} = -(g + kv^2)$ where g ms^{-2} is the acceleration due to gravity. Assume $k = 10$ and the acceleration due to gravity is 10 ms^{-2}. Find an expression for the time taken as a function of velocity.

5 **i** Find the constants a, b and c such that:

$$\frac{300x}{1000 + x^3} = \frac{a}{10 + x} + \frac{bx + c}{100 - 10x + x^2}$$

ii A particle of mass m kg is projected vertically upwards in a highly resistive medium at a velocity of 5 m/s. The particle is subjected to the force of gravity and to a resistance due to the medium of magnitude $\frac{mv^3}{100}$ Newtons. Given the acceleration due to gravity is 10 ms^{-2},

$(\boldsymbol{\alpha})$ State the equation of motion (if upwards is the positive direction)

$(\boldsymbol{\beta})$ Hence find the maximum height reached by the particle, (giving your answer correct to 1 decimal place).

6 A particle of unit mass falls from rest under gravity and the resistance to its motion is kv^2, where v is its speed and k is a positive constant. Prove $v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$.

7 A rock of mass m is dropped under gravity g, from rest, at the top of a cliff. The vertical distance travelled is represented by x in time t. Air resistance is proportional to the velocity v of the rock, $R = -kv$.

i Explain why $\frac{dv}{dt} = g - \frac{k}{m}v$

ii Show that $v = \frac{g}{k}\left(1 - e^{-\frac{k}{m}t}\right)$ when $t \geq 0$.

iii Show that $x = -\frac{m}{k}v + \frac{m^2 g}{k^2}\log_e\left(\frac{mg}{mg - kv}\right)$

8 A body of mass m in falling from rest, experiences air resistance of magnitude kv^2 per unit mass, where k is a positive constant.

i Write the equation of motion of the body and find the value of the terminal velocity V of the body in terms of k and g (acceleration due to gravity).

ii If w is the velocity of the body when it reaches the ground, show that the distance S fallen is given by $S = -\frac{1}{2k}\ln\left(1 - \frac{w^2}{V^2}\right)$

iii With air resistance remaining the same, prove that if the body is projected vertically upwards from the ground with velocity U, then it will attain its greatest height H where

$H = \frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right)$ and return to the ground with velocity w given by $w^{-2} = U^{-2} + V^{-2}$

9 A particle A of unit mass travels horizontally through a viscous medium. When $t = 0$, the particle is at point O with initial speed u. The resistance on particle A due to the medium is kv^2, where v is the velocity of the particle at time t and k is a positive constant. When $t = 0$, a second particle B of equal mass is projected vertically upwards from O with the same initial speed u through the same medium. It experiences both a gravitational force and a resistance due to the medium. The resistance on particle B is kw^2, where w is the velocity of the particle B at time t. The acceleration due to gravity is g.

i Show that the velocity v of particle A is given by $\frac{1}{v} = kt + \frac{1}{u}$

ii By considering the velocity w of particle B, show that

$$t = \frac{1}{\sqrt{gk}}\left(\tan^{-1}\left(\mathrm{u}\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(\mathrm{w}\sqrt{\frac{k}{g}}\right)\right)$$

iii Show that the velocity V of particle A when particle B is at rest is given by

$$\frac{1}{V} = \frac{1}{u} + \sqrt{\frac{k}{g}}\,tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

iv Hence, if u is very large, explain why $\mathrm{V} \approx \frac{2}{\pi}\sqrt{\frac{g}{k}}$

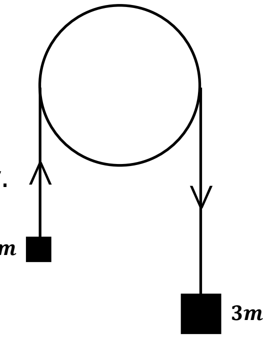
10 A particle of unit mass is projected vertically upwards. The resistance to the motion is proportional to the square of the velocity. The velocity of projection is V.

i Show that the acceleration is given by: $\ddot{x} = -(g + kv^2)$.

ii Show that the maximum height H reached is: $H = \dfrac{1}{2k}\ln\left\{\dfrac{g + kV^2}{g}\right\}$

iii Show that T, the time taken to reach H is: $T = \dfrac{1}{\sqrt{kg}}\tan^{-1}\left(\dfrac{\sqrt{k}}{\sqrt{g}}V\right)$

11 Particles of mass m and $3m$ kilograms are connected by a light inextensible string which passes over a smooth fixed pulley. The string hangs vertically on each side, as shown in the diagram.



The particles are released from rest and move under the influence of gravity. The air resistance on each particle is kv Newtons, when the speed of the particles is v ms^{-1} and the acceleration due to gravity is g ms^{-2} and is taken as positive throughout the question and is assumed to be constant. k is a positive constant.

i Draw diagrams to show the forces acting on each particle.

ii Show that the equation of motion is: $\ddot{x} = \dfrac{mg - kv}{2m}$

iii Find the terminal velocity V or maximum speed of the system stating your answer in terms of m, g and k.

iv Prove that the time since the beginning of the motion is given by: $t = \dfrac{2m}{k}\ln\left(\dfrac{mg}{mg - kv}\right)$

v If the bodies attain a velocity equal to half of the terminal speed, show by using the results in iii. and iv. that the time elapsed is equal to $\frac{V}{g}\ln 4$, where V is the terminal velocity.

12 A weight is oscillating on the end of a spring under water. Because of the resistance by the water (proportional to speed), the equation of the particle is : $\ddot{x} = -4x - 2\sqrt{3}\dot{x}$, where x is the distance in metres above equilibrium position at time t seconds. Initially the particle is at the equilibrium position, moving upwards with a speed of 3 ms^{-1}

i Find the first and second derivatives of $x = Ae^{-\sqrt{3}t}\sin t$, where A is the constant, and hence show that $x = Ae^{-\sqrt{3}t}\sin t$, is a solution of the differential equation, $\ddot{x} = -4x - 2\sqrt{3}\dot{x}$, then substitute the initial conditions to find A.

ii At what times during the first 2π seconds is the particle moving downwards?

1 constant velocity when force is zero

$$0 = g - kx$$
$$kx = g$$
$$x = \frac{g}{k}$$

2

$$\ddot{x} = g - kv^2$$
$$0 = g - kV_T^2 \Rightarrow V_T^2 = \frac{g}{k} \Rightarrow V_T = \sqrt{\frac{g}{k}}$$
$$v\frac{dv}{dx} = g - kv^2$$
$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$
$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$
$$x = \int_0^{\frac{1}{2}\sqrt{\frac{g}{k}}} \frac{v}{g - kv^2}\,dv$$
$$= \frac{1}{2k}\Big[\ln(g - kv^2)\Big]_{\frac{1}{2}\sqrt{\frac{g}{k}}}^{0}$$
$$= \frac{1}{2k}\left(\ln g - \ln\left(g - k\left(\frac{g}{4k}\right)\right)\right)$$
$$= \frac{1}{2k}\left(\ln g - \ln\frac{3g}{4}\right)$$
$$= \frac{1}{2k}\ln\left(\frac{4}{3}\right)$$

3

$$m\ddot{x} = -mg - kv$$
$$\frac{dv}{dt} = -\frac{mg + kv}{m}$$
$$\frac{dt}{dv} = -\frac{m}{mg + kv}$$
$$t = -\int_u^0 \frac{m}{mg + kv}\,dv$$
$$= \frac{m}{k}\Big[\ln(mg + kv)\Big]_0^u$$
$$= \frac{m}{k}\Big(\ln(mg + ku) - \ln mg\Big)$$
$$= \frac{m}{k}\ln\left(\frac{mg + ku}{mg}\right)$$

4

$$\frac{dv}{dt} = -(10 + 10v^2)$$
$$\frac{dt}{dv} = -\frac{1}{10} \times \frac{1}{1 + v^2}$$
$$t = -\frac{1}{10}\int_u^v \frac{1}{1 + v^2}\,dv$$
$$= \frac{1}{10}\Big[\tan^{-1}(v)\Big]_v^u$$
$$= \frac{1}{10}\Big(\tan^{-1} u - \tan^{-1} v\Big)$$

5

$$a = \frac{300(-10)}{100 - 10(-10) + (-10)^2} = -10$$

equating coefficients of x^2: $a + b = 0 \Rightarrow$
$b = 10$

equating constants: $100a + 10c = 0 \Rightarrow$
$c = 100$

$$\therefore \frac{300x}{1000 + x^3} = -\frac{10}{10 + x} + \frac{10x + 100}{100 - 10x + x^2}$$

$\boldsymbol{\alpha})$ $m\ddot{x} = -mg - \frac{mv^3}{100}$

$$\ddot{x} = -\left(g + \frac{v^3}{100}\right)$$
$$= -\left(10 + \frac{v^3}{100}\right)$$

5

$$\boldsymbol{\beta})\ v\frac{dv}{dx} = -\left(10 + \frac{v^3}{100}\right)$$
$$\frac{dv}{dx} = -\frac{1000 + v^3}{100v}$$
$$\frac{dx}{dv} = -\frac{100v}{1000 + v^3}$$
$$x = -\int_5^0 \frac{100v}{1000 + v^3}\,dv$$
$$= \frac{1}{3}\int_0^5 \left(-\frac{10}{10 + v} + \frac{10v + 100}{100 - 10v + v^2}\right)dv$$
$$= \frac{1}{3}\int_0^5 \left(-\frac{10}{10 + v} + \frac{5(2v - 10) + 150}{100 - 10v + v^2}\right)dv$$
$$= \frac{1}{3}\int_0^5 \left(-\frac{10}{10 + v} + \frac{5(2v - 10)}{100 - 10v + v^2} + \frac{150}{(v - 5)^2 + \left(\sqrt{75}\right)^2}\right)dv$$
$$= \frac{1}{3}\left[-10\ln(10 + v) + 5\ln(100 - 10v + v^2) + \frac{150}{\sqrt{75}}\tan^{-1}\left(\frac{v - 5}{\sqrt{75}}\right)\right]_0^5$$
$$= \frac{1}{3}\left((-10\ln 15 + 5\ln 75 + 0) - \left(-10\ln 10 + 5\ln 100 + \frac{150}{\sqrt{75}}\tan^{-1}\left(-\frac{5}{\sqrt{75}}\right)\right)\right)$$
$$= 1.19197\ldots$$
$$= 1.2\text{ m}\ \ (1\text{ dp})$$

6

$$\ddot{x} = g - kv^2$$
$$v\frac{dv}{dx} = g - kv^2$$
$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$
$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$
$$x = \int_0^v \frac{v}{g - kv^2}\,dv$$
$$= -\frac{1}{2k}\Big[\ln(g - kv^2)\Big]_0^v$$
$$= \frac{1}{2k}\left(\ln g - \ln(g - kv^2)\right)$$
$$= \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$
$$e^{2kx} = \frac{g}{g - kv^2}$$
$$g - kv^2 = ge^{-2kx}$$
$$kv^2 = g\left(1 - e^{-2kx}\right)$$
$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$$

7

i

$$m\ddot{x} = mg - kv$$
$$\therefore \frac{dv}{dt} = g - \frac{k}{m}v$$

ii

$$\therefore \frac{dt}{dv} = \frac{m}{mg - kv}$$
$$t = \int_0^v \frac{m}{mg - kv}$$
$$= -\frac{m}{k}\Big[\ln(mg - kv)\Big]_0^v$$
$$= \frac{m}{k}\Big(\ln mg - \ln(mg - kv)\Big)$$
$$\frac{k}{m}t = \ln\frac{mg}{mg - kv}$$
$$e^{\frac{k}{m}t} = \frac{mg}{mg - kv}$$
$$mge^{kt} - kve^{\frac{k}{m}t} = mg$$
$$kve^{\frac{k}{m}t} = mg\left(e^{\frac{k}{m}t} - 1\right)$$
$$v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

iii

$$v\frac{dv}{dx} = \frac{mg - kv}{m}$$
$$\frac{dv}{dx} = \frac{mg - kv}{mv}$$
$$\frac{dx}{dv} = \frac{mv}{mg - kv}$$
$$x = m\int_0^v \frac{v}{mg - kv}\,dv$$
$$= -\frac{m}{k}\int_0^v \frac{mg - kv - mg}{mg - kv}\,dv$$
$$= -\frac{m}{k}\int_0^v \left(1 + \frac{mg}{k} \times \frac{-k}{mg - kv}\right)dv$$
$$= \frac{m}{k}\left[v + \frac{mg}{k}\ln(mg - kv)\right]_v^0$$
$$= \frac{m}{k}\left(\left(0 + \frac{mg}{k}\ln mg\right)\right.$$
$$\left. - \left(v + \frac{mg}{k}\ln(mg - kv)\right)\right)$$
$$= -\frac{m}{k}v + \frac{m^2 g}{k^2}\ln\left(\frac{mg}{mg - kv}\right)$$

8

i

$$m\ddot{x} = mg - mkv^2$$
$$\ddot{x} = g - kv^2$$
$$\therefore 0 = g - kV^2$$
$$V^2 = \frac{g}{k}$$
$$V = \sqrt{\frac{g}{k}}$$

ii

$$v\frac{dv}{dx} = g - kv^2$$
$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$
$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$
$$S = \int_0^w \frac{v}{g - kv^2}\,dv$$
$$= -\frac{1}{2k}\Big[\ln(g - kv^2)\Big]_0^w$$
$$= -\frac{1}{2k}\Big(\ln(g - kw^2) - \ln g\Big)$$
$$= -\frac{1}{2k}\ln\left(\frac{g - kw^2}{g}\right)$$
$$= -\frac{1}{2k}\left(1 - \frac{k}{g}w^2\right)$$
$$= -\frac{1}{2k}\ln\left(1 - \frac{w^2}{V^2}\right) \quad \text{from (i)}$$

iii

$$m\ddot{x} = -mg - mkv^2$$
$$\ddot{x} = -(g + kv^2)$$
$$v\frac{dv}{dx} = -(g + kv^2)$$
$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$
$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$
$$H = -\int_U^0 \frac{v}{g + kv^2}\,dv$$
$$= \frac{1}{2k}\Big[\ln(g + kv^2)\Big]_0^U$$
$$= \frac{1}{2k}(\ln(g + kU^2) - \ln g)$$
$$= \frac{1}{2k}\ln\left(\frac{g + kU^2}{g}\right)$$
$$= \frac{1}{2k}\ln\left(1 + \frac{k}{g}U^2\right)$$
$$= \frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right)$$

but $H = S$

$$\therefore \frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right) = -\frac{1}{2k}\ln\left(1 - \frac{w^2}{V^2}\right)$$
$$\ln\left(1 + \frac{U^2}{V^2}\right) = -\ln\left(1 - \frac{w^2}{V^2}\right)$$
$$\therefore \frac{V^2 + U^2}{V^2} = \frac{V^2}{V^2 - w^2}$$
$$V^4 + U^2V^2 - V^2w^2 - U^2w^2 = V^4$$
$$U^2V^2 = V^2w^2 + U^2w^2$$
$$\frac{1}{w^2} = \frac{1}{U^2} + \frac{1}{V^2}$$
$$\therefore w^{-2} = U^{-2} + V^{-2}$$

9 **i**

$$\frac{dv}{dt} = -kv^2$$
$$\frac{dt}{dv} = -\frac{1}{k} \times \frac{1}{v^2}$$
$$t = -\frac{1}{k}\int_u^v v^{-2}\,dv$$
$$= \frac{1}{k}\left[\frac{1}{v}\right]_u^v$$
$$= \frac{1}{k}\left(\frac{1}{v} - \frac{1}{u}\right)$$
$$kt = \frac{1}{v} - \frac{1}{u}$$
$$\frac{1}{v} = kt + \frac{1}{u}$$

ii

$$\frac{dw}{dt} = -g - kw^2$$
$$\frac{dt}{dw} = -\frac{1}{g + kw^2}$$
$$t = -\int_w^u \frac{1}{g + kw^2}\,dw$$
$$= \frac{1}{k}\int_w^u \frac{1}{\left(\sqrt{\frac{g}{k}}\right)^2 + w^2}\,dw$$
$$= \frac{1}{k}\left[\sqrt{\frac{k}{g}}\tan^{-1}\left(\sqrt{\frac{k}{g}}w\right)\right]_w^u$$
$$= \frac{1}{\sqrt{kg}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right)$$

iii

Let $w = 0$

$$t = \frac{1}{\sqrt{kg}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)\right)$$
$$\therefore \frac{1}{v} = k\left(\frac{1}{\sqrt{kg}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)\right)\right) + \frac{1}{u}$$
$$= \frac{1}{u} + \sqrt{\frac{k}{g}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

iv

as $u \to \infty$ $\frac{1}{u} \to 0$ and $\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) \to \frac{\pi}{2}$

$$\frac{1}{v} \to \frac{\sqrt{k}}{\sqrt{g}}\tan^{-1}\left(u\frac{\sqrt{k}}{\sqrt{g}}\right) = \frac{\pi\sqrt{k}}{2\sqrt{g}}$$
$$\therefore v \to \frac{2}{\pi}\sqrt{\frac{g}{k}}$$

10 **i**

$$m\ddot{x} = -mg - mkv^2$$
$$\ddot{x} = -(g + kv^2)$$

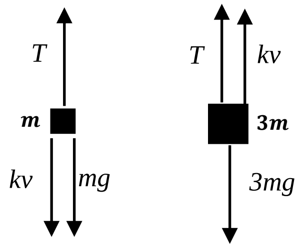
ii

$$v\frac{dv}{dx} = -(g + kv^2)$$
$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$
$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$
$$H = -\int_v^0 \frac{v}{g + kv^2}$$
$$= \frac{1}{2k}\left[\ln(g + kv^2)\right]_0^V$$
$$= \frac{1}{2k}\left(\ln(g + kV^2) - \ln g\right)$$
$$= \frac{1}{2k}\ln\left\{\frac{g + kV^2}{g}\right\}$$

iii

$$\frac{dv}{dt} = -(g + kv^2)$$
$$\frac{dt}{dv} = -\frac{1}{g + kv^2}$$
$$t = -\int_V^0 \frac{1}{g + kv^2}\,dv$$
$$= \frac{1}{k}\int_0^V \frac{1}{\left(\sqrt{\frac{g}{k}}\right)^2 + v^2}\,dv$$
$$= \frac{1}{k}\left[\sqrt{\frac{k}{g}}\tan^{-1}\left(\sqrt{\frac{k}{g}}v\right)\right]_0^V$$
$$= \frac{1}{\sqrt{kg}}\tan^{-1}\left(\frac{\sqrt{k}}{\sqrt{g}}V\right)$$

11 **i**



ii

$m\ddot{x} + 3m\ddot{x}$
$= 3mg - kv - T - mg - kv + T$

$$4m\ddot{x} = 2mg - 2kv$$
$$\ddot{x} = \frac{mg - kv}{2m}$$

iii

Let $\ddot{x} = 0$

$\therefore mg - kV = 0$

$$V = \frac{mg}{k}$$

iv

$$\frac{dv}{dt} = \frac{mg - kv}{2m}$$
$$\frac{dt}{dv} = \frac{2m}{mg - kv}$$
$$t = 2m\int_0^v \frac{1}{mg - kv}\,dv$$
$$= -\frac{2m}{k}\Big[\ln(mg - kv)\Big]_0^v$$
$$= \frac{2m}{k}\Big(\ln(mg) - \ln|(mg - kv)|\Big)$$
$$= \frac{2m}{k}\ln\left(\frac{mg}{mg - kv}\right)$$

v

Let $v = \frac{V}{2} = \frac{mg}{2k}$

$$t = \frac{2m}{k}\ln\left(\frac{mg}{mg - k\left(\frac{mg}{2k}\right)}\right)$$
$$= \frac{2m}{k}\ln\left(\frac{mg}{mg - \frac{mg}{2}}\right)$$
$$= \frac{2m}{k}\ln 2$$
$$= \frac{m}{k}\ln 4$$
$$= \frac{V}{g}\ln 4 \quad \left(\text{since } V = \frac{mg}{k}\right)$$

12 **i**

$x = Ae^{-\sqrt{3}t}\sin t$

$\dot{x} = A\left(e^{-\sqrt{3}t}\cos t + \sin t\left(-\sqrt{3}e^{-\sqrt{3}t}\right)\right)$

$\quad = Ae^{-\sqrt{3}t}\left(\cos t - \sqrt{3}\sin t\right)$

$\ddot{x} = A\left(e^{-\sqrt{3}t}\left(-\sin t - \sqrt{3}\cos t\right) + \left(\cos t - \sqrt{3}\sin t\right)\left(-\sqrt{3}e^{-\sqrt{3}t}\right)\right)$

$\quad = 2Ae^{-\sqrt{3}t}\left(\sin t - \sqrt{3}\cos t\right)$

$-4x - 2\sqrt{3}\dot{x}$

$= -4\left(Ae^{-\sqrt{3}t}\sin t\right) - 2\sqrt{3}\left(Ae^{-\sqrt{3}t}\left(\cos t - \sqrt{3}\sin t\right)\right)$

$= -4Ae^{-\sqrt{3}t}\sin t - 2\sqrt{3}Ae^{-\sqrt{3}t}\cos t + 6Ae^{-\sqrt{3}t}\sin t$

$= 2Ae^{-\sqrt{3}t}\left(\sin t - \sqrt{3}\cos t\right)$

$= \ddot{x}$

$\therefore x = Ae^{-\sqrt{3}t}\sin t$ is a solution of $\ddot{x} = -4x - 2\sqrt{3}\dot{x}$

ii

Let $\dot{x} < 0$

$3e^{-\sqrt{3}t}\left(\cos t - \sqrt{3}\sin t\right) < 0$

since $3e^{-\sqrt{3}t} > 0$ for all t, velocity is negative when

$\cos t - \sqrt{3}\sin t < 0$

Let $\cos t - \sqrt{3}\sin t = r\cos(t + \alpha)$ where

$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ and $\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$\therefore 2\cos\left(t + \frac{\pi}{3}\right) < 0$

$$\cos\left(t + \frac{\pi}{3}\right) < 0$$
$$\frac{\pi}{2} < t + \frac{\pi}{3} < \frac{3\pi}{2}$$
$$\frac{\pi}{6} < t < \frac{7\pi}{6}$$

The particle is moving downwards between $t = \frac{\pi}{6}$ and $t = \frac{7\pi}{6}$ seconds.

CARTESIAN EQUATIONS

The Cartesian equation describes the entire trajectory of the particle, while the parametric equations for horizontal and vertical displacement describe the position only at a point in time. Each form is useful for different types of questions. For example in computers to make an object appear to move along the trajectory (a Cartesian equation) requires the programming to have horizontal and vertical displacement as functions of time (parametric equations).

In Projectile Motion in Extension 1 we mainly solved questions using the parametric equations for the horizontal and vertical components of motion. We might also have to find simple Cartesian equations (like the example below) that involve only numbers and the variable t (time), so where no other constants (such as gravitational acceleration, angles or velocity) are involved. This follows from the earlier Extension 1 work on Parametric Equations.

Example 1 (Extension 1 level)

A stone is thrown from the top of a cliff. Its position t seconds after projection is given by

$\underset{\sim}{r} = \begin{pmatrix} 10\sqrt{3}t \\ 20 + 2t - 5t^2 \end{pmatrix}$. What is its Cartesian equation?

Solution

$x = 10\sqrt{3}t \qquad (1)$

$y = 20 + 2t - 5t^2 \qquad (2)$

$t = \dfrac{x}{10\sqrt{3}} \qquad \text{from } (1)$

$$\text{sub in (2):} \qquad y = 20 + 2\left(\frac{x}{10\sqrt{3}}\right) - 5\left(\frac{x}{10\sqrt{3}}\right)^2$$

$$= 20 + \frac{x}{5\sqrt{3}} - \frac{x^2}{60}$$

$$= 20 + \frac{\sqrt{3}x}{15} - \frac{x^2}{60}$$

CARTESIAN EQUATIONS IN EXTENSION 2

In Extension 2 we will combine the parametric equations involving g, V and θ to create the Cartesian equation.

The Cartesian equation of a particle fired from the origin with velocity V and angle to the horizontal θ, where the gravitational acceleration is g, where air resistance is negligible, is given by:

$$\boxed{y = -\frac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta}$$

As with all projectile motion equations you will need to derive them in the exam, or they will be given to you in the question, so the only advantage to students of memorising the formula would be for checking their answers.

Proof 2

Given the parametric equations $x = Vt\cos\theta$ and $y = -\frac{gt^2}{2} + Vt\sin\theta$, prove the Cartesian equation of motion of a projectile fired from the Origin is $y = -\frac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$.

This question would be the same if we were given the position vector $\underset{\sim}{r} = \begin{pmatrix} Vt\cos\theta \\ -\frac{gt^2}{2} + Vt\sin\theta \end{pmatrix}$

Solution

$$x = Vt\cos\theta \qquad (1)$$

$$y = -\frac{gt^2}{2} + Vt\sin\theta \qquad (2)$$

From (1):

$$t = \frac{x}{V\cos\theta}$$

Substituting into (2):

$$y = -\frac{g\left(\frac{x}{V\cos\theta}\right)^2}{2} + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$$

$$y = -\frac{gx^2}{2V^2\cos^2\theta} + x\tan\theta$$

$$y = -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta$$

$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$$

A common question using Cartesian equations involves having to clear an object at a given horizontal and vertical position. Usually this will involve a given initial velocity where you have to find the initial angle of projection. There will be two answers - one above $45°$ and one below $45°$. If the object has to clear the object (rather than just clear it) then any angle in between is sufficient.

In the next example we are given the Cartesian equation while in the example after that it must be derived first.

Example 3

A cricket player hits a ball at a velocity of $40\ \mathrm{ms}^{-1}$ and the ball just clears a 1 metre high fence which is $60\ \mathrm{m}$ away. Find the two possible angles at which the ball could have been hit, to the nearest degree. Assume there is no air resistance and that $g = 10\ \mathrm{ms}^{-2}$.

The equation of motion is $y = -\dfrac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$

Solution

$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$$

$$1 = -\frac{10 \times 60^2}{2 \times 40^2}(1 + \tan^2\theta) + 60\tan\theta$$

$$1 = -11.25(1 + \tan^2\theta) + 60\tan\theta$$

$$11.25\tan^2\theta - 60\tan\theta + 12.25 = 0$$

$$\tan\theta = \frac{60 \pm \sqrt{60^2 - 4(11.25)(12.25)}}{2(11.25)}$$

$$= 5.1207, \qquad 0.2126$$

$$\theta = 79°, \qquad 12°$$

The ball can be hit at angle of $12°$ or $79°$ and just clear the fence.

Example 4

A stone is thrown from the top of a 40 m high cliff and lands in the sea 20 m from the base. If the stone was thrown at a velocity of 10 ms^{-1} what are the possible angles of projection? Assume negligible air resistance and $g = -9.8 \text{ ms}^{-2}$.

Solution

$$\underset{\sim}{a} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\underset{\sim}{v} = \begin{pmatrix} c_1 \\ -9.8t + c_2 \end{pmatrix}$$

See next page for an alternative solution using dot notation and definite integrals.

$$\text{At } t = 0\ \underset{\sim}{v}_0 = \begin{pmatrix} 10\cos\theta \\ 10\sin\theta \end{pmatrix} \rightarrow \mathrm{c}_1 = 10\cos\theta, c_2 = 10\sin\theta$$

$$\therefore \underset{\sim}{v} = \begin{pmatrix} 10\cos\theta \\ -9.8t + 10\sin\theta \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} 10t\cos\theta + c_3 \\ -4.9t^2 + 10t\sin\theta + c_4 \end{pmatrix}$$

$$\text{At } t = 0\ \underset{\sim}{r}_0 = \begin{pmatrix} 0 \\ 40 \end{pmatrix} \rightarrow \mathrm{c}_3 = 0, c_4 = 40$$

$$\therefore \underset{\sim}{r} = \begin{pmatrix} 10t\cos\theta \\ -4.9t^2 + 10t\sin\theta + 40 \end{pmatrix}$$

At impact $x = 20, y = 0$

$$\therefore 10t\cos\theta = 20 \rightarrow t = \frac{2}{\cos\theta}$$

$$0 = -4.9\left(\frac{2}{\cos\theta}\right)^2 + 10\left(\frac{2}{\cos\theta}\right)\sin\theta + 40$$

$$0 = -4.9 \times 4\sec^2\theta + 20\tan\theta + 40$$

$$0 = -19.6(\tan^2\theta + 1) + 20\tan\theta + 40$$

$$0 = 19.6\tan^2\theta - 20\tan\theta - 20.4$$

$$\tan\theta = \frac{20 \pm \sqrt{(-20)^2 - 4(19.6)(-20.4)}}{2 \times 19.6}$$

$$\tan\theta = -0.6305, 1.6508,$$

$$\theta = -32°14', 58°48$$

The stone can be thrown up at an angle of $58°48'$ or down at an angle of $32°14'$.

Alternatively, using dot notation and definite integrals:

$$\dot{x} = 10\cos\theta$$
$$x = 10\cos\theta\, t \qquad (D = S \times T)$$
$$\text{Let } x = 20$$
$$20 = 10t\cos\theta$$
$$t = \frac{2}{\cos\theta}$$
$$\ddot{y} = -9.8$$
$$\dot{y} - 10\sin\theta = -9.8\int_0^t dt$$
$$\dot{y} - 10\sin\theta = -9.8\Big[t\Big]_0^t$$
$$\dot{y} = -9.8t + 10\sin\theta$$
$$y - 40 = \int_0^t (-9.8t + 10\sin\theta)\,dt$$
$$= \Big[-4.9t^2 + 10t\sin\theta\Big]_0^t$$
$$= -4.9t^2 + 10t\sin\theta$$
$$y = -4.9t^2 + 10t\sin\theta + 40$$
$$\text{Let } t = \frac{2}{\cos\theta}, y = 0$$
$$0 = -4.9\left(\frac{2}{\cos\theta}\right)^2 + 10\left(\frac{2}{\cos\theta}\right)\sin\theta + 40$$
$$0 = -4.9 \times 4\sec^2\theta + 20\tan\theta + 40$$
$$0 = -19.6(\tan^2\theta + 1) + 20\tan\theta + 40$$
$$0 = 19.6\tan^2\theta - 20\tan\theta - 20.4$$
$$\tan\theta = \frac{20 \pm \sqrt{(-20)^2 - 4(19.6)(-20.4)}}{2 \times 19.6}$$
$$\tan\theta = -0.6305, 1.6508,$$
$$\theta = -32°14', 58°48$$

The stone can be thrown up at an angle of $58°48'$ or down at an angle of $32°14'$.

Another similar type of question involves the projectile being projected onto sloping ground, and we need to find the point of impact. The ground will often have a slope of ± 1 to make calculations easier.

Example 5

A projectile is fired from the base of a hill which has a angle of $\frac{\pi}{4}$ to the horizontal. The projectile is fired at a velocity of $20\ \mathrm{ms}^{-1}$ at an angle to the horizontal of $\frac{\pi}{3}$. How far up the hill will the projectile land? Assume there is no air resistance and that $g = 10\ \mathrm{ms}^{-2}$.

The equation of motion is $y = -\dfrac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$

Solution

Let the hill be represented by the line $y = x$.

substitute $y = x, g = 10, V = 20, \theta = \frac{\pi}{3}$ into the equation of motion

$$x = -\frac{10 \times x^2}{2 \times 20^2}\left(1 + \left(\sqrt{3}\right)^2\right) + \sqrt{3}x$$

$$x = -\frac{x^2}{20} + \sqrt{3}x$$

$$\frac{x^2}{20} + \left(1 - \sqrt{3}\right)x = 0$$

$$x\left(\frac{x}{20} + 1 - \sqrt{3}\right) = 0$$

$$x = 0, 20\left(\sqrt{3} - 1\right)$$

$\therefore$ at impact $x = 20\left(\sqrt{3} - 1\right)$

Using exact triangles the distance to impact is $20\sqrt{2}(\sqrt{3} - 1)$ metres.

HARDER PROJECTILE MOTION

Although it was rarely tested, the old Extension 2 Mechanics syllabus actually included projectile motion and simple harmonic motion. In 2019, the last year the old syllabus was tested, we saw a question that does not involve Cartesian equations, so could be included in the HSC for the new Extension 1 course, but may indicate the level of difficulty that is intended for the new Extension 2 course.

Example 6 (2019 HSC)

Two objects are projected from the same point on a horizontal surface. Object 1 is projected with an initial velocity of $20\ \mathrm{ms}^{-1}$ directed at an angle of $\frac{\pi}{3}$ to the horizontal. Object 2 is projected 2 seconds later.

The equations of an object projected from the origin with initial velocity v at an angle θ to the x-axis are $x = vt\cos\theta$ and $y = -4.9t^2 + vt\sin\theta$, where t is the time after projection of the object. Do NOT prove these equations.

i Show that Object 1 will land at a distance $\frac{100\sqrt{3}}{4.9}$ m from the point of projection.

ii The two objects hit the horizontal plane at the same place and time. Find the initial speed and the angle of projection of Object 2, giving your answer correct to 1 decimal place.

Solution

i

Let $y = 0$

$$\therefore 0 = -4.9t^2 + 20t\sin\frac{\pi}{3}$$

$$0 = -4.9t^2 + 10\sqrt{3}t$$

$$t\left(4.9t - 10\sqrt{3}\right) = 0$$

$$t = 0, \frac{10\sqrt{3}}{4.9}$$

$\therefore$ the time of flight is $\frac{10\sqrt{3}}{4.9}$ seconds

At impact

$$x = 20\left(\frac{10\sqrt{3}}{4.9}\right)\cos\frac{\pi}{3}$$

$$= \frac{100\sqrt{3}}{4.9}\ \mathrm{m}$$

ii

Let Object 2 have time of flight $t-2=\frac{10\sqrt{3}}{4.9}-2=\frac{10\sqrt{3}-9.8}{4.9}$ and be projected at an angle of α with initial velocity U.

At impact:

$$\frac{100\sqrt{3}}{4.9}=U\left(\frac{10\sqrt{3}-9.8}{4.9}\right)\cos\alpha \qquad (1)$$

$$0=-4.9\left(\frac{10\sqrt{3}-9.8}{4.9}\right)^2+U\left(\frac{10\sqrt{3}-9.8}{4.9}\right)\sin\alpha$$

$$4.9\left(\frac{10\sqrt{3}-9.8}{4.9}\right)^2=U\left(\frac{10\sqrt{3}-9.8}{4.9}\right)\sin\alpha \qquad (2)$$

$(2)\div(1)$:

$$\frac{4.9\left(\frac{10\sqrt{3}-9.8}{4.9}\right)^2}{100\sqrt{3}/_{4.9}}=\tan\alpha$$

$$\begin{aligned}\alpha&=\tan^{-1}\left(\frac{\left(10\sqrt{3}-9.8\right)^2}{100\sqrt{3}}\right)\\&=18.1^\circ \quad (1\text{ dp})\end{aligned}$$

sub in (1):

$$\frac{100\sqrt{3}}{4.9}=U\left(\frac{10\sqrt{3}-9.8}{4.9}\right)\cos 18.1^\circ$$

$$\begin{aligned}U&=\frac{100\sqrt{3}}{\left(10\sqrt{3}-9.8\right)\cos 18.1^\circ}\\&=24.2\text{ ms}^{-1} \qquad (1\text{ dp})\end{aligned}$$

For Object 2 the initial angle and velocity are unrelated to those of Object 1, so we use new variables, not V and θ. The time of Object 2 is related to the time of Object 1, so we can still use t as a variable for time. The range of the two Objects is the same, but that does not mean the two angles are the complements of each other, as the velocities are not the same.

BASIC

1 Given the parametric equations $x = Vt\cos\theta$ and $y = -\frac{gt^2}{2} + Vt\sin\theta$, prove the Cartesian equation of motion of a projectile fired from the Origin is $y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$

2 A stone is thrown from the top of a cliff. Its parametric equations of motion are $x = 5\sqrt{2}t$ and $y = 10 + 2\sqrt{2}t - 5t^2$. What is its Cartesian equation?

3 A cricket player hits a ball at a velocity of $30\ \mathrm{ms^{-1}}$ and the ball just clears a 1 metre high fence which is $50\ \mathrm{m}$ away. Find the two possible angles at which the ball could have been hit, to the nearest degree. Assume there is no air resistance and that $g = 10\ \mathrm{ms^{-2}}$).

The equation of motion is $y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$

MEDIUM

4 A ball is kicked on level ground to clear a fence 2 metres high and 40 metres away. The initial velocity is 30 metres per second and the angle of projection is α. The displacement equations are $x = 30t\cos\alpha$ and $y = -5t^2 + 30t\sin\alpha$. (Do NOT prove these).

i Show that $y = -\frac{x^2}{180}\sec^2\alpha + x\tan\alpha$

ii Hence, or otherwise, find the angles of projection that allow the ball to clear the fence. Answer to the nearest degree.

5 A stone is thrown from the top of a $50\ \mathrm{m}$ high cliff and lands in the sea $30\ \mathrm{m}$ from the base. If the stone was thrown at a velocity of $20\ \mathrm{ms^{-1}}$ what are the possible angles of projection? Assume $g = -9.8\ \mathrm{ms^{-2}}$.

6 A paint ball is fired at a velocity of $20\ \mathrm{ms^{-1}}$, at an angle of θ to the horizontal, at a target $2.5\ \mathrm{m}$ above the ground which is $25\ \mathrm{m}$ horizontally from the point of projection. The paint ball is fired from a height of $1.5\ \mathrm{m}$. Assume $g = -9.8\ \mathrm{ms^{-2}}$.

i The equation of horizontal motion is given by $x = 20t\cos\theta$.
Derive the equations of vertical motion.

ii To avoid overhead power lines the paintball must be fired at an angle less than $45°$.
At what does it need to fired to hit the target?

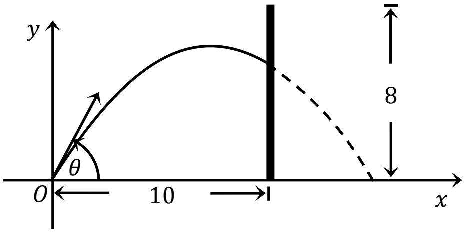
7 A ball is hit at a velocity of $50\ \mathrm{ms^{-1}}$ at an angle of projection θ where $\tan\theta = \frac{3}{4}$.

i Taking the origin as the point of projection and $g = 10\ \mathrm{ms^{-2}}$ show that $\dot{x} = 40$ and $\dot{y} = -10t + 30$, and then find x and y in terms of t.

ii A tall building is $100\ \mathrm{m}$ from where the ball is hit on horizontal ground. If the ball passes through a small open window in the building find the height of the window.

iii Find the velocity and angle that the ball makes with the horizontal as it passes through the window.

8 A small paintball is fired from the origin with initial velocity 14 metres per second towards an eight metre high barrier. The origin is at ground level 10 metres from the base of the barrier. The equations of motion are $x = 14t\cos\theta$ and $y = 14t\sin\theta - 4.9t^2$ where θ is the angle to the horizontal at which the paintball is fired and t is the time in seconds. (Do NOT prove these equations of motion.)



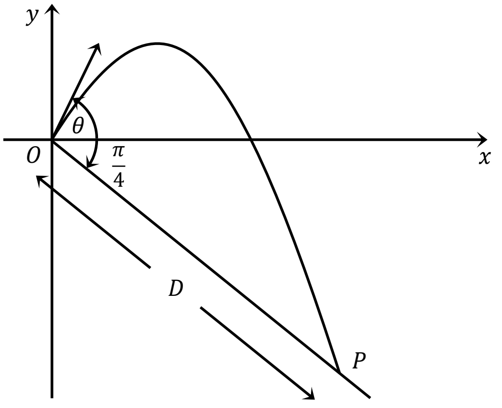
i Show that the equation of trajectory of the paintball is $y = mx - \left(\dfrac{1+m^2}{40}\right)x^2$ where $m = \tan\theta$.

ii Show that the paintball hits the barrier at height h metres when $m = 2 \pm \sqrt{3 - 0.4h}$

iii There is a large hole in the barrier. The bottom of the hole is 3.9 metres above the ground and the top of the hole is 5.9 metres above the ground. The paintball passes through the hole if m is in one of two intervals. One interval is $2.8 \le m \le 3.2$. Find the other interval.

iv Show that, if the paintball passes through the hole, the range is $\dfrac{40m}{1+m^2}$ metres. Hence find the width of the two intervals in which the paintball can land at ground level on the other side of the barrier.

9 The take-off point O on a ski jump is located at the top of a downslope. The angle between the downslope and the horizontal is $\frac{\pi}{4}$. A skier takes off from O with velocity $V\ \mathrm{ms}^{-1}$ at angle θ to the horizontal, where $0 \leq \theta < \frac{\pi}{4}$. The skier lands on the downslope at some point P, a distance D metres from O.



The flight path of the skier is given by $\underset{\sim}{r} = \begin{pmatrix} Vt\cos\theta \\ -\frac{1}{2}gt^2 + Vt\sin\theta \end{pmatrix}$, where t is the time in seconds after take-off. (Do NOT prove this.)

i Show that the Cartesian equation of the flight path of the skier is given by

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

ii Show that $D = 2\sqrt{2}\dfrac{V^2}{g}\cos\theta\left(\cos\theta + \sin\theta\right)$

iii Show that $\dfrac{dD}{d\theta} = 2\sqrt{2}\dfrac{V^2}{g}\left(\cos 2\theta - \sin 2\theta\right)$

iv Show that D has a maximum value and find the value of θ for which this occurs.

1

$$x = Vt\cos\theta \qquad (1)$$

$$y = -\frac{gt^2}{2} + Vt\sin\theta \qquad (2)$$

From (1):

$$t = \frac{x}{V\cos\theta}$$

Substituting into (2):

$$y = -\frac{g\left(\frac{x}{V\cos\theta}\right)^2}{2} + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$$

$$y = -\frac{gx^2}{2V^2\cos^2\theta} + x\tan\theta$$

$$y = -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta$$

$$y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$$

2

$$x = 5\sqrt{2}t \qquad (1)$$

$$y = 10 + 2\sqrt{2}t - 5t^2 \qquad (2)$$

$$t = \frac{x}{5\sqrt{2}} \qquad \text{from (1)}$$

sub in (2):

$$y = 10 + 2\sqrt{2}\left(\frac{x}{5\sqrt{2}}\right) - 5\left(\frac{x}{5\sqrt{2}}\right)^2$$

$$= 10 + \frac{2x}{5} - \frac{x^2}{10}$$

3

$$y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$$

$$1 = -\frac{10\times 50^2}{2\times 30^2}(1+\tan^2\theta) + 50\tan\theta$$

$$1 = -13.\dot{8}(1+\tan^2\theta) + 50\tan\theta$$

$$13.\dot{8}\tan^2\theta - 50\tan\theta + 14.\dot{8} = 0$$

$$\tan\theta = \frac{50 \pm \sqrt{50^2 - 4(13.\dot{8})(14.\dot{8})}}{2(13.\dot{8})}$$

$$= 3.2724, \qquad 0.3276$$

$$\theta = 89°, \qquad 18°$$

4

i

$$x = 30t\cos\alpha \;\rightarrow\; t = \frac{x}{30\cos\alpha}$$

$$\therefore y = -5\left(\frac{x}{30\cos\alpha}\right)^2 + 30\left(\frac{x}{30\cos\alpha}\right)\sin\alpha$$

$$= -\frac{5x^2}{900}\sec^2\alpha + x\tan\alpha$$

$$= -\frac{x^2}{180}\sec^2\alpha + x\tan\alpha$$

ii

Let $x = 40, y = 2$

$$2 = -\frac{40^2}{180}(\tan^2\alpha + 1) + 40\tan\alpha$$

$$2 = -\frac{80}{9}\tan^2\alpha - \frac{80}{9} + 40\tan\alpha$$

$$80\tan^2\alpha - 360\tan\alpha + 98 = 0$$

$$\tan\alpha = \frac{360 \pm \sqrt{(-360)^2 - 4(80)(98)}}{2(80)}$$

$$= -0.2910, 4.2090$$

$$= 16°, 77°$$

The ball will clear the fence for any angle from 16° to 77°.

5

$\underset{\sim}{a} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$

$\underset{\sim}{v} = \begin{pmatrix} c_1 \\ -9.8t + c_2 \end{pmatrix}$

At $t = 0$ $\underset{\sim}{v}_0 = \begin{pmatrix} 20\cos\theta \\ 20\sin\theta \end{pmatrix} \ \rightarrow\ c_1 = 20\cos\theta\,, c_2 = 20\sin\theta$

$\therefore \underset{\sim}{v} = \begin{pmatrix} 20\cos\theta \\ -9.8t + 20\sin\theta \end{pmatrix}$

$\underset{\sim}{r} = \begin{pmatrix} 20t\cos\theta + c_3 \\ -4.9t^2 + 20t\sin\theta + c_4 \end{pmatrix}$

At $t = 0$ $\underset{\sim}{r}_0 = \begin{pmatrix} 0 \\ 50 \end{pmatrix} \ \rightarrow\ c_3 = 0, c_4 = 50$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 20t\cos\theta \\ -4.9t^2 + 20t\sin\theta + 50 \end{pmatrix}$

At impact $x = 30, y = 0$

$\therefore 20t\cos\theta = 30 \ \rightarrow t = \dfrac{3}{2\cos\theta}$

$0 = -4.9\left(\dfrac{3}{2\cos\theta}\right)^2 + 20\left(\dfrac{3}{2\cos\theta}\right)\sin\theta + 50$

$0 = -\dfrac{4.9 \times 9}{4}\sec^2\theta + 30\tan\theta + 50$

$0 = -11.025(\tan^2\theta + 1) + 30\tan\theta + 50$

$0 = 11.025\tan^2\theta - 30\tan\theta - 38.975$

$\tan\theta = \dfrac{30 \pm \sqrt{(-30)^2 - 4(11.025)(-38.975)}}{2 \times 11.025}$

$\tan\theta = -0.9603, 3.6814,$

$\theta = -43°50', 74°48$

The stone can be thrown up at an angle of $74°48'$ or down at an angle of $43°50'$.

6 **i**

$\ddot{y} = -9.8$

$\dot{y} - 20\sin\theta = -9.8\displaystyle\int_0^t dt$

$= -9.8\Big[t\Big]_0^t$

$= -9.8t$

$\dot{y} = -9.8t + 20\sin\theta$

$y - 1.5 = \displaystyle\int_0^t (-9.8t + 20\sin\theta)\,dt$

$= \Big[-4.9t^2 + 20t\sin\theta\Big]_0^t$

$= -4.9t^2 + 20t\sin\theta - 0$

$y = -4.9t^2 + 20t\sin\theta + 1.5$

ii

At impact $x = 25, y = 2.5$

$t = \dfrac{25}{20\cos\theta} = \dfrac{5}{4\cos\theta}$

$2.5 = -4.9\left(\dfrac{5}{4\cos\theta}\right)^2 + 20\left(\dfrac{5}{4\cos\theta}\right)\sin\theta + 1.5$

$= -\dfrac{245}{32}\sec^2\theta + 25\tan\theta + 1.5$

$80 = -245(\tan^2\theta + 1) + 800\tan\theta + 48$

$245\tan^2\theta - 800\tan\theta + 277 = 0$

$\tan\theta = \dfrac{800 \pm \sqrt{(-800)^2 - 4(24)(277)}}{2(245)}$

$= 0.3937, 2.8716$

$\theta = 21°, 71°$

The paintball needs to be fired at an angle of $21°$ to miss the power lines and hit the target.

7 **i**

$$\underset{\sim}{v}_0 = \begin{pmatrix} 50\cos\left(\tan^{-1}\frac{3}{4}\right) \\ 50\sin\left(tan^{-1}\frac{3}{4}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 50 \times \frac{4}{\sqrt{3^2+4^2}} \\ 50 \times \frac{3}{\sqrt{3^2+4^2}} \end{pmatrix}$$

$$= \begin{pmatrix} 40 \\ 30 \end{pmatrix}$$

$$\underset{\sim}{a} = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$$

$$\underset{\sim}{v} = \begin{pmatrix} c_1 \\ -10t + c_2 \end{pmatrix}$$

At $t = 0$ $\underset{\sim}{v}_0 = \begin{pmatrix} 40 \\ 30 \end{pmatrix}$

$\rightarrow \quad c_1 = 40, c_2 = 30$

$$\therefore \underset{\sim}{v} = \begin{pmatrix} 40 \\ -10t + 30 \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} 40t + c_3 \\ -5t^2 + 30t + c_4 \end{pmatrix}$$

At $t = 0$ $\underset{\sim}{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\rightarrow \quad c_3 = 0, c_4 = 0$

$$\therefore \underset{\sim}{r} = \begin{pmatrix} 40t \\ -5t^2 + 30t \end{pmatrix}$$

ii

$$t = \frac{x}{40}$$

$$\therefore y = -5\left(\frac{x}{40}\right)^2 + 30\left(\frac{x}{4}\right)$$

$$= -5\left(\frac{100}{40}\right)^2 + 30\left(\frac{100}{40}\right)$$

$$= 43.75 \text{ m}$$

iii

$$t = \frac{100}{40} = 2.5$$

When $t = 2.5$

$$\dot{x} = 40$$

$$\dot{y} = -10(2.5) + 30 = 5$$

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$= \sqrt{40^2 + 5^2}$$

$$= 5\sqrt{65}$$

$$\tan\alpha = \frac{\dot{y}}{\dot{x}}$$

$$= \frac{5}{40}$$

$$\alpha = 7°8'$$

8 **i**

$$x = 14t\cos\theta \rightarrow t = \frac{x}{14\cos\theta}$$

$$y = 14\left(\frac{x}{14\cos\theta}\right)\sin\theta - 4.9\left(\frac{x}{14\cos\theta}\right)^2$$

$$= \tan\theta\, x - \frac{4.9}{14^2}\sec^2\theta\, x^2$$

$$= \tan\theta\, x - \frac{1}{40}(1 + \tan^2\theta)x^2$$

$$= mx - \frac{1+m^2}{40}x^2$$

ii

Let $x = 10, y = h$:

$$h = 10m - (1 + m^2) \times \frac{5}{2}$$

$$2h = 20m - 5 - 5m^2$$

$$5m^2 - 20m + 5 + 2h = 0$$

$$m = \frac{20 \pm \sqrt{(-20)^2 - 4(5)(5+2h)}}{2(5)}$$

$$= \frac{20 \pm \sqrt{300 - 40h}}{10}$$

$$= 2 \pm \sqrt{3 - 0.4h}$$

iii

$$2 - \sqrt{3 - 0.4 \times 3.9} \leq m \leq 2 - \sqrt{3 - 0.4 \times 5.9}$$

$$0.8 \leq m \leq 1.2$$

iv)

Let $y = 0$:

$$0 = mx - \frac{1+m^2}{40}x^2$$

$$x\left(m - \frac{1+m^2}{40}x\right) = 0$$

$$\therefore x = 0 \text{ or } m - \frac{1+m^2}{40}x = 0$$

$$\frac{1+m^2}{40}x = m$$

$$x = \frac{40m}{1+m^2} \text{ metres}$$

$m = 0.8$ Range $= \dfrac{40(0.8)}{1+0.8^2} = 19.51\ m$

$m = 1$ $\text{Range}_{\text{max}} = \dfrac{40(1)}{1+1^2} = 20\ m$

$m = 1.2$ Range $= \dfrac{40(1.2)}{1+1.2^2} = 19.67\ m$

$m = 2.8$ Range $= \dfrac{40(2.8)}{1+2.8^2} = 12.67\ m$

$m = 3.2$ Range $= \dfrac{40(3.2)}{1+3.2^2} = 11.39\ m$

The paintball can land for 49 cm from 19.51 to 20 m, or 128 cm from 11.39 to 12.67 m.

9 **i**

$x = Vt\cos\theta \ \to t = \dfrac{x}{V\cos\theta}$

$$y = -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$$

$$= -\frac{gx^2}{2V^2\cos^2\theta} + \frac{\sin\theta\, x}{\cos\theta}$$

$$= x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

ii

Let P be $\left(\frac{D}{\sqrt{2}}, -\frac{D}{\sqrt{2}}\right)$, since it lies on the line $y = -x$, and $OP = D$.

$$\therefore -\frac{D}{\sqrt{2}} = \frac{D}{\sqrt{2}}\tan\theta - \frac{g\left(\frac{D}{\sqrt{2}}\right)^2}{2V^2}\sec^2\theta$$

$$-\frac{D}{\sqrt{2}} = \frac{D}{\sqrt{2}}\tan\theta - \frac{gD^2}{4V^2}\sec^2\theta$$

$$-\frac{D}{\sqrt{2}}\cos^2\theta = \frac{D}{\sqrt{2}}\sin\theta\cos\theta - \frac{gD^2}{4V^2}$$

$$\frac{gD^2}{4V^2} - \frac{D}{\sqrt{2}}(\sin\theta\cos\theta + \cos^2\theta) = 0$$

$$D\left(\frac{gD}{4V^2} - \frac{1}{\sqrt{2}}\cos\theta\,(\sin\theta + \cos\theta)\right) = 0$$

$$\therefore D = 0 \ \ or \ \ \frac{gD}{4V^2} = \frac{1}{\sqrt{2}}\cos\theta\,(\sin\theta + \cos\theta)$$

$$D = 2\sqrt{2}\frac{V^2}{g}\cos\theta\,(\sin\theta + \cos\theta)$$

iii

$$\frac{dD}{d\theta} = 2\sqrt{2}\frac{V^2}{g}\Big[(\cos\theta + \sin\theta)(-\sin\theta) + (\cos\theta)(-\sin\theta + \cos\theta)\Big]$$

$$= 2\sqrt{2}\frac{V^2}{g}\Big[-\cos\theta\sin\theta - \sin^2\theta - \sin\theta\cos\theta + \cos^2\theta\Big]$$

$$= 2\sqrt{2}\frac{V^2}{g}\Big[\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta\Big]$$

$$= 2\sqrt{2}\frac{V^2}{g}\Big[\cos 2\theta - \sin 2\theta\Big]$$

iv

$$\frac{dD^2}{d\theta^2} = 2\sqrt{2}\frac{V^2}{g}\Big[-2\sin 2\theta - 2\cos 2\theta\Big]$$

$$= -4\sqrt{2}\frac{V^2}{g}\Big[\sin 2\theta + \cos 2\theta\Big] < 0 \quad \text{for } 0 \le \theta < \frac{\pi}{4}$$

$\therefore D$ has a maximum value

Let $\dfrac{dD}{d\theta} = 0$:

$$2\sqrt{2}\frac{V^2}{g}\Big[\cos 2\theta - \sin 2\theta\Big] = 0$$

$\cos 2\theta = \sin 2\theta$

$\tan 2\theta = 1$

$2\theta = \dfrac{\pi}{4}$

$\theta = \dfrac{\pi}{8}$

The maximum value of D occurs when $\theta = \frac{\pi}{8}$

We finish the chapter off by looking at projectile motion in a resisted medium. We look at particles that are projected vertically, which follows closely from our work in Lesson 6. We then look at particles projected at an angle to the horizontal, which mixes our work from differential equations and exponential growth and decay with projectile motion.

Exam questions on particles that rise then fall tend to have many parts which can put people off, but the individual parts are simple once you make the proper start.

Questions involving particles that rise then fall tend to involve distance and velocity, as the maximum height (usually denoted H) the velocity is zero. Less commonly they can involve time and velocity.

Looking at the next example we will see:

- In part (i) we find an expression for terminal velocity, by substituting into the equation of motion.
- In part (ii) we find a result involving displacement and velocity, so let acceleration equal $v\frac{dv}{dx}$.
- In part (iii) we find a second result involving displacement and velocity, and have to equate it to the expression from part (ii) to achieve the result.

Example 1 (2013 HSC)

A ball of mass m is projected vertically into the air from the ground with initial velocity u. After reaching the maximum height H it falls back to the ground. While in the air, the ball experiences a resistive force kv^2, where v is the velocity of the ball and k is a constant. The equation of motion when the ball falls can be written as $\mathrm{m}\dot{v} = \mathrm{mg} - \mathrm{k}v^2$ (Do NOT prove this.)

i Show that the terminal velocity v_T of the ball when it falls is $\sqrt{\dfrac{mg}{k}}$

ii Show that when the ball goes up, the maximum height H is $\mathrm{H} = \dfrac{v_T^2}{2g}\ln\left(1 + \dfrac{u^2}{v_T^2}\right)$

iii When the ball falls from height H it hits the ground with velocity w. Show that $\dfrac{1}{w^2} = \dfrac{1}{u^2} + \dfrac{1}{v_T^2}$

Solution

i

At terminal velocity $\dot{v} = 0$

$\therefore mg - kv_T^2 = 0$

$$v_T = \sqrt{\frac{mg}{k}}$$

ii

$m\dot{v} = -(mg + kv^2)$

$$\therefore v\frac{dv}{dx} = -\frac{(mg + kv^2)}{m}$$

$$\frac{dv}{dx} = -\frac{mg + kv^2}{mv}$$

$$\frac{dx}{dv} = -\frac{mv}{mg + kv^2}$$

$$\therefore H = -\int_u^0 \frac{mv}{mg + kv^2}\,dv$$

$$= -\frac{m}{2k}\Big[\ln(mg + kv^2)\Big]_u^0$$

$$= -\frac{m}{2k}\left(\ln mg - \ln(mg + ku^2)\right)$$

$$= \frac{m}{2k}\ln\frac{mg + ku^2}{mg}$$

$$= \frac{m}{2\left(\dfrac{mg}{V_T^2}\right)}\ln\frac{mg + \left(\dfrac{mg}{V_T^2}\right)u^2}{mg}$$

$$= \frac{V_T^2}{2g}\ln\left(1 + \frac{u^2}{V_T^2}\right)$$

iii

$$mv\frac{dv}{dx} = mg - \frac{mg}{V_T^2}v^2$$

$$\frac{dv}{dx} = \frac{g(V_T^2 - v^2)}{v.V_T^2}$$

$$\frac{dx}{dv} = \frac{V_T^2}{g} \times \frac{v}{V_T^2 - v^2}$$

$$\therefore\ H = \frac{V_T^2}{g}\int_0^w \frac{v}{V_T^2 - v^2}\,dv$$

$$= -\frac{V_T^2}{2g}\Big[\ln(V_T^2 - v^2)\Big]_0^w$$

$$= -\frac{V_T^2}{2g}\left(\ln(V_T^2 - w^2) - \ln(V_T^2)\right)$$

$$= \frac{V_T^2}{2g}\ln\left(\frac{V_T^2}{V_T^2 - w^2}\right)$$

$$\therefore\ \frac{v_T^2}{2g}\ln\left(1 + \frac{u^2}{v_T^2}\right) = \frac{v_T^2}{2g}\ln\left(\frac{v_T^2}{v_T^2 - w^2}\right)$$

$$\therefore 1 + \frac{u^2}{v_T^2} = \frac{v_T^2}{v_T^2 - w^2}$$

$$\frac{v_T^2 + u^2}{v_T^2} = \frac{v_T^2}{v_T^2 - w^2}$$

$v_T^4 - w^2v_T^2 + u^2v_T^2 - u^2w^2 = v_T^4$

$u^2v_T^2 = w^2v_T^2 + w^2u^2$

$$\frac{1}{w^2} = \frac{1}{u^2} + \frac{1}{v_T^2}$$ (÷ both sides by $u^2V_T^2w^2$)

Example 2

A capsule of mass m is fired straight up at speed u ms^{-1}. On the upward flight air resistance is negligible and the magnitude of the acceleration due to gravity is g.

i Use calculus to show that the maximum height attained by the capsule is $H = \frac{u^2}{2g}$

The capsule then deploys a parachute and falls back to earth, with equation of motion $\ddot{x} = g - kv^2$, where the origin is taken at the point it begins falling and downward motion is positive.

ii Show that the motion can be written in the form $\ddot{x} = k(\alpha^2 - v^2)$, where $\alpha^2 = \frac{g}{k}$.
iii Let U be the impact speed of the package. Prove $U^2 = \alpha^2\left(1 - e^{-2kH}\right)$
iv Assume that the package is launched at speed $u = \alpha$. Find the impact speed as a percentage of the launch speed.

Solution

i

$$\ddot{x} = -g$$
$$v\frac{dv}{dx} = -g$$
$$\frac{dv}{dx} = -\frac{g}{v}$$
$$\frac{dx}{dv} = -\frac{v}{g}$$
$$H = -\frac{1}{g}\int_u^0 v\,dv$$
$$= -\frac{1}{g}\left[\frac{v^2}{2}\right]_u^0$$
$$= -\frac{1}{g}\left(0 - \frac{u^2}{2}\right)$$
$$\therefore\ H = \frac{u^2}{2g}$$

ii

$$\ddot{x} = g - kv^2$$
$$= k\left(\frac{g}{k} - v^2\right)$$
$$= k(\alpha^2 - v^2)$$

iii

$$v\frac{dv}{dx} = k(\alpha^2 - v^2)$$
$$\frac{dx}{dv} = \frac{1}{k} \times \frac{v}{\alpha^2 - v^2}$$
$$H = \frac{1}{k}\int_0^U \frac{v}{\alpha^2 - v^2}\,dv$$
$$= -\frac{1}{2k}\Big[\ln(\alpha^2 - v^2)\Big]_0^U$$
$$-2kH = \ln(\alpha^2 - U^2) - \ln(\alpha^2)$$
$$-2kH = \ln\left(\frac{\alpha^2 - U^2}{\alpha^2}\right)$$
$$e^{-2kH} = \frac{\alpha^2 - U^2}{\alpha^2}$$
$$\alpha^2 e^{-2kH} = \alpha^2 - U^2$$
$$U^2 = \alpha^2\left(1 - e^{-2kH}\right)$$

iv

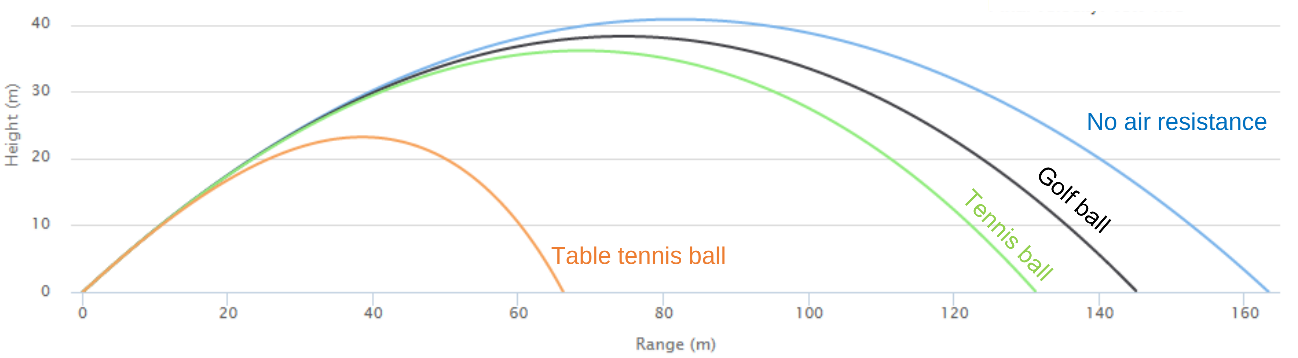
$$\frac{U}{u} \times 100\%$$
$$= \frac{\sqrt{\alpha^2\left(1 - e^{-2\left(\frac{g}{\alpha^2}\right)\left(\frac{\alpha^2}{2g}\right)}\right)}}{\alpha} \times 100\%$$
$$= \frac{100\alpha\sqrt{1 - e^{-1}}}{\alpha}\%$$
$$= 100\sqrt{\frac{e-1}{e}}$$
$$\approx 79.5\%$$

The impact speed is 79.5% of the launch speed.

RESISTED PROJECTILE MOTION AT AN ANGLE TO THE HORIZONTAL

In all the previous work we have done with projectile motion at an angle the only force involved has been gravity. Since gravity acts vertically it only affects the vertical motion, and so we have seen that the horizontal velocity remains constant. This has allowed us to create parametric or Cartesian equations without too much trouble and use them to solve a wide range of questions.

In real life the drag caused by air resistance has a large impact on the motion of the particle, as we have seen with horizontal and vertical resisted motion. The screenshot below is taken from a simulator, and shows the motion of 4 particles each with the same initial velocity ($50\ \mathrm{ms}^{-1}$) and angle of projection ($40°$), where air resistance is proportional to the square of velocity.



Source: http://galileoandeinstein.physics.virginia.edu/more_stuff/Applets/Projectile/projectile.html

The blue line shows a particle with no air resistance and its path is a parabola. The other lines represent balls where air resistance has an increasing affect as the ratio of surface area to mass increases - if we decrease mass and/or increase surface area then air resistance has more affect. The lines approximate a golf ball (black), a tennis ball (green) and a table tennis ball (orange). The table below shows how different aspects of flight are affected by air resistance.

Compared to no air resistance	Time of Flight	Maximum Height	Final Velocity	Range	Angle at Impact
Golf Ball	97%	94%	89%	89%	110%
Tennis Ball	94%	88%	82%	80%	118%
Table Tennis Ball	75%	57%	49%	41%	150%

In summary we can say that with air resistance the particle:

- has a lower maximum height
- has a shorter range
- has a shorter time of flight
- descends more steeply that it climbed
- lands at a lower velocity.

Air resistance affects the range and final velocity the most, while the time of flight is least affected.

The highest point is closer to the point of impact than the point of projection - the RHS of the path is compressed horizontally more than the LHS. Although not part of the course, it can be shown that air resistance does not affect the trajectory much until after approximately $t = \frac{V_T}{g}$, where V_T is the terminal velocity and g the gravitational acceleration. Terminal velocity still occurs when the vertical air resistance balances gravity.

Example 3

A particle is projected with velocity 40 ms^{-1} at an angle of $30°$ to the horizontal in a resistive medium. It reached a maximum height of 15.5 m and lands 87.4 m away from its projection point. Which of the following statements cannot be true?

A The maximum height occurs when the particle has travelled 48 m horizontally
B The velocity at impact is 24.0 ms^{-1}.
C The angle to the horizontal at impact is $24°$

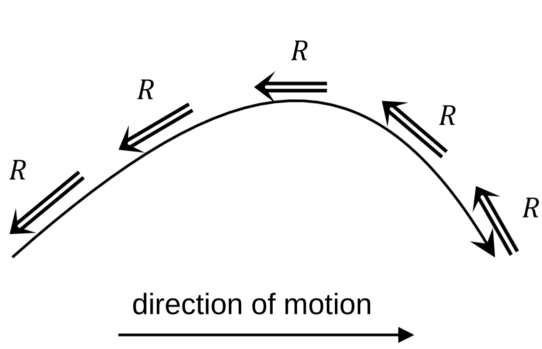
Solution

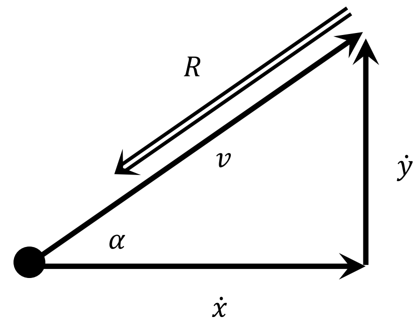
A The maximum height occurs more than half way along the range, so is possible
B The impact velocity is less than the projection velocity, so is possible.
C The impact angle is less than the projection angle, which is impossible.

ANSWER (C)

SPLITTING AIR RESISTANCE INTO COMPONENTS

Air resistance acts in the opposite direction to the motion of the particle, so as the particle changes direction and magnitude along its trajectory, the direction and magnitude of the air resistance have to change as well. To create equations of motion we have to be able to split air resistance into horizontal and vertical components.



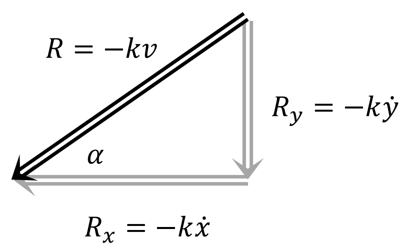


LINEAR DRAG

For linear drag we can do this easily, and so most questions on projectile motion with resistance will use linear drag. For linear drag we have overall air resistance of $-kv$ which can be split into $-k\dot{x}$ and $-k\dot{y}$ using trigonometry or vectors.

Proof 4

A particle is moving with velocity v in a medium where resistance to motion is proportional to velocity and in the opposite direction, so $R = -kv$ where k is a positive constant. If v has horizontal and vertical components $\dot{x}$ and $\dot{y}$ respectively, prove resistance has horizontal and vertical components $-k\dot{x}$ and $-k\dot{y}$ respectively.



Solution

Using trigonometry

Let the angle of motion to the horizontal be α

$\dot{x} = v\cos\alpha\,, \dot{y} = v\sin\alpha$

$R_x = -kv\cos\alpha$

$\quad = -k\dot{x}$

$R_y = -kv\sin\alpha$

$\quad = -k\dot{y}$

Alternatively using vectors

$R = -kv$

$$= -k\left(\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}\right)$$

$$= -k\dot{x}\underset{\sim}{i} - k\dot{y}\underset{\sim}{j}$$

$\therefore R_x = -k\dot{x}$ and $R_y = -k\dot{y}$

$\therefore$ Linear drag has horizontal and vertical components $-k\dot{x}$ and $-k\dot{y}$ respectively.

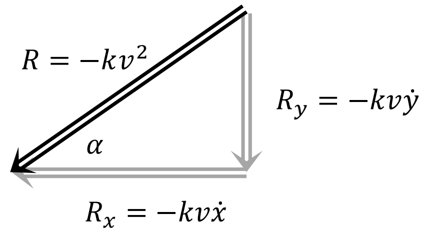
QUADRATIC DRAG

For Quadratic Drag we can only partially separate the horizontal and vertical components, so we cannot use calculus to find equations for horizontal and vertical velocity and displacement, despite what at least one textbook says!

For quadratic drag we have overall air resistance of $-kv^2$ which can be split into horizontal and vertical components $-kv\dot{x}$ and $-kv\dot{y}$ respectively. Since $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ we cannot separate the vertical and horizontal components.

Proof 5

A particle is moving with velocity v in a medium where resistance to motion is proportional to the square of velocity and in the opposite direction, so $R = -kv^2$ where k is a positive constant. If v has horizontal and vertical components $\dot{x}$ and $\dot{y}$ respectively, prove resistance has horizontal and vertical components $-kv\dot{x}$ and $-kv\dot{y}$ respectively.



Solution

Using trigonometry

Let the angle to the horizontal be α

$\dot{x} = v\cos\alpha \,, \dot{y} = v\sin\alpha$

$$\begin{aligned} R_x &= -kv^2\cos\alpha \\ &= -kv\,(v\cos\alpha) \\ &= -kv\dot{x} \end{aligned}$$

$$\begin{aligned} R_y &= -kv^2\sin\alpha \\ &= -kv\,(v\sin\alpha) \\ &= -kv\dot{y} \end{aligned}$$

Alternatively using vectors

$$\begin{aligned} R &= -kv^2 \\ &= -k \times |v|^2 \times \underset{\sim}{\hat{v}} \\ &= -k \times (v \cdot v) \times \frac{\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}}{\sqrt{|\dot{x}|^2 + |\dot{y}|^2}} \\ &= -k(\dot{x}^2 + \dot{y}^2)\frac{\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \\ &= -k\sqrt{\dot{x}^2 + \dot{y}^2}\left(\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}\right) \\ &= -k\,v\left(\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}\right) \\ &= -kv\dot{x}\underset{\sim}{i} - kv\dot{y}\underset{\sim}{j} \end{aligned}$$

$\therefore$ Quadratic drag has horizontal and vertical components $-kv\dot{x}$ and $-kv\dot{y}$ respectively

VERTICAL AND HORIZONTAL ACCELERATION

We have seen that without air resistance the equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -g$. We can adjust these to account for air resistance:

Linear Drag	Quadratic Drag
$\ddot{x} = -k\dot{x}$	$\ddot{x} = -kv\dot{x}$
$\ddot{y} = -g - k\dot{y}$	$\ddot{y} = -g - kv\dot{y}$

We can only use the formulae for quadratic drag to find the acceleration or forces at a point in the flight, as we cannot integrate them to find equations for velocity or displacement.

The formula for linear drag are differential equations, and we solve them in a similar way to our work in exponential growth and decay. There we solved equations like $\frac{dN}{dt} = k(N - A)$, so the first derivative is a function of the original function.

In projectile motion with linear drag we have the second derivative as a function of the first derivative, so to find an expression for displacement we will integrate twice. Just as in exponential growth and decay we find that the solution involves exponential functions, so the trajectory is not a parabola as we have seen in projectile motion without air resistance.

So we seen that we cannot completely separate the components of quadratic drag, so cannot use questions requiring calculus. Let's have a look at a simple question that would still be within the syllabus.

Terminal velocity still occurs when the vertical air resistance balances gravity.

Example 6

A particle of mass $10\,\mathrm{kg}$ is moving through a medium where resistance is proportional to the square of velocity. At a point in its flight $\dot{x} = 12\,\mathrm{ms}^{-1}$ and $\dot{y} = 5\,\mathrm{ms}^{-1}$. If its terminal velocity is $100\,\mathrm{ms}^{-1}$ and $g = 10$, find the horizontal and vertical acceleration of the particle at that point in its flight. Comment on the results.

Solution

As the particle falls $\dot{x} \to 0$ and so $v \to |\dot{y}|$.
The particle will approach terminal velocity and so $\ddot{y} \to 0$.

$m\ddot{y} = -mg - kv\dot{y}$

Let $\ddot{y} = 0, m = 10, g = 10, \dot{y} = -100, v_T = 100$

$\therefore 0 = -10 \times 10 - k \times 100 \times (-100)$

$\therefore k = \dfrac{100}{100^2} = 0.01$

$m\ddot{x} = -kv\dot{x}$

$\ddot{x} = -0.01 \times \sqrt{12^2 + 5^2} \times 12 \div 10 = -0.156\,\mathrm{ms}^{-2}$

$m\ddot{y} = -mg - kv\dot{y}$

$\ddot{y} = \left(-10 \times 10 - 0.01 \times \sqrt{12^2 + 5^2} \times 5\right) \div 10 = -10.065\,\mathrm{ms}^{-2}$

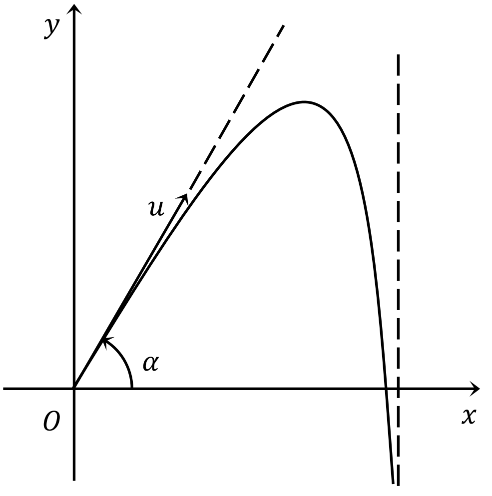
The particle has a large mass and small coefficient of drag, so its horizontal motion is only slowing slightly while its vertical motion is slowing a bit more than with just gravity.

Example 7 (2003 HSC)

A particle of mass m is thrown from the top, O, of a very tall building with an initial velocity u at an angle α to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -k\dot{x} \quad \text{and} \quad \ddot{y} = -k\dot{y} - g,$$

where k is a constant and the acceleration due to gravity is g. (You are NOT required to show these.)



i Derive the result $\dot{x} = ue^{-kt}\cos\alpha$ from the relevant equation of motion.

ii Verify that $\dot{y} = \frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)$ satisfies the appropriate equation of motion and initial condition.

iii Find the value of t when the particle reaches its maximum height.

iv What is the limiting value of the horizontal displacement of the particle?

In this past HSC question the results for horizontal and vertical velocity were given, though we could have derived them using our knowledge of differential equations. We could also integrate each of them again to find the equations for displacement, then combine them for the Cartesian equation. We will see questions like this in the exercises.

Solution

i

$$\ddot{x} = -k\dot{x}$$

$$\frac{d\dot{x}}{dt} = -k\dot{x}$$

$$\frac{dt}{d\dot{x}} = -\frac{1}{k} \times \frac{1}{\dot{x}}$$

$$t = -\frac{1}{k}\int_{u\cos\alpha}^{\dot{x}} \frac{d\dot{x}}{\dot{x}}$$

$$-kt = \Big[\ln(\dot{x})\Big]_{u\cos\alpha}^{\dot{x}}$$

$$= \ln(\dot{x}) - \ln(u\cos\alpha)$$

$$e^{-kt} = \frac{\dot{x}}{u\cos\alpha}$$

$$\dot{x} = ue^{-kt}\cos\alpha$$

ii

$$\frac{d}{dt}\left(\frac{1}{k}\Big((ku\sin\alpha + g)e^{-kt} - g\Big)\right)$$

$$= -(ku\sin\alpha + g)e^{-kt}$$

$$= -k\left(\frac{1}{k}\Big((ku\sin\alpha + g)e^{-kt} - g\Big)\right) - g$$

$$= -k\dot{y} - g$$

Let $t = 0$

$$\dot{y} = \frac{1}{k}\Big((ku\sin\alpha + g)\Big) - g$$

$$= u\sin\alpha + g - g$$

$$= u\sin\alpha$$

$\therefore\ \dot{y} = \frac{1}{k}\Big((ku\sin\alpha + g)e^{-kt} - g\Big)$ satisfies

$\ddot{y} = -k\dot{y} - g$ and the initial conditions

iii

At maximum height $\dot{y} = 0$

$$\frac{1}{k}\Big((ku\sin\alpha + g)e^{-kt} - g\Big) = 0$$

$$(ku\sin\alpha + g)e^{-kt} = g$$

$$e^{kt} = \frac{ku\sin\alpha + g}{g}$$

$$kt = \ln\left(\frac{ku\sin\alpha + g}{g}\right)$$

$$t = \frac{1}{k}\ln\left(\frac{ku\sin\alpha + g}{g}\right)$$

iv

$$x = \int_0^\infty \dot{x}\,dt$$

$$= u\cos\alpha\int_0^\infty e^{-kt}\,dt$$

$$= -\frac{u\cos\alpha}{k}\int_0^\infty \left(-ke^{-kt}\right)dt$$

$$= -\frac{u\cos\alpha}{k}\Big[e^{-kt}\Big]_0^\infty$$

$$= -\frac{u\cos\alpha}{k}(0 - 1)$$

$$= \frac{u\cos\alpha}{k}$$

The past HSC questions and the Sample question both use linear drag, and start with equations of motion for the horizontal and vertical acceleration and stop at finding expressions for horizontal and vertical velocity. This might give us an indication of how far we can expect you to go in exams.

Without too much extra effort we can create the equations of motion, including variables for mass, and go through to find the Cartesian equation, all assuming that drag is linear. You can attempt this in the second last question in the exercise for this lesson.

BASIC

1 A particle is projected upwards from ground level with initial velocity $\frac{1}{2}\sqrt{\frac{g}{k}}$ ms^{-1}, where g is the acceleration due to gravity and k is a positive constant. The particle moves through the air with speed v ms^{-1} and experiences a resistive force. The acceleration of the particle is given by $\ddot{x} = -g - kv^2$. Do NOT prove this. The particle reaches a maximum height, H, before returning to the ground.

Using $\ddot{x} = v\frac{dv}{dx}$, or otherwise, show that $H = \dfrac{1}{2k}\log_e\left(\dfrac{5}{4}\right)$ metres.

2 A particle is projected with velocity 60 ms^{-1} at an angle of $30°$ to the horizontal in a resistive medium. It reached a maximum height of 17.9 m and lands 75.7 m away from its projection point. Which of the following statements cannot be true?

A The maximum height occurs when the particle has travelled 47 m horizontally
B The velocity at impact is 60.0 ms^{-1}.
C The angle to the horizontal at impact is $58°$

3 A particle of unit mass is projected in a medium where air resistance is proportional to velocity, at 50 ms^{-1} at an angle of θ to the horizontal where $\tan\theta = \frac{3}{4}$. The vertical equation of motion of is $\dot{y} = 230e^{-\frac{t}{20}} - 200$, where $\dot{y}$ is in metres per second.

Find the time taken to reach maximum height, to 1 decimal place.

CHALLENGING

4 A ball of mass m is projected vertically into the air from the ground with initial velocity u. After reaching the maximum height H it falls back to the ground. While in the air, the ball experiences a resistive force kv^2, where v is the velocity of the ball and k is a constant. The equation of motion when the ball falls can be written as $m\dot{v} = mg - kv^2$ (Do NOT prove this.)

i Show that the terminal velocity v_T of the ball when it falls is $\sqrt{\dfrac{mg}{k}}$

ii Show that when the ball goes up, the maximum height H is $H = \dfrac{v_T^2}{2g}\ln\left(1 + \dfrac{u^2}{v_T^2}\right)$

iii When the ball falls from height H it hits the ground with velocity w.

Show that $\dfrac{1}{w^2} = \dfrac{1}{u^2} + \dfrac{1}{v_T^2}$

5 A particle of mass 1 kg is projected vertically upwards from the ground with a speed of 20 m/s. The particle is under the effect of both gravity (g) and an air resistance of magnitude $\frac{1}{40}v^2$ where v is the current velocity of the particle at any time. The effects of gravity and air resistance remain the same no matter which direction the particle is going.

i Explain why the acceleration of the particle at any time whilst traveling upwards is given by $\ddot{x} = -g - \frac{1}{40}v^2$

(For the remainder of this question you may use $g = 10$ ms^{-2})

ii Calculate the greatest height reached by the particle

iii Write an expression for the acceleration of the particle as it returns to earth.

iv Find the speed of the particle just before it strikes the ground.

6 A rubber ball of mass 7 kg, falls from rest, from the top of a building. While falling the ball experiences a resistive force $\frac{7v^2}{10}$, where v is the velocity of the ball. Take g, acceleration due to gravity, as $g = 10$ ms^{-2}.

i Show that $\ddot{x} = 10 - \frac{v^2}{10}$, where x is the distance the ball has fallen.

ii Find the terminal velocity of the ball as it falls.

iii Show that $v^2 = 100\left(1 - e^{-\frac{x}{5}}\right)$

iv After hitting the ground the ball rises vertically such that $\ddot{X} = -10 - \frac{V^2}{10}$, where V is the velocity of the ball as it rises and X is the distance the ball rises. Find the time that it takes for the ball to rise to its maximum height if initially $V = \frac{10}{\sqrt{3}}$ ms^{-1}.

7 A particle is projected from the origin with an initial velocity $60\,\mathrm{ms}^{-1}$ at $30°$ to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -\frac{\dot{x}}{10} \quad \text{and} \quad \ddot{y} = -\frac{\dot{y}}{10} - 10,$$

(You are NOT required to show these.)

i Find an expression for horizontal displacement as a function of time.

ii Find an expression for vertical displacement as a function of time.

iii Find the Cartesian equation of the trajectory of the particle

iv Find the value of t when the particle reaches its maximum height. to 1 decimal place.

8 A body of unit mass is projected vertically upwards in a medium that has a constant gravitational force g and a resistance $\frac{v}{10}$, where v is the velocity of the projectile at a given time t. The initial velocity is $10(20-g)$.

i Show that the equation of motion of the projectile is $\dfrac{dv}{dt}=-g-\dfrac{v}{10}$

ii Show that the time T for the particle to reach its greatest height is given by $T=10\ln\left(\dfrac{20}{g}\right)$

iii Show that the maximum height H is given by $H=2000-10g[10+T]$

iv If the particle then falls from this height, find the terminal velocity in this medium.

9 A particle of mass m is projected from the origin with an initial velocity $V\ \mathrm{ms}^{-1}$ at an angle of θ to the horizontal. The particle experiences the effect of gravity and a resistance proportional to its velocity in both the horizontal and vertical directions.

Prove the following results, where k is the coefficient of drag and g is gravitational acceleration.

i $\dot{x}=V\cos\theta\, e^{-\frac{k}{m}t}$

ii $x=\dfrac{mV\cos\theta}{k}\left(1-e^{-\frac{k}{m}t}\right)$

iii $\dot{y}=\left(\dfrac{mg}{k}+V\sin\theta\right)e^{-\frac{k}{m}t}-\dfrac{mg}{k}$

iv $y=\dfrac{m}{k}\left(\dfrac{mg}{k}+V\sin\theta\right)\left(1-e^{-\frac{k}{m}t}\right)-\dfrac{mgt}{k}$

v $y=\left(\dfrac{mg}{kV\cos\theta}+\tan\theta\right)x+\dfrac{m^2g}{k^2}\ln\left(1-\dfrac{kx}{mV\cos\theta}\right)$

10 A particle is moving in a medium where resistance to motion is proportional to the square of velocity, so $R=-kv^2$. At some point in its flight $\dot{x}=7\ \mathrm{ms}^{-1}$ and $\dot{y}=24\ \mathrm{ms}^{-1}$.

a Use similar triangles to find the horizontal and vertical components of resistance at the point, and prove that the total resistance and its components satisfy Pythagoras' Theorem.

b Show that the horizontal and vertical components at the point can be found using $R_x=-kv\dot{x}$ and $R_y=-kv\dot{y}$

c Show that $R_x=-k\dot{x}^2$ and $R_y=-k\dot{y}^2$ do not match the horizontal and vertical components at the point.

d Prove that $R=-kv^2$ cannot be split into horizontal and vertical components of $R_x=-k\dot{x}^2$ and $R_y=-k\dot{y}^2$ if the particle is moving at angle to the horizontal (ie unless $\dot{x}=0$ and/or $\dot{y}=0$ so the particle is moving vertically or horizontally, or is stationary).

1

$$v\frac{dv}{dx} = -(g + kv^2)$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$x = -\frac{1}{2k}\int_{\frac{1}{2}\sqrt{\frac{g}{k}}}^{0} \frac{2kv}{g + kv^2}\,dv$$

$$= \frac{1}{2k}\Big[\ln(g + kv^2)\Big]_0^{\frac{1}{2}\sqrt{\frac{g}{k}}}$$

$$= \frac{1}{2k}\left(\ln\left(g + k\left(\frac{g}{4k}\right)\right) - \ln g\right)$$

$$= \frac{1}{2k}\ln\left(1 + \frac{1}{4}\right)$$

$$= \frac{1}{2k}\ln\frac{5}{4} \text{ metres}$$

2

A The maximum height occurs more than half way along the range, so is possible

B The impact velocity is equal to the projection velocity, which is impossible.

C The impact angle is greater than the projection angle, so is possible.

ANSWER (B)

3

Let $\dot{y} = 0$

$$\therefore 230e^{-\frac{t}{20}} - 200 = 0$$

$$230e^{-\frac{t}{20}} = 200$$

$$e^{-\frac{t}{20}} = \frac{200}{230}$$

$$e^{\frac{t}{20}} = \frac{230}{200}$$

$$\frac{t}{20} = \ln\frac{230}{200}$$

$$t = 20\ln\frac{230}{200}$$

$= 2.8$ seconds (1 dp)

4

i At terminal velocity $\dot{v} = 0$

$$\therefore mg - kv_T^2 = 0$$

$$v_T = \sqrt{\frac{mg}{k}}$$

ii

$$m\dot{v} = -(mg + kv^2)$$

$$\therefore v\frac{dv}{dx} = -\frac{(mg + kv^2)}{m}$$

$$\frac{dv}{dx} = -\frac{mg + kv^2}{mv}$$

$$\frac{dx}{dv} = -\frac{mv}{mg + kv^2}$$

$$\therefore H = -\int_u^0 \frac{mv}{mg + kv^2}\,dv$$

$$= -\frac{m}{2k}\Big[\ln(mg + kv^2)\Big]_u^0$$

$$= -\frac{m}{2k}\left(\ln mg - \ln(mg + ku^2)\right)$$

$$= \frac{m}{2k}\ln\frac{mg + ku^2}{mg}$$

$$= \frac{m}{2\left(\frac{mg}{V_T^2}\right)}\ln\frac{mg + \left(\frac{mg}{V_T^2}\right)u^2}{mg}$$

$$= \frac{V_T^2}{2g}\ln\left(1 + \frac{u^2}{V_T^2}\right)$$

iii

$$mv\frac{dv}{dx} = mg - \frac{mg}{V_T^2}v^2$$

$$\frac{dv}{dx} = \frac{g(V_T^2 - v^2)}{v.V_T^2}$$

$$\frac{dx}{dv} = \frac{V_T^2}{g} \times \frac{v}{V_T^2 - v^2}$$

$$\therefore H = \frac{V_T^2}{g}\int_0^w \frac{v}{V_T^2 - v^2}\,dv$$

$$= -\frac{V_T^2}{2g}\Big[\ln(V_T^2 - v^2)\Big]_0^w$$

$$= -\frac{V_T^2}{2g}\left(\ln(V_T^2 - w^2) - \ln(V_T^2)\right)$$

$$= \frac{V_T^2}{2g}\ln\left(\frac{V_T^2}{V_T^2 - w^2}\right)$$

$$\therefore \frac{v_T^2}{2g}\ln\left(1 + \frac{u^2}{v_T^2}\right) = \frac{v_T^2}{2g}\ln\left(\frac{v_T^2}{v_T^2 - w^2}\right)$$

$$\therefore 1 + \frac{u^2}{v_T^2} = \frac{v_T^2}{v_T^2 - w^2}$$

$$\frac{v_T^2 + u^2}{v_T^2} = \frac{v_T^2}{v_T^2 - w^2}$$

$$v_T^4 - w^2v_T^2 + u^2v_T^2 - u^2w^2 = v_T^4$$

$$u^2v_T^2 = w^2v_T^2 + w^2u^2$$

$$\frac{1}{w^2} = \frac{1}{u^2} + \frac{1}{v_T^2}$$

$(\div$ both sides by $u^2V_T^2w^2)$

5 **i**

$$m\ddot{x} = -mg - R$$
$$1 \times \ddot{x} = -1 \times g - \frac{1}{40}v^2$$
$$\ddot{x} = -g - \frac{1}{40}v^2$$

ii

$$v\frac{dv}{dx} = -\left(g + \frac{v^2}{40}\right)$$
$$\frac{dv}{dx} = -\frac{40g + v^2}{40v}$$
$$\frac{dx}{dv} = -\frac{40v}{40g + v^2}$$
$$x = -\int_{20}^{0} \frac{40v}{40g + v^2}\,dv$$
$$= 20\left[\ln(40g + v^2)\right]_0^{20}$$
$$= 20(\ln 800 - \ln 400)$$
$$= 20\ln 2$$

iii

$$\ddot{x} = g - \frac{1}{40}v^2$$

iv

Let w be the velocity just before impact.

$$v\frac{dv}{dx} = g - \frac{1}{40}v^2$$
$$\frac{dv}{dx} = \frac{40g - v^2}{40v}$$
$$\frac{dx}{dv} = \frac{40v}{40g - v^2}$$
$$x = \int_0^w \frac{40v}{40g - v^2}\,dv$$
$$= 20\left[\ln(40g - v^2)\right]_w^0$$
$$= 20\left(\ln(40g) - \ln(40g - w^2)\right)$$
$$= 20\ln\left(\frac{400}{400 - w^2}\right)$$
$$\therefore 20\ln 2 = 20\ln\left(\frac{400}{400 - w^2}\right)$$
$$\frac{400}{400 - w^2} = 2$$
$$400 = 800 - 2w^2$$
$$2w^2 = 400$$
$$w = \sqrt{200}$$
$$= 10\sqrt{2}\text{ ms}^{-1}$$

6 **i**

$$m\ddot{x} = mg - R$$
$$7\ddot{x} = 7 \times 10 - \frac{7v^2}{10}$$
$$\ddot{x} = 10 - \frac{v^2}{10}$$

ii

$$\ddot{x} = 0$$
$$0 = 10 - \frac{v_T^2}{10}$$
$$V_T^2 = 100$$
$$V_T = 10\text{ ms}^{-1}$$

iii

$$v\frac{dv}{dx} = 10 - \frac{v^2}{10}$$
$$\frac{dv}{dx} = \frac{100 - v^2}{10v}$$
$$\frac{dx}{dv} = \frac{10v}{100 - v^2}$$
$$x = \int_0^v \frac{10v}{100 - v^2}\,dv$$
$$= -5\left[\ln(100 - v^2)\right]_0^v$$
$$= 5\ln\left(\frac{100}{100 - v^2}\right)$$
$$e^{\frac{x}{5}} = \frac{100}{100 - v^2}$$
$$100e^{\frac{x}{5}} - e^{\frac{x}{5}}v^2 = 100$$
$$e^{\frac{x}{5}}v^2 = 100\left(e^{\frac{x}{5}} - 1\right)$$
$$v^2 = 100\left(1 - e^{-\frac{x}{5}}\right)$$

iv

$$\frac{dV}{dt} = -\left(10 + \frac{V^2}{10}\right)$$
$$\frac{dt}{dV} = -\frac{10}{100 + V^2}$$
$$t = -10\int_{\frac{10}{\sqrt{3}}}^{0} \frac{1}{100 + V^2}\,dV$$
$$= 10 \times \frac{1}{10}\left[\tan^{-1}\frac{V}{10}\right]_0^{\frac{10}{\sqrt{3}}}$$
$$= \tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0$$
$$= \frac{\pi}{6}$$

7 **i**

$$\frac{d\dot{x}}{dt} = -\frac{\dot{x}}{10}$$
$$\frac{dt}{d\dot{x}} = -\frac{10}{d\dot{x}}$$
$$t = -10\int_{60\cos 30°}^{\dot{x}} \frac{1}{\dot{x}}\,d\dot{x}$$
$$-\frac{t}{10} = \Big[\ln \dot{x}\Big]_{30\sqrt{3}}^{\dot{x}}$$
$$\ln|\dot{x}| - \ln\left(30\sqrt{3}\right) = -\frac{t}{10}$$
$$\ln|\dot{x}| = \ln(30\sqrt{3}) - \frac{t}{10}$$
$$\dot{x} = 30\sqrt{3}e^{-\frac{t}{10}}$$
$$x = 30\sqrt{3}\int_0^t e^{-\frac{t}{10}}\,dt$$
$$= -300\sqrt{3}\Big[e^{-\frac{t}{10}}\Big]_0^t$$
$$= -300\sqrt{3}\left(e^{-\frac{t}{10}} - 1\right)$$
$$= 300\sqrt{3}\left(1 - e^{-\frac{t}{10}}\right)$$

ii

$$\frac{d\dot{y}}{dt} = -\frac{\dot{y}}{10} - 10$$
$$= -\frac{\dot{y} + 100}{10}$$
$$\frac{dt}{d\dot{y}} = -\frac{10}{\dot{y} + 100}$$
$$t = -10\int_{60\sin 30°}^{\dot{y}} \frac{1}{\dot{y} + 100}\,d\dot{y}$$
$$-\frac{t}{10} = \Big[\ln(\dot{y} + 100)\Big]_{30}^{\dot{y}}$$
$$\ln|\dot{y} + 100| - \ln(130) = -\frac{t}{10}$$
$$\ln|\dot{y} + 100| = \ln 130 - \frac{t}{10}$$
$$\dot{y} + 100 = 130e^{-\frac{t}{10}}$$
$$\dot{y} = 130e^{-\frac{t}{10}} - 100$$
$$y = \int_0^t \left(130e^{-\frac{t}{10}} - 100\right)dt$$
$$= \Big[-1300e^{-\frac{t}{10}} - 100t\Big]_0^t$$
$$= -1300e^{-\frac{t}{10}} - 100t + 1300$$
$$= 1300 - 1300e^{-\frac{t}{10}} - 100t$$

7.. **iii**

$$\frac{x}{300\sqrt{3}} = 1 - e^{-\frac{t}{10}} \qquad (1)$$

$$e^{-\frac{t}{10}} = 1 - \frac{x}{300\sqrt{3}}$$

$$= \frac{300\sqrt{3} - x}{300\sqrt{3}}$$

$$e^{\frac{t}{10}} = \frac{300\sqrt{3}}{300\sqrt{3} - x}$$

$$t = 10\ln\left(\frac{300\sqrt{3}}{300\sqrt{3} - x}\right) \qquad (2)$$

from (1) and (2):

$$y = 1300\left(1 - e^{-\frac{t}{10}}\right) - 100t$$

$$= 1300\left(\frac{x}{300\sqrt{3}}\right)$$

$$= \frac{13x}{3\sqrt{3}} + 1000\ln\left(\frac{300\sqrt{3} - x}{300\sqrt{3}}\right)$$

iv

$$\text{Let } \dot{y} = 0$$

$$130e^{-\frac{t}{10}} - 100 = 0$$

$$e^{-\frac{t}{10}} = \frac{100}{130}$$

$$e^{\frac{t}{10}} = \frac{130}{100}$$

$$t = 10\ln 1.3$$

$$= 2.6 \text{ sec}$$

8 **i**

$$m\ddot{x} = -mg - \frac{mv}{10}$$

$$\therefore \frac{dv}{dt} = -g - \frac{v}{10}$$

ii

$$\frac{dt}{dv} = -\frac{10}{10g + v}$$

$$t = -\int_{10(20-g)}^{v} \frac{10}{10g + v}\,dv$$

$$= 10\Big[\ln(10g + v)\Big]_{v}^{10(20-g)}$$

$$= 10\Big(\ln(10g + 200 - 10g) - \ln(10g + v)\Big)$$

$$= 10\ln\left(\frac{200}{10g + v}\right)$$

when $t = T, v = 0$

$$\therefore T = 10\ln\left(\frac{200}{10g}\right)$$

$$= 10\ln\left(\frac{20}{g}\right)$$

iii

$$v\frac{dv}{dx} = -\frac{10g + v}{10}$$

$$\frac{dv}{dx} = -\frac{10g + v}{10v}$$

$$\frac{dx}{dv} = -\frac{10v}{10g + v}$$

$$H = -10\int_{10(20-g)}^{0} \frac{v}{10g + v}\,dv$$

$$= 10\int_{0}^{10(20-g)} \frac{10g + v - 10g}{10g + v}\,dv$$

$$= 10\int_{0}^{10(20-g)} \left(1 - \frac{10g}{10g + v}\right)dv$$

$$= 10\Big[v - 10g\ln(10g + v)\Big]_{0}^{10(20-g)}$$

$$= 10\big((200 - 10g - 10g\ln(200))$$
$$- (0 - 10g\ln(10g))\big)$$

$$= 10\left(200 - 10g\left(1 + \ln\left(\frac{20}{g}\right)\right)\right)$$

$$= 2000 - 10g\left[10 + 10\ln\left(\frac{20}{g}\right)\right]$$

$$= 2000 - 10g[10 + T]$$

iv

$$\ddot{x} = g - \frac{v}{10}$$

$$0 = g - \frac{V_T}{10}$$

$$V_T = 10g$$

9 **i**

$$m\ddot{x} = -k\dot{x}$$
$$\frac{d\dot{x}}{dt} = -\frac{k}{m}\dot{x}$$
$$\frac{dt}{d\dot{x}} = -\frac{m}{k} \times \frac{1}{\dot{x}}$$
$$t = -\frac{m}{k}\int_{V\cos\theta}^{\dot{x}} \frac{d\dot{x}}{\dot{x}}$$
$$-\frac{k}{m}t = \Big[\ln\dot{x}\Big]_{V\cos\theta}^{\dot{x}}$$
$$= \ln\dot{x} - \ln(V\cos\theta)$$
$$\ln\dot{x} = \ln(V\cos\theta) - \frac{k}{m}t$$
$$\dot{x} = V\cos\theta\, e^{-\frac{k}{m}t}$$

ii

$$\frac{dx}{dt} = V\cos\theta\, e^{-\frac{k}{m}t}$$
$$x = V\cos\theta \int_0^t e^{-\frac{k}{m}t}\,dt$$
$$= V\cos\theta \times \left(-\frac{m}{k}\right)\left[e^{-\frac{k}{m}t}\right]_0^t$$
$$= \frac{mV\cos\theta}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

iii

$$m\ddot{y} = -k\dot{y} - mg$$
$$\frac{d\dot{y}}{dt} = -\frac{k}{m}\dot{y} - g$$
$$= -\frac{k\dot{y} + mg}{m}$$
$$\frac{dt}{d\dot{y}} = -\frac{m}{k\dot{y} + mg}$$
$$t = -\frac{m}{k}\int_{V\sin\theta}^{\dot{y}} \frac{k}{k\dot{y} + mg}\,d\dot{y}$$
$$= -\frac{m}{k}\Big[\ln(k\dot{y} + mg)\Big]_{V\sin\theta}^{\dot{y}}$$
$$= -\frac{m}{k}\left(\ln(k\dot{y} + mg) - \ln(kV\sin\theta + mg)\right)$$
$$-\frac{k}{m}t = \ln(k\dot{y} + mg) - \ln(kV\sin\theta + mg)$$
$$\ln(k\dot{y} + mg) = \ln(kV\sin\theta + mg) - \frac{k}{m}t$$
$$k\dot{y} + mg = (kV\sin\theta + mg)e^{-\frac{k}{m}t}$$
$$\dot{y} = \left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$$

iv

$$\frac{dy}{dt} = \left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$$
$$y = \int_0^t \left(\left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}\right)dt$$
$$= \left[-\frac{m}{k}\left(\left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t}\right) - \frac{mgt}{k}\right]_0^t$$
$$= -\frac{m}{k}\left(\left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t}\right) - \frac{mgt}{k} + \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right)$$
$$= \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right)\left(1 - e^{-\frac{k}{m}t}\right) - \frac{mgt}{k}$$

9.. **v**

$$x = \frac{mV\cos\theta}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$\frac{kx}{mV\cos\theta} = 1 - e^{-\frac{k}{m}t} \qquad (1)$$

$$e^{-\frac{k}{m}t} = 1 - \frac{kx}{mV\cos\theta}$$

$$-\frac{k}{m}t = \ln\left(1 - \frac{kx}{mV\cos\theta}\right)$$

$$t = -\frac{m}{k}\ln\left(1 - \frac{kx}{mV\cos\theta}\right) \qquad (2)$$

sub (1), (2) in (iv):

$$y = \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right)\left(\frac{kx}{mV\cos\theta}\right) + \frac{mg}{k}\left(\frac{m}{k}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)\right)$$

$$= \frac{m}{k}\left(\frac{mg + kV\sin\theta}{k}\right)\left(\frac{kx}{mV\cos\theta}\right) + \frac{m^2 g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)$$

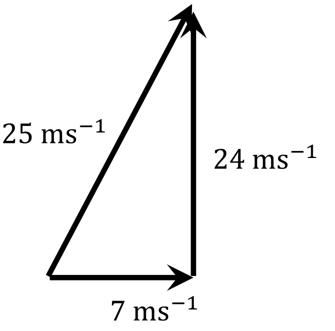
$$= \left(\frac{mg + kV\sin\theta}{kV\cos\theta}\right)x + \frac{m^2 g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)$$

$$= \left(\frac{mg}{kV\cos\theta} + \tan\theta\right)x + \frac{m^2 g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)$$

10 **a**

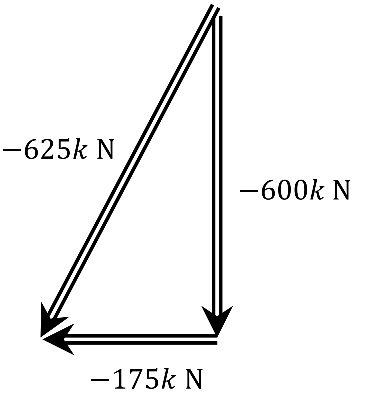
$v = \sqrt{7^2 + 24^2} = 25 \text{ ms}^{-1}$

$\therefore R = -k \times 25^2 = -625k \text{ N}$



Now the triangle representing velocity and its components (above) and resistance and its components (below) must be similar.

The scale is $1: -25k$ so the horizontal component of resistance is $-175k$ N and the vertical component is $-600k$ N.



Using the magnitudes of the resistance and its components

$$175^2 + 600^2 = 390625$$
$$25^2 = 390625$$

$\therefore$ the resistance and its components satisfy Pythagoras' Theorem.

b

$R_x = -kv\dot{x} = -k \times 25 \times 7 = -175k$

$R_y = -kv\dot{y} = -k \times 25 \times 24 = -600k$

$\therefore$ the horizontal and vertical components at the point can be found using $R_x = -kv\dot{x}$ and $R_y = -kv\dot{y}$

c

$R_x = -k\dot{x}^2 = -k \times 7^2 = -49k \neq -175k$

$R_y = -k\dot{y}^2 = -k \times 24^2 = -576k \neq -600k$

$\therefore$ the horizontal and vertical components at the point cannot be found using $R_x = -k\dot{x}^2$ and $R_y = -k\dot{y}^2$

d

If $-kv^2$ can be split into horizontal and vertical components $R_x = -k\dot{x}^2$ and $R_y = -k\dot{y}^2$ then their magnitudes must satisfy Pythagoras' Theorem

$$\therefore (kv^2)^2 = (k\dot{x}^2)^2 + (k\dot{y}^2)^2$$
$$k^2(\dot{x}^2 + \dot{y}^2)^2 = k^2(\dot{x}^4 + \dot{y}^4)$$
$$\dot{x}^4 + 2\dot{x}^2\dot{y}^2 + \dot{y}^4 = \dot{x}^4 + \dot{y}^4$$
$$\therefore 2\dot{x}^2\dot{y}^2 = 0$$
$$\therefore \dot{x} = 0 \text{ and/or } \dot{y} = 0$$

$\therefore$ the resistance of $R = -kv^2$ cannot only be split into horizontal and vertical components $R_x = -k\dot{x}^2$ and $R_y = -k\dot{y}^2$ if the particle is moving vertically or horizontally, or is stationary